On the Hausdorff pantachy existence problem

Vladimir Kanovei [1] Vassily Lyubetsky [1]

[1] Laboratory of mathematical methods and models in bioinformatics, Institute for Information Transmission Problems, Russian Academy of Sciences,
kanovei@rambler.ru, lyubetsk@iitp.ru
http://lab6.iitp.ru/

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In the second half of XX Century two fundamental discoveries in mathematics were made; they might not be known in detail to the audience.

1. Two very different types of the real numbers (=reals) were discovered: **constructible reals** and **random reals**.
2. Some **absolutely undecidable problems** were discovered within one unique axiomatics (= the axiomatics of set theory = ZFC axiomatics).

As Luzin said on the latter (1925), "the answers to these problems are not known and will never be known".
Apparently even a general idea of such undecidable problems did not exist before Luzin. Moreover, opposite views prevailed.
I begin this talk with a short explanation of the two discoveries mentioned.

First discovery

A real number x (which we understood as, e.g., a set of natural numbers) is **constructible**, if (roughly speaking): there is a **formula** $\varphi(x, x_1, \ldots, x_n)$ and an n-tuple $\alpha_1, \ldots, \alpha_n$ of **well-ordered sets** of the **rational numbers**, such that

$$k \in x \iff \varphi(x, \alpha_1, \ldots, \alpha_n), \qquad \forall\, k\,.$$

Such well-orderings are called **ordinals** or transfinite numbers. Briefly, every constructible real x can be defined by an **individual formula** φ and an n-tuple $\alpha_1, \ldots, \alpha_n$ of **ordinals** α_i .
I will omit the accurate definition: it can be done by induction introduced by Goedel and extensively studied in the domain of real numbers by P.S. Novikov.

First discovery

I recall that, by definition, **Borel sets** (of reals) form the smallest σ-ring of sets containing all intervals. Every **Borel set** B of reals can be easily **coded by a real number**, say d, such that

$$B = B_d.$$

I will not give a precise definition of the coding but, for instance:

1. any open interval (a, b) is coded by a "fusion" of a and b (so that all digits of the real a occupy even positions, while the digits of b occupy odd positions);
2. an open set is coded as a union of the form $\bigcup_n (a_n, b_n)$: so that codes d_n of the intervals (a_n, b_n) are joined in the real d in the same manner as a matrix with the row d_n by a diagonal enumeration of all its elements.

And so on, for all Borel sets.

First discovery

Definition

A random real is any real number x, which avoids all Borel sets B_d **with null Lebesgue measure** and a **constructible code** d, i.e.

$$x \notin B_d\,, \quad \text{for all constructible codes } d \text{ with } \mu(B_d) = 0\,.$$

Second discovery

The axiomatics of Zermelo–Fraenkel ZFC was designed (1908 – 1925) with the aim to make mathematical proofs more accurate, and to codify all principles involved in the proofs. After many years of studies in all areas of mathematics it is firmly established that any mathematical argument can be converted to a proof based on the ZFC axioms, or, briefer, a proof in ZFC.

The language of ZFC (it contains a single symbol $\in$) and the axiomatics ZFC are unique in the following sense. Any mathematical argument and any construction can be formally expressed in ZFC in the same manner as any theorem or construction of the elementary geometry can be expressed in the language and axiomatics of the Euclidean geometry.

Second discovery

Moreover, the axiomatics of ZFC has some **computability properties**: there is a recursive function theory for arbitrary sets. We won't discuss it here.

In addition, some **effectiveness properties** in ZF $(= \mathrm{ZFC}$ sans the axiom of choice) have been discovered: we can prove $\exists x\, \psi(x)$, then there is a formal definition of such a set x by a formula φ, i.e., $\exists x \, \{\psi(x) \wedge [y \in x \Longleftrightarrow \varphi(y),\ \forall y]\}$.

This **concludes Hilbert's program of formalization of mathematics**.

Second discovery

Thus, any statement P for which P is unprovable and $\neg P$ is unprovable in ZFC, is absolutely undecidable.

Such statements exist, often among the simplest and natural — we'll give some examples below. How to to understand this situation is unclear, in fact they relate to the nature of the cognizing subject.

Remark

Below, by a "model" we understand "a model of ZFC".

It is clear that if there exist models M_1 and M_2, such that

$$\text{a sentence } P \text{ is true in } M_1 \text{ but false in } M_2$$

— then P is absolutely undecidable.

Remark

One cannot consider in ZFC, e.g., "the set of all groups" (since this leads to well-known paradoxes), yet one can consider "the set of all groups with the support (underlying set) of a restricted cardinality". This is pretty sufficient in typical cases.

RESULTS

As said before: there exist absolutely undecidable problems! This means the following, for example:

Theorem

There exist models M_1 and M_2 such that it is true in M_1 that "all reals are constructible", and it is true in M_2 that "the set of all random reals has the full Lebesgue measure (=measure 1 inside the unit interval [0,1])".

Later we consider some other theorems of this kind. But now we have a theorem about random reals:

RESULTS

Random reals are **locally undistinguishable from each other** in the following sense:

Theorem

Let x be a random real and $P(x)$ be any its property. Then there is a Borel set X of positive measure such that:
$x \in X$,
and every random real $x' \in X$ satisfies the same property $P(x')$.

Definition

Projective sets are those sets, which can be obtained from Borel sets by any finite sequence of operations of projection and complement.

Basically all "conventional" (e.g., known from the analysis, geometry, algebra and so on) sets of reals are projective.

Theorem

There exist models M_1 and M_2 such that:
it is true in M_1 that "there are non-measurable projective sets", while in M_2 "all projective sets are measurable".

RESULTS

We let:

- **CA** be all sets, which can be obtained from Borel sets by **just a single operation of projection and just a single operation of complement**,
- $\mathbf{A}_2$ be all sets which can be obtained from **CA** sets by **just a single operation of projection**.

A more precise form of the last theorem is as follows:

Theorem

There exist models M_1 and M_2 such that:
it is true in M_1 that "there is a non-measurable individually defined projective set $\mathbf{A}_2$",
while in M_2 "all projective sets are measurable".

RESULTS

Consider the following three properties of a family T of sets of reals:

T **has the kernel property** $\iff$ any set $X \in T$ is either countable or it contains a subset Y, Borel isomorphic to the interval $[0,1]$;

T **has the measurability property** $\iff$ all sets $X \in T$ are Lebesgue measurable;

T **has the Baire property** $\iff$ all sets $X \in T$ have the Baire property.

The latter means that X coincides with a Borel set modulo a set Y of the 1st category (a meager set by Bourbaki), that is, Y is a countable union of nowhere dense sets.

RESULTS

It turns out that the "imaginary" notion of random reals essentially helps to prove some theorems in any model (that is, independently of any axioms). For example:

Theorem (Lyubetsky)

"$\mathbf{CA}$ *sets have the kernel property*" $\Longrightarrow$
$\Longrightarrow$ "$\mathbf{A}_2$ *sets have the measurability property*".

Theorem (Lyubetsky, Mansfield, Solovay, Stern)

"$\mathbf{CA}$ *sets have the kernel property*" $\Longrightarrow$
$\Longrightarrow$ "$\mathbf{A}_2$ *sets have the measurability property*" $\Longrightarrow$
$\Longrightarrow$ "$\mathbf{A}_2$ *sets have the Baire property*".

RESULTS

Theorem (by many set theorists)

There is no other implication between the three sentences:

"$\mathbf{CA}$ *sets have the kernel property*",

"$\mathbf{A}_2$ *sets have the measurability property*",

"$\mathbf{A}_2$ *sets have the Baire property*",

provable in ZFC.

That is, **any other combination between them fails in an appropriate model**.

They similarly define the notions of:

- a real x "**constructible relatively** to a given real a", and
- a real x "**random relatively** to a given real a".

If a is constructible itself then any real constructible relatively to a is simply constructible, and any real random relatively to a is simply random. Any real a is constructible relatively to itself, but no real a is random relatively to itself.

Theorem

"The set of all random reals relatively to any real a is a set of full measure" $\Longleftrightarrow$ "all $\mathbf{A}_2$ sets are measurable".

Thus, the property of measurability of $\mathbf{A}_2$ sets and random reals are closely related.

Let us return to the notion of randomness once again. Any other family of "small" sets of reals can be considered instead of the family of null sets. This may lead to notions of randomness really different from the one above.

Definition (Example 1)

A generic real is any real number x which avoids all Borel meager sets B_d with a constructible code d.

Theorem

There exist models M_1 and M_2 such that:
it is true in M_1 that "all reals are constructible (thus, neither random no generic)",
and it is true in M_2 that "the set of all random reals has the full measure and the set of all generic reals is co-meager".

Generic reals are **locally undistinguishable from each other** in the same sense as random reals:

Theorem

Let x be random and $P(x)$ be any its property. Then there is a Borel co-meager set X such that $x \in X$ and every generic real $x' \in X$ satisfies the same property P(x').

Definition (Example 2)

A Martin-Löf real is any real number x which avoids all Borel sets B_d with a **computable code** d and B_d **has the form**: $B_d = \bigcap_n U_n$, where $\{U_n\}$ is a sequence of open sets.

This notion is among the fundamentals of the computability theory.

Definition (Example 3)

All "random graphs", that is, those countable graphs which, roughly speaking, avoid typical singularities, are isomorphic to each other, so there is basically a single random graph.

There exist other meaningful examples related to other families of "small sets of reals".

We recall a theorem above, which shows that measurability of $\mathbf{A}_2$ sets and random reals are closely related:

Theorem

"The set of all random reals relatively to any real a is a set of full measure" $\Longleftrightarrow$ "all $\mathbf{A}_2$ sets are measurable".

A similar result holds for generic reals:

Theorem

"The set of all generic reals relatively to any real a is co-meager" $\Longleftrightarrow$ "all $\mathbf{A}_2$ sets have the Baire property".

Thus, the Baire property of $\mathbf{A}_2$ sets and generic reals are closely related.

Analogously:

Theorem

"The set of all constructible reals relatively to any real a is countable" $\iff$ *"all* $\mathbf{CA}$ *sets have the kernel property".*

Thus, the kernel property of $\mathbf{CA}$ sets and constructible reals are closely related.

Our original result

Now we turn to our original result presented here.

Definition

Let $(X, \leq)$ be a partial quasi-order, that is, any transitive and reflexive relation on a set X ("**partial order**"). A **partial order** $(X, \leq)$ is **Borel** if it is a Borel set as a set of pairs in $X \times X$. By Hausdorff, a **pantachy** in $(X, \leq)$ is any maximal totally ordered subset Y of X.

In other words, a **pantachy** is a maximal chain in $(X, \leq)$.

Theorem (Hausdorff 1906, in ZFC)

Any partial order $(X, \leq)$ contains a pantachy.

In fact, this was one of the first applications of the axiom of choice **AC** in mathematics.

Such partial orders can be pretty meaningful mathematically, for instance:

Example (eventual domination, Paul Du Bois Reymond, 1870)

Let S be the set of all infinite sequences $\{x_n\}$ of reals. Define the **eventual domination order** $\leq^*$ on S so that $\{x_n\} \leq^* \{y_n\}$ if $x_n \leq y_n$ for all but finite n. This means

$$\{x_n\} \leq^* \{y_n\} \Longleftrightarrow \exists\, m \,\forall\, n \geq m\, (x_n \leq y_n).$$

Then $(S, \leq^*)$ belongs to P. There is a lot of other partial orders in P.

Our original result

Haudorff wrote in 1907:

> *"to legitimate construct a pantachy seems completely hopeless; the pantachy existence cannot be proved without* **AC***".*

The Hausdorff conjecture can be rendered, in modern terms, as follows:

1. taking **AC** for granted, the pantachy existence cannot be proved effectively for some partial orders $(X, \leq)$;
2. without **AC**, the pantachy existence cannot be proved for some partial orders $(X, \leq)$.

We have proved both parts 1 and 2. I will consider part 1 of the Hausdorff conjecture, and then part 2.

Our original result

Definition

ROD = real-ordinal definable set X is a set which can be defined by any formula containing only ordinals and reals (as parameters), that is,

$$x \in X \Longleftrightarrow \varphi(x, \alpha_1, \ldots, \alpha_n, b_1, \ldots, b_m), \quad \forall x$$

where α_i – ordinals, b_j – reals.

The class of all **ROD** sets is considered as the largest class of effective sets in modern set theory. It contains, for instance, all Borel sets and even all projective sets.

Theorem (Kanovei – Lyubetsky, a vague formulation)

There is a model M *and a reasonably large class* P *of Borel partial orders such that no order in* P *has* **ROD** *pantachies in* M.

Now we **define the class** P.

Definition

A partial order $(X, \leq)$ is **locally bounded** if for any countable subset Y of X there exists an element $x \in X$ such that $y < x$ strictly for all $y \in Y$.

Let P be the class of all Borel locally bounded partial orders defined on a subset of the reals (or any other complete separable metrizable space).

Our original result

Theorem (Kanovei – Lyubetsky, a precise formulation)

There is a model M such that no order in the class P has **ROD** *pantachies in M.*

This result applies for the eventual domination order $(S, \leq^*)$ and for many other similar partial orders.

Conclusion

In support of the Hausdorff conjecture: it is established that there is a model in which many typical Borel orderings do not have ROD pantachies.

Our original result

Now about part 2 of the Hausdorff conjecture:

Theorem (Kanovei – Lyubetsky)

There is a model of ZF *(*$= \mathrm{ZFC}$ *without the axiom of choice!) in which the eventual domination partial order* $(S, \leq^*)$ *has no pantachy of any kind.*

Comments

Fundamentally interesting is the case when an **explicitly defined set** T does not contain **explicitly defined elements**.

Example

The set T of all sections (=selectors) of the quotient $\mathbb{R}/\mathbb{Q}$ (reals / rationals) is easily definable but it doesn't include any definable elements. In other words: the type of $T = \mathbb{R}/\mathbb{Q}$ is very simple, but if $Y \in T$ then Y is not a Borel set, and moreover, Y is not even projective or Lebesgue measurable in some model.

Our original result presented above also belongs to this category. The set T of all maximal chains in a given partial order $\leq$ is easily definable, but in some cases one cannot define any particular maximal chain.

Comments to the proof

To prove the original results above, we employ a model known as the Solovay model. This model M is famous by the fact that it is true in M that all projective sets are Lebesgue measurable, and they also have the Baire property and the kernel property.

Our proof is based on two auxiliary theorems (known since 1990s) on the Solovay model.

Comments to the proof

The first of those auxiliary theorems is as follows.

The class of all Borel sets can be divided into classes, that is, “0 class”, “1 class” and so on, where

- “0 class” is the class of all open and closed sets (of reals),
- “1 class” is the class of all countable unions and intersections of the “0 class” sets,

and so on by transfinite induction.

Theorem (Stern)

In the Solovay model: for any Borel class K there is no uncountable **ROD** *sequences of pairwise different sets in K.*

Comments to the proof

The second of the auxiliary theorems also needs a few definitions. The set $2^{<\omega_1}$ of all transfinite countable dyadic sequences is naturally ordered by inclusion (that is, the extension of sequences).

- An antichain in $2^{<\omega_1}$ is any set A of sequences in which no sequence is an extension of another sequence in A.

Any antichain A is then linearly ordered by the lexicographical order.

Theorem (Kanovei)

In the Solovay model: any Borel linear quasi-order is order isomorphic to a **ROD** *antichain, ordered lexicographically, in the set* $2^{<\omega_1}$ *of all transfinite countable dyadic sequences ordered by inclusion.*

The end

Thank you for your attention