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Notes on the Equiconsistency of ZFC Without the Power Set Axiom and Second-Order Arithmetic

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Abstract

We demonstrate that theories $\mathbf{Z}^-$, $\mathbf{ZF}^-$, $\mathbf{ZFC}^-$ (minus means the absence of the Power Set axiom) and $\mathbf{PA}_2$, $\mathbf{PA}_2^-$ (minus means the absence of the Countable Choice schema) are equiconsistent to each other. The methods used include the interpretation of a power-less set theory in $\mathbf{PA}_2^-$ via well-founded trees, as well as the Gödel constructibility in said power-less set theory.

Keywords: constructibility; theories without the PS axiom; second-order arithmetic; consistency

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1. Introduction

This paper contains a proof of the following theorem.

Theorem 1. *Theories* $\mathbf{PA}_2^-$, $\mathbf{PA}_2$, $\mathbf{Z}^-$, $\mathbf{ZFC}^-$, $\mathbf{ZF}^-$ *are equiconsistent.*

Here, $\mathbf{PA}_2$, resp., $\mathbf{PA}_2^-$ is a second-order Peano arithmetic with, resp., without the (countable) **AC**, whereas $\mathbf{Z}^-$ is Zermelo set theory without the well-orderability axiom **WOA**, and $\mathbf{ZFC}^-$ / $\mathbf{ZF}^-$ are Zermelo–Fraenkel set theories resp. with/without **WOA**, and all three of them without the **Power Set** axiom. See the exact definitions in Section 2 related to the second-order Peano arithmetic and to power-less set theories. We recall that the **Power Set** axiom claims the existence of the power set of any given set, leading to set theories much stronger than the second-order Peano arithmetic. Thus, the significance of power-less set theories is related to the fact that they combine a rich set theoretic environment with the foundational strength equal to a second-order arithmetic.

In fact, Theorem 1 has been known since at least the late 1960s; see, for example, [1]. However, no self-contained and more or less complete proof has apparently ever been published (see the brief discussion in *Mathoverflow* around [2]). In fact, significant fragments of the proof turned out to be scattered across various unrelated publications, from which the overall picture of their interaction in obtaining the final result does not immediately become clear. The first goal of this paper is to finally present these fragments in a coherent and easy-to-read proof that includes all the necessary details, particularly those related to the Gödel constructibility.

The proof of Theorem 1 consists of two parts. For the **first part**, we define (Sections 2 and 3) a set theory **TMC**, which extends $\mathbf{Z}^-$ by (1) the existence of transitive closures, (2) an axiom saying that any well-founded relation on ω admits a transitive model, and (3) the **Countability** axiom. This is a subtheory of $\mathbf{ZF}^-$ + **Countability**, which turns out to be strong enough to prove the schema of **Replacement** in the case when the range of the function declared to exist is a transitive class (Lemma 1). The second goal of this paper is to highlight the foundational role of **TMC** as the theory of the set theoretic hull over a universe of $\mathbf{PA}_2^-$ and a straightforward set theoretic counterpart of $\mathbf{PA}_2^-$—in the same way as $\mathbf{ZFC}^-$ is the theory of the set theoretic hull over a universe of $\mathbf{PA}_2$ (with the countable **AC**) and a straightforward set theoretic counterpart of $\mathbf{PA}_2$.

Theorem 4 below provides interpretations of **TMC** in $\mathbf{PA}_2^-$, as well as of $\mathbf{ZFC}^-$ in $\mathbf{PA}_2$, obtained by using well-founded subtrees of $\omega^{<\omega}$ as the domain of interpretation. This is a well-known method, presented in [3–6] among other papers, as well as in Sections VII.3–6 of Simpson [7], and in [8] with respect to second-order set theory. The tree structure $\mathbb{V}$, related to this interpretation, is defined and studied in Section 4. The ensuing Corollary 3 claims the existence of two groups of mutually interpretable and equiconsistent theories, which include $\mathbf{PA}_2^-$, $\mathbf{Z}^-$, **TMC** (group 1) and $\mathbf{PA}_2$, $\mathbf{ZF}^-$, $\mathbf{ZFC}^-$ (group 2).

The **second part** of the proof of Theorem 1 presents an interpretation of $\mathbf{ZFC}^-$ in **TMC**, contained in the following theorem, which is our **second key result** here. This theorem involves Gödel's class $\mathbf{L} = \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{L}_\alpha$ of all constructible sets.

Theorem 2 (**TMC**)**.** *The following set or class satisfies* $\mathbf{ZFC}^-$:

$$\mathbf{L}^* = \begin{cases} \mathbf{L}\,, & \text{in case when the ordinal } \omega_1^{\mathbf{L}} \text{ does not exist,} \qquad \text{(a)} \\ \mathbf{L}_\Omega = \bigcup_{\alpha<\Omega} \mathbf{L}_\alpha\,, & \text{in case when } \omega_1^{\mathbf{L}} = \Omega \text{ does exist.} \qquad \text{(b)} \end{cases}$$

Theorem 2 provides an interpretation (namely, $\mathbf{L}^*$) of $\mathbf{ZFC}^-$ in **TMC**, hence connecting groups 1 and 2 above, thereby **implying the equiconsistency result of Theorem 1.** This interpretation is close to an interpretation defined by Simpson [7] (VII.4). We review some other interpretations, including an early one defined in [9], in Section 12. Note the additional advantage of Theorem 2: it gives a *transitive "standard"* (that is, with the true membership) interpretations of $\mathbf{ZFC}^-$ in **TMC**, a theory apparently weaker than $\mathbf{ZFC}^-$.

Theorem 2 is proven in Sections 7 and 8 on the basis of Gödel's constructibility, as developed in Sections 5 and 6 in the context of **TMC**. In particular, Section 7 contains Theorem 6, a key result saying that, in **TMC**, a class of the form $K = \bigcup_{\alpha\in\Omega} \mathbf{L}_\alpha$ satisfies $\mathbf{ZFC}^-$ under certain conditions. This leads to the proofs of Theorems 2 and 1 in Section 8.

Regarding the class $\mathbf{L}$ as a whole, we may note that $\mathbf{L}$ does not necessarily satisfy $\mathbf{ZFC}^-$ under **TMC**, as Example 1 shows. Therefore, option (b) of Theorem 2 definitely cannot be abandoned. Nevertheless, we prove the following theorem in Sections 9 and 10:

Theorem 3 (**TMC**)**.**

(I) $\mathbf{L} \cap \mathscr{P}(\omega)$ *satisfies* $\mathbf{PA}_2$.

(II) $\mathbf{L}$ *itself satisfies* $\mathbf{Z}^-$, *in particular, thus satisfying the schema of* **Separation**.

The third goal of this paper is to present this new result.

The ensuing Corollary 5 states that, under $\mathbf{PA}_2^-$, $\mathbf{L} \cap \mathscr{P}(\omega)$ satisfies $\mathbf{PA}_2$. Saying it differently, $\mathbf{L} \cap \mathscr{P}(\omega)$ is an interpretation of $\mathbf{PA}_2$ in $\mathbf{PA}_2^-$.

Our proof of Theorem 1 leaves open the following question: is there a way to interpret $\mathbf{PA}_2$ in $\mathbf{PA}_2^-$, thus avoiding substantial use of set theoretic concepts and methods such as constructibility? A possible approach to this goal, based on the ramified analytical hierarchy, is outlined in Section 13.

Overall, this is a research and survey article, the purpose of which is to provide proofs of such fundamentally important results, as indicated in Theorems 1–3, in a fairly self-contained and easy-to-read form.

2. Preliminaries

Second-order arithmetic. Recall that second-order arithmetic $\mathbf{PA}_2$ is a theory in the language $\mathcal{L}(\mathbf{PA}_2)$ with two sorts of variables: for natural numbers and for sets of them.

We will use j, k, m, n for variables over ω and x, y, z for variables over $\mathscr{P}(\omega)$, reserving capital letters for subsets of $\mathscr{P}(\omega)$ and other sets.

The axioms of $\mathbf{PA}_2$ are the Peano axioms for numbers plus the following:

- **Induction:** $\forall\, x\, \big(0 \in x \wedge \forall\, n\, (n \in x \implies n+1 \in x) \implies \forall\, n\, (n \in x)\big)$.
- **Extensionality** for sets: $\forall\, x, y\, \big(\forall\, k\, (k \in x \iff k \in y) \implies x = y\big)$.
- **Comprehension CA**: $\exists\, x\, \forall\, k\, (k \in x \iff \Phi(k))$—for every formula Φ in which x does not occur, and in Φ, we allow parameters, that is, free variables other than k.
- **Countable Choice $\mathbf{AC}_\omega$**: $\forall\, n\, \exists\, x\, \Phi(n, x) \implies \exists\, x\, \forall\, n\, \Phi(n, (x)_n))$—for any formula Φ with parameters, where $(x)_n = \{j : (\!|n, j|\!) \in x\}$, and $(\!|n, j|\!) = 2^n(2j+1) - 1$ is a standard bijection $\omega \times \omega$ onto ω.

The theory $\mathbf{PA}_2$ is also known as A_2 (see, for instance an early survey [3]), as Z_2 (in [10] or elsewhere). See also [1,7,11]. Let $\mathbf{PA}_2^-$ be $\mathbf{PA}_2$ sans $\mathbf{AC}_\omega$.

Coding in second-order arithmetic. It can be viewed as a certain disadvantage that $\mathbf{PA}_2^-$ does not explicitly treat such objects as pairs, tuples, and finite sets of numbers, as well as trees of tuples at the next level. However, these and similar (and, in fact, even more complex) mathematical objects can be effectively encoded as single natural numbers or sets of them. We refer to [7], Chap. I, and especially Section II.2, with respect to many examples.

Recall that $\mathrm{SEQ} = \omega^{<\omega}$, the set of all tuples (finite sequences) of numbers in ω. If $s \in \mathrm{SEQ}$ and $j < \omega$, then $s^\frown j \in \mathrm{SEQ}$ is obtained by adjoining j as the rightmost term. Let $\mathtt{lh}\, s$ denote the length (the number of terms).

Let $s_0 = \Lambda$ (the empty tuple), and, by induction, if $n = (\!|m, j|\!) + 1 \geq 1$ then, $s_n = s_m{}^\frown j$. Clearly, $\mathrm{SEQ} = \{s_n : n < \omega\}$ and, in fact, $n \longmapsto s_n$ is a bijection onto SEQ. Subsequently, $n = n(s)$ is viewed as *the code* of any $s = s_n \in \mathrm{SEQ}$, and a set $x \subseteq \omega$ is viewed as *the code* of $\{s_n : n \in x\} \subseteq \mathrm{SEQ}$. Following [7] (esp. II.2), this enables us to freely consider tuples and sets of them as if they properly exist, but still on the basis of $\mathbf{PA}_2^-$.

Similarly, still based on $\mathbf{PA}_2^-$, we can treat sets $X \subseteq \omega \times \omega$, $H \subseteq \mathrm{SEQ} \times \mathrm{SEQ}$, and the like as properly existing.

Finite and infinite sequences of subsets of ω are within reach in $\mathbf{PA}_2^-$ as well, because each set $x \subseteq \omega$ is a code of the infinite sequence of sets $(x)_n = \{j : (\!|n, j|\!) \in x\}$ (see the formulation of $\mathbf{AC}_\omega$ above). Thus, they are, for instance infinite sequences of subsets of SEQ.

Power-less set theories. We recall that *the power-less set theory* $\mathbf{ZFC}^-$ is a subtheory of $\mathbf{ZFC}$ obtained so that the following are achieved:

(I) The Power Set axiom **PS** is excluded—symbolized by the upper minus.

(II) The usual set theoretic Axiom of Choice **AC** of **ZFC** is removed (as it does not work properly without **PS**), and instead the *well-orderability axiom* **WOA** is added, which claims that every set can be well-ordered.

(III) The Separation schema **Sep** is preserved, but the Replacement schema **Repl** (too weak in the absence of **PS**) is substituted with the *Collection* schema:

$$\mathbf{Coll}: \forall\, X\, \big(\forall\, x \in X\, \exists\, y\, \Phi(x, y) \implies \exists\, Y\, \forall\, x \in X\, \exists\, y \in Y\, \Phi(x, y)\big).$$

Note that $\mathbf{Coll} + \mathbf{Sep} \implies \mathbf{Repl}$.

See [12–14] for a comprehensive account of main features of $\mathbf{ZFC}^-$.

See [15,16] and [17] (Sect. 2) or elsewhere for different but equivalent formulations of Collection, such as in the following form in [15] (Chap. 6):

$$\mathbf{Coll}' : \forall X \, \exists Y \, \forall x \in X \, \big(\exists y \, \varphi(x,y) \implies \exists y \in Y \, \varphi(x,y)\big).$$

This is apparently stronger than **Coll** above, but in fact, **Coll**$'$ is a consequence of **Coll**, for $\Phi(x,y) := \varphi(x,y) \vee \big(y = 0 \wedge \neg \exists y \, \varphi(x,y)\big)$ in **Coll**.

- $\mathbf{ZF}^-$ is $\mathbf{ZFC}^-$ without the well-orderability axiom **WOA**;
- $\mathbf{Z}^-$ is $\mathbf{ZF}^-$ without the Collection schema **Coll**.

Let **TMC** be $\mathbf{Z}^-$ plus the following three axioms **TrSups**, **MoClps**, **Countability**:

- **Transitive superset, TrSups:** For any X, there is a transitive superset $Y \supseteq X$.
- **Mostowski Collapse, MoClps:** Any well-founded relation A on a set $D = \mathbf{fld}\, A := \mathtt{dom}\, A \cup \mathtt{ran}\, A$ admits a transitive set X and $\boldsymbol{\mu} : D$ onto X, satisfying, for all $d \in D$, (*) $\boldsymbol{\mu}(d) = \{\boldsymbol{\mu}(j) : j \, A \, d\}$. By standard arguments, the map $\boldsymbol{\mu}$ and the set X are unique.
- **Countability:**$\forall x \, \exists f \, (f : x \to \omega$ is 1–1$)$, that is, all sets are at their most countable.

The name **TMC** reflects the initial letters of the additional axioms. Quite obviously, $\mathbf{TMC} \subsetneq \mathbf{ZF}^- + \mathbf{Countability}$; see [15] (Theorem 6.15) for a proof of **MoClps** from **Repl**.

It follows from **TrSups** by **Sep** that the transitive closure $\mathrm{TC}(X)$ of any set X properly exists. Recall that Y is *transitive* if $\forall x \, \forall y \, (x \in y \in Y \implies x \in Y)$, and the *transitive closure* of X is the intersection of all transitive supersets of X.

The axiom **MoClps** is called *Axiom Beta* in [7] (Def. VII.3.8). It follows the ideas first put forward by Mostowski [18,19]. Its different aspects were discussed in [9,20–22]. The idea of using **MoClps** as an axiom in weak set theoretic systems is due to Simpson [23].

Recall that a binary relation A on $D = \mathbf{fld}\, A$ is *well-founded* if any set $\varnothing \neq Y \subseteq D$ contains some $y \in Y$ with $\forall x \in Y \, \neg \, (x \, A \, y)$. Applying **MoClps** for $A = \in \restriction D$, we obtain:

Corollary 1 (**TMC**, transitive collapse)**.** *Let D be any set. There is a unique transitive set X and a unique collapse map $\tau : D$ onto X satisfying $\tau(x) = \{\tau(y) : y \in x \cap D\}$ for all $x \in D$.*

Simpson's approach. Simpson [7] (VII.3.3 and VII.3.8) considers a related theory $\mathbf{ATR}_0^{\mathrm{set}}$ in the $\in$-language, containing the following axioms:

(a) Axiom of Equality: $=$ is an equivalence relation and $\in$ is $=$ -invariant;

(b) Axioms of Extensionality and Infinity in their usual forms;

(c) Axiom of Rudimentary Closure, which asserts, for all u, v, w, the proper existence of $\{u, v\}$, $u \smallsetminus v$, $u \times v$, $\bigcup u$, $\in \restriction u$, and the following:

$$\begin{aligned} u^{-1} \quad = \quad & \{\langle x,y\rangle : \langle y,x\rangle \in u\}, \\ & \{\langle y, \langle x,z\rangle\rangle : \langle y,x\rangle \in w \wedge z \in u\}, \\ & \{\langle y, \langle z,x\rangle\rangle : \langle y,x\rangle \in w \wedge z \in u\}, \\ & \{v : \exists x \, (x \in u \wedge v = w''\{x\}\}. \end{aligned}$$

(d) Axiom of Regularity in its usual form;

(e) Axioms **TrSups**, **MoClps**, **Countability**, as above.

Quite obviously, we have $\mathbf{TMC} \smallsetminus \text{Separation} \subseteq \mathbf{ATR}_0^{\mathrm{set}} \subseteq \mathbf{TMC}$. Indeed, regarding the second $\subseteq$, all operations, listed in (c) above, are properly convergent within any transitive finite-subset-closed set. Now refer to Lemma 2 below.

Therefore, **TMC** as a whole coincides with $\mathbf{ATR}_0^{\mathrm{set}} + \text{Separation}$.

3. Development of the Intermediate Power-Less Theory

We proceed with a few simple results in **TMC** hardly available in $\mathbf{Z}^-$.

Let a *class-map* be a (definable) class that satisfies the standard definition of a function (that is, consists of sets that are ordered pairs, etc.).

Lemma 1 (**TMC**)**.** *Let F be a class-map, $D = \mathtt{dom}\, F$ any set. Then, F and the image $R = F''D = \{F(x) : x \in D\}$ are sets in each of the two cases*: (1) *R is transitive,* (2) *there is a set Y such that $R \subseteq \mathscr{P}(Y)$.*

Proof. (1) By **Countability** we can without any loss of generality assume that $D \subseteq \omega$. We can also assume that F is 1–1; otherwise, replace D by the set

$$D' = \{k \in D : \forall j \in D\,(j < k \Longrightarrow F(j) \neq F(k))\}.$$

Then, the relation $A = \{\langle j, k\rangle : j, k \in D \wedge F(j) \in F(k)\}$ is well-founded as isomorphic to $\in\restriction R$. On the other hand, by **MoClps**, A is isomorphic to $\in\restriction Y$, where Y is a transitive *set*. It follows that Y and R are $\in$-isomorphic, and hence $R = Y$ is a set. Finally, $F \subseteq X \times R$ is a set by **Separation**.

(2) We, without any loss of generality assume that Y is transitive by **TrSups**. We can assume as well that $D \cap Y = \varnothing$; otherwise, put $D' = D \times \{Y\}$ and change F accordingly. Under these assumptions, put $D_1 = D \cup Y$ and extend F to F_1 by the identity on Y. Then, the image $F_1{}''D_1 = R \cup Y$ is transitive; hence, a set by (1). Now $R \subseteq F_1{}''D_1$ is a set by **Sep**. □

A set Y is called *finite-subset closed* if $\forall z \subseteq Y\,(z \text{ finite } \Longrightarrow z \in Y)$. For any set X, let the *finite-closure* $\mathrm{FC}(X)$ be the least finite-subset closed superset $Y \supseteq X$, if it exists.

Lemma 2 (**TMC**)**.** *For any set X, $\mathrm{FC}(X)$ properly exists.*

Proof. To handle the case $X = \omega$, let p_k be kth prime, so $p_1 = 2$, $p_2 = 3$, and so on. Let $A = \{\langle k, n\rangle : k \geq 1 \wedge p_k \text{ divides } n\}$. Then, $\mathbf{fld}\, A = \omega \smallsetminus \{0\}$, A is well-founded (since $k\,A\,n \Longrightarrow k < n$), and (†) for any finite $u \subseteq \mathbf{fld}\, A$, there is $n \in \mathbf{fld}\, A$ satisfying $u = \{k : k\,A\,n\}$. By **MoClps** there is a map $\boldsymbol{\mu} : \mathbf{fld}\, A$ onto a transitive set R, satisfying (*) $\boldsymbol{\mu}(n) = \{\boldsymbol{\mu}(k) : k\,A\,n)\}$, for all $n \in \mathbf{fld}\, A$. Then, easily $R = \mathrm{FC}(\omega)$ by (†).

To handle the general case, we may assume that X is transitive, by **TrSups**. Let $h : \omega$ onto X, by **Countability**. Then, h can be extended to a class-map H defined on the bigger set $R = \mathrm{FC}(\omega)$ so that $H\restriction\omega = h$, and if $u \in R \smallsetminus \omega$, then $H(u) = \{H(n) : n \in u\}$. Then, $\mathtt{ran}\, H = \mathrm{FC}(X)$ (so far a class), and hence $\mathtt{ran}\, H$ is transitive and so is X. It follows by Lemma 1 that both H and $\mathtt{ran}\, H = \mathrm{FC}(X)$ are proper sets. □

Lemma 3 (**TMC**)**.** *Let U, V be any sets. Then, $U \times V$, $\mathscr{P}_{\mathtt{fin}}(U)$, $U^{<\omega}$ properly exist (as sets).*

Proof. $X = U \cup V = \bigcup\{U, V\}$ is a set by $\mathbf{Z}^-$. Now, $\mathrm{FC}(X)$ is a set by Lemma 2, hence $U \times V \subseteq \mathrm{FC}(X)$ is a set by **Sep**. To prove the other claims, note that $\mathscr{P}_{\mathtt{fin}}(U), U^{<\omega} \subseteq \mathrm{FC}(U)$ and use Lemma 2 and **Sep**. □

Thus, **TMC** proves the existence of Cartesian products. Note that $\mathbf{Z}^-$ does not prove even the existence of $\omega \times \omega$!

Lemma 4 (**TMC**)**.** *Let E be a strict well-ordering of a set U. Then, there is an ordinal λ and an order isomorphism of $\langle U\,; E\rangle$ onto $\langle \lambda\,; \in\rangle$.*

Proof. By **Countability** we can without any loss of generality assume that $U \subseteq \omega$. Then, E is a well-founded relation with $\mathbf{fld}\, E \subseteq \omega$. Apply **MoClps**. Then, $\lambda = X$ is a transitive set well-ordered by $\in$, that is, an ordinal. □

Corollary 2 (**TMC**)**.** *If α, β are ordinals, then there exist (as sets) ordinals $\alpha+\beta$, $\alpha\cdot\beta$, α^β* (in the sense of the ordinal arithmetic.)

Proof. We have to define well-ordered sets, which represent the mentioned orders. For instance, the Cartesian product $\alpha\times\beta$ (a set by Lemma 3), ordered lexicographically, represents $\alpha\cdot\beta$. The exponent α^β is represented by the set

$$W = \{f : D \to \alpha \smallsetminus \{0\} : D \subseteq \beta \text{ is finite}\},$$

ordered lexicographically, with the understanding that each $f \in D$ is by default extended by $f(\xi) = 0$ for all $\xi \in \beta \smallsetminus D$. Note that $W \subseteq \mathrm{FC}(\beta\times\alpha)$ is a set by Lemma 2. □

4. The Set Theoretic Tree Hull over Second-Order Arithmetic

Following [7] (VII.3), we consider the collection WFT of all well-founded trees $\varnothing \neq T \subseteq \mathrm{SEQ} = \omega^{<\omega}$. Recall the following:

- Λ is the empty tuple, $\langle k\rangle$ is the tuple with k as the single term;
- $T \subseteq \mathrm{SEQ}$ is a tree if $s^\frown j \in T \Longrightarrow s \in T$;
- T is well-founded if $\neg\exists g : \omega\to\omega\,\forall m\,(g\restriction m \in T)$;
- $s^\frown j$ is obtained by adding $j\in\omega$ to $s \in \mathrm{SEQ}$ as the rightmost term, and if $s,t\in\mathrm{SEQ}$, then $s^\frown t \in \mathrm{SEQ}$ is the *concatenation*;
- If T is a tree and $s \in T$, then put $T^s = \{t \in \mathrm{SEQ} : s^\frown t \in T\}$; thus, T^s is a tree as well, and if T is well-founded then so is T^s.

Definition 1 (**$\mathbf{PA}_2^-$**)**.** *Let $S, T \in$ WFT.*

A set $H \subseteq S\times T$ is an S,T-bisimulation, if, for all $s\in S$ and $t\in T$,

$$s\,H\,t \iff \forall s' = s^\frown j \in S\;\exists t' = t^\frown k \in T\;(s'\,H\,t') \;\wedge\; \wedge\;\forall t' = t^\frown k \in T\;\exists s' = s^\frown j \in S\;(s'\,H\,t')\,. \tag{1}$$

Define $S \cong T$ if there is an S,T-bisimulation H such that $\Lambda\,H\,\Lambda$.

Define $S \mathrel{\tilde{\in}} T$ if $S \cong T^u$ for some $u \in T$ with $\mathtt{lh}\,u = 1$.

The structure $\mathbb{V} = \langle \mathrm{WFT}\,; \cong, \tilde{\in}\rangle$ *is considered in* $\mathbf{PA}_2^-$.

The $\mathbb{V}$-interpretation $\lceil\Phi\rceil^{\mathbb{V}}$ *of an $\in$-formula Φ (with parameters in* WFT*) is naturally defined in the sense of interpreting $=,\in$ as resp. $\cong, \tilde{\in}$, and relativizing the quantifiers to* WFT*. Thus, for instance $\lceil x=y\rceil^{\mathbb{V}}$ is $x\cong y$.*

Note that the bisimulation relation $\cong$ between trees in WFT, and subsequently the derived relation $\tilde{\in}$ as well, are naturally formalized in $\mathbf{PA}_2^-$ in the frameworks of the approach based on coding; see Section 2. It follows that, for any $\in$-formula Φ with parameters in WFT, the $\mathbb{V}$-interpretation $\lceil\Phi\rceil^{\mathbb{V}}$ of is a $\mathcal{L}(\mathbf{PA}_2)$-formula.

The next theorem is a version of the interpretation results known since at least Kreisel [1] and published somewhat later in [3–5,7] or elsewhere. The $\mathbf{PA}_2$ part of the theorem is essentially Theorem 5.5 in [3]. The $\mathbf{PA}_2^-$ part is close to Theorem 1.1 and Corollary 1.1 in [4] or VII.3.24 in [7].

Theorem 4 (**$\mathbf{PA}_2^-/\mathbf{PA}_2$**)**.** *$\mathbb{V}$ is a well-defined structure: $\cong$ is an equivalence on* WFT*, $\tilde{\in}$ is a binary relation on* WFT *invariant with respect to $\cong$.*

Moreover, $\mathbb{V}$ satisfies resp. **TMC**/**ZFC**$^-$*. In other words, if Φ is an axiom of* **TMC***, resp.,* **ZFC**$^-$*, then $\lceil\Phi\rceil^{\mathbb{V}}$ is a theorem of resp.* $\mathbf{PA}_2^-$, $\mathbf{PA}_2$.

Proof. Besides the papers cited above, the bulk of the theorem was established in [7] (VII.3). Namely, using just $\mathbf{ATR}^0$ as the basis theory (which is a small part of $\mathbf{PA}_2^-$), Lemma VII.3.20

in [7] proves that if Φ is an axiom of $\mathbf{ATR}_0^{\text{set}}$, then $\ulcorner\Phi\urcorner^{\mathbb{V}}$ is a theorem of $\mathbf{ATR}^0$ (and then of $\mathbf{PA}_2^-$ as well). Thus, to prove the $\mathbf{PA}_2^-$ part of Theorem 4, it suffices to check **Sep** in $\mathbb{V}$.

Arguing in $\mathbf{PA}_2^-$, assume that $S \in \text{WFT}$, $X = \{k : \langle k\rangle \in S\}$, and $\Phi(x)$ is an $\in$-formula with parameters in WFT and with x as the only free variable. Trees of the form $S^k = \{t \in \text{SEQ} : k^\frown t \in S\}$, $k \in X$, belong to $\mathbb{V}$ and are the only (modulo $\cong$) $\widetilde{\in}$-elements of S in $\mathbb{V}$. Now, using the $\mathbf{PA}_2^-$ **Comprehension**, we let $Y = \{k \in X : \ulcorner\Phi(S^k)\urcorner^{\mathbb{V}}\}$. The set $T = \{\Lambda\} \cup \{t \in S : t(0) \in X\}$ is a tree in WFT. We claim that $\ulcorner T = \{x \in S : \Phi(x)\}\urcorner^{\mathbb{V}}$.

Indeed, assume that $C \in \text{WFT}$, $C \widetilde{\in} S$, and $\ulcorner\Phi(C)\urcorner^{\mathbb{V}}$. Then, $C \cong S^k$ for some $k \in X$, so that $\ulcorner\Phi(S^k)\urcorner^{\mathbb{V}}$ holds, and hence $k \in Y$. It follows that $C \cong T^k = S^k \widetilde{\in} T$. The proof of the inverse implication is similar.

Finally, we prove **the** $\mathbf{PA}_2$ **part** of the theorem. **Arguing in** $\mathbf{PA}_2$, we have to additionally check **Coll** in $\mathbb{V}$. Thus, let $S \in \text{WFT}$ and let $\Phi(x,y)$ be an $\in$-formula with parameters in WFT, satisfying $\ulcorner\forall x \in S\, \exists y\, \Phi(x,y)\urcorner^{\mathbb{V}}$, that is,

$$\forall A \in \text{WFT}\, \exists B \in \text{WFT}\, \big(A \widetilde{\in} S \Longrightarrow \ulcorner\Phi(A,B)\urcorner^{\mathbb{V}}\big). \tag{2}$$

But $\widetilde{\in}$-elements of S are, modulo $\cong$, all trees $S^k = \{s \in S : k^\frown s \in T\}$, where $k \in K = \{k \in \omega : \langle k\rangle \in T\}$, and only them. Thus, (2) implies

$$\forall k \in K\, \exists B \in \text{WFT}\, \big(\ulcorner\Phi(S^k, B)\urcorner^{\mathbb{V}}\big).$$

Using $\mathbf{AC}_\omega$ of $\mathbf{PA}_2$, we obtain a (coded, see Section 2) sequence of trees $B_k \in \text{WFT}$ with $\ulcorner\Phi(S^k, B_k)\urcorner^{\mathbb{V}}$ for all k. Now, $T = \langle\Lambda\rangle \cup \bigcup_{k\in K} k^\frown B_k \in \text{WFT}$, and each B_k is an $\widetilde{\in}$-element of T. Thus, we have

$$\forall k \in K\, \exists B \widetilde{\in} T\, \big(\ulcorner\Phi(S^k, B)\urcorner^{\mathbb{V}}\big), \ \text{ that is, } \ \ulcorner\forall x \in S\, \exists y \in T\, \Phi(S^k, B)\urcorner^{\mathbb{V}},$$

as required. □

Corollary 3 (of Theorem 4)**.** *Theories* $\mathbf{PA}_2^-$, $\mathbf{Z}^-$, $\mathbf{TMC}$ *are mutually interpretable and hence equiconsistent to each other. Theories* $\mathbf{PA}_2$, $\mathbf{ZF}^-$, $\mathbf{ZFC}^-$ *are mutually interpretable and equiconsistent as well.*

Corollary 3 is the first part of the proof of Theorem 1. The remainder of the proof involves the ideas and technique of Gödel's constructibility, and **the goal will be Theorem 2**, which provides an interpretation of $\mathbf{ZFC}^-$ in $\mathbf{TMC}$.

5. Constructible Sets in the Intermediate Theory

We will make use of some keynote definitions and results related to constructible sets as given in [7] (Sect. VII.4). We present these results based on **TMC**, whereas Simpson works in $\mathbf{ATR}_0^{\text{set}}$ and in some other sub-theories of **TMC** in [7], which is not our intention here.

Lemma 5 (**TMC**, VII.4.1 in [7])**.** *Let* X *be a nonempty transitive set. There exists a unique set* **Def** X *consisting of all sets* $Y \subseteq X$*, definable over* X *by an* $\in$*-formula with parameters from* X*.*

This set **Def** X *is obviously transitive, and* $X \cup \{X\} \subseteq \mathbf{Def}\, X$*.*

Lemma 6 (**TMC**, [7], Lemma VII.4.2)**.** *Let* u *be a transitive set and* $\beta \in \text{Ord}$*. There is a unique function* $f = \mathbb{f}^u_\beta$ *such that* $\mathtt{dom}\, f = \beta$, $f(0) = u$, $f(\alpha+1) = \mathbf{Def}\, f(\alpha)$ *whenever* $\alpha + 1 < \beta$*, and* $f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha)$ *for all limit* $\lambda < \beta$*.*

The lemma enables us to define $\mathbf{L}_\alpha[u] = \mathbb{f}^u_{\alpha+1}(\alpha)$ in **TMC**, legitimizing the standard definition of relative constructible hierarchy for any set $u \subseteq \omega$:

$$
\left.\begin{array}{lcl}
\mathbf{L}_0[u] & = & \omega \cup \{u\} - \text{to keep it transitive}, \\
\mathbf{L}_{\alpha+1}[u] & = & \mathbf{Def}\ \mathbf{L}_\alpha[u] \ \text{ for all } \alpha\,, \\
\mathbf{L}_\lambda[u] & = & \bigcup_{\alpha<\lambda} \mathbf{L}_\alpha[u] \ \text{ for all limit } \lambda\,, \\
\mathbf{L}[u] & = & \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{L}_\alpha[u] \ = \text{ all sets constructible in } u, \\
\mathbf{L}_\alpha & = & \mathbf{L}_\alpha[\varnothing], \\
\mathbf{L} & = & \mathbf{L}[\varnothing].
\end{array}\right\} \qquad (3)
$$

Theorem 5 (**TMC**)**.** *Suppose that $u \subseteq \omega$, then the following conditions apply:*

(i) *Each $\mathbf{L}_\alpha[u]$ is transitive and $\alpha \subseteq \mathbf{L}_\alpha[u]$;*

(ii) *If $\alpha < \beta$ then $\mathbf{L}_\alpha[u] \in \mathbf{L}_\beta[u]$ and $\mathbf{L}_\alpha[u] \subseteq \mathbf{L}_\beta[u]$;*

(iii) *If λ is the limit, then $\mathbf{L}_\lambda[u]$ is closed under the rudimentary operations (c) in Section 2;*

(iv) (I) *If $\lambda \in \mathrm{Ord}$ is the limit, then the map $\alpha \longmapsto \mathbf{L}_\alpha[u]$ $(\alpha < \lambda)$ is definable over $\mathbf{L}_\lambda[u]$ with u as the only parameter;* (II) *the class-map $\alpha \longmapsto \mathbf{L}_\alpha[u]$ $(\alpha \in \mathrm{Ord})$ is definable over $\mathbf{L}[u]$, with u as the only parameter.*

Proof. See [7], Theorem VII.4.3 on (i), (ii), (iii). Regarding (iv), see Theorem VII.4.8 in [7] or [24] (B.5, Lemma 4.1) in case $u = \varnothing$. □

What kind of set theory is provided in $\mathbf{L}[u]$ by **TMC**?

Lemma 7 (**TMC**)**.** *Let $u \subseteq \omega$. All axioms of $\mathbf{Z}^-$, except perhaps for the* **Separation** *schema, hold in $\mathbf{L}[u]$ and in any set $\mathbf{L}_\lambda[u]$, where $\lambda \in \mathrm{Ord}$ is the limit.*

Proof (sketch)**.** This does not differ from the full-**ZF** case. Consider, for instance the Union axiom. Let $X \in \mathbf{L}[u]$, so that $X \in \mathbf{L}_\alpha[u]$, $\alpha \in \mathrm{Ord}$. As $\mathbf{L}_\alpha[u]$ is transitive, the union $Y = \bigcup X \subseteq \mathbf{L}_\alpha[u]$ is definable over $\mathbf{L}_\alpha[u]$, hence $Y \in \mathbf{L}_{\alpha+1}[u] = \mathbf{Def}\ \mathbf{L}_\alpha[u]$. □

On the other hand, axioms of **TMC** do not imply that the schemata of **Replacement/Collection** necessarily hold in $\mathbf{L}$, as the next example shows.

Example 1. *Arguing in the full-**ZF** theory, let $\mathfrak{M} = \mathbf{L}_\vartheta$, where $\vartheta = (\aleph_\omega)^{\mathbf{L}}$. Let $\mathfrak{N}$ be the forcing extension of $\mathfrak{M}$ by ajoining a generic sequence of (generic) maps $f_n : \omega$ onto $(\aleph_n)^{\mathbf{L}}$. Then, $\mathfrak{N}$ is a model of* **TMC**. *However, $(\mathbf{L})^{\mathfrak{N}} = \mathfrak{M}$, and* **Repl/Coll** *definitely fail in $\mathfrak{M}$.*

Unlike **Repl**/**Coll**, the **Separation** schema always holds in $\mathbf{L}$ under the **TMC** axioms in the background set universe by Theorem 3(II), as proven in Section 10.

6. Definability and Well-orderings

Our goal here is to prove a few more delicate results related to the constructible hierarchy. The next lemma presents a key definability result.

Lemma 8 (**TMC**)**.** *Let $u \subseteq \omega$, λ be the limit, and $Y \in \mathbf{L}_\lambda[u]$. Then, Y is definable over $\mathbf{L}_\lambda[u]$* (i) *by a formula with parameters $\mathbf{L}_\delta[u]$, $\delta < \lambda$;* (ii) *by a formula with parameters $\delta < \lambda$ and u.*

Proof. (i) By definition, $Y = \{y \in \mathbf{L}_\alpha[u] : \mathbf{L}_\alpha[u] \models \varphi(y)\}$, where $\alpha < \lambda$ and φ may contain parameters in $\mathbf{L}_\alpha[u]$. Arguing by induction on α, let $\varphi(y)$ be $\varphi(p, y)$, where $p \in \mathbf{L}_\alpha[u]$ is a parameter. Then, $p \in \mathbf{L}_{\gamma+1}[u]$ for some $\gamma < \alpha$ by (3) above. According to the inductive hypothesis, we have $p = \{z \in \mathbf{L}_\gamma[u] : \mathbf{L}_\lambda[u] \models \psi(z)\}$, where ψ has only sets $\mathbf{L}_\delta[u]$, $\delta < \lambda$, as parameters. Then, $Y = \{y \in \mathbf{L}_\alpha[u] : \mathbf{L}_\lambda[u] \models \Phi(y)\}$, where

$$
\Phi(y) := \exists\, p\, \big(y, p \in \mathbf{L}_\alpha[u] \wedge p = \{z : z \in \mathbf{L}_\gamma[u] \wedge \psi(z)\} \wedge \varphi(p, y)^{\mathbf{L}_\alpha[u]}\big),
$$

and $\varphi(p,y)^{\mathbf{L}_\alpha[u]}$ means the formal relativization to $\mathbf{L}_\alpha[u]$, that is, all quantifiers $\exists a$, $\forall a$ are changed to resp. $\exists a \in \mathbf{L}_\alpha[u]$, $\forall a \in \mathbf{L}_\alpha[u]$. Then, φ' has only the sets $\mathbf{L}_\gamma[u]$, $\mathbf{L}_\alpha[u]$, and some $\mathbf{L}_\delta[u]$, $\delta < \lambda$, as parameters. This proves part (i). We now infer part (ii) applies to Theorem 5(iv). □

Lemma 9 (**TMC**)**.** *Let $u \subseteq \omega$ and λ be the limit. There is a map $H : D = \omega \times \lambda \times \lambda^{<\omega}$ onto $\mathbf{L}_\lambda[u]$, definable over $\mathbf{L}_\lambda[u]$ with u as the only parameter.*

Proof. By Lemma 8, each $Y \in \mathbf{L}_\lambda[u]$ has the form $Y = \{y \in \mathbf{L}_\alpha[u] : \mathbf{L}_\lambda[u] \models \varphi(y)\}$ for some $\alpha < \lambda$, where φ contains parameters $\delta < \lambda$ and u.

Given a triple of n, α, p of $n \in \omega$, $\alpha < \lambda$, and $p = \langle \delta_1, \dots, \delta_k \rangle \in \lambda^k$, let φ_n be the n-th parameter-free $\in$-formula. If

(†) $\delta_1 \dots, \delta_k < \lambda$ and φ_n is $\varphi_n(v_1, \dots, v_k, v)$ with $k+1$ free variables,

then define the set

$$H(n, \alpha, p) = \{y \in \mathbf{L}_\alpha[u] : \mathbf{L}_\lambda[u] \models \varphi(\delta_1, \dots, \delta_k, y)\}.$$

If (†) fails, then put $H(n,\alpha,p) = \varnothing$. Then, H is definable over $\mathbf{L}_\lambda[u]$ with u as a parameter by Theorem 5(iv) since it is defined in terms of the definable map $\alpha \longmapsto \mathbf{L}_\alpha[u]$. □

Lemma 10 (**TMC**)**.** *Let $u \subseteq \omega$. There is a well-ordering $<_{\mathbf{L}[u]}$ of $\mathbf{L}[u]$ definable over $\mathbf{L}[u]$ with u as the only parameter. If $\lambda \in \mathrm{Ord}$ is the limit, then there is a well-ordering $<_{\mathbf{L}_\lambda[u]}$ of $\mathbf{L}_\lambda[u]$ definable over $\mathbf{L}_\lambda[u]$ with u as the only parameter.*

Proof. In the λ-case, let the map $H : D \xrightarrow{\text{onto}} \mathbf{L}_\lambda[u]$ be given by Lemma 9. The set $D = \omega \times \lambda \times \lambda^{<\omega} \subseteq \mathbf{L}_\lambda[u]$ is parameter-free definable over $\mathbf{L}_\lambda[u]$. Thus, to define $<_{\mathbf{L}_\lambda[u]}$, it suffices to show that D admits a well-ordering $<_D$ parameter-free definable over $\mathbf{L}_\lambda[u]$. For that purpose, if

$$d = \langle n, \alpha, u = \langle \gamma_1, \dots, \gamma_m \rangle \rangle \in D, \quad d' = \langle n', \alpha', u' = \langle \gamma'_1, \dots, \gamma'_{m'} \rangle \rangle \in D,$$

then let $\mu(d) = \max\{\alpha, \gamma_1, \dots, \gamma_m\}$ and define $d <_D d'$, if and only if, any of the following conditions are met:

(‡) $\mu(d) < \mu(d')$;
$\mu(d) = \mu(d')$ and $m < m'$;
$\mu(d) = \mu(d')$, $m = m'$, and $u < u'$ lexicographically in λ^m;
$\mu(d) = \mu(d')$, $m = m'$, $u = u'$, and $n < n'$.

The well-ordering $<_{\mathbf{L}[u]}$ of $\mathbf{L}[u]$ is then defined so that $x <_{\mathbf{L}[u]} y$ if either (1) $\lambda_x < \lambda_y$, where λ_x is the least limit ordinal with $x \in \mathbf{L}_{\lambda_x}$, or (2) $\lambda_x = \lambda_y$ and $x <_{\mathbf{L}_\lambda[u]} y$. □

7. The Key Technical Theorem

The purpose of this section is to formulate a convenient necessary condition for obtaining $\mathbf{ZFC}^-$ in some constructible domains. This will be Theorem 6 below, the key theorem of the title. To simplify formalities, we define the following formula:

Definition 2 (**TMC**)**.** *Let $\mathfrak{A}(u, \Omega, K)$ be $u \subseteq \omega$, and either the following conditions are met;*

- (A) $\Omega = \mathrm{Ord}$, $K = \mathbf{L}[u]$, *and* $\omega_1^{\mathbf{L}[u]}$ *does not exist; in other words, every ordinal is countable in* $\mathbf{L}[u]$,
- (B) *the ordinal* $\Omega = \omega_1^{\mathbf{L}[u]}$ *exists, and* $K = \mathbf{L}_\Omega[u] = \mathbf{L}_{\omega_1^{\mathbf{L}[u]}}[u]$.

Thus, $K = \bigcup_{\alpha \in \Omega} \mathbf{L}_\alpha[u]$ *in both cases* (A), (B).

Lemma 11 (**TMC** $+ \mathfrak{A}(u, \Omega, K)$)**.** *If $\alpha \in \Omega$, then $\mathbf{L}_\alpha[u]$ is ctble in $\mathbf{L}[u]$.*

Proof. Let $\alpha \in \Omega$ be the limit. By Definition 2, there is a map $f \in \mathbf{L}[u]$, $f : \omega$ onto α. Lemma 8 provides a set $D = \omega \times \alpha \times \alpha^{<\omega} \in \mathbf{L}[u]$ and a map $H \in \mathbf{L}[u]$, $H : D$ onto $\mathbf{L}_\alpha[u]$. We obtain a map $h \in \mathbf{L}[u]$, $h : \omega$ onto $\mathbf{L}_\alpha[u]$ by combining f and H in $\mathbf{L}[u]$. $\square$

Lemma 12 ($\mathbf{TMC} + \mathfrak{A}(u, \Omega, K)$)**.** *Let $X \in K$, and $F : X \to K$ be a class-map definable over $\mathbf{L}[u]$. Then, $\mathtt{ran}\, F = \{F(x) : x \in X\} \subseteq \mathbf{L}_\gamma[u]$ for some $\gamma \in \Omega$; hence F, $\mathtt{ran}\, F$ are sets.*

Proof. By Lemma 11, we without any loss of generality suppose that $X = \omega$. For any $k < \omega$, let δ_k be the least $\delta \in \Omega$ satisfying $F(k) \in \mathbf{L}_\delta[u]$. Assume towards the contrary that $\{\delta_k : k < \omega\}$ is <u>un</u>bounded in Ω. Then, $\Omega = \bigcup_{k<\omega} \delta_k$.

In case (A), for any k, there are functions $h \in \mathbf{L}[u]$, $h : \omega$ onto δ_k; let h_k be the $<_{\mathbf{L}[u]}$-least of them. If $n = 2^k(2j+1) - 1$, then put $G(n) = h_k(j)$. Then, G is a definable class-map from ω onto $\Omega = \mathrm{Ord}$ by construction. Thus, Ω and G are sets by Lemma 1 since Ω is transitive. This is a contradiction since Ord is not a set in **TMC**.

In case (B), $\Omega = \omega_1^{\mathbf{L}[u]}$. Define h_k and G using the well-ordering $<_{\mathbf{L}_\Omega[u]}$ of $\mathbf{L}_\Omega[u]$ instead of $<_{\mathbf{L}[u]}$. Then G is a class-map from ω onto $\Omega = \omega_1^{\mathbf{L}[u]}$, definable over $\mathbf{L}_\Omega$ since $<_{\mathbf{L}_\Omega[u]}$. Thus, $G \in \mathbf{L}_{\Omega+1}[u] \subseteq \mathbf{L}[u]$, and hence the ordinal Ω is countable in $\mathbf{L}[u]$. This is a contradiction. $\square$

Corollary 4 ($\mathbf{TMC} + \mathfrak{A}(u, \Omega, K)$)**.** *Assume that $\alpha \in \Omega$, $m < \omega$, and $G_1, \ldots, G_m : K \to K$ be class-maps definable over $\mathbf{L}[u]$. There is a limit ordinal $\beta \in \Omega$, $\beta > \alpha$, satisfying $G_k{}''\mathbf{L}_\beta[u] \subseteq \mathbf{L}_\beta[u]$ for all $k = 1, \ldots, m$.*

Proof. Put $G(x) = \langle G_1(x), \ldots, G_m(x) \rangle$. Use Lemma 12 to obtain a class-sequence $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \ldots$ of ordinals in Ω satisfying $G''\mathbf{L}_{\alpha_n}[u] \subseteq \mathbf{L}_{\alpha_{n+1}}[u]$, $\forall n$. Apply Lemma 12 again to show that $\beta = \sup_n \alpha_n \in \Omega$. $\square$

Assume $\mathfrak{A}(u, \Omega, K)$. Say that $\beta \in \Omega$ *reflects* a formula $\varphi(x_1, \ldots, x_n)$, if the equivalence $\varphi^K(x_1, \ldots, x_n) \iff \varphi^{\mathbf{L}_\beta[u]}(x_1, \ldots, x_n)$ holds for all $x_j \in \mathbf{L}_\beta$. The following *reflection lemma* is a standard consequence of Corollary 4.

Lemma 13 ($\mathbf{TMC} + \mathfrak{A}(u, \Omega, K)$)**.** *If $\alpha \in \Omega$ and φ is a parameter-free formula, then there exists a limit ordinal $\beta \in \Omega$, $\beta > \alpha$ which reflects φ and every subformula of φ.*

Proof (sketch)**.** We, without any loss of generality assume that φ does not contain $\forall$ (otherwise, replace $\forall$ with $\neg \exists \neg$). Let us enumerate $\psi_1, \ldots, \psi_n$ all the sub-formulas of φ (including possibly φ itself) beginning with $\exists$. If $j \le n$, then we define a class-map G_j as follows.

Let $j \le n$ and ψ_j be $\exists y\, \chi_j(y, x_1, \ldots, x_m)$. If $p = \langle x_1, \ldots, x_m \rangle \in K$ and there is $y \in K$ satisfying $\chi_j^K(y, x_1, \ldots, x_m)$, then let $G_j(p)$ be the $<_{\mathbf{L}[u]}$-least of these y. Otherwise let $G_j(p) = \varnothing$. Each class-map G_j is definable over $\mathbf{L}[u]$, such is the well-ordering $<_{\mathbf{L}[u]}$.

By Corollary 4, there is an ordinal $\beta \in \Omega$, $\beta > \alpha$, satisfying $G_j{}''\mathbf{L}_\beta[u] \subseteq \mathbf{L}_\beta[u]$ for all $j = 1, \ldots, n$. Now, it easily goes by induction on the number of logical symbols that β reflects every subformula of φ. In particular, it reflects φ itself, as required. $\square$

Theorem 6 ($\mathbf{TMC} + \mathfrak{A}(u, \Omega, K)$)**.** *The schemata of* **Separation** *and* **Collection** *hold in K.*

Therefore, $\mathbf{ZFC}^-$ *as a whole holds in K by Lemma 7.*

Proof. Separation. Assume that $\varphi(x, y)$ is a parameter-free formula, $\alpha \in \Omega$, $p \in X = \mathbf{L}_\alpha[u]$. We have to prove that $Y = \{x \in X : \varphi^K(x, p)\} \in K$. Let, by Lemma 13, a limit ordinal $\beta \in \Omega$, $\beta > \alpha$ reflect $\varphi(x, y)$, so that

$$Y = \{x \in X : \varphi^{\mathbf{L}_\beta[u]}(x, p)\} = \{x \in X : \mathbf{L}_\beta[u] \models \varphi(x, p)\} \in \mathbf{L}_{\beta+1}[u] \subseteq K.$$

Collection. Assume that $\varphi(x,y,z)$ is a parameter-free formula, $\alpha \in \Omega$, $p \in X = \mathbf{L}_\alpha[u]$, and we have $\forall\, x \in X\, \exists\, y \in K\, \varphi^K(x,y,p)$. By Lemma 13, there exists a limit ordinal $\beta \in \Omega$, $\beta > \alpha$, which reflects $\exists\, y\, \varphi(x,y,z)$, with all its subformulas, including $\varphi(x,y,z)$, so that

$$\forall\, x \in X\, \exists\, y \in \mathbf{L}_\beta[u]\, \varphi^{\mathbf{L}_\beta[u]}(x,y,p), \text{ and } \forall\, x \in X\, \exists\, y \in \mathbf{L}_\beta[u]\, \varphi^K(x,y,p). \qquad \square$$

8. Proof of Theorems 1 and 2

Theorem 1 is an elementary consequence of Theorem 2, so we concentrate on the latter. In fact, all the necessary work has already been done.

Case (b) of Theorem 2. Arguing in **TMC**, we have case (B) of Definition 2 with $u = \varnothing$, $\Omega = \omega_1^{\mathbf{L}}$, $K = \mathbf{L}^* = \mathbf{L}_{\omega_1^{\mathbf{L}}}$. Then, $\mathfrak{A}(\varnothing, \omega_1^{\mathbf{L}}, \mathbf{L}^*)$ holds, and hence $\mathbf{L}^*$ satisfies $\mathbf{ZFC}^-$ by Theorem 6.

Case (a) of Theorem 2. Similar, but via case (A) of Definition 2.

9. Proof of Theorem 3(I)

We may note that item (I) of Theorem 3 is a simple corollary of item (II), as proven below in Section 10. However, we present here a different proof based on Theorem 6 above.

We argue in TMC. Prove that $\mathbf{L} \cap \mathscr{P}(\omega)$ satisfies $\mathbf{PA}_2$.

Case 1: There is $u \subseteq \omega$ such that $\omega_1^{\mathbf{L}[u]}$ does not exist. Then, $\mathbf{ZFC}^-$ holds in $\mathbf{L}[u]$ by Theorem 6; hence, $\mathbf{ZFC}^-$ holds in $\mathbf{L}$ as well. This implies $\mathbf{PA}_2$ in $\mathbf{L} \cap \mathscr{P}(\omega)$, as required.

Case 2: $\omega_1^{\mathbf{L}[u]} \in \mathrm{Ord}$ exists for all $u \subseteq \omega$. In particular, $\Omega = \omega_1^{\mathbf{L}} \in \mathrm{Ord}$ exists, and $\mathbf{L}_\Omega$ is a model of $\mathbf{ZFC}^-$ by Theorem 2. Therefore, it suffices to prove that $\mathbf{L} \cap \mathscr{P}(\omega) \subseteq \mathbf{L}_\Omega$.

This is a well-known result in **ZFC** and $\mathbf{ZFC}^-$, a part of Gödel's proof of **CH** in **L**. Gödel's reasoning is doable in **TMC**, and a close claim is established in [7] in the course of the proof of Theorem VII.4.34. However, the proof there involves quite special arguments. For instance, the Σ_1-theory of constructible hierarchy, which we do not plan to use in our proof. Yet, there is a much simpler way to achieve the same goal, which is by reduction to the $\mathbf{ZFC}^-$ environment.

Thus, let $x \in \mathbf{L} \cap \mathscr{P}(\omega)$. Then, $x \in \mathbf{L}_\lambda$ for some $\lambda \in \mathrm{Ord}$. We assert that

$(*)$ there is an ordinal $\vartheta > \lambda$ such that $\mathbf{L}_\vartheta$ is a model of $\mathbf{ZFC}^-$.

Indeed, by the axiom of **Countability** in **TMC**, there is a bijection $h : \omega$ onto λ. Let $u = \{2^j \cdot 3^k : h(j) < h(k)\}$. Thus, $u \subseteq \omega$ codes h. Note that $\vartheta = \omega_1^{\mathbf{L}[u]} \in \mathrm{Ord}$ by the Case 2 assumption, and $\mathbf{L}_\vartheta[u]$ is a model of $\mathbf{ZFC}^-$ by Theorem 6; hence, $\mathbf{L}_\vartheta \models \mathbf{ZFC}^-$ as well. Thus, it suffices to show that $\lambda \le \vartheta$.

Suppose to the contrary that $\vartheta < \lambda$. Then, $\mathbf{L}_\vartheta[u] \models \mathbf{ZFC}^-$, as stated above. In addition, $\mathbf{L}_\vartheta[u]$ is a model of ordinal height ϑ, and $u \in \mathbf{L}_\omega[u] \subseteq \mathbf{L}_\vartheta[u]$, by construction. But u effectively codes the ordinal $\lambda > \vartheta$, which is a contradiction. This completes the proof of $(*)$.

Choose ϑ by $(*)$. Thus, $x \in \mathbf{L}_\vartheta$. We do not claim that $\Omega = \omega_1^{\mathbf{L}_\vartheta}$, but Ω obviously remains a regular uncountable cardinal in $\mathbf{L}_\vartheta$. This implies that $\mathbf{L}_\vartheta \cap \mathscr{P}(\omega) \subseteq \mathbf{L}_\Omega$ by a standard collapse argument by Gödel. We conclude that $x \in \mathbf{L}_\Omega$, as required.

10. Proof of Theorem 3(II), Sketch

We argue in TMC. Due to Lemma 7, it suffices to check the **Separation** schema in **L**.

We will make use of a series of deep results in [25], particularly those related to countable *index ordinals*, that is, ordinals α satisfying $(\mathbf{L}_{\alpha+1} \setminus \mathbf{L}_\alpha) \cap \mathscr{P}(\omega) \neq \varnothing$.

It is asserted in [25] that there exists a parameter-free closed $\in$-formula σ such that, for any transitive set M, σ^M (the formal relativization) holds if $M = \mathbf{L}_\lambda$ for some limit ordinal λ, and in addition $\sigma^{\mathbf{L}}$ holds as well. Basically, σ says that all sets are constructible and there is no largest ordinal. The required property is based on the absoluteness of Gödel's construction for transitive sets satisfying some simple conditions [15]. It is explained in [15] (Ch. 13) between Theorem 13.16 and Lemma 13.17 how such a formula σ can be constructed, satisfying the desired property (13.13) there. See also [26] with a complete argument.

Now, suppose to the contrary that **Sep** fails in $\mathbf{L}$, that is, there exist the following: a transitive set $B \in \mathbf{L}$ (say $B = \mathbf{L}_\alpha$ for some α) and a formula $\varphi(p, x)$ with a parameter $p \in \mathbf{L}$, such that $Y = \{b \in B : \varphi^{\mathbf{L}}(p, b)\} \notin \mathbf{L}$ (Y is a set in the **TMC** universe by **Sep**). Taking the $<_{\mathbf{L}}$-least B and p with these properties, we reduce the general case to the following:

(†) $B = \{b \in \mathbf{L} : \vartheta^{\mathbf{L}}(b)\}$ is parameter-free definable in $\mathbf{L}$, and $\varphi(x)$ is a parameter-free formula, still satisfying $Y = \{b \in B : \varphi^{\mathbf{L}}(b)\} \notin \mathbf{L}$.

Assuming that the formulas φ and σ do not contain the quantifier $\forall$ (replaced by $\neg \exists \neg$), we let $f_1, \ldots, f_m$ be the Skolem functions for all existential subformulas of the formulas

(‡) σ, $\varphi(x)$, and the formula '$B = \{b \in \mathbf{L} : \vartheta(b)\}$',

defined in $\mathbf{L}$ on the basis of the parameter-free definable well-ordering $<_{\mathbf{L}}$.

Consider the closure M of $B \cup \{B\}$ under $f_1, \ldots, f_m$. By a standard combinatorial argument, there is a class-map Φ defined on the set $U = B^{<\omega} \times \omega^{<\omega}$, such that $M = \Phi'' U$. Let $\tau : M$ onto a transitive class N be a collapse map, that is, $\tau(x) = \{\tau(y) : y \in x \cap M\}$ for all $x \in M$. (To define N, τ apply Corollary 1 for sets $M_\alpha = M \cap \mathbf{L}_\alpha$, $\alpha \in \mathrm{Ord}$, and let τ be the union of all partial collapse maps $\tau_\alpha : M_\alpha$ onto a transitive set N_α.)

Using Lemma 1 for the superposition of Φ and τ, we conclude that N is a set. Moreover, as B is transitive, we have $B = \tau(B) \in N$.

On the other hand, the class or set M is an elementary submodel of $\mathbf{L}$ with respect to formulas (‡) by construction. In particular, $M \models \sigma$, hence $N \models \sigma$ as well, and we conclude by the choice of σ that $N = \mathbf{L}_\lambda$ for some limit λ.

By the same argument (and because $B = \tau(B)$), we conclude that $Y = \{b \in B : \varphi^{\mathbf{L}_\lambda}(b)\} \in \mathbf{L}_{\lambda+1} \subseteq \mathbf{L}$, which contradicts (†).

11. A Corollary in the Domain of Reals

Theorem 2 being proven implies the following corollary.

Corollary 5 ($\mathbf{PA}_2^-$)**.** *$\mathbf{L} \cap \mathscr{P}(\omega)$ satisfies $\mathbf{PA}_2$.*

Saying it differently, $\mathbf{L} \cap \mathscr{P}(\omega)$ is an interpretation of $\mathbf{PA}_2$ in $\mathbf{PA}_2^-$.

Proof (sketch)**.** Here, $\mathbf{L} \cap \mathscr{P}(\omega)$ essentially means $\{x \subseteq \omega : \mathbf{constr}(x)\}$, where $\mathbf{constr}(x)$ is a certain Σ^1_2 formula of $\mathcal{L}(\mathbf{PA}_2)$ that expresses the constructibility of $x \subseteq \omega$ by referring to the existence of a real that encodes (similar to for instance encoding by trees in WFT) a set theoretic structure that indicates the constructibility of x. Such a formula was explicitly defined by Addison [27,28], but it implicitly can be found in studies by Gödel [29] and Novikov [30].

As for the proof itself, recall that the $\mathbf{PA}_2^-$ structure $\mathbb{V}$ satisfies **TMC** by Theorem 4. Therefore, we have $\ulcorner \mathbf{L} \cap \mathscr{P}(\omega)$ satisfies $\mathbf{PA}_2 \urcorner^{\mathbb{V}}$ by Theorem 2. Yet, the $\mathbb{V}$-reals are isomorphic to the true reals in the background $\mathbf{PA}_2^-$ universe. We conclude that, in $\mathbf{PA}_2^-$, $\mathbf{L} \cap \mathscr{P}(\omega)$ satisfies $\mathbf{PA}_2$. □

Corollary 5 can be compared with its better-known **ZF** version:

Proposition 1 (**ZF**, Theorem 1.5 in [4])**.** *If $X \subseteq \mathscr{P}(\omega)$ is a β-model of $\mathbf{PA}_2^-$, then $X \cap \mathbf{L}$ is a β-model of $\mathbf{PA}_2$ plus constructibility.*

The proof of the proposition in [4] involves Lemma 1.4, which cites Theorem 1 in [25], as presented in Proposition 2(ii) below. Another path to Proposition 1, quite complicated in its own way, is given in [31,32]. It is definitely tempting to accomodate these proofs of Proposition 1 to the case $X = \mathscr{P}(\omega)$ towards Corollary 5 under the **TMC** axioms. Yet, we are not going to pursue this plan here as it will definitely involve more complex arguments than the above proof of Theorems 2 and 3.

12. Some Other Models

Here, we briefly describe three other interpretations of $\mathbf{ZFC}^-$ in **TMC**, which are designed rather similar to $\mathbf{L}^*$ of Theorem 2.

Model 1. Consider the least ordinal Λ such that the set $\mathbf{L}_\Lambda$ is not countable in $\mathbf{L}_{\Lambda+1}$—provided such ordinals exist, and otherwise $\Lambda =$ all ordinals. Put $\mathbf{L}^\dagger = \bigcup_{\alpha\in\Lambda} \mathbf{L}_\alpha$. It is demonstrated in [9] that $\mathbf{L}^\dagger$ is an interpretation of $\mathbf{ZFC}^-$ in **TMC**.

Model 2: version of Model 1. Consider the least ordinal Ξ such that the difference $\mathbf{L}_{\Xi+1} \smallsetminus \mathbf{L}_\Xi$ contains no sets $x \subseteq \omega$—the first *index ordinal* as defined in [25]—provided such ordinals exist, and otherwise $\Xi =$ all ordinals. Arguments close to those in [9] show that $\mathbf{L}^\ddagger = \bigcup_{\alpha\in\Xi} \mathbf{L}_\Xi$ $\mathbf{L}^\dagger$ is an interpretation of $\mathbf{ZFC}^-$ in **TMC**.

Model 3. Simpson defines in [7] (VII.4.22) the set or class **HCL** of all sets x that belong to transitive sets $X \in \mathbf{L}$, countable in $\mathbf{L}$, and proves that **HCL** is an interpretation of $\mathbf{ZFC}^-$ in **TMC** yet again. But it looks like **HCL** is just equal to $\mathbf{L}^*$ of Theorem 2.

13. Ramified Analytical Hierarchy—A Shortcut?

Cutting Theorem 1 to the equiconsistency of $\mathbf{PA}_2$ and $\mathbf{PA}_2^-$ (second-order arithmetic with, resp., without the countable Choice $\mathbf{AC}_\omega$), one may want to manufacture a true second-order arithmetical proof, not involving set theories like $\mathbf{Z}^-$, $\mathbf{ZFC}^-$, $\mathbf{ZF}^-$, **TMC**. The above proof (Section 8) definitely does not belong to this type, since it involves **TMC** in a quite significant way. In this section, we survey a possible approach to this problem.

Using earlier ideas of Kleene [33] and Cohen [34], a transfinite sequence of countable sets $\mathbf{A}_\alpha \subseteq \mathscr{P}(\omega)$ is defined in, for instance [25], (§ 3) by induction so that

$$\left.\begin{array}{lcl} \mathbf{A}_0 & = & \mathscr{P}_{\mathtt{fin}}(\omega) = \text{all finite sets } x \subseteq \omega \\ \mathbf{A}_{\alpha+1} & = & \mathbf{Def}\ \mathbf{A}_\alpha \ \text{ for all } \alpha \\ \mathbf{A}_\lambda & = & \bigcup_{\alpha<\lambda} \mathbf{A}_\alpha \ \text{ for all limit } \lambda \\ \mathbf{A} & = & \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{A}_\alpha = \text{all } \textit{ramified analytic} \text{ sets} \end{array}\right\}, \tag{4}$$

where $\mathbf{Def}\,\mathbf{A}_\alpha = \{x \subseteq \omega : x \text{ is definable over } \mathbf{A}_\alpha \text{ with parameters}\}$ in the second line. Thus, a set $x \subseteq \omega$ belongs to $\mathbf{Def}\,\mathbf{A}_\alpha$ if $x = \{n : \mathbf{A}_\alpha \models \varphi(n)\}$ for some formula φ of $\mathcal{L}(\mathbf{PA}_2)$ with parameters in $\mathbf{A}_\alpha$, and $X \models \dots$ means the formal truth in the $\mathcal{L}(\mathbf{PA}_2)$-structure $\langle \omega\,; X\rangle$. The following is routine.

Lemma 14. *If $x \in \mathbf{A}_\alpha$ and $y \subseteq \omega$ is arithmetical in x, then $y \in \mathbf{A}_\alpha$.*

In spite of obvious similarities with the Gödel constructible hierarchy (3), the ramified analytic hierarchy is collapsing below ω_1:

Lemma 15 (Cohen). *There is an ordinal $\beta_0 < \omega_1^{\mathbf{L}}$ such that $\mathbf{A}_{\beta_0} = \mathbf{A}_{\beta_0+1} = \mathbf{A}_\gamma$ for all $\gamma > \beta_0$. Then, obviously, $\mathbf{A} = \mathbf{A}_{\beta_0}$ and $\mathbf{A} \models \mathbf{PA}_2^-$.*

Proof. By the cardinality argument, there is an ordinal β with $\mathbf{A}_\beta = \mathbf{A}_{\beta+1}$. Then, $\mathbf{A}_\beta \models$ **Sep**. Let $\kappa = \beta^+$, the least cardinal bigger than β. Consider a countable elementary submodel M of $\mathbf{L}_\kappa$ containing β, and let $H : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse. Let $\beta_0 = H(\beta)$; then, $\beta_0 < \lambda$. As the construction of the sets $\mathbf{A}_\alpha$ is obviously absolute for $\mathbf{L}$, we have $\mathbf{A}_{\beta_0} \models \mathbf{Sep}$ as well, and then $\mathbf{A}_{\beta_0} = \mathbf{A}_{\beta_0+1}$, as required. □

The following theorem is essentially Lemma 2.2 in [35].

Theorem 7 (**ZF**)**.** $\mathbf{A} = \mathbf{A}_{\beta_0}$ *satisfies* $\mathbf{PA}_2$ *with the choice schema* $\mathbf{AC}_\omega$.

Proof. To sketch a proof of this profound result, we need to have a look at the ramified analytic hierarchy from a somewhat different angle. This involves a "shift" in Gödel's hierarchy and ensuing classification of ordinals:

- Let $\mathbf{M}_\alpha = \mathbf{L}_{\omega+\alpha}$ for all α. In particular, $\mathbf{M}_0 = \mathbf{L}_\omega =$ all hereditarily finite sets, but still, similarly to (3), $\mathbf{M}_{\alpha+1} = \mathbf{Def}\ \mathbf{M}_\alpha$, $\forall\,\alpha$, and the union is taken at limit steps. (See, for instance note 2 on p. 499 in [25] or Section 5 in [36], where "$\mathbf{L}_0 =$ hereditarily finite sets" is defined outright.) Needless to say that $\mathbf{M}_\alpha = \mathbf{L}_\alpha$ for all $\alpha \geq \omega^2$.
- An ordinal α is an *index* if $(\mathbf{M}_{\alpha+1} \smallsetminus \mathbf{M}_\alpha) \cap \mathscr{P}(\omega) \neq \varnothing$.

We will refer to a result established in [25], using Theorems 1 and 9 by a complex mixture of set theoretic and recursion theoretic methods. A set $E \subseteq \omega \times \omega$ is *a code* (or *arithmetical copy*, as in [25,37]) of $\mathbf{M}_\alpha$ if it is isomorphic to $\in\restriction\mathbf{M}_\alpha$ via a bijection of **fld** E onto $\mathbf{M}_\alpha$. □

Proposition 2.

(i) *If* $\alpha \leq \beta_0 + 1$ *then* $\mathbf{A}_\alpha = \mathbf{M}_\alpha \cap \mathscr{P}(\omega)$.

(ii) *If* β *is an index then there is a code of* $\mathbf{M}_\beta$ *in* $\mathbf{M}_{\beta+1}$.

Proof (sketch)**.** (ii) Suppose that β is the limit, as argued in Section 10 with $B = \omega$ and $\mathbf{M}_\beta = \mathbf{L}_{\omega+\beta}$ instead of $\mathbf{L}$, so that $Y = \{k \in \omega : \varphi^{\mathbf{M}_\beta}(k)\} \notin \mathbf{M}_\beta$. In the notation of Section 10, we still have $N = \mathbf{M}_\lambda$ for a limit λ. Note that $\lambda < \beta$ is impossible since $Y \in \mathbf{M}_{\lambda+1} \smallsetminus \mathbf{M}_\beta$. $\lambda > \beta$ is impossible as well since N is the transitive collapse of $M \subseteq \mathbf{M}_\beta$.

Thus, $\lambda = \beta$, and hence $\mathbf{M}_\beta$ is $\in$-isomorphic to M.

On the other hand, $M \in \mathbf{M}_{\beta+1}$ as a definable subset of $\mathbf{M}_\beta$. Moreover, the inductive construction of M as the closude of ω under a finite list of functions definable over $\mathbf{M}_\beta$ can be represented as a construction of a relation $E \subseteq \omega \times \omega$, still definable over $\mathbf{M}_\beta$, and such that $\langle \omega ; E \rangle$ is isomorphic to $\langle M ; \in \rangle$ and hence to $\langle \mathbf{M}_\beta ; \in \rangle$ by the above.

In other words, $E \in \mathbf{M}_{\beta+1}$ is a code of $\mathbf{M}_\beta$, as required.

If $\beta = \nu + k$, where ν is the limit and $1 \leq k < \omega$, then we have to go back to Section 10 and, using σ, define a closed formula σ_k by induction on k, such that, for any transitive set M, $(\sigma_k)^M$ holds if $M = \mathbf{L}_{\nu+k}$ for some limit ordinal ν. Namely, put $\sigma_0 := \sigma$ as in Section 10, then let σ_{k+1} say: "there is a transitive set X with $(\sigma_k)^X$ and (all sets) = **Def** X".

Then, go through the arguments in the limit case, *mutatis mutandis*.

(i) This claim goes by induction, using (ii) as the key argument. See [25] for details.

□ (Proposition)

Beginning the proof of Theorem 7 itself, note that the equality $\mathbf{A}_{\beta_0} = \mathbf{A}_{\beta_0+1}$ immediately implies **Comprehension** in $\mathbf{A}_{\beta_0}$. The proof of $\mathbf{AC}_\omega$ takes more effort. We claim the following:

(I) β_0 *is not an index, whereas each* $\alpha < \beta_0$ *is an index;*

(II) β_0 *is a limit ordinal*—Lemma 2.5 in [35].

To prove (I), note that, by the choice of β_0 and Proposition 2(i), β_0 is not an index since $(\mathbf{M}_{\beta_0+1} \smallsetminus \mathbf{M}_{\beta_0}) \cap \mathscr{P}(\omega) = (\mathbf{A}_{\beta_0+1} \smallsetminus \mathbf{A}_{\beta_0}) \cap \mathscr{P}(\omega) = \varnothing$, whereas every $\alpha < \beta_0$ is an index by similar reasons.

To verify (II), suppose to the contrary that $\beta_0 = \alpha + 1$. By (I) and Proposition 2(ii), there is a code $x \subseteq \omega$ of $\mathbf{M}_\alpha$ in $\mathbf{M}_{\beta_0}$ and hence in $\mathbf{A}_{\beta_0}$ by Proposition 2(i). In particular, x codes all sets in $\mathbf{M}_\alpha \cap \mathscr{P}(\omega)$. Therefore, we can extract a part $y \subseteq \omega$ of x, which codes all those sets so that

$$\mathbf{M}_\alpha \cap \mathscr{P}(\omega) = \{(y)_n : n < \omega\}, \tag{5}$$

(see Section 2 on $(x)_n$), and in addition, y is arithmetical in x.

Then, $y \in \mathbf{A}_{\beta_0}$ by Lemma 14. But each $z \in \mathbf{A}_{\beta_0}$ is arithmetical in y by (5). This is a contradiction since $\mathbf{A}_{\beta_0} \models \mathbf{PA}_2^-$ by Lemma 15.

Now, coming to $\mathbf{AC}_\omega$, we are going to prove that

$$\forall n \, \exists x \, \Phi(n, x) \implies \exists y \, \forall n \, \Phi(n, (y)_n) \tag{6}$$

holds in $\mathbf{A}_{\beta_0}$, where Φ is a $\mathbf{PA}_2$ formula possibly with parameters in $\mathbf{A}_{\beta_0}$.

By Lemma 10, there exists a well-ordering $<_{\mathbf{L}_{\beta_0}}$ of $\mathbf{M}_{\beta_0}$, definable over $\mathbf{M}_{\beta_0}$. (β_0 is limit by (II).) Assuming that the left-hand side of (6) holds in $\mathbf{A}_{\beta_0}$, we let x_n be the $<_{\mathbf{L}_{\beta_0}}$-least element $x \in \mathbf{A}_{\beta_0} = \mathbf{M}_{\beta_0} \cap \mathscr{P}(\omega)$ satisfying $\mathbf{A}_{\beta_0} \models \Phi(n, x)$.

The set $y = \{\langle\!|n, j|\!\rangle : j \in x_n\}$ is then definable over $\mathbf{M}_{\beta_0}$, hence $y \in \mathbf{Def}\ \mathbf{M}_{\beta_0} = \mathbf{M}_{\beta_0+1}$. We conclude that $y \in \mathbf{A}_{\beta_0+1}$ by Proposition 2(i). Finally $y \in \mathbf{A}_{\beta_0}$, because $\mathbf{A}_{\beta_0} = \mathbf{A}_{\beta_0+1}$ by the choice of β_0. Thus y witnesses the right-hand side of (6) since $(y)_n = x_n$ by construction. □ (Theorem 7)

It remains to note that the construction of the ramified analytical hierarchy is purely analytical and can be described by suitable $\mathcal{L}(\mathbf{PA}_2)$ formulas. In principle, the proof of Theorem 7 remains valid in **TMC** *mutatis mutandis*. For instance, as ω_1 may not exist in **TMC**, the case $\beta_0 = \mathrm{Ord}$ has to be taken care of. Let

$$\beta_0 = \begin{cases} \text{the least } \beta \text{ with } \mathbf{A}_\beta = \mathbf{A}_{\beta+1} & - \quad \text{if such ordinals } \beta \text{ exist,} \\ \mathrm{Ord}, \text{ the class of all ordinals} & - \quad \text{otherwise,} \end{cases} \tag{7}$$

so that $\mathbf{A} = \bigcup_{\alpha \in \beta_0} \mathbf{A}_\alpha$ in both cases. It can be an interesting problem to maintain the construction and the proof of Theorem 7 entirely by analytical means on the base of $\mathbf{PA}_2^-$, thereby giving a pure analytical proof of the ensuing equiconsistency of $\mathbf{PA}_2^-$ and $\mathbf{PA}_2$.

14. Conclusions and Problems

In this study, the methods of second-order arithmetic and set theory were employed to giving a full, and self-contained in major details, proof of Theorem 1 on the formal equiconsistency of such theories as second-order arithmetic $\mathbf{PA}_2^-$ and Zermelo–Fraenkel $\mathbf{ZFC}^-$ without the Power Set axiom (Theorem 1). In addition, Theorems 2 and 3 contain new results related to constructible sets.

The following problems arise from our study.

Problem 1. *Regarding the axiom* **TrSups** *(Transitive superset, Section 2), is it really independent of the rest of* **TMC** *axioms? On the other hand, can* **TrSups** *be eliminated from the above proofs of the main results?*

Problem 2. *Find a purely analytical proof of Theorem 7 in* $\mathbf{PA}_2^-$ *that does not involve* $\mathbb{V}$ *of Definition 1, or any similar derived set theoretic structure, explicitly or implicitly.*

We expect that the methods and results of this paper can be used to strengthen and further develop Cohen's set theoretic forcing method in its recent applications to theories $\mathbf{ZFC}^-$ and $\mathbf{PA}_2$ in [38]. The technique of definable generic forcing notions has been recently applied for some definability problems in modern set theory, including the following applications:

- A model of **ZFC** in [39], in which minimal collapse functions $\omega \xrightarrow{\text{onto}} \omega_1^{\mathbf{L}}$ first appear at a given projective level;
- A model of **ZFC** in [40], in which the Separation principle fails for a given projective class $\mathbf{\Sigma}^1_n$, $n \geq 3$;
- A model of **ZFC** in [41], in which the full basis theorem holds in the absence of analytically definable well-orderings of the reals;
- A model of **ZFC** in [42], in which the Separation principle holds for a given effective class Σ^1_n, $n \geq 3$.

It is a common problem, in relation to to all these results, to establish their $\mathbf{PA}_2$-consistency versions similar to Theorem 1.

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