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GENERALIZATION OF ONE CONSTRUCTION BY SOLOVAY

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Abstract: The well-known Σ-construction in forcing by Solovay is generalized to the case of intermediate sets that are not subsets of the initial model. Our method gives a more transparent construction of a forcing over an intermediate model than that in the classical paper [1] by Grigorieff on intermediate models.

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1. Introduction

The well-known Σ-construction by Solovay [2, 4.4] (also see [3, 13.3]) makes it possible to prove that if $\mathbb{P} \in \mathfrak{M}$ is a forcing in a countable transitive model $\mathfrak{M}$, $t \in \mathfrak{M}$ is a $\mathbb{P}$-name, and $X \subseteq \mathfrak{M}$ is an arbitrary set then there exists $\Sigma(X, t) \subseteq \mathbb{P}$ such that

(I) $\Sigma(X, t) \neq \varnothing$ if and only if there exists a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ over $\mathfrak{M}$ such that $X = t[G]$;

(II) if $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set over $\mathfrak{M}$ and $t[G] = X$ then $G \subseteq \Sigma(X, t)$, $\Sigma(X, t) \in \mathfrak{M}[X]$ and the set G is also $\Sigma(X, t)$-generic over $\mathfrak{M}[X]$;

(III) the model $\mathfrak{M}[G]$ is thus a $\Sigma(X, t)$-extension of the model $\mathfrak{M}[X]$ in (II);

(IV) in addition, in (II), if a set $G \subseteq \Sigma(X, t)$ is $\Sigma(X, t)$-generic over $\mathfrak{M}$ then $t[G] = X$.

There naturally appears the problem of generalizing this Σ-construction to arbitrary sets X not necessary satisfying the relation $X \subseteq \mathfrak{M}$. This article is devoted to the problem: We obtain the desired generalization, basing on the auxiliary forcing $\Sigma^+(X, t)$ that consists of "superconditions"—pairs of the form $\langle p, a \rangle$, where $p \in \mathbb{P}$ and a is the function with finite domain connecting the elements of X with their names. This generalization (Theorem 10) is direct with respect to conditions (I) and (III) and partial as regards (II) and (IV). The technique used enables us to obtain more successful generalizations than those in [1]. For example, (III) was obtained in [1] by a convolution forcing in combination with $\mathbb{P}$ (see Section 8); for this reason, the structure of the final forcing for (III) has no definite, clear description in [1] which would be comparable to our $\Sigma^+(X, t)$. We assume that these results will find applications in the study of such modern models of set theory as models with encoding effect in [4].

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2. The Main Assumptions and Definitions

Definition 1. Henceforth, we assume that

$\mathfrak{M}$ is a countable transitive model of **ZFC**;

$\mathbb{P} \in \mathfrak{M}$ is a forcing (i.e., a partially ordered set); the elements $p \in \mathbb{P}$ are called conditions, and the relation $p \leq q$ means that p is a stronger condition;

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$t \in \mathfrak{M}$ is the $\mathbb{P}$-name of a transitive set (so that $\mathbb{P}$ forces the assertion: "the set t is transitive);

X is a finite and countable transitive set not necessarily satisfying $X \in \mathfrak{M}$ or $X \subseteq \mathfrak{M}$.[1)]

As regards forcing, we keep to the notations of [3, 9.2]. In particular, $\breve{x}$ is the canonical name of $x \in \mathfrak{M}$. If t is a name and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then $t[G]$ stands for the G-interpretation of t, so that

$$\mathfrak{M}[G] = \{t[G] : t \in \mathfrak{M} \text{ is a name}\}.$$

Denote by $\Vdash$ the forcing relation $\Vdash_{\mathbb{P}}^{\mathfrak{M}}$ over $\mathfrak{M}$.

A condition $p \in \mathbb{P}$ *solves* a formula Φ if either $p \Vdash \Phi$ or $p \Vdash \neg\, \Phi$.

DEFINITION 2. If $G \subseteq \mathfrak{M}$ is generic over $\mathfrak{M}$, t is a name, and $X = t[G]$; then $\mathfrak{M}(X)$ is the least transitive model of the theory **ZF** (not necessarily of **ZFC**) containing X (and all elements of the transitive closure of X) and all sets from $\mathfrak{M}$.[2)] Clearly, $\mathfrak{M} \subseteq \mathfrak{M}(X) \subseteq \mathfrak{M}[G]$.

For $\mathbb{P}$-names s and t, the relation $s \prec t$ will mean that s occurs in t as the name of a possible element of $t[G]$. Then $\mathrm{PE}_t = \{s : s \prec t\}$ (the set of all "potential elements" of $t[G]$) belongs to $\mathfrak{M}$, and if $G \subseteq \mathbb{P}$ is generic over $\mathfrak{M}$ then

$$t[G] = \{s[G] : s \in \mathrm{PE}_t \wedge \exists p \in G (p \Vdash s \in t)\}.$$

Let $d \subseteq \mathrm{PE}_t$. A condition $p \in \mathbb{P}$ is called *d-complete* if

(1) $p \Vdash s \in t$ for all $s \in d$,

(2) p solves each formula of the form $s \in s'$ and $s = s'$, where $s, s' \in d$.

3. Superconditions and the Set Σ^+

In this section, we give the definitions leading to the main theorem.

DEFINITION 3. Denote by $\mathbb{P}^+(X, t)$ the set of all pairs $\langle p, a \rangle$ (called *superconditions*) such that $p \in \mathbb{P}$, a is a function, $\operatorname{dom} a \subseteq \mathrm{PE}_t$ is a finite set, $\operatorname{ran} a \subseteq X$, the condition p is $(\operatorname{dom} a)$-complete, and, moreover, $a(\breve{x}) = x$ for all $x \in \mathfrak{M}$ with $\breve{x} \in \operatorname{dom} a$.

Order $\mathbb{P}^+(X, t)$ as follows: $\langle p, a \rangle \le \langle p', a' \rangle$ (i.e., the supercondition $\langle p, a \rangle$ is stronger) if $p \le p'$ in $\mathbb{P}$ and a extends a' as a function.

For example, if $p \in \mathbb{P}$ then $\langle p, \varnothing \rangle \in \mathbb{P}^+(X, t)$.

If $\langle p, a \rangle \in \mathbb{P}^+(X, t)$ then refer to the component p as a *condition* and to the component a as an *assignment* because a assigns names to sets. Superconditions need not belong to $\mathfrak{M}$ but the forcing $\mathbb{P}^+(X, t)$ obviously belongs to the wider model $\mathfrak{M}(X)$.

Lemma 4. *If $\langle p, a \rangle \in \mathbb{P}^+(X, t)$, $q \in \mathbb{P}$, $q \le p$, then $\langle q, a \rangle \in \mathbb{P}^+(X, t)$.*

The following definition introduces the set $\Sigma^+(X, t)$ of all superconditions $\langle p, a \rangle$ that do not force any assertions incompatible with the assumption of the existence of a generic set $G \subseteq \mathbb{P}$ over $\mathfrak{M}$ such that $X = t[G]$ and $a(s) = s[G]$ for all $s \in \operatorname{dom} a$.

DEFINITION 5. Define $\Sigma^+_\gamma(X, t) \subseteq \mathbb{P}^+(X, t)$ by transfinite induction on $\gamma \in \mathbf{Ord}$. The definition includes the three items:

1. The set $\Sigma^+_0(X, t)$ consists of all superconditions $\langle p, a \rangle \in \mathbb{P}^+(X, t)$ for which the following holds: if $s, s' \in \operatorname{dom} a$ then the relations $p \Vdash s \in (\text{or} =)\, s'$ and $a(s) \in$ (respectively, $=$) $a(s')$ are equivalent.

2. If $\gamma \in \mathbf{Ord}$ then $\Sigma^+_{\gamma+1}(X, t)$ consists of all superconditions $\langle p, a \rangle \in \Sigma^+_\gamma(X, t)$ such that, for every set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, dense in $\mathbb{P}$, every $s \in \mathrm{PE}_t$, and every $x \in X$, there exists a supercondition $\langle q, b \rangle \in \Sigma^+_\gamma(X, t)$ for which

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1)In fact, the assumption of the transitivity of X does not diminish generality since every set X is effectively encoded by the transitive closure $\mathrm{TC}(\{X\})$, i.e. the least transitive set containing X as an element. Namely, X is the only element of $\mathrm{TC}(\{X\})$ that belongs to no element of this set.

2)The notation $\mathfrak{M}(X)$ instead of $\mathfrak{M}[X]$, adopted in forcing theory, stresses that the axiom of choice need not hold in this model in contrast to $\mathfrak{M}[G]$.

(a) $\langle q,b\rangle \le \langle p,a\rangle$ (i.e., $\langle q,b\rangle$ is a stronger condition) and $q\in D$,

(b) $x\in \operatorname{ran} b$, and either $s\in \operatorname{dom} b$ or $q \Vdash s\notin t$.

3. Finally, if an ordinal λ is defined then $\Sigma^+_\lambda(X,t)=\bigcap_{\gamma<\lambda}\Sigma^+_\gamma(X,t)$.

The sequence of sets $\Sigma^+_\lambda(X,t)$ decreases by construction. Therefore, there exists an ordinal $\lambda = \lambda(X,t)$ such that $\Sigma^+_{\lambda+1}(X,t)=\Sigma^+_\lambda(X,t)$; put $\Sigma^+(X,t)=\Sigma^+_\lambda(X,t)$.

Lemma 6. *If $\langle p,a\rangle \in \Sigma^+(X,t)$, the set $D\in \mathfrak{M}$, $D\subseteq \mathbb{P}$, is dense in $\mathbb{P}$, and $s\in \mathrm{PE}_t$, $x\in X$ then there exists a pair $\langle q,b\rangle\in\Sigma^+(X,t)$ for which $\langle q,b\rangle\le\langle p,a\rangle$, $q\in D$, $x\in\operatorname{ran} b$, and either $s\in\operatorname{dom} b$ or $q\Vdash s\notin t$.*

PROOF. $\Sigma^+(X,t)=\Sigma^+_{\lambda(X,t)}(X,t)=\Sigma^+_{\lambda(X,t)+1}(X,t)$. □

Let us prove that $\Sigma^+(X,t)$ is closed under weakening.

Lemma 7. *Suppose that $\langle p,a\rangle\in\Sigma^+(X,t)$. Then*

(i) *if $\langle q,b\rangle\in\Sigma^+_0(X,t)$ and $\langle p,a\rangle\le\langle q,b\rangle$ then $\langle q,b\rangle\in\Sigma^+(X,t)$;*

(ii) *if $q\in\mathbb{P}$, $q\ge p$, and $\langle q,a\rangle\in\mathbb{P}^+(X,t)$, then $\langle q,a\rangle\in\Sigma^+(X,t)$.*

PROOF. (i) Prove that $\langle q,b\rangle\in\Sigma^+_\gamma(X,t)$ by induction on γ. The case of $\gamma=0$ and the limit step are rather simple. Consider the step $\gamma\to\gamma+1$. By the induction hypothesis, $\langle q,b\rangle\in\Sigma^+_\gamma(X,t)$. Assume that $D\in\mathfrak{M}$, $D\subseteq\mathbb{P}$, is dense in $\mathbb{P}$, $s\in\mathrm{PE}_t$, and $x\in X$. Since $\langle p,a\rangle\in\Sigma^+_{\gamma+1}(X,t)$, by construction, there is a stronger supercondition $\langle r,c\rangle\in\Sigma^+_\gamma(X,t)$ for which $\langle r,c\rangle\le\langle p,a\rangle$, $r\in D$, $x\in\operatorname{ran} c$, and either $s\in\operatorname{dom} b$ or $r\Vdash s\notin t$. But then $\langle r,c\rangle\le\langle q,b\rangle$; therefore, the supercondition $\langle r,c\rangle$ also guarantees that $\langle q,b\rangle\in\Sigma^+_{\gamma+1}(X,t)$.

(ii) It follows from $\langle q,a\rangle\in\mathbb{P}^+(X,t)$ that the supercondition $\langle q,a\rangle$ belongs to $\Sigma^+_0(X,t)$ together with $\langle p,a\rangle$. It remains to refer to (i). □

4. The Main Theorem

One more definition precedes our main result:

DEFINITION 8. If $G\subseteq\mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then denote by $a[G]$ the function defined on the set $\operatorname{dom} a[G]=\mathrm{PE}_t[G]=\{s\in\mathrm{PE}_t : s[G]\in t[G]\}$ by the equality $a[G](s)=s[G]$ for all names $s\in\mathrm{PE}_t[G]$. If $\Gamma\subseteq\mathbb{P}^+(X,t)$ then put

$$\Gamma{\downarrow}=\{p\in\mathbb{P}:\exists a(\langle p,a\rangle\in\Gamma)\}\quad \text{(the projection of }\Gamma\text{ to }\mathbb{P}),$$

$$A[\Gamma]=\{a:\exists p(\langle p,a\rangle\in\Gamma)\}\quad \text{(all assignments in pairs of }\Gamma),$$

$$a[\Gamma]=\bigcup A[\Gamma]\quad \text{(the union of all assignments in }\Gamma).$$

Lemma 9. *If $G\subseteq\mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then $\operatorname{ran} a[G]=t[G]$.*

Let us formulate the main theorem of the article.

Theorem 10 (cf. (I)–(IV) of the Introduction). *Under the assumptions of Definition* 1,

(i) *$\Sigma^+(X,t)\ne\varnothing$ is equivalent to the fact that there exists a $\mathbb{P}$-generic set $G\subseteq\mathbb{P}$ over $\mathfrak{M}$ for which $X=t[G]$;*

(ii) *if $G\subseteq\mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $t[G]=X$ then*

$$G^+=\{\langle p,a\rangle\in\Sigma^+(X,t):p\in G\wedge a\subset a[G]\}$$

is a $\Sigma^+(X,t)$-generic set over $\mathfrak{M}(X)$ and $G=G^+{\downarrow}$;

(iii) *hence, in* (ii), *$\mathfrak{M}[G]$ is a $\Sigma^+(X,t)$-generic extension of $\mathfrak{M}(X)$;*

(iv) *if $\Gamma\subseteq\Sigma^+(X,t)$ is a $\Sigma^+(X,t)$-generic set over $\mathfrak{M}(X)$ then $H=G^+{\downarrow}\subseteq\mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$; moreover, $t[H]=X$ and $a[\Gamma]=a[H]$.*

This theorem will be proved in Sections 5 and 6.

5. The Boundedness Lemma

Here we prove part (i) of Theorem 10. In particular, we will prove that if $X = t[G]$, where $G \subseteq \mathbb{P}$ is a generic set then the transfinite length of the construction of Definition 5 is an ordinal in $\mathfrak{M}$ (the key Lemma 13). Continue arguing under the conditions of Lemma 1.

Lemma 11. *Suppose that $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $t[G] = X$. If $\langle p, a\rangle \in \mathbb{P}^+(X,t)$, $a \subseteq a[G]$, and $p \in G$ then $\langle p, a\rangle \in \Sigma^+(X,t)$. In particular, if $p \in G$ then $\langle p, \varnothing\rangle \in \Sigma^+(X,t)$.*

Proof. Let us prove by induction on γ that $\langle p, a\rangle \in \Sigma^+_\gamma(X,t)$. Let $\gamma = 0$. In view of the $(\operatorname{dom} a)$-completeness of p, if $s, s' \in \operatorname{dom} a$ then p solves the formula $s \in s'$. If $p \Vdash s \in s'$ then $s[G] \in s'[G]$; therefore, $a(s) \in a(s')$ because $a \subseteq a[G]$. Similarly, if $p \Vdash s \notin s'$ then $a(s) \notin a(s')$.

The step $\gamma \to \gamma + 1$. Suppose on the contrary that $\langle p, a\rangle \notin \Sigma^+_{\gamma+1}(X,t)$ but $\langle p, a\rangle \in \Sigma^+_\gamma(X,t)$ by the induction hypothesis. By definition, there exist $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, dense in $\mathbb{P}$ and elements $s \in \mathrm{PE}_t$, $x \in X$ such that no supercondition $\langle q, b\rangle \in \Sigma^+_\gamma(X,t)$ satisfies all relations $\langle q, b\rangle \leq \langle p, a\rangle$, $q \in D$, $x \in \operatorname{ran} b$, and either $s \in \operatorname{dom} b$ or $q \Vdash s \notin t$. By genericity, there is $q \in G \cap D$ such that $q \leq p$. Since $t[G] = X$, there is a finite assignment $b : (\operatorname{dom} b \subseteq \mathrm{PE}_t) \to X$ satisfying

(1) $a \subseteq b$, $x \in \operatorname{ran} b$,

(2) $r[G] \in t[G]$ and $b(r) = r[G]$ for every name $r \in \operatorname{dom} b$,

(3) either $s[G] \notin t[G]$ or $s \in \operatorname{dom} b$.

There is a stronger condition $q' \in G \cap D$ such that if $s[G] \notin t[G]$ then $q' \Vdash s \notin t$ and, moreover, the condition q' is $(\operatorname{dom} b)$-complete. Then $\langle q', b\rangle \in \Sigma^+_\gamma(X,t)$ by the induction hypothesis; a contradiction.

The limit step is obvious. □

Lemma 12. *If $\langle p, a\rangle \in \Sigma^+(X,t)$ then there exists a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ over $\mathfrak{M}$ such that $p \in G$ and $t[G] = X$.*

Proof. Both $\mathfrak{M}$ and X are countable; therefore, Lemma 6 enables us to construct some increasing sequence of superconditions

$$\langle p, a\rangle = \langle p_0, a_0\rangle \geq \langle p_1, a_1\rangle \geq \langle p_2, a_2\rangle \geq \cdots$$

in $\Sigma^+(X,t)$ such that the sequence $\{p_n\}_{n\in\omega}$ intersects each set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, dense in $\mathbb{P}$. Then $G = \{p \in \mathbb{P} : \exists n (p_n \leq p)\}$ is a generic set; moreover, the union $\varphi = \bigcup_n a_n : \operatorname{dom}\varphi \to X$ of all assignments a_n satisfies the conditions (1) $\operatorname{ran}\varphi = X$, $\operatorname{dom}\varphi \subseteq \mathrm{PE}_t$; and (2) for every $s \in \mathrm{PE}_t$, either $s \in \operatorname{dom}\varphi$, and then $s[G] \in t[G]$; or $q \Vdash s \notin t$ for some $q \in G$, and then $s[G] \notin t[G]$. In view of the transitivity of both sets $t[G] = \{s[G] : s \in \operatorname{dom}\varphi\}$ and $X = \operatorname{ran}\varphi$, for deducing $t[G] = X$ it suffices to check that $\varphi(s) \in \varphi(s') \iff s[G] \in s'[G]$ for all $s, s' \in \operatorname{dom}\varphi$. By the construction of φ, there is an index n for which $s, s' \in \operatorname{dom} a_n$. By definition, $p_n \in G$ is $(\operatorname{dom} a_n)$-complete, and so p_n solves $s \in s'$.

If $p_n \Vdash s \in s'$ then $s[G] \in s'[G]$; on the other hand, we have $\varphi(s) = a_n(s) \in a_n(s') = \varphi(s')$ since $\langle p_n, a_n\rangle \in \Sigma^+_0(X,t)$.

Similarly, if $q \Vdash s \notin s'$ then $s[G] \notin s'[G]$ and $\varphi(s) \notin \varphi(s')$. □

By the key Lemma 13, if $\Sigma^+(X,t) \neq \varnothing$ then the ordinals of the form $\lambda(X,t)$ as in Definition 5 are bounded in the model $\mathfrak{M}$.

Lemma 13 (the boundedness lemma). *There exists an ordinal $\lambda^*(t) \in \mathfrak{M}$ such that $\lambda(t[G], t) < \lambda^*(t)$ for every $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ over $\mathfrak{M}$. Therefore, $\Sigma^+(t[G], t) \in \mathfrak{M}$ for every such set G.*

Proof. Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $\mathfrak{M}$. Then $X = t[G] \in \mathfrak{M}[G]$, and so $\lambda(X,t)$ is an ordinal in $\mathfrak{M}$ and there are an ordinal $\lambda_p(t) \in \mathfrak{M}$ and a condition $p \in G$ such that $\lambda(X,t) = \lambda_p(t)$. Put $\lambda^*(t) = \sup_{p\in\mathbb{P}} \lambda_p(t)$. The second assertion of the lemma follows form the first, since $\Sigma^+(X,t)$ is the result of the inductive construction of the length $\lambda^*(t)$ in $\mathfrak{M}$. □

Corollary 14 (Theorem 10(i)). *Under the conditions of Theorem* 10, *the following three propositions are equivalent:*

(i) *there exists a* $\mathbb{P}$*-generic set* $G \subseteq \mathbb{P}$ *over* $\mathfrak{M}$ *for which* $t[G] = X$;

(ii) $\Sigma^+(X,t) \neq \varnothing$;

(iii) $\Sigma^+_{\lambda^*(t)}(X,t) = \Sigma^+_{\lambda^*(t)+1}(X,t) \neq \varnothing$.

Proof. Use Lemmas 11–13. □

6. Proof of the Main Theorem

Let us prove items (ii)–(iv) of Theorem 10. Continue arguing under the assumptions of Definition 1.

Lemma 15 (Theorem 10(iv)). *If* $\Gamma \subseteq \Sigma^+(X,t)$ *is a* $\Sigma^+(X,t)$*-generic set over* $\mathfrak{M}(X)$ *then* $H = \Gamma{\downarrow} \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$*; moreover,* $t[H] = X$ *and* $a[\Gamma] = a[H]$.

Proof. By Lemma 6, if $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, is dense in $\mathbb{P}$ then $D^* = \{\langle p, a\rangle \in \Sigma^+(X,t) : p \in D\}$ is dense in $\Sigma^+(X,t)$ and belongs to $\mathfrak{M}(X)$. This implies that H is generic.

Further, if $\langle p, a\rangle \in \Gamma \subseteq \Sigma^+(X,t)$ then, by definition, $\operatorname{dom} a \subseteq \mathrm{PE}_t$ is a finite set, and if $s \in \operatorname{dom} a$ then $p \Vdash s \in t$. Since $p \in H$, we have $s[H] \in t[H]$, whence $s \in \mathrm{PE}_t[H]$. On the other hand, if $s \in \mathrm{PE}_t[H]$ and $x \in X$ then, by Lemma 6, there is a supercondition $\langle q, b\rangle \in \Gamma$ for which $s \in \operatorname{dom} b$ and $x \in \operatorname{ran} b$. Therefore, $a[\Gamma]$ maps $\mathrm{PE}_t[H]$ onto X.

By definition, if $\langle p, a\rangle \in \Gamma$ and $s, s' \in \operatorname{dom} a$ then p solves each of the formulas $s \in s'$ and $s = s'$; moreover, the relation $p \Vdash s \in s'$ is equivalent to the fact that $a(s) \in a(s')$; the same holds for $=$. Consequently, if $s, s' \in \mathrm{PE}_t[H]$ then the relation $s[H] = s'[H]$ is equivalent to $a[\Gamma](s) = a[\Gamma](s')$. We conclude that $a[\Gamma] = a[H]$.

Finally, $t[H] = \operatorname{ran} a[H] = \operatorname{ran} a[\Gamma] = X$. □

Lemma 16 (Theorem 10(ii)). *If* $G \subseteq \mathbb{P}$ *is a* $\mathbb{P}$*-generic set over* $\mathfrak{M}$ *and* $X = t[G]$ *then*

$$G^+ = \{\langle p, a\rangle \in \Sigma^+(X,t) : p \in G \wedge a \subset a[G]\}$$

is $\Sigma^+(X,t)$*-generic over* $\mathfrak{M}(X)$*; moreover,* $G = G^+{\downarrow}$.

Proof. Suppose that there is a condition $p_0 \in G$ that forces the opposite; i.e., for every $\mathbb{P}$-generic set $H \subseteq \mathbb{P}$ over $\mathfrak{M}$, if $X = t[H]$ and $p_0 \in H$ then the set H^+ is not $\Sigma^+(X,t)$-generic over $\mathfrak{M}(X)$. By Lemma 11, $\langle p_0, \varnothing\rangle \in \Sigma^+(X,t)$.

Suppose that $\Gamma \subseteq \Sigma^+(X,t)$ is $\Sigma^+(X,t)$-generic over $\mathfrak{M}(X)$ and contains $\langle p_0, \varnothing\rangle$. Then $H = \Gamma{\downarrow}$ is $\mathbb{P}$-generic over $\mathfrak{M}(X)$ and $t[H] = X$ by Lemma 15. It remains to check that $\Gamma = H^+$; i.e., to deduce for every supercondition $\langle p, a\rangle \in \Sigma^+(X,t)$ that $\langle p, a\rangle \in \Gamma$ is equivalent to $p \in H \wedge a \subset a[H]$.

If $\langle p, a\rangle \in \Gamma$ then, by definition, $p \in H = \Gamma{\downarrow}$ and $a \subseteq a[\Gamma]$ but $a[\Gamma] = a[H]$ by Lemma 15.

Conversely, suppose that $\langle p, a\rangle \in \Sigma^+(X,t)$, $p \in H$, and $a \subset a[H] = a[\Gamma]$. We assert that $\langle p, a\rangle \in \Gamma$. If $s \in \operatorname{dom} a$ then $a \in \operatorname{dom} a[\Gamma]$; therefore, by definition, there exists a supercondition $\langle p_s, a_s\rangle \in \Gamma$ for which $a \in \operatorname{dom} a_s$. Consequently, there also exists a supercondition $\langle q, b\rangle \in \Gamma$ such that $q \leq p$ and $\operatorname{dom} a \subseteq \operatorname{dom} b$. Then $a \subset b$ since $a, b \subset a[\Gamma]$. Thus, the supercondition $\langle q, b\rangle \in \Gamma$ is stronger than $\langle p, a\rangle \in \Sigma^+(X,t)$, whence $\langle p, a\rangle \in \Gamma$. □

Lemma 17 (Theorem 10(iii)). *If* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$ *and* $X = t[G]$ *then* $\mathfrak{M}[G]$ *is a* $\Sigma^+(X,t)$*-generic extension of* $\mathfrak{M}(X)$.

Proof. $\mathfrak{M}[G] = \mathfrak{M}(X)[G^+]$ under the conditions of Lemma 16. □

This finishes the proof of Theorem 10.

7. An Example

Continue arguing under the assumptions of Definition 1. Put

$$\Sigma(X,t) = \Sigma^+(X,t){\downarrow} = \{p \in \mathbb{P} : \langle p, \varnothing\rangle \in \Sigma^+(X,t)\}.$$

Then $\Sigma(X,t) \subseteq \mathbb{P}$, and if $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $X = t[G]$ then $G \subseteq \Sigma(X,t)$ by Lemma 11. But is it true, by analogy with assertion (II) in the Introduction, that G is $\Sigma(X,t)$-generic over $\mathfrak{M}(X)$? The following example answers this in the negative.

Example 18. Take as $\mathbb{P}$ the countable power of a Cohen forcing with finite support. Thus, the conditions $p \in \mathbb{P}$ are functions with finite domain of definition $\operatorname{dom} p \subseteq \omega \times \omega$ and range $\operatorname{ran} p \subseteq \omega$. Each generic set $G \subseteq \mathbb{P}$ adjoins the objects $x_n[G] \in \omega^\omega$ defined so that $x_n[G](i) = r$ when there exists a condition $p \in G$ such that $p(n,i) = r$. Denote by $\breve{x}_n$ the canonical name for $x_n[G] = \breve{x}_n[G]$ and designate as t the name of the whole set of these objects $t[G] = \{\breve{x}_n[G] : n \in \omega\}$. In other words, $\mathfrak{M}(t[G])$ is a symmetric Cohen extension in which the axiom AC fails and $t[G] \subseteq \omega^\omega$ is an infinite Dedekind finite set in the model $\mathfrak{M}(t[G])$.

The sets of the form $t[G]$ are nontransitive; therefore, to fit Definition 1, define the transitive closure $U(X) = X \cup U$, where

$$U = \omega \cup \{\{m,n\} : m,n \in \omega\} \cup \{\langle m,n\rangle : m,n \in \omega\}$$

for every $X \subseteq \omega^\omega$, and denote by t' the canonical name of the transitive set $t'[G] = \{\breve{x}_n[G] : n \in \omega\} \cup U$.

Since $U \in \mathfrak{M}$, we may identify each element $u \in U$ with its canonical name $\breve{u}$. Then $\mathrm{PE}_{t'} = \{\breve{x}_n : n \in \omega\} \cup U$.

Lemma 19. *If $p \in \mathbb{P}$ and $n,k,r \in \omega$ then $p \Vdash \breve{x}_n[G](k) = r$ is equivalent to the fact that $\langle n,k\rangle \in \operatorname{dom} p$ and $p(n,k) = r$.*

If $X \subseteq \omega^\omega$ then, by Definition 3, the set of superconditions $\mathbb{P}^+(X \cup U, t')$ consists of all pairs $\langle p,a\rangle$ such that

(1) $p \in \mathbb{P}$,

(2) a is a function, and $\operatorname{dom} a \subseteq \{\breve{x}_n : n \in \omega\} \cup U$ is a finite set,

(3) $\operatorname{ran} a \subseteq X \cup U$,

(4) $a(u) = u$ for all $u \in U \cap \operatorname{dom} a$ and $a(\breve{x}_n) \in X$ for all $\breve{x}_n \in \operatorname{dom} a$,

(5) (completeness by Definition 3) if a name $\breve{x}_n$ and a pair $\langle k,r\rangle$ $(n,k,r \in \omega)$ belong to $\operatorname{dom} a$ then the condition p solves the formula "$\breve{x}_n[G](k) = r$," or, which is equivalent, $\langle n,k\rangle \in \operatorname{dom} p$.

By Definition 8, if a set $G \subseteq \mathbb{P}$ is generic over $\mathfrak{M}$ then

$$a[G] : \{\breve{x}_n : n \in \omega\} \cup U \xrightarrow{\text{on}} X \cup U$$

is defined through $a[G](u) = u$ for $u \in U$ but $a[G](\breve{x}_n) = \breve{x}_n[G]$ for all n.

Recall that $\Sigma(X \cup U, t') = \{p \in \mathbb{P} : \langle p, \varnothing\rangle \in \Sigma^+(X \cup U, t')\}$.

Lemma 20. *In the case under consideration, if $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set over $\mathfrak{M}$ and $X = t[G]$ then*

(i) $\Sigma(X \cup U, t') = \mathbb{P}$;

(ii) $\Sigma^+(X \cup U, t')$ *consists of all superconditions $\langle p,a\rangle \in \mathbb{P}^+(X \cup U, t')$ that if a name $\breve{x}_n$ and a pair $\langle k,r\rangle$ belong to $\operatorname{dom} a$ then (1) $\langle n,k\rangle \in \operatorname{dom} p$; (2) $p(n,k) = r$ is equivalent to $a(\breve{x}_n)(k) = r$.*

Proof. (i) Let $p \in \mathbb{P}$. By Lemma 11, for inferring $p \in \Sigma(X \cup U, t')$, we must find a $\mathbb{P}$-generic set $G' \subseteq \mathbb{P}$ over $\mathfrak{M}$ such that $t[G'] = X$ and $p \in G'$. The set $t[G] = \{\breve{x}_m[G] : m \in \omega\}$ is topologically dense in ω^ω. Hence, there is a bijection $\pi : N = \{n : \exists k(\langle n,k\rangle \in \operatorname{dom} p)\} \to \omega$ such that if $\langle n,k\rangle \in \operatorname{dom} p$ (and so $n \in N$) then $\breve{x}_{\pi(n)}[G](k) = p(n,k)$.

Using the invariance of $\mathbb{P}$ under permutations, we obtain a generic set $G' \subseteq \mathbb{P}$ such that $\breve{x}_{\pi(n)}[G] = \breve{x}_n[G']$ for all $n \in N$, $t[G'] = t[G] = X$, and even $x_m[G] = x_m[G']$ for all but possibly finitely many indices $m \in \omega$. Then $p \in G'$, q.e.d.

(ii) The proof is similar. $\square$

By (i), the forcing $\Sigma(X \cup U, t')$ is identical to the given forcing $\mathbb{P}$. But G cannot be $\mathbb{P}$-generic over $\mathfrak{M}(X)$ and even over any smaller model $\mathfrak{M}[\breve{x}_n[G]]$ because $X = t[G] = \{\breve{x}_n[G] : n \in \omega\}$. This implies the negative answer to the question at the beginning of the section.

We can prove using (ii) that $\Sigma^+(X \cup U, t')$ contains a coinitial subset in $\mathfrak{M}(X)$ that is order isomorphic to $\mathbf{BColl}(\{\breve{x}_n : n \in \omega\}, X)$ (a bijective convolution forcing) consisting of all partial bijections $\{\breve{x}_n : n \in \omega\} \to X$.

Corollary 21. *In the case under consideration, the whole model* $\mathfrak{M}[G]$ *is a* $\mathbf{BColl}(\{\breve{x}_n : n \in \omega\}, X)$*-generic extension of* $\mathfrak{M}(X)$.

8. On Grigorieff's Proof

For comparing our approach with the approach to intermediate models in [1], here we briefly expose the proof of the following more abstract analog of Lemma 17, which was given by Grigorieff.

Theorem 22. *Under the assumptions of Definition* 1*, if* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$ *and* $X = t[G]$ *then* $\mathfrak{M}[G]$ *is a generic extension of* $\mathfrak{M}(X)$.

The proof is based on the following lemma (Theorem 2 in [1, 2.14]):

Lemma 23. *Suppose that* $\mathbb{P}$ *is a forcing in* $\mathfrak{M}$ *and* $G \subseteq \mathbb{P}$ *is generic over* $\mathfrak{M}$*. If* $x \in \mathfrak{M}[G]$ *and* $x \subseteq \mathfrak{M}$ *then* $\mathfrak{M}[x]$ *is a generic extension of* $\mathfrak{M}$.

Proof (a short one; see [1, 2.13 and 2.14] for details). Take a name $t \in \mathfrak{M}$ for which $x = t[G]$. Assume without loss of generality that the forcing $\mathbb{P}$ is equal to $\mathscr{B} \setminus \{0\}$, where $\mathscr{B}$ is a complete Boolean algebra (CBA) in $\mathfrak{M}$. Denote by $\mathscr{A}$ the complete subalgebra generated by t; this is the least CBA $\mathscr{A} \subseteq \mathscr{B}$ containing all sets of the form $[[\check{y} \in t]]$, $y \in \mathfrak{M}$. Let $\mathbb{Q} = \mathscr{A} \setminus \{0\}$. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then $H = G \cap \mathbb{Q}$ is $\mathbb{Q}$-generic over $\mathfrak{M}$ and $\mathfrak{M}[x] = \mathfrak{M}[H]$. $\square$

Proof of Theorem 22. Take an ordinal $\alpha \in \mathfrak{M}$ greater than the von Neumann rank of X, and let $Y = V_\alpha \cap \mathfrak{M}(X)$ (and $X \subseteq Y$). Consider the set $H \subseteq \mathbb{C} = \mathbf{Coll}(\omega, Y)$ generic over $\mathfrak{M}[G]$. Then $\mathfrak{M}[G][H]$ is a generic extension of $\mathfrak{M}$ by the theorem on double forcing; moreover, $\mathfrak{M}(X)[H] = \mathfrak{M}[r]$ for some $r \in \omega^\omega$. Applying assertion (III) of the Introduction, we conclude that the whole model $\mathfrak{M}[G][H]$ is a generic extension of $\mathfrak{M}[r]$. But $\mathfrak{M}[r] = \mathfrak{M}(X)[H]$ is a generic extension of $\mathfrak{M}(X)$, and so $\mathfrak{M}[G][H]$ is a generic extension of $\mathfrak{M}(X)$ by the same theorem on double forcing.

We have $G \subseteq \mathfrak{M}(X)$ and $\mathfrak{M}(X) \subseteq \mathfrak{M}(X)[G] = \mathfrak{M}[G] \subseteq \mathfrak{M}[r][H]$. Thus, $\mathfrak{M}[G]$ is an intermediate model between $\mathfrak{M}(X)$ and its generic extension $\mathfrak{M}[r]G[H]$. It remains to apply Lemma 23 to the models $\mathfrak{M}(X) \subseteq \mathfrak{M}[G] \subseteq \mathfrak{M}[G][H]$ as the models $\mathfrak{M} \subseteq \mathfrak{M}[x] \subseteq \mathfrak{M}[G]$. $\square$

9. An Application: Borelness of Some Sets

Our results can be used for proving some assertions on the Borelness of some sets for which the usual methods only give the membership in the wider class $\mathbf{\Sigma}^1_1$ of analytic (or Suslin, see [3, Chapter 4; 5]) sets. We confine exposition to one result which we first state informally:

($*$) *If* $\mathfrak{M}$ *is a countable transitive model of* **ZFC** *then the set* $S(\mathfrak{M})$ *of all sets belonging to the generic extensions of* $\mathfrak{M}$ *is Borel.*

The informality is that, for the exact definition of the space in which we assert the Borelness of $S(\mathfrak{M})$ would require more space than we can share here. Therefore, consider only the special (but rather representative) case of the set $S'(\mathfrak{M}) = S(\mathfrak{M}) \cap \mathscr{P}(\omega^\omega)$ consisting of all (necessarily at most countable) sets $R \subseteq \mathscr{P}(\omega^\omega)$ lying in the generic extensions of $\mathfrak{M}$.

We will need to encode countable sets $R \subseteq \mathscr{P}(\omega^\omega)$. If $y \in \omega^\omega$ then let $R_y = \{(y)_n : n \in \omega\} \setminus \{(y)_0\}$, where $(y)_n \in \omega^\omega$ and $(y)_n(k) = y(2^n(2k+1)-1)$ for all n and k. Thus, $\{R_y : y \in \omega^\omega\}$ is the set of all at most countable sets $R \subseteq \omega^\omega$ (including the empty set).

Theorem 24. *If* $\mathfrak{M}$ *is a countable transitive model of* **ZFC** *then the set* $W = \{y \in \omega^\omega : R_y \in S(\mathfrak{M})\}$ *of all codes of the sets* $X \subseteq \mathscr{P}(\omega^\omega)$ *belonging to the generic extensions of* $\mathfrak{M}$ *is a Borel set.*

Proof. Let ϑ be the least ordinal not belonging to $\mathfrak{M}$. Define the set $U \in \mathfrak{M}$ as in Example 18. By Corollary 15, if $y \in \omega^\omega$ then for the fulfillment of $y \in W$ it is necessary and sufficient that there exist a forcing $\mathbb{P} \in \mathfrak{M}$ and a $\mathbb{P}$-name $t \in \mathfrak{M}$ satisfying one of the following two equivalent requirements:

(A) there exist an ordinal $\lambda < \vartheta$ and a sequence of sets $\Sigma^+_\gamma(X, t')$, $\gamma \leq \lambda + 1$, where $X = R_y \cup U$, as in Definition 5 such that $\Sigma^+_\lambda(X, t') = \Sigma^+_{\lambda+1}(X, t') \neq \varnothing$;

(B) for every ordinal $\lambda < \vartheta$ and every sequence of sets $\Sigma^+_\gamma(X,t')$, $\gamma \le \lambda+1$, where $X = R_y \cup U$, as in Definition 5, if $\Sigma^+_\lambda(X,t') = \Sigma^+_{\lambda+1}(X,t')$ then $\Sigma^+_\lambda(X,t') \neq \varnothing$.

Requirement (A) gives the $\mathbf{\Sigma}^1_1$-definition for W, and (B) gives its $\mathbf{\Pi}^1_1$-definition. Since the model $\mathfrak{M}$ is countable, the quantifiers with respect to $\mathbb{P}$, t, and λ do not violate Borelness. □

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