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Parameterfree Comprehension Does Not Imply Full Comprehension in Second Order Peano Arithmetic

Abstract. The parameter-free part $\mathbf{PA}_2^*$ of $\mathbf{PA}_2$, second order Peano arithmetic, is considered. We make use of a product/iterated Sacks forcing to define an ω-model of $\mathbf{PA}_2^* + \mathbf{CA}(\Sigma_2^1)$, in which an example of the full Comprehension schema $\mathbf{CA}$ fails. Using Cohen's forcing, we also define an ω-model of $\mathbf{PA}_2^*$, in which not every set has its complement, and hence the full $\mathbf{CA}$ fails in a rather elementary way.

Keywords: Peano arithmetic, Comprehension, Parameters, Iterated forcing, Sacks forcing.

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1. Introduction

Discussing the structure and deductive properties of second order Peano arithmetic $\mathbf{PA}_2$, Kreisel [18, Section III, page 366] wrote that the selection of subsystems "is a central problem". In particular, Kreisel noted that

> [...] if one is convinced of the significance of something like a given axiom schema, it is natural to study details, such as the effect of parameters.

Recall that *parameters* in this context are free variables in various axiom schemata in $\mathbf{PA}$, $\mathbf{ZFC}$, and other similar theories. Thus the most obvious way to study "the effect of parameters" is to compare the strength of a given axiom schema S with its parameter-free subschema S^*. (The asterisk will mean the parameter-free subschema in this paper.)

Some work in this direction was done in the early years of modern set theory. In particular Guzicki [10] proved that the Levy-style generic collapse

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(see, *e.g.*, Levy [20] and Solovay [27]) of all cardinals $\omega_\alpha^{\mathbf{L}}$, $\alpha < \omega_1^{\mathbf{L}}$, results in a generic extension of $\mathbf{L}$ in which the (countable) choice schema $\mathbf{AC}_\omega$, in the language of $\mathbf{PA}_2$, fails but its parameter-free subschema $\mathbf{AC}_\omega^*$ holds, so that $\mathbf{AC}_\omega^*$ is strictly weaker than $\mathbf{AC}_\omega$. This can be compared with an opposite result for the *dependent choice* schema $\mathbf{DC}$, in the language of $\mathbf{PA}_2$, which is equivalent to its parameter-free subschema $\mathbf{DC}^*$ by a simple argument given in [10]. (See Section 2 on $\mathbf{AC}_\omega$ and $\mathbf{DC}$.)

Some results related to parameter-free versions of the Separation and Replacement axiom schemata in $\mathbf{ZFC}$ also are known from [3, 21, 23].

This paper is devoted to the role of parameters in the *comprehension schema* $\mathbf{CA}$ of $\mathbf{PA}_2$. Let $\mathbf{PA}_2^*$ be the subtheory of $\mathbf{PA}_2$ in which the full schema $\mathbf{CA}$ is replaced by its parameter-free version $\mathbf{CA}^*$, and the Induction principle is formulated as a schema rather than one sentence. The following Theorems 1 and 2 are our main results.

THEOREM 1. *Suppose that $\langle x_i \rangle_{i<\omega}$ is a Cohen-generic sequence over $\mathbf{L}$, the constructible universe. Let $X = (\mathscr{P}(\omega) \cap \mathbf{L}) \cup \{x_i : i < \omega\}$. Then $\langle \omega\,; X \rangle$ is a model of $\mathbf{PA}_2^*$, but not a model of $\mathbf{CA}$ as X does not contain the complements $\omega \smallsetminus x_i$.*

Thus $\mathbf{CA}$, even in the particular form claiming that every set has its complement, is not provable in $\mathbf{PA}_2^*$.

It is quite obvious that a subtheory like $\mathbf{PA}_2^*$, that does not allow such a fundamental thing as the complement formation, is unacceptable. This is why we adjoin $\mathbf{CA}(\mathbf{\Sigma}_2^1)$, *i.e.*, the full $\mathbf{CA}$ restricted to $\mathbf{\Sigma}_2^1$ formulas with parameters, in the next theorem, to obtain a more plausible subsystem.

THEOREM 2. *There is a generic extension $\mathbf{L}[G]$ of $\mathbf{L}$ and a set $X \in \mathbf{L}[G]$, such that $\mathscr{P}(\omega) \cap \mathbf{L} \subsetneq X \subsetneq \mathscr{P}(\omega)$ and $\langle \omega\,; X \rangle$ is a model of $\mathbf{PA}_2^* + \mathbf{CA}(\mathbf{\Sigma}_2^1)$ but not a model of $\mathbf{PA}_2$. Therefore $\mathbf{CA}$ is not provable even in $\mathbf{PA}_2^* + \mathbf{CA}(\mathbf{\Sigma}_2^1)$.*

Theorem 2 will be established by means of a complex product/iteration of the Sacks forcing and the associated coding by degrees of constructibility, approximately as discussed in [22, page 143], around Theorem T3106.

Identifying the theories with their deductive closures, we may present the concluding statements of Theorems 1 and 2 as resp.

$$\mathbf{PA}_2^* \subsetneqq \mathbf{PA}_2 \quad \text{and} \quad (\mathbf{PA}_2^* + \mathbf{CA}(\mathbf{\Sigma}_2^1)) \subsetneqq \mathbf{PA}_2.$$

Studies on subsystems of $\mathbf{PA}_2$ have discovered many cases in which $S \subsetneqq S'$ holds for a given pair of subsystems S, S', see *e.g.* [26]. And it is a rather typical case that such a strict extension is established by demonstrating that

S' proves the consistency of S. One may ask whether this is the case for the results in the displayed line above. The answer is in the negative: namely *the theories* $\mathbf{PA}_2^*$, $\mathbf{PA}_2^* + \mathbf{CA}(\mathbf{\Sigma}_2^1)$, *and the* full $\mathbf{PA}_2$, *are equiconsistent*, by a result in [4], also mentioned in [24]. This equiconsistency result also follows from a somewhat sharper theorem in [25, 1.5].[1]

Generally topics in subsystems of second order arithmetic remain of big interest in modern studies, see *e.g.* [7], and our paper contributes to this research line.

2. Preliminaries

Following [1,18,26] we define second order Peano arithmetic $\mathbf{PA}_2$ as a theory in the language $\mathcal{L}(\mathbf{PA}_2)$ with two sorts of variables – for natural numbers and for sets of them. We use j, k, m, n for variables over ω and x, y, z for variables over $\mathscr{P}(\omega)$, reserving capital letters for subsets of $\mathscr{P}(\omega)$ and other sets. The axioms are as follows:

(1) Peano's axioms for numbers.

(2) The Induction schema $\Phi(0) \wedge \forall\, k\,(\Phi(k) \Longrightarrow \Phi(k+1)) \implies \forall\, k\, \Phi(k)$, for every formula $\Phi(k)$ in $\mathcal{L}(\mathbf{PA}_2)$ where we allow parameters (free variables other than k).[2]

(3) Extensionality for sets.

(4) The Comprehension schema $\mathbf{CA}$: $\exists\, x\, \forall\, k\,(k \in x \iff \Phi(k))$, for every formula Φ in which the variable x does not occur, and in Φ we allow parameters.

We let $\mathbf{CA}(\mathbf{\Sigma}_2^1)$ be the full $\mathbf{CA}$ restricted to $\mathbf{\Sigma}_2^1$ formulas Φ with parameters.[3]

We let $\mathbf{CA}^*$ be the parameter-free sub-schema of $\mathbf{CA}$ (that is, $\Phi(k)$ contains no free variables other than k).

We let $\mathbf{PA}_2^*$ be the subsystem of $\mathbf{PA}_2$ with $\mathbf{CA}$ replaced by $\mathbf{CA}^*$.

Remark 3. In spite of Theorem 1, $\mathbf{PA}_2^*$ proves $\mathbf{CA}$ with parameters over ω (but not over $\mathscr{P}(\omega)$) allowed. Indeed suppose that Φ is $\Phi(k, m)$ in (4)

[1] We are thankful to Ali Enayat for the references to [4,24,25] in matters of this equiconsistency result.

[2] We cannot use Induction as one sentence here because the Comprehension schema $\mathbf{CA}$ is not assumed in full generality in the context of Theorem 1.

[3] A $\mathbf{\Sigma}_2^1$ formula is any $\mathcal{L}(\mathbf{PA}_2)$ formula of the form $\exists\, x\, \forall\, y\, \Psi$, where Ψ is *arithmetic* *i.e.*, does not contain quantified variables over $\mathscr{P}(\omega)$.

and Φ has no other free variables. Arguing in $\mathbf{PA}_2$, assume towards the contrary that the formula $\psi(m) := \exists\, x\, \forall\, k\, (k \in x \iff \Phi(k,m))$ holds not for all m. By Induction, take the least m for which $\psi(m)$ *fails*. This m is definable, and therefore it can be eliminated, and hence we have $\psi(m)$ for this m by $\mathbf{CA}^*$. This is a contradiction. ■

The following schemata are not assumed to be parts of $\mathbf{PA}_2$, yet they are often considered in the context of and in the connection with $\mathbf{PA}_2$.

> **The Schema of Choice $\mathbf{AC}_\omega$:** $\forall\, k\, \exists\, x\, \Phi(k,x) \implies \exists\, x\, \forall\, k\, \Phi(k,(x)_k))$, for every formula Φ in which we allow parameters, where as usual $(x)_k = \{j : 2^k(2j+1) - 1 \in x\}$.

We use $\mathbf{AC}_\omega$ instead of $\mathbf{AC}$, more common in $\mathbf{PA}_2$ studies, because $\mathbf{AC}$ is the general axiom of choice in the $\mathbf{ZF}$ context.

> **Dependent Choices DC**: $\forall\, x\, \exists\, y\, \Phi(x,y) \implies \exists\, x\, \forall\, k\, \Phi((x)_k,(x)_{k+1}))$, for every formula Φ, and in Φ we allow parameters.

3. Extension by Cohen Reals

Here we prove Theorem 1. We assume some knowledge of forcing and generic models, as *e.g.* in Kunen [19], especially Section IV.6 there on the "forcing over the universe" approach.

Recal that the Cohen forcing notion $\mathsf{Cohen} = 2^{<\omega}$ consists of all finite dyadic tuples including the empty tuple Λ. If $u,v \in 2^{<\omega}$ then $u \subset v$ means that v is a proper extension of u, whereas $u \subseteq v$ means $u \subset v \vee u = v$. The *finite-support product* $\mathbf{P} = (2^{<\omega})^\omega$ consists of all maps $p : \omega \to 2^{<\omega}$ such that $p(i) = \Lambda$ (the empty tuple) for all but finite $i < \omega$. The set $\mathbf{P}$ is ordered opposite to the componentwise extension, so that $p \le q$ (p is *stronger* as a forcing condition) iff $q(i) \subseteq p(i)$ for all $i < \omega$. The condition Λ^ω defined by $\Lambda^\omega(i) = \Lambda$, $\forall\, i$, is the $\le$- largest (the weakest) element of $\mathbf{P}$.

We consider the set $\mathbf{Perm}$ of all idempotent *permutations* of ω, that is, all bijections $\pi : \omega \xrightarrow{\text{onto}} \omega$ such that $\pi = \pi^{-1}$ and *the domain of nontriviality* $|\pi| = \{i : \pi(i) \ne i\}$ is finite. If $\pi \in \mathbf{Perm}$ and p is a function with $\mathtt{dom}\pi = \omega$, then πp is defined by $\mathtt{dom}(\pi p) = \omega$ and $(\pi p)(\pi(i)) = p(i)$ for all $i < \omega$, so formally $\pi p = p \circ \pi^{-1} = p \circ \pi$ (the superposition). In particular if $p \in \mathbf{P}$ then $\pi p \in \mathbf{P}$ and $|\pi p| = \pi``|p| = \{\pi(i) : i \in |p|\}$.

Proof of Theorem 1. We make use of Gödel's *constructible universe* $\mathbf{L}$ as *the ground model* for our forcing constructions. Suppose that $G \subseteq \mathbf{P}$ is a set $\mathbf{P}$- generic over $\mathbf{L}$. If $i < \omega$ then we define:

$$
\begin{aligned}
G_i &= \{p(i) : p \in G\} \subseteq 2^{<\omega}, \quad \text{a set } 2^{<\omega}\text{-generic (Cohen generic) over } \mathbf{L},\\
a_i[G] &= \bigcup G_i \in 2^{\omega}, \quad \text{a real Cohen generic over } \mathbf{L},\\
x_i[G] &= \{n : a_i(n) = 1\} \subseteq \omega, \quad \text{a subset of } \omega \text{ Cohen generic over } \mathbf{L}, \quad \text{and}\\
X &= X[G] = (\mathscr{P}(\omega) \cap \mathbf{L}) \cup \{x_i[G] : i < \omega\}.
\end{aligned}
$$

Thus $X[G] \in \mathbf{L}[G]$ and $X[G]$ consists of all subsets of ω already in $\mathbf{L}$ and all Cohen-generic sets $x_i[G]$, $i < \omega$.

We assert that the model $\langle \omega\,; X[G]\rangle$ proves Theorem 1.

The only thing to check is that $\langle \omega\,; X[G]\rangle$ satisfies $\mathbf{CA}^*$. For that purpose, assume that $\Phi(k)$ is a parameter-free $\mathcal{L}(\mathbf{PA}_2)$ formula with k the only free variable. Consider the set $y = \{k < \omega : \langle \omega\,; X[G]\rangle \models \Phi(k)\}$; then $y \in \mathbf{L}[G]$, $y \subseteq \omega$.

We claim that in fact y belongs to $\mathbf{L}$, and hence to $X[G]$.

Let $\Vdash$ be the forcing relation associated with $\mathbf{P}$. In particular, if $p \in \mathbf{P}$ and ψ is a parameter-free formula then $p \Vdash \psi$ iff ψ holds in any $\mathbf{P}$- generic extension $\mathbf{L}[H]$ of $\mathbf{L}$ such that $p \in H$.

Let $\underline{G}$ be a canonical $\mathbf{P}$-name for G. We assert that

$$y = \{k < \omega : \Lambda^{\omega} \Vdash \text{“}\langle \omega\,; X[\underline{G}]\rangle \models \Phi(k)''\}. \tag{1}$$

To prove $\supseteq$, assume that the condition Λ^{ω} $\mathbf{P}$- forces “$\langle \omega\,; X[\underline{G}]\rangle \models \Phi(k)$”. But $\Lambda^{\omega} \in G$ since Λ^{ω} is the weakest condition in $\mathbf{P}$. Therefore $\langle \omega\,; X[G]\rangle \models \Phi(k)$ by the forcing theorem, thus $k \in y$, as required.

To prove the converse, let $k \in y$. By the forcing theorem there is a condition $p \in G$ forcing “$\langle \omega\,; X[\underline{G}]\rangle \models \Phi(k)$”. We claim that then Λ^{ω} forces the same sentence.

Indeed otherwise there is a condition $q \in \mathbf{P}$ which forces “$\langle \omega\,; X[\underline{G}]\rangle \models \neg\,\Phi(k)$”. There is a permutation $\pi \in \mathbf{Perm}$ satisfying $|r| \cap |p| = \varnothing$, where $r = \pi q \in \mathbf{P}$. We claim that r forces “$\langle \omega\,; X[\underline{G}]\rangle \models \neg\,\Phi(k)$”. Indeed assume that $H \subseteq \mathbf{P}$ is a set $\mathbf{P}$- generic over $\mathbf{L}$, and $r \in H$. We have to prove that $\langle \omega\,; \nVdash_{\rangle} \models \neg\,\Phi(k)$. The set $K = \{\pi r' : r' \in H\}$ is $\mathbf{P}$- generic over $\mathbf{L}$ along with H since $\pi \in \mathbf{L}$. Moreover K contains q. It follows that $\langle \omega\,; X[K]\rangle \models \neg\,\Phi(k)$ by the forcing theorem and the choice of q. However the sequence $\langle x_i[K]\rangle_{i<\omega}$ is equal to the permutation of the sequence $\langle x_i[H]\rangle_{i<\omega}$ by π. It follows that $\nVdash_{=} X[K]$, and hence $\langle \omega\,; \nVdash_{\rangle} \models \neg\,\Phi(k)$, as required. Thus indeed r forces “$\langle \omega\,; X[\underline{G}]\rangle \models \neg\,\Phi(k)$”.

However p forces “$\langle \omega\,; X[\underline{G}]\rangle \models \Phi(k)$”, and p, r are compatible in $\mathbf{P}$ because $|r| \cap |p| = \varnothing$. This is a contradiction.

We conclude that Λ^{ω} forces $\langle \omega\,; X[\underline{G}]\rangle \models \Phi(k)$, and this completes the proof of (1).

But it is known that the forcing relation $\Vdash$ is expressible in $\mathbf{L}$, the ground model. Therefore it follows from (1) that $y \in \mathbf{L}$, hence $y \in X[G]$, as required. ■

4. Generalized Sacks Iterations

Here we begin the proof of Theorem 2. The proof involves the engine of generalized product/iterated Sacks forcing developed in [11,12] on the base of earlier papers [2,9] and others. We still consider the constructible universe $\mathbf{L}$ as the ground model for the extension, and define, in $\mathbf{L}$, the set

$$\boldsymbol{I} \;=\; (\omega_1 \times 2^{<\omega}) \cup \omega_1; \quad \boldsymbol{I} \in \mathbf{L}, \tag{2}$$

partially ordered so that $\langle \gamma, s\rangle \preccurlyeq \langle \beta, t\rangle$ iff $\gamma = \beta$ and $s \subseteq t$ in $2^{<\omega}$, while the ordinals in ω_1 (the second part of $\boldsymbol{I}$) remain $\preccurlyeq$- incomparable inside $\boldsymbol{I}$.

Our plan is to define a product/iterated generic Sacks extension $\mathbf{L}[\vec{a}]$ of $\mathbf{L}$ by an array $\vec{a} = \langle a_{\boldsymbol{i}}\rangle_{\boldsymbol{i}\in\boldsymbol{I}}$ of reals $a_{\boldsymbol{i}} \in 2^\omega$, in which the structure of "sacksness" is determined by this set $\boldsymbol{I}$, so that in particular each $a_{\boldsymbol{i}}$ is Sacks-generic over the submodel $\mathbf{L}[\langle a_{\boldsymbol{j}}\rangle_{\boldsymbol{j}\prec\boldsymbol{i}}]$.

Then we define the set $\boldsymbol{J} \in \mathbf{L}[\vec{a}]$ of all elements $\boldsymbol{i} \in \boldsymbol{I}$ such that:

- Either $\boldsymbol{i} = \langle \gamma, 0^m\rangle$, where $\gamma < \omega_1$ and $m < \omega$,
- Or $\boldsymbol{i} = \langle \gamma, 0^m{}^\frown 1\rangle$, where $\gamma < \omega_1$ and $m < \omega$, $a_\gamma(m) = 1$.

Thus any $\boldsymbol{i} = \langle \gamma, 0^m\rangle \in \boldsymbol{J}$ is a splitting node in $\boldsymbol{J}$ iff $a_\gamma(m) = 1$, or in other words

$$a_\gamma(m) = 1 \quad \text{iff} \quad \langle \gamma, 0^m\rangle \text{ is a splitting node in } \boldsymbol{J}, \tag{3}$$

We'll finally prove that the according set

$$W = \mathscr{P}(\omega) \cap \bigcup_{\boldsymbol{i}_1,\dots,\boldsymbol{i}_n \in \boldsymbol{J}} \mathbf{L}[a_{\boldsymbol{i}_1},\dots,a_{\boldsymbol{i}_n}] \tag{4}$$

leads to the model $\langle \omega\,; W\rangle$ for Theorem 2. The reals a_γ will not belong to M by the choice of $\boldsymbol{J}$, but will be definable in $\langle \omega\,; M\rangle$ (with $a_{\langle\gamma,\Lambda\rangle} \subseteq \omega$ as a parameter) via the characterization of the splitting nodes in $\boldsymbol{J}$ by (3).

5. Iterated Perfect Sets

Arguing in L in this section, we define $\boldsymbol{I} = \langle \boldsymbol{I}\,; \preccurlyeq\rangle$ as above.

Let $\boldsymbol{\Xi}$ be the set of all countable (including finite) initial segments $\zeta \subseteq \boldsymbol{I}$.

Greek letters ξ, η, ζ, ϑ will denote sets in $\mathbf{\Xi}$, and generally countable subsets of $\boldsymbol{I}$.

Characters $\boldsymbol{i}$, $\boldsymbol{j}$ are used to denote *elements* of $\boldsymbol{I}$.

For any $\boldsymbol{i} \in \zeta \in \mathbf{\Xi}$, we consider initial segments $\zeta[\prec\boldsymbol{i}] = \{\boldsymbol{j} \in \zeta : \boldsymbol{j} \prec \boldsymbol{i}\}$ and $\zeta[\not\succcurlyeq\boldsymbol{i}] = \{\boldsymbol{j} \in \zeta : \boldsymbol{j} \not\succcurlyeq \boldsymbol{i}\}$, and $\zeta[\preccurlyeq\boldsymbol{i}]$, $\zeta[\not\succ\boldsymbol{i}]$ defined analogously.

Further, ω^ω is the *Baire space*. Points of ω^ω will be called *reals*.

Let $\mathscr{D} = 2^\omega \subseteq \omega^\omega$ be the *Cantor space*. For any countable set ξ, $\mathscr{D}^\xi$ is the product of ξ- many copies of $\mathscr{D}$ with the product topology. Then every $\mathscr{D}^\xi$ is a compact space, homeomorphic to $\mathscr{D}$ itself unless $\xi = \varnothing$.

Assume that $\eta \subseteq \xi \in \mathbf{\Xi}$. If $x \in \mathscr{D}^\xi$ then let $x \restriction \eta \in \mathscr{D}^\eta$ denote the usual restriction. If $X \subseteq \mathscr{D}^\xi$ then let $X \restriction \eta = \{x \restriction \eta : x \in X\}$. To save space, let $X \restriction_{\prec\boldsymbol{i}}$ mean $X \restriction \xi[\prec\boldsymbol{i}]$, $X \restriction_{\not\succcurlyeq\boldsymbol{i}}$ mean $X \restriction \xi[\not\succcurlyeq\boldsymbol{i}]$, *etc.*

But if $Y \subseteq \mathscr{D}^\eta$ then we put $Y \restriction^{-1} \xi = \{x \in \mathscr{D}^\xi : x \restriction \eta \in Y\}$.

To describe the idea behind iterated perfect sets, recall that the Sacks forcing consists of perfect subsets of $\mathscr{D}$, that is, sets of the form $H\,"\mathscr{D} = \{H(a) : a \in \mathscr{D}\}$, where $H : \mathscr{D} \xrightarrow{\text{onto}} X$ is a homeomorphism.

To get a product Sacks model, with two factors (the case of a two-element unordered set as the length of iteration), we have to consider sets $X \subseteq \mathscr{D}^2$ of the form $X = H\,"\mathscr{D}^2$ where H, a homeomorphism defined on $\mathscr{D}^2$, splits in obvious way into a pair of one-dimentional homeomorphisms.

To get an iterated Sacks model, with two stages of iteration (the case of a two-element ordered set as the length of iteration), we have to consider sets $X \subseteq \mathscr{D}^2$ of the form $X = H\,"\mathscr{D}^2$, where H, a homeomorphism defined on $\mathscr{D}^2$, satisfies the following: if $H(a_1, a_2) = \langle x_1, x_2 \rangle$ and $H(a_1', a_2') = \langle x_1', x_2' \rangle$ then $a_1 = a_1' \iff x_1 = x_1'$.

The combined product/iteration case results in the following definition.

Definition 4. (iterated perfect sets, [11,12]) For any $\zeta \in \mathbf{\Xi}$, $\mathbf{Perf}_\zeta$ is the collection of all sets $X \subseteq \mathscr{D}^\zeta$ such that there is a homeomorphism $H : \mathscr{D}^\zeta \xrightarrow{\text{onto}} X$ satisfying

$$x_0 \restriction \xi = x_1 \restriction \xi \iff H(x_0) \restriction \xi = H(x_1) \restriction \xi$$

for all x_0, $x_1 \in \mathtt{dom} H$ and $\xi \in \mathbf{\Xi}$, $\xi \subseteq \zeta$. Homeomorphisms H satisfying this requirement will be called *projection–keeping*. In other words, sets in $\mathbf{Perf}_\zeta$ are images of $\mathscr{D}^\zeta$ via projection–keeping homeomorphisms.

Remark 5. Note that $\varnothing$, the empty set, formally belongs to $\mathbf{\Xi}$, and then $\mathscr{D}^\varnothing = \{\varnothing\}$, and we easily see that $\mathbf{1} = \{\varnothing\}$ is the only set in $\mathbf{Perf}_\varnothing$. ■

For the convenience of the reader, we now present five lemmas on sets in $\mathbf{Perf}_\zeta$ established in [11,12], with according references.

Lemma 6. (Proposition 4 in [11]) *Let* $\zeta, \xi, \eta \in \boldsymbol{\Xi}$. *Every set* $X \in \mathbf{Perf}_\zeta$ *is closed and satisfies the following properties*:

(1) If $\boldsymbol{i} \in \zeta$ *and* $z \in X{\restriction}_{<\boldsymbol{i}}$ *then* $D_{Xz}(\boldsymbol{i}) = \{x(\boldsymbol{i}) : x \in X \wedge x{\restriction}_{<\boldsymbol{i}} = z\}$ *is a perfect set in* $\mathscr{D}$.

(2) If $\xi \subseteq \zeta$, *and a set* $X' \subseteq X$ *is open in* X *(in the relative topology) then the projection* $X' \restriction \xi$ *is open in* $X \restriction \xi$. *In other words, the projection from* X *to* $X \restriction \xi$ *is an open map.*

(3) If $\xi, \eta \subseteq \zeta$, $x \in X \restriction \xi$, $y \in X \restriction \eta$, *and* $x \restriction (\xi \cap \eta) = y \restriction (\xi \cap \eta)$, *then* $x \cup y \in X \restriction (\xi \cup \eta)$.

Proof (sketch). Clearly $\mathscr{D}^\zeta$ satisfies (1), (2), (3), and one easily shows that projection–keeping homeomorphisms preserve the requirements.

Lemma 7. ([11], Lemma 6) *If* $\xi \subseteq \zeta$ *belong to* $\boldsymbol{\Xi}$ *and* $X \in \mathbf{Perf}_\zeta$ *then* $X \restriction \xi \in \mathbf{Perf}_\xi$.

Lemma 8. (Lemma 8 in [11]) *If* $\zeta \in \boldsymbol{\Xi}$, $X \in \mathbf{Perf}_\zeta$, *a set* $X' \subseteq X$ *is open in* X, *and* $x_0 \in X'$, *then there is a set* $X'' \in \mathbf{Perf}_\zeta$, $X'' \subseteq X'$, *clopen in* X *and containing* x_0.

Lemma 9. (Lemma 10 in [11]) *Suppose that* $\eta \subseteq \zeta$ *belong to* $\boldsymbol{\Xi}$, $X \in \mathbf{Perf}_\zeta$, $Y \in \mathbf{Perf}_\eta$, *and* $Y \subseteq X \restriction \eta$. *Then* $Z = X \cap (Y \restriction^{-1} \zeta)$ *belongs to* $\mathbf{Perf}_\zeta$.

Lemma 10. (Lemma 10 in [12]) *Suppose that* $\xi \subseteq \zeta$ *belong to* $\boldsymbol{\Xi}$, $X \in \mathbf{Perf}_\xi$. *Then* $X \restriction^{-1} \zeta$ *belongs to* $\mathbf{Perf}_\zeta$.

6. The Forcing and the Basic Extension

This section introduces the forcing notion we consider and the according generic extension called the basic extension.

We continue to argue in L. Recall that a partially ordered set $\boldsymbol{I} \in \mathbf{L}$ is defined by (2) in Section 4, and $\boldsymbol{\Xi}$ is the set of all at most countable initial segments $\xi \subseteq \boldsymbol{I}$ in $\mathbf{L}$. For any $\zeta \in \boldsymbol{\Xi}$, let $\mathbb{P}_\zeta = (\mathbf{Perf}_\zeta)^{\mathbf{L}}$.

The set $\mathbb{P} = \mathbb{P}_{\boldsymbol{I}} = \bigcup_{\zeta \in \boldsymbol{\Xi}} \mathbb{P}_\zeta \in \mathbf{L}$ will be the *forcing notion*.

To define the order, we put $\|X\| = \zeta$ whenever $X \in \mathbb{P}_\zeta$. Now we set $X \le Y$ (*i.e.* X is *stronger* than Y) iff $\zeta = \|Y\| \subseteq \|X\|$ and $X \restriction \zeta \subseteq Y$.

Remark 11. We may note that the set $\mathbf{1} = \{\varnothing\}$ as in Remark 5 belongs to $\mathbb{P}$ and is the $\le$- largest (*i.e.*, the weakest) element of $\mathbb{P}$. ■

Now let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$- generic set (filter) over $\mathbf{L}$.

Remark 12. If $X \in \mathbb{P}_\zeta$ in $\mathbf{L}$ then X is not even a closed set in $\mathscr{D}^\zeta$ in $\mathbf{L}[G]$. However we can transform it to a perfect set in $\mathbf{L}[G]$ by the closure operation. Indeed the topological closure $X^\#$ of such a set X in $\mathscr{D}^\zeta$ taken in $\mathbf{L}[G]$ belongs to $\mathbf{Perf}_\zeta$ from the point of view of $\mathbf{L}[G]$. ■

It easily follows from Lemma 8 that there exists a unique array $\mathbf{a}[G] = \langle \mathbf{a}_{\boldsymbol{i}}[G] \rangle_{\boldsymbol{i} \in \boldsymbol{I}}$, all $\mathbf{a}_{\boldsymbol{i}}[G]$ being elements of 2^ω, such that $\mathbf{a}[G] \restriction \xi \in X^\#$ whenever $X \in G$ and $\|X\| = \xi \in \boldsymbol{\Xi}$. Then $\mathbf{L}[G] = \mathbf{L}[\langle \mathbf{a}_{\boldsymbol{i}}[G] \rangle_{\boldsymbol{i} \in \boldsymbol{I}}] = \mathbf{L}[\mathbf{a}[G]]$ is a $\mathbb{P}$-*generic extension* of $\mathbf{L}$.

Theorem 13. (Theorems 24, 31 in [11]) *Every cardinal in* $\mathbf{L}$ *remains a cardinal in* $\mathbf{L}[G]$. *Every* $\mathbf{a}_{\boldsymbol{i}}[G]$ *is Sacks generic over the model* $\mathbf{L}[\mathbf{a}[G] \restriction_{\prec \boldsymbol{i}}]$.

Here follows a list of several lemmas on reals in $\mathbb{P}$- generic models $\mathbf{L}[G]$, established in [11]. In the lemmas, we let $G \subseteq \mathbb{P}$ be an arbitrary set $\mathbb{P}$-generic over $\mathbf{L}$.

Lemma 14. (Lemma 22 in [11]) *Suppose that finite or countable sets* $\eta, \xi \subseteq \boldsymbol{I}$ *in* $\mathbf{L}$ *satisfy* $\forall\, \boldsymbol{j} \in \eta\, \exists\, \boldsymbol{i} \in \xi\, (\boldsymbol{j} \preccurlyeq \boldsymbol{i})$. *Then* $\mathbf{a}[G] \restriction \eta \in \mathbf{L}[\mathbf{a}[G] \restriction \xi]$. ■

Lemma 15. (Lemma 26 in [11]) *Suppose that* $\boldsymbol{K} \in \mathbf{L}$ *is an initial segment in* $\boldsymbol{I}$, *and* $\boldsymbol{i} \in \boldsymbol{I} \setminus \boldsymbol{K}$. *Then* $\mathbf{a}_{\boldsymbol{i}}[G] \notin \mathbf{L}[\mathbf{a}[G] \restriction \boldsymbol{K}]$. ■

Lemma 16. (Corollary 27 in [11]) *If* $\boldsymbol{i} \neq \boldsymbol{j}$ *then* $\mathbf{a}_{\boldsymbol{i}}[G] \neq \mathbf{a}_{\boldsymbol{j}}[G]$ *and even* $\mathbf{L}[\mathbf{a}_{\boldsymbol{i}}[G]] \neq \mathbf{L}[\mathbf{a}_{\boldsymbol{j}}[G]]$. ■

Lemma 17. (Lemma 29 in [11]) *If* $\boldsymbol{K} \in \mathbf{L}$ *is an initial segment of* $\boldsymbol{I}$, *and* r *is a real in* $\mathbf{L}[G]$, *then either* $r \in \mathbf{L}[\mathbf{a} \restriction \boldsymbol{K}]$ *or there is* $\boldsymbol{i} \notin \boldsymbol{K}$ *such that* $\mathbf{a}_{\boldsymbol{i}}[G] \in \mathbf{L}[r]$. ■

We apply these lemmas in the proof of the next key theorem. Let $\leq_{\mathbf{L}}$ denote the relative constructibility ordering on 2^ω, so that $x \leq_{\mathbf{L}} y$ iff $x \in \mathbf{L}[y]$. Let $x <_{\mathbf{L}} y$ mean that $x \leq_{\mathbf{L}} y$ but $y \not\leq_{\mathbf{L}} x$, and accordingly $x \equiv_{\mathbf{L}} y$ mean that $x \leq_{\mathbf{L}} y$ and $y \leq_{\mathbf{L}} x$.

Theorem 18. *Assume that* $\boldsymbol{i} \in \boldsymbol{I}$ *and* $r \in \mathbf{L}[G] \cap 2^\omega$. *Then*

(i) *if* $\boldsymbol{j} \in \boldsymbol{I}$ *and* $\boldsymbol{j} \preccurlyeq \boldsymbol{i}$ *then* $\mathbf{a}_{\boldsymbol{j}}[G] \leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}}[G]$;

(ii) *if* $\boldsymbol{j} \in \boldsymbol{I}$ *and* $\boldsymbol{j} \not\preccurlyeq \boldsymbol{i}$ *then* $\mathbf{a}_{\boldsymbol{j}}[G] \not\leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}}[G]$;

(iii) *if* $r \leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}}[G]$ *then* $r \in \mathbf{L}$ *or* $r \equiv_{\mathbf{L}} \mathbf{a}_{\boldsymbol{j}}[G]$ *for some* $\boldsymbol{j} \in \boldsymbol{I}$, $\boldsymbol{j} \preccurlyeq \boldsymbol{i}$;

(iv) *if* $\boldsymbol{i} = \langle \gamma, s \rangle \in \boldsymbol{I}$, $e = 0, 1$, *and* $\boldsymbol{i}^\frown e = \langle \gamma, s^\frown e \rangle$ *then* $\mathbf{a}_{\boldsymbol{i}^\frown e}[G]$ *is a* **true successor** *of* $\mathbf{a}_{\boldsymbol{i}}[G]$ *in the sense that* $\mathbf{a}_{\boldsymbol{i}}[G] <_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown e}[G]$ *and any real* $y \in 2^\omega$ *satisfies* $y <_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown e}[G] \Longrightarrow y \leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}}[G]$;

(v) if $\boldsymbol{i} = \langle \gamma, s\rangle \in \boldsymbol{I}$*, and* $x \in 2^\omega \cap \mathbf{L}[G]$ *is a true successor of* $\mathbf{a}_{\boldsymbol{i}}[G]$ *in the sense of (iv), then there is* $e = 0$ *or* 1 *such that* $x \equiv_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown e}[G]$*.*

Proof. (i) Apply Lemma 14 with $\eta = \{\boldsymbol{j}\}$ and $\xi = \{\boldsymbol{i}\}$.
(ii) Apply Lemma 15 with $\boldsymbol{K} = [\preccurlyeq \boldsymbol{i}]$.
(iii) If there are elements $\boldsymbol{j} \in \mathcal{I}$, $\boldsymbol{j} \preccurlyeq \boldsymbol{i}$, such that $\mathbf{a}_{\boldsymbol{j}}[G] \in \mathbf{L}[r]$, then let $\boldsymbol{j}$ be the largest such one, and let $\xi = [\preccurlyeq \boldsymbol{j}]$ (a finite initial segment of $\boldsymbol{I}$). Then, by Lemma 17, either $r \in \mathbf{L}[\mathbf{a}[G]\restriction \xi]$, or there is $\boldsymbol{i}' \notin \xi$ such that $\mathbf{a}_{\boldsymbol{i}'}[G] \in \mathbf{L}[r]$.

In the "either" case, we have $r \in \mathbf{L}[\mathbf{a}_{\boldsymbol{j}}[G]]$ by (i), so that $\mathbf{L}[r] = \mathbf{L}[\mathbf{a}_{\boldsymbol{j}}[G]]$ by the choice of $\boldsymbol{j}$. In the "or" case we have $\mathbf{a}_{\boldsymbol{i}'}[G] \in \mathbf{L}[a_{\boldsymbol{i}}[G]]$, hence $\boldsymbol{i}' \preccurlyeq \boldsymbol{i}$ by (ii). But this contradicts the choice of $\boldsymbol{j}$ and $\boldsymbol{i}'$.

Finally if there is no $\boldsymbol{j} \in \mathcal{I}$, $\boldsymbol{j} \preccurlyeq \boldsymbol{i}$, such that $\mathbf{a}_{\boldsymbol{j}}[G] \in \mathbf{L}[r]$, then the same argument with $\xi = \varnothing$ gives $r \in \mathbf{L}$.

(iv) The relation $\mathbf{a}_{\boldsymbol{j}}[G] <_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown e}[G]$ is implied by Lemmas 14 and 15. If now $y <_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown e}[G]$ then $y \in \mathbf{L}$ or $y \equiv_{\mathbf{L}} \mathbf{a}_{\boldsymbol{j}}[G]$ for some $\boldsymbol{j} \preccurlyeq \boldsymbol{i}^\frown e$ by (iii), and in the latter case in fact $\boldsymbol{j} \prec \boldsymbol{i}^\frown e$, hence $\boldsymbol{j} \preccurlyeq \boldsymbol{i}$, and then $y \leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}}[G]$.

(v) By (iv), it suffices to prove that $x \leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown 0}[G]$ or $x \leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown 1}[G]$. Assume that $x \not\leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown 0}[G]$. Then by Lemma 17 there is an element $\boldsymbol{j} \in \boldsymbol{I}$ such that $\boldsymbol{j} \not\preccurlyeq \boldsymbol{i}^\frown 0$ and $\mathbf{a}_{i_0}[G] \leq_{\mathbf{L}} x$. If $\mathbf{a}_j[G] <_{\mathbf{L}} x$ strictly then $\mathbf{a}_j[G] \leq_{\mathbf{L}} \mathbf{a}_i[G]$ by the true successor property, hence $\boldsymbol{i}_0 \preccurlyeq \boldsymbol{i}$, contrary to $\boldsymbol{i}_0 \not\preccurlyeq \boldsymbol{i}^\frown 0$, see above. Therefore in fact $\mathbf{a}_{i_0}[G] \equiv_{\mathbf{L}} x$. Then we must have $\boldsymbol{i}_0 = \boldsymbol{i}^\frown 0$ or $\boldsymbol{i}_0 = \boldsymbol{i}^\frown 1$ as x is a true successor, but then $\boldsymbol{i}_0 = \boldsymbol{i}^\frown 1$, as $x \not\leq_{\mathbf{L}} \mathbf{a}_{\boldsymbol{i}^\frown 0}[G]$ was assumed, and we are done. ■

7. The Subextension

Following the arguments above, assume that $G \subseteq \mathbb{P}$ is a set $\mathbb{P}$- generic over $\mathbf{L}$, and consider the set $\boldsymbol{J}[G] \in \mathbf{L}[G]$ of all elements $\boldsymbol{i} \in \boldsymbol{I}$ such that:

either $\boldsymbol{i} = \langle \gamma, 0^m\rangle$, where $\gamma < \omega_1$, $m < \omega$, $0^m = \langle 0, 0, \dots, 0\rangle$ (m terms equal to 0),

or $\boldsymbol{i} = \langle \gamma, 0^{m\frown} 1\rangle$, where $\gamma < \omega_1$ and $m < \omega$, $\mathbf{a}_\gamma[G](m) = 1$.

Following (4), we define

$$W[G] \;=\; \mathscr{P}(\omega) \cap \bigcup_{\boldsymbol{i}_1,\dots,\boldsymbol{i}_n \in \boldsymbol{J}[G]} \mathbf{L}[a_{\boldsymbol{i}_1}[G], \dots, a_{\boldsymbol{i}_n}[G]], \tag{5}$$

Lemma 19. *If* $\boldsymbol{i} \notin \boldsymbol{J}[G]$ *then* $\mathbf{a}_{\boldsymbol{i}}[G] \notin W[G]$*.*

Proof. This is not immediately a case of Lemma 15 because $\boldsymbol{J}[G] \notin \mathbf{L}$. However the set $\boldsymbol{K} = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{i} \not\preccurlyeq \boldsymbol{j}\}$ belongs to $\mathbf{L}$ and satisfies $\boldsymbol{J}[G] \subseteq \boldsymbol{K} \subseteq \boldsymbol{I}$. We have $\boldsymbol{i} \notin \boldsymbol{K}$, and hence $\mathbf{a}_{\boldsymbol{i}}[G] \notin \mathbf{L}[\mathbf{a}[G] \restriction \boldsymbol{K}]$ by Lemma 15. On the other hand, we easily check $W[G] \subseteq \mathbf{L}[\mathbf{a}[G] \restriction \boldsymbol{K}]$, and we are done. ∎

We are going to prove that $\langle \omega\,; W[G] \rangle$ is a model of $\mathbf{PA}_2^* + \mathbf{CA}(\boldsymbol{\Sigma}_2^1)$, but the full $\mathbf{CA}$ fails in $\langle \omega\,; W[G] \rangle$.

Part 1: $\langle \omega\,; W[G] \rangle$ is a model of all axioms of $\mathbf{PA}_2$ except for $\mathbf{CA}$, trivial.

Part 2: $\langle \omega\,; W[G] \rangle$ is a model of $\mathbf{CA}(\boldsymbol{\Sigma}_2^1)$ (with parameters). This is also easy by the Shoenfield absoluteness theorem.

Part 3: $\langle \omega\,; W[G] \rangle$ fails to satisfy the full $\mathbf{CA}$. Here we need some work. Let $\gamma < \omega_1^{\mathbf{L}}$, so that both γ and each pair $\langle \gamma, s \rangle$, $s \in 2^{<\omega}$, belong to $\boldsymbol{I}$ by (2) in Section 4, in particular $\boldsymbol{i}_0 = \langle \gamma, \Lambda \rangle \in \boldsymbol{I}$, where Λ is the empty tuple. In addition γ (as an element of $\boldsymbol{I}$) does not belong to $\boldsymbol{J}[G]$. Our plan is to prove that $\mathbf{a}_\gamma[G] \notin W[G]$ but $\mathbf{a}_\gamma[G]$ is definable in $\langle \omega\,; W[G] \rangle$.

Subpart 3.1: $\mathbf{a}_\gamma[G] \notin W[G]$ by Lemma 19 just because $\gamma \notin \boldsymbol{J}[G]$.

Subpart 3.2: $\mathbf{a}_\gamma[G]$ is definable in $\langle \omega\,; W[G] \rangle$ with $\mathbf{a}_{\boldsymbol{i}_0}[G]$ as a parameter, where $\boldsymbol{i}_0 = \langle \gamma, \Lambda \rangle \in \boldsymbol{J}[G]$. Namely we claim that for any $m < \omega$:

$\mathbf{a}_\gamma[G](m) = 1$ iff there is an array of reals $b_0, b_1, \dots, b_m, b_{m+1}$ and b'_{m+1} in 2^ω such that $b_0 = \mathbf{a}_{\boldsymbol{i}_0}$, each b_{k+1} is a true successor of b_k $(k \le m)$, b'_{m+1} is a true successor of b_m as well, and $b'_{m+1} \not\equiv_{\mathbf{L}} b_{m+1}$. (6)

The formula in the right-hand side of (6) is based on the Gödel canonical Σ_2^1 formula for $\le_{\mathbf{L}}$, which is absolute for $W[G]$ by the definition of $W[G]$. Therefore (6) implies that $\mathbf{a}_\gamma[G]$ is definable in $\langle \omega\,; W[G] \rangle$ with $\mathbf{a}_{\boldsymbol{i}_0}[G]$ as a parameter. Thus it remains to establish (6).

Direction $\Longrightarrow$. Assume that $\mathbf{a}_\gamma[G](m) = 1$. Then $\boldsymbol{J}[G]$ contains the elements $\boldsymbol{i}_k = \langle \gamma, 0^k \rangle$, $k \le m+1$, along with an element $\boldsymbol{i}'_{m+1} = \langle \gamma, 0^m {}^\frown 1 \rangle$. Therefore the reals $b_k = \mathbf{a}_{\boldsymbol{i}_k}[G]$, $k \le m+1$, and $b'_{m+1} = \mathbf{a}_{\boldsymbol{i}'_{m+1}}[G]$ belong to $W[G]$. Now Theorem 18(iv),(ii) implies that the reals b_k and b'_{m+1} satisfy the right-hand side of (6), as required.

Direction $\Longleftarrow$. Assume that the reals b_k, $k \le m+1$, and b'_{m+1} satisfy the right-hand side of (6). By Theorem 18(v), there is an array of bits $e_1, \dots, e_m, e_{m+1}$ and e'_{m+1} such that $b_k = \mathbf{a}_{\boldsymbol{i}_k}[G]$ for all $k \le m+1$ and $b'_{m+1} = \mathbf{a}_{\boldsymbol{i}'_{m+1}}[G]$, where $\boldsymbol{i}_k = \langle \gamma, \langle e_1, \dots, e_k \rangle \rangle$ and $\boldsymbol{i}'_{m+1} = \langle \gamma, \langle e_1, \dots, e_m, e'_{m+1} \rangle \rangle$.

However $\boldsymbol{i}_k \in \boldsymbol{J}[G]$ for all $k \le m+1$, and $\boldsymbol{i}'_{m+1} \in \boldsymbol{J}[G]$, by Lemma 19, since the reals b_k and b'_{m+1} belong to $W[G]$. Then obviously $e_1 = \dots = e_m = 0$ while $e_{m+1} = 0$ and $e'_{m+1} = 1$ or vice versa $e_{m+1} = 1$ and $e'_{m+1} = 0$.

In other words, the elements $\langle\gamma, 0^{m+1}\rangle$ and $\langle\gamma, 0^m{}^\frown 1\rangle$ belong to $\boldsymbol{J}[G]$. This implies $\mathbf{a}_\gamma[G](m) = 1$.

Part 4: $\langle\omega\,; W[G]\rangle$ satisfies the parameter-free schema $\mathbf{CA}^*$. This is rather similar to the verification of $\mathbf{CA}^*$ in $\langle\omega\,; X[G]\rangle$ in Section 3.

Assume that $\Phi(k)$ is a parameter-free $\mathcal{L}(\mathbf{PA}_2)$ formula with k the only free variable. Consider the set $y = \{k < \omega : \langle\omega\,; W[G]\rangle \models \Phi(k)\}$; then $y \in \mathbf{L}[G]$, $y \subseteq \omega$. We claim that y even belongs to $\mathbf{L}$, and hence to $W[G]$.

Let $\Vdash$ be the forcing relation associated with $\mathbb{P}$, over $\mathbf{L}$ as the ground model. Thus if $X \in \mathbb{P}$ and $k < \omega$ then $X \Vdash \Phi(k)$ iff $\Phi(k)$ holds in any $\mathbb{P}$-generic extension $\mathbf{L}[H]$ of $\mathbf{L}$ such that $X \in H$.[4] Let $\underline{G}$ be a canonical $\mathbb{P}$-name for G. We assert that

$$y = \{k < \omega : \mathbf{1} \Vdash \text{“}\langle\omega\,; W[\underline{G}]\rangle \models \Phi(k)''\}. \tag{7}$$

(See Remark 11 on $\mathbf{1}$, the weakest condition in $\mathbb{P}$.)

In the nontrivial direction, assume that $k \in y$. Then by the forcing theorem there is a condition $X \in G$ forcing $\langle\omega\,; W[\underline{G}]\rangle \models \Phi(k)$. We claim that then $\mathbf{1}$ forces the same as well.

To prove this reduction, we define, **still in L**, the set $\mathbf{Perm} \in \mathbf{L}$ that consists of all bijections $\pi : \omega_1 \xrightarrow{\text{onto}} \omega_1$ such that $\pi = \pi^{-1}$ and *the domain of nontriviality* $|\pi| = \{\alpha : \pi(\alpha) \neq \alpha\}$ is at most countable, *i.e.*, bounded in ω_1. Any $\pi \in \mathbf{Perm}$ acts on:

- Elements $\boldsymbol{i} = \gamma$ or $\boldsymbol{i} = \langle\gamma, s\rangle$ of $\boldsymbol{I}$, by $\pi\boldsymbol{i} = \pi(\gamma)$, resp. $\boldsymbol{i} = \langle\pi(\gamma), s\rangle$;
- Maps g with $\mathtt{dom}g \subseteq \boldsymbol{I}$, by $\mathtt{dom}(\pi g) = \pi\text{''}\mathtt{dom}g$ and $(\pi g)(\pi(\alpha)) = g(\alpha)$ for all $\alpha \in \mathtt{dom}g$;
- Thus if $\xi \subseteq \boldsymbol{I}$ and $x \in \mathscr{D}^\xi$ then $\pi x \in \mathscr{D}^{\pi\text{''}\xi}$ and $(\pi x)(\pi(\alpha)) = x(\alpha)$;
- Sets $X \in \mathbf{Perf}_\xi$, $\xi \in \mathbf{\Xi}$, by $\pi X = \{\pi x : x \in X\} \in \mathbf{Perf}_{\pi\text{''}\xi}$.

We return to the nontrivial direction $\Longrightarrow$ of (7), where we have to prove that the condition $\mathbf{1}$ forces “$\langle\omega\,; W[\underline{G}]\rangle \models \Phi(k)$”. Let this be not the case.

Then there is a condition $Y \in \mathbb{P}$ which forces “$\langle\omega\,; W[\underline{G}]\rangle \models \neg\,\Phi(k)$”. There is a permutation $\pi \in \mathbf{Perm}$ satisfying $\|Z\| \cap \|X\| = \varnothing$, where $Z = \pi Y \in \mathbb{P}$. We claim that Z forces “$\langle\omega\,; W[\underline{G}]\rangle \models \neg\,\Phi(k)$”. Indeed assume that $H \subseteq \mathbb{P}$ is a set $\mathbb{P}$- generic over $\mathbf{L}$, and $Z \in H$. We have to prove that $\langle\omega\,; W[H]\rangle \models \neg\,\Phi(k)$. The set $K = \{\pi Z' : Z' \in H\}$ is $\mathbf{P}$- generic over $\mathbf{L}$ along with H since $\pi \in \mathbf{L}$. Moreover K contains Y. It follows that $\langle\omega\,; M[K]\rangle \models \neg\,\Phi(k)$ by the forcing theorem and the choice of Y.

[4] See Kunen [19] on forcing, especially Section IV.6 there on the “forcing over the universe” approach.

However the array $\mathbf{a}[K]$ is equal to the permutation of the array $\mathbf{a}[H]$ by π. It follows that $W[H] = M[K]$, and hence $\langle \omega ; W[H] \rangle \models \neg\, \Phi(k)$, as required. Thus indeed Z forces "$\langle \omega ; W[\underline{G}] \rangle \models \neg\, \Phi(k)$".

Recall that X forces "$\langle \omega ; W[\underline{G}] \rangle \models \Phi(k)$". On the other hand, X, Z are compatible in $\mathbb{P}$ because $\|Z\| \cap \|X\| = \varnothing$. This is a contradiction.

We conclude that $\mathbf{1}$ forces "$\langle \omega ; W[\underline{G}] \rangle \models \Phi(k)$", and this completes the proof of (7). But it is known that the forcing relation $\Vdash$ is expressible in $\mathbf{L}$, the ground model. Therefore it follows from (7) that $y \in \mathbf{L}$, hence $y \in W[G]$, as required.

8. Discussion

We present several remarks and questions related to possible extensions of Theorem 2.

Problem 1. Is the parameter-free countable choice schema $\mathbf{AC}^*_\omega$ in the language $\mathcal{L}(\mathbf{PA}_2)$ true in the models $\langle \omega ; W[G] \rangle$ defined in Section 7?

Models of $\mathbf{ZF}$ separating $\mathbf{AC}^*_\omega$ from the full $\mathbf{AC}_\omega$ are defined in [10] (via a cardinal-collapse below $\aleph_{\omega_1}$) and in our recent paper [13] (by a cardinal-preserving model rather close to the model $\mathbf{L}[G]$ as in Section 6).

Problem 2. Can we sharpen the result of Theorem 2 by specifying that $\mathbf{CA}(\mathbf{\Sigma}^1_3)$ is violated? The combination $\mathbf{CA}(\mathbf{\Sigma}^1_2) + \neg\, \mathbf{CA}(\mathbf{\Sigma}^1_3)$ would be optimal. The counterexample to $\mathbf{CA}$ defined in Section 7 (Part 3) definitely is more complex than $\mathbf{\Sigma}^1_3$.

According to recent advances in this direction partially outlined in [5,8], Jensen's iterated forcing introduced in [6], may lead to a solution. Such a construction makes use of the consecutive "jensenness", known to be a $\mathit{\Pi}^1_2$ relation, instead of the consecutive "sacksness", as in this paper, which can help to define a counterexample required at the minimally possible level $\mathbf{\Sigma}^1_3$.

Problem 3. As a generalization of the above, prove that, for any $n \geq 2$, $\mathbf{PA}^*_2 + \mathbf{CA}(\mathbf{\Sigma}^1_n)$ does not imply $\mathbf{CA}(\mathbf{\Sigma}^1_{n+1})$. (Compare to Problem 9 in [1, § 11].) Such a result would imply that the full schema $\mathbf{CA}$ is not finitely axiomatizable over $\mathbf{PA}^*_2$.

We expect that methods of inductive construction of forcing notions in $\mathbf{L}$, that are similar to the iterated Jensen forcing as in [6] but carry hidden automorphisms, recently developed in our recent papers [14–16] and some others, may lead to the solution of Problem 3.

Problem 4. [Communicated by Ali Enayat] A natural question is whether the results of this note also hold for second order set theory (the Kelley-Morse theory of classes). This may involve a generalization of the Sacks forcing to uncountable cardinals, as in Kanamori [17], that have been recently further developed in [5,8].

As a concluding remark, we expect that the methods developed for this research can also be useful in creating computational algorithmic models, of various complexity in terms of the second order Peano arithmetic, that represent the evolution of cell types and are related to the storage and processing of genomic information.

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