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On the vertical ordering of planar curves

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Abstract

We prove that every ordinal $\alpha < \omega_2$ is the order type of a certain system of Borel curves (uniform planar sets) in the sense of a natural vertical ordering relation between planar curves. This result gives a positive answer to a problem posed by Luzin in 1935.

Keywords: uniform sets; Borel curves; transfinite sequences; Borel coding

MSC: 03E25; 03E35; 03F35; 03E15

int

1. Introduction

In the first half of the 1930s, a number of profound and interesting results were obtained in descriptive set theory – then a rather new branch of mathematics. The book [1] by Nicolas Luzin, a leader of this research area at that time, was devoted to the presentation and thoroughful analysis of these results. A significant part of the book presented some results obtained by Petr Novikov, then a young mathematician and a student of Luzin. In addition to those of Novikov's results, which, at the time of writing [1], had already been or would soon be published in such works as [2–5], Luzin surveyed those studies carried out in the doctoral thesis of Novikov, which have never been published by Novikov himself, and are known only from their presentation and analysis in [1].

In particular, § 32 and sections I–IV of § 33 of [1] analyzed well-ordered families of planar *curves*, *i.e.*, uniform planar sets, that Novikov proposed for "geometric" representation of ordinals $\alpha < \omega_2$ (transfinite numbers of the third class, in the notation of that time). Investigating order types of such families, Luzin proved that these types are strictly smaller than the orginal ω_2. Luzin asked (ibid., sec. III) whether conversely, every ordinal $\mu < \omega_2$ is equal to the length (= the order type) of some well-ordered sequence of planar curves.

Our main result, Theorem 1 in § 2, gives a positive answer to this question. In addition, the curves we define for the proof of the theorem will be Borel sets having effectively defined Gödel-constructible Borel codes.

prel

2. Preliminaries and the main theorem

Given a *planar set* $P \subseteq \mathbb{R} \times \mathbb{R}$, its *projection* is the *linear* set

$$\mathbf{proj}\, P = \{x : \exists\, y\, P(x,y)\} \subseteq \mathbb{R},$$

and if $x \in \mathbb{R}$ then we consider the *(vertical) section*

$$P\restriction_x = \{x : \exists\, y\, (\langle x,y\rangle \in P)\} \subseteq \mathbb{R},$$

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so that $\mathbf{proj}\, P = \{x : P\restriction_x \neq \varnothing\}$. A planar set P is *uniform*, or simply a *curve*, if each its sections $P\restriction_x$ contains at most one element. This unique element will be denoted $P(x)$. Thus, $P\restriction_x = \{P(x)\}$, and P is basically (the graph of) a function $P : \mathbf{proj}\, P \to \mathbb{R}$.

P trl Q If $P, Q \subseteq \mathbb{R} \times \mathbb{R}$ are curves then define $P \lhd Q$ (P is lower than Q) iff

uf1 (1) $\varnothing \neq \mathbf{proj}\, P \subseteq \mathbf{proj}\, Q$ or $\varnothing \neq \mathbf{proj}\, Q \subseteq \mathbf{proj}\, P$, and

uf2 (2) $P(x) < Q(x)$ for all $x \in \mathbf{proj}\, P \cap \mathbf{proj}\, Q$.

Here the relation $P \lhd Q$ does not say which one of the two options in (1) holds.

The relation $\lhd$ is not transitive, as simple examples show. Yet we will consider families of planar curves that are well-ordered by $\lhd$ — we will call them *chains*, — as well as their *lengths*, *i.e.* order types. Writing $\langle P_\alpha \rangle_{\alpha<\mu}$, we'll mean that this is a well-ordered chain of planar curves of length $\mu \in \mathrm{Ord}$, so that $P_\alpha \lhd P_\beta$ holds for each pair $\alpha < \beta < \mu$.

tl **Proposition 1** (Novikov, Luzin [1], § 33-II – see § 3 below)**.** *The length μ of any well-ordered chain $\langle P_\alpha \rangle_{\alpha<\mu}$ of curves P_α is strictly less than the ordinal ω_2.*

This result led Luzin [1], § 33-III to the following **reverse problem**:

pro **Problem 1.** *For any ordinal $\mu < \omega_2$, find out, whether there exists a well-ordered chain of curves of length exactly μ.*

Item (A) of the following theorem provides a positive solution to this problem. Moreover, this solution is found in the domain of *Borel* curves P, and precisely those satisfying $P \subseteq \mathbb{R} \times \mathbb{Q}$, *i.e.* only with rational ordinates. Items (B), (C), (D) provide additional material concerning the encoding of these Borel curves and efficiency of this encoding. In particular, claim (C) concerns the case when $\mu < \Omega$, where Ω is the first $\mathbf{L}$-cardinal above the true ω_1, *i.e.* if $\omega_1 = \omega_\gamma^{\mathbf{L}}$ then $\Omega = \omega_{\gamma+1}^{\mathbf{L}}$. We recall that $\mathbf{L}$ is the class of all *constructible sets*. For the Gödel constructibility and related concepts, see, for example, [6], [7, § 2], or [8, ch. 8].

tm **Theorem 1** (the main theorem)**.** *Suppose that $\mu < \omega_2$. Then*:

tm1 (A) *there exists a chain well-ordered by $\lhd$, of length μ, of Borel curves $P_\alpha \subseteq \mathbb{R} \times \mathbb{Q}$ $(\alpha < \mu)$, such that in addition*:

tm2 (B) *all these sets P_α have constructible Borel codes* (see § 6 on Borel codes);

tm3a (C) *if $\mu < \Omega$ then the Borel codes for P_α as above can be chosen to form a constructible sequence,*

tm3b (D) *if $\mu \geq \Omega$ then no constructible sequence of pairwise different Borel codes of length μ exists.*

It follows from (C), (D) that the ordinal Ω divides the domain of effectivity (understood as the existence of a constructible sequence of Borel codes for the curves in a given $\lhd$-chain) and the domain of non-effectivity. The existence of such a critical ordinal was discussed by Novikov; see Luzin [1, § 33.X]. We may observe however that both relations $\Omega < \omega_2$ and $\Omega = \omega_2$ are compatible with the axioms of the Zermelo-Fraenkel theory **ZFC**.

About the organization of the article. The proof of Proposition 1 is given in § 3. Theorem 1 in part (A) is established in § 4 for the case of $\mu = \omega_1$, and in § 5 in the general case. The Borel encoding is introduced in § 6. We prove Theorem 1 in parts (B), (C), (D) in § 7. We finish with some concluding remarks and a problem in § 8.

3. Restriction of the length of increasing chains

ogr For the convenience of the reader, we present here a fairly short, though by no means obvious **proof of Proposition 1** given by Luzin in [1], § 33–II. Note that Proposition 1 **is not used** in the proof of Theorem 1.

Thus let $\langle P_\alpha\rangle_{\alpha<\mu}$ be a well-ordered chain of curves (uniform sets) $P_\alpha\subseteq\mathbb{R}\times\mathbb{R}$, *i.e.* $P_\alpha\lhd P_\beta$ whenever $\alpha<\beta<\mu$. We have to prove that $\mu<\omega_2$.

Let $P=\bigcup_{\alpha<\mu}P_\alpha$. Each section $P{\restriction}_x=\{y:\langle x,y\rangle\in P\}\subseteq\mathbb{R}$ is well-ordered, and its order type $\mu_x<\omega_1$, since there are no strictly increasing ω_1-sequences of reals. Thus $P{\restriction}_x=\{y_{x\xi}:\xi<\mu_x\}$, where the numbering of the reals $y\in P{\restriction}_x$ is given in the ascending real number ordering. Let $Q_\xi=\{\langle x,y_{x\xi}\rangle:\xi<\mu_x\}$ for all $\xi<\omega_1$, so that

ogr1 (3) the sets Q_ξ are curves, $Q_\xi\lhd Q_\eta$, $\mathbf{proj}\,Q_\eta\subseteq\mathbf{proj}\,Q_\xi$ for $\xi<\eta<\omega_1$, and in addition $P=\bigcup_{\alpha<\mu}P_\alpha=\bigcup_{\xi<\omega_1}Q_\xi$.

We claim that

ogr2 (4) If $\xi<\omega_1$ then the set $A_\xi=\{\alpha<\mu:Q_\xi\cap P_\alpha\neq\varnothing\}$ is countable.

Suppose otherwise. Let $\xi_0<\omega_1$ be the least ordinal such that A_{ξ_0} *is uncountable*, so all A_ξ, $\xi<\xi_0$, are countable. Thus by removing a countable number of indices from A_{ξ_0}, we get an uncountable set $A\subseteq A_{\xi_0}$ such that

ogr3 (5) if $\xi<\xi_0$ then $Q_\xi\cap P_\alpha=\varnothing$ for all $\alpha\in A$.

We assert that

ogr4 (6) if $\alpha<\beta$ belong to A then $\mathbf{proj}\,P_\alpha\subseteq\mathbf{proj}\,P_\beta$.

Indeed, as $\beta\in A$, there is a pair $\langle x,y\rangle\in Q_{\xi_0}\cap P_\beta$, and then $x\in\mathbf{proj}\,P_\beta$. Let's prove that $x\notin\mathbf{proj}\,P_\alpha$. Indeed otherwise $\langle x,y'\rangle\in P_\alpha$ holds for some real y'. In this case, $y'<y$ is impossible, because then it would be $\langle x,y'\rangle\in Q_\xi$ for some $\xi<\xi_0$ according to (3), which contradicts (5). The equality $y'=y$ is impossible as well, since $P_\beta\cap P_\alpha=\varnothing$ for $\alpha\neq\beta$. Finally, $y<y'$ is impossible, since it contradicts the fact that $P_\alpha\lhd P_\beta$ for $\alpha<\beta$. So, in fact, $x\notin\mathbf{proj}\,P_\alpha$. This implies $\mathbf{proj}\,P_\beta\not\subseteq\mathbf{proj}\,P_\alpha$, which means that $\mathbf{proj}\,P_\alpha\subseteq\mathbf{proj}\,P_\beta$ according to (1), so that (6) is verified.

Continuing the proof of (4), we take the least ordinal $\alpha_0\in A$ and any $x\in\mathbf{proj}\,A_{\alpha_0}$. According to (6), $x\in\mathbf{proj}\,A_\alpha$ holds for all $\alpha\in A$. This means that there is a family of reals y_α, $\alpha\in A$, for which $\langle x,y_\alpha\rangle\in P_\alpha$, and at the same time $y_\alpha<y_\beta$ for $\alpha<\beta$, since $\alpha<\beta$ implies $P_\alpha\lhd P_\beta$. So, we have an uncountable strictly increasing sequence of reals y_α, $\alpha\in A$, in $\mathbb{R}$, which is impossible. This contradiction completes the proof of (4).

However, it follows from (4) that this chain $\langle P_\alpha\rangle_{\alpha<\mu}$ contains no more than $\aleph_1$ sets, which means $\mu<\omega_2$. This ends the proof of Proposition 1.

w1 ## 4. The main theorem for the first uncountable ordinal

Here we present the **proof of Theorem 1**, only in part (A), **for the case** $\mu=\omega_1$. This result will be an important step for the general case.

As we consider mainly Borel sets $B\subseteq\mathbb{R}\times\mathbb{Q}$, the *decomposition* of such sets will be used, onto *horizontal sections*, *i.e.*, also obviously Borel sets:

Wq $$[B]_r=\{x:\langle x,r\rangle\in B\}\subseteq\mathbb{R},\ \text{ where } r\in\mathbb{Q},\ \text{so } B=\textstyle\bigcup_{r\in\mathbb{Q}}\big([B]_r\times\{r\}\big).\tag{7}$$

We begin with a standard lemma. As usual, $\mathbb{Q}=$ rational numbers.

dotL1 **Lemma 1.** *There is such a Borel set* $G\subseteq\mathbb{R}\times\mathbb{Q}$ *that for every* $Z\subseteq\mathbb{Q}$ *there exists a real* $x\in\mathbb{R}$ *satisfying* $Z=G{\restriction}_x:=\{q\in\mathbb{Q}:\langle x,q\rangle\in G\}$.

Proof. We fix a recursive enumeration of rational numbers, $\mathbb{Q}=\{r_n:n<\omega\}$. For $x\in\mathbb{R}$, let $A_x\subseteq\omega$ be the set of all n such that the decomposition of the fractional part of x into a binary fraction has 0 at position $2n+1$. The set $G=\{\langle x,r_n\rangle:n\in A_x\}$ is as required. □

We fix such a set G for further consideration.

We may notice that G is a form of the *Lebesgue binary sieve* [9], for which see *e.g.* [1], §17. Define, for each ordinal $\xi < \omega_1$,

UD
$$\left.\begin{array}{rcl} D_\xi &=& \{x \in \mathbb{R} : \text{the section } G\restriction_x \text{ has a well-ordered} \\ && \qquad \text{initial segment of length} > \xi\}; \\ U_\xi &=& \{\langle x, r\rangle \in D_\xi \times \mathbb{Q} : r \text{ is the } \xi\text{th largest real in } C\restriction_x\}; \end{array}\right\} \tag{8}$$

and finally $\mathscr{D} = \{D_\xi : \xi < \omega_1\}$.

The next lemma implies Theorem 1, claim (A), in case $\mu = \omega_1$:

dotB **Lemma 2.** *Sets $U_\xi \subseteq \mathbb{R} \times \mathbb{Q}$, $\xi < \omega_1$, are curves; they form a $\lhd$-increasing chain $\langle U_\xi\rangle_{\xi<\omega_1}$ of lengths ω_1; also $D_\xi = \mathbf{proj}\, U_\xi$, and $D_\eta \subseteq D_\xi$ for $\xi < \eta$.*

All sets U_ξ and D_ξ are nonempty and Borel.

Proof. Only the Borelness needs to be proved, the rest is obvious. We prove that U_ξ is Borel by induction on ξ. Assume for simplicity that the variables p, q, r denote only rational numbers. For $\xi = 0$:

bo1
$$x \in [U_0]_r \iff x \in [G]_r \wedge \forall\, q < r\, \big(x \notin [G]_q\big), \tag{9}$$

which implies the Borelness of all sections $[U_0]_r$ and U_0 itself as well, since the set G is Borel by Lemma 1, and the quantifier $\forall\, q$ on the right-hand side is limited to the countable set $\mathbb{Q}$.

Now assume that $\xi > 0$, and all sets U_η, $\eta < \xi$ are Borel. Then

bo2
$$\left.\begin{array}{rcl} x \in [U_\xi]_r &\iff& x \in [G]_r \wedge \forall\, \eta < \xi\, \exists\, q < r\, \big(x \in [U_\eta]_q \wedge \\ && \qquad \wedge\, \forall\, p\, (q' < p < q \implies x \notin [G]_p)\big) \end{array}\right\} \tag{10}$$

in case $\xi = \eta + 1$, while if ξ is a limit ordinal then

bo3
$$\left.\begin{array}{rcl} x \in [U_\xi]_r &\iff& x \in [G]_r \wedge \forall\, \eta < \xi\, \exists\, q < r\, \big(x \in [U_\eta]_r\big) \wedge \\ && \qquad \wedge\, \forall\, q < r\, \exists\, \eta < \xi\, \big(x \in [G]_q \implies x \in [U_\eta]_q\big). \end{array}\right\} \tag{11}$$

In both cases U_ξ is Borel since such are both G and all sets U_η, $\eta < \xi$.

Finally, each set $D_\xi = \mathbf{proj}\, U_\xi$ is Borel because

bo44
$$x \in D_\xi \iff \exists\, r \in \mathbb{Q}\, \big(x \in [U_\xi]_r\big), \tag{12}$$

which completes the proof of Lemma 2. □

dot
5. Proof of the main theorem, part (A), in the general case

We begin with an important special case of *transitivity* of the relation $\lhd$.

tran **Lemma 3.** *Assume that $P \lhd U \lhd Q$ are curves. Then the conjunction of (13) below and (1) above in §2 suffices for $P \lhd Q$ to hold:*

tran2 (13) $\mathbf{proj}\, P \subseteq \mathbf{proj}\, U$ *and/or* $\mathbf{proj}\, Q \subseteq \mathbf{proj}\, U$.

Proof. To prove (2) take any $x \in \mathbf{proj}\, P \cap \mathbf{proj}\, Q$ by (1). Then $x \in \mathbf{proj}\, U$ by (13), and we are done. □

Say that a chain of planar curves P_α has the *property of $\mathscr{D}$-projections* if $\mathbf{proj}\, P_\alpha \in \mathscr{D}$ for all α. The chain $\langle U_\xi\rangle_{\xi<\omega_1}$ obviously has this property.

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dotL3 **Lemma 4.** *Let $\mu < \omega_2$. There is a $\lhd$-increasing chain of Borel curves $P \subseteq \mathbb{R}\times\mathbb{Q}$, of length μ, with the property of $\mathscr{D}$-projections.*

Proof. According to lemma 2, the chain $\langle U_\xi\rangle_{\xi<\omega_1}$ handles the case $\mu \le \omega_1$. Next, we argue by induction.

Beginning of the inductive step. Fix an ordinal μ; $\omega_1 < \mu < \omega_2$.

fp (14) Fix an enumeration $\mu = \{\nu_\xi : \xi < \omega_1\}$ of all smaller ordinals $\nu < \mu$. (We will return to the analysis of this action below.)

For each ordinal ν_ξ, by the inductive hypothesis, there is

FF (15) a $\lhd$-increasing chain $\langle F^\xi_\alpha\rangle_{\alpha<\nu_\xi}$, of length ν_ξ, of Borel curves $F^\xi_\alpha \subseteq \mathbb{R}\times\mathbb{Q}$, with the property of $\mathscr{D}$-projections.

Next, we're going to insert each chain $\langle F^\xi_\alpha\rangle_{\alpha<\nu_\xi}$ between the curves U_ξ and $U_{\xi+1}$ by a vertical shift. To this end, we first define

defQ
$$\left.\begin{array}{l} Q^\xi_\alpha = F^\xi_\alpha \restriction D_{\xi+1} := P^\xi_\alpha \cap (D_{\xi+1}\times\mathbb{Q}) \subseteq \mathbb{R}\times\mathbb{Q}\,, \\ \text{or equivalently,}\quad [Q^\xi_\alpha]_r = [F^\xi_\alpha]_r \cap D_{\xi+1}\,,\ \forall\, r\,, \end{array}\right\} \tag{16}$$

for all $\alpha < \nu_\xi$. As $D_{\xi+1} \in \mathscr{D}$, and the chain $\langle F^\xi_\alpha\rangle_{\alpha<\nu_\xi}$ has the property of $\mathscr{D}$-projections, all reduced sets Q^ξ_α are nonempty and form $\lhd$-ascending chain $\langle Q^\xi_\alpha\rangle_{\alpha<\nu_\xi}$ of the same length ν_ξ and with the property of $\mathscr{D}$ projections, and $\mathbf{proj}\, Q^\xi_\alpha \subseteq D_{\xi+1} \subseteq D_\xi$, $\forall\,\alpha$.

All sets Q^ξ_α are Borel, since such are all F^ξ_α and D_ξ (by Lemma 2).

Let $\xi < \omega_1$ and $x \in D_{\xi+1}$. By Lemma 2, there are unique pairs $\langle x, u^x_\xi\rangle \in U_\xi$ and $\langle x, u^x_{\xi+1}\rangle \in U_{\xi+1}$, moreover, $u^x_\xi < u^x_{\xi+1}$ are rational.

For any pair $u < v$ of rational numbers, define an *order-preserving* bijection $H[u,v] : (u,v) \overset{\text{onto}}{\longrightarrow} \mathbb{R}$ of an open interval (u,v) onto $\mathbb{R}$ as follows:

huv
$$H[u,v](y) = \left\{\begin{array}{ll} \frac{y-c}{v-y} & \text{in case}\quad c \le y < v\,; \\ \frac{y-c}{u-y} & \text{in case}\quad u < y \le c\,; \end{array}\right| \quad \text{where } c = \tfrac{u+v}{2}\,. \tag{17}$$

Clearly $H[u,v]$ preserves the rationality, so that $r \in \mathbb{Q} \iff H[u,v](r) \in \mathbb{Q}$.

Let again $\xi < \omega_1$. If $\alpha < \nu_\xi$, then the curve $Q^\xi_\alpha \subseteq \mathbb{R}\times\mathbb{Q}$ satisfies $\mathbf{proj}\, Q^\xi_\alpha \subseteq D_{\xi+1}$ by the above. Define the *vertical shift*

defS
$$S^\xi_\alpha = \{\langle x, r\rangle \in D_{\xi+1}\times\mathbb{Q} : u^x_\xi < r < u^x_{\xi+1} \wedge \langle x, H[u^x_\xi, u^x_{\xi+1}](r)\rangle \in Q^\xi_\alpha\} \tag{18}$$

of Q^ξ_α into the area between U_ξ and $U_{\xi+1}$. As the abscissas do not change, and the ordinates maintain the order and rationality, we conclude that the sets $S^\xi_\alpha \subseteq D_{\xi+1}\times\mathbb{Q}$, $\alpha < \nu_\xi$, are uniform, hence curves, and form a $\lhd$-increasing chain $\langle S^\xi_\alpha\rangle_{\alpha<\nu_\xi}$ of length ν_ξ and with the property of $\mathscr{D}$-projections. In addition, we have

mez
$$\mathbf{proj}\, S^\xi_\alpha = \mathbf{proj}\, Q^\xi_\alpha \subseteq D_{\xi+1} \subseteq D_\xi \quad\text{and}\quad U_\xi \lhd S^\xi_\alpha \lhd U_{\xi+1} \tag{19}$$

for all $\alpha < \nu_\xi$ by construction.

Prove that these new sets S^ξ_α are Borel. Indeed it holds by construction that

sxa
$$[S^\xi_\alpha]_r = \bigcup_{u,v,w\in\mathbb{Q}\wedge u<v\wedge w=H[u,v](r)} [U_\xi]_u \cap [U_{\xi+1}]_v \cap [Q^\xi_\alpha]_w. \tag{20}$$

for all $r \in \mathbb{Q}$. Yet the sets U_ξ, $U_{\xi+1}$, $Q^\xi_\alpha \subseteq \mathbb{R} \times \mathbb{Q}$ are Borel, therefore their horizontal sections $[U_\xi]_u$, $[U_{\xi+1}]_v$, $[Q^\xi_\alpha]_w$ are Borel as well. Therefore, according to (20), the sections $[S^\xi_\alpha]_r$ are Borel, too. This implies that the sets S^ξ_α themselves are Borel by (7), as required.

Let's now analyze the whole family

ww
$$\mathbf{S}_\mu = \{U_\xi : \xi < \omega_1\} \cup \{S^\xi_\alpha : \xi < \omega_1 \wedge \alpha < \nu_\xi\} \tag{21}$$

of curves in $\mathbb{R} \times \mathbb{Q}$, considered for the inductive step μ. Prove that

21 (22) $\mathbf{S}_\mu$ is a $\lhd$-chain of length $\mu' = \sum_{\xi<\omega_1}(1+\nu_\xi) \geq \mu$.

Fact 1. We already know that both $\langle U_\xi\rangle_{\xi<\omega_1}$, and $\langle S^\xi_\alpha\rangle_{\alpha<\nu_\xi}$ for any $\xi < \omega_1$, are $\lhd$-increasinging chains of curves of corresponding lengths, and with the property of $\mathscr{D}$-projections, and in addition (19) holds.

Fact 2. If $P, Q \in \mathbf{S}_\mu$ then the projections $\mathbf{proj}\,P$ and $\mathbf{proj}\,Q$ belong to $\mathscr{D}$, hence $\mathbf{proj}\,P \subseteq \mathbf{proj}\,Q$ or $\mathbf{proj}\,Q \subseteq \mathbf{proj}\,P$ by (1) of the definition of $\lhd$.

Fact 3. If $\xi < \eta < \omega_1$ and $\alpha < \nu_\xi$ then $S^\xi_\alpha \lhd U_{\xi+1} \lhd U_\eta$ by Fact 1, $\mathbf{proj}\,U_\eta \subseteq \mathbf{proj}\,U_{\xi+1}$ by Lemma 2, thus $S^\xi_\alpha \lhd U_\eta$ by Lemma 3 and Fact 2.

Fact 4. If $\xi < \eta < \omega_1$ and $\beta < \nu_\eta$ then $U_\xi \lhd U_\eta \lhd S^\eta_\beta$ by Fact 1, and $\mathbf{proj}\,S^\eta_\beta \subseteq \mathbf{proj}\,U_\eta$ by (19), so that $S^\xi_\alpha \lhd U_\eta$ by Lemma 3 and Fact 2.

Fact 5. If $\xi < \eta < \omega_1$ and $\alpha < \nu_\xi$, $\beta < \nu_\eta$, then $S^\xi_\alpha \lhd S^\eta_\beta$. Here $S^\xi_\alpha \lhd U_\eta \lhd S^\eta_\beta$ by Fact 3, and $\mathbf{proj}\,S^\eta_\beta \subseteq \mathbf{proj}\,U_\eta$ by (19), whence $S^\xi_\alpha \lhd S^\eta_\beta$ holds again by Lemma 3 and Fact 2.

As a consequence of Facts 1–5, we get (22) for the family (21), *i.e.*

wwp
$$\mathbf{S}_\mu = \{P_\alpha : \alpha < \mu'\} \tag{23}$$

in the ascending order of $\lhd$. However, $\mu' \geq \mu$ is obvious, so that a chain of length exactly μ is obtained by simply cutting the "tail" $\mathbf{S}_\mu$ in (23) from μ to μ'. This **completes the inductive step** and the proof of Lemma 4. □

Finally, Lemma 4 obviously **implies Theorem 1, in part** (A). □

kbm
6. Encoding Borel sets

Here we review those basic concepts related to Borel codes which appeared in the formulations and will be used in **the proofs of the statements** (B), (C), (D) **of Theorem 1** below. Let $\omega_1^{<\omega}$ be the set of all tuples (finite sequences) of countable ordinals. Further:

- $s \subset t$ means that the tuple t is a proper extension of the tuple s,
- $\langle\rangle$ is the empty tuple, $\langle\alpha_1, \ldots, \alpha_n\rangle$ is the tuple with terms $\alpha_1, \ldots, \alpha_n$,
- $s^\frown\alpha$ is a tuple obtained by adjoining the rightmost term α to the tuple s,
- a set $T \subseteq \omega_1^{<\omega}$ is a *tree*, if $s \subset t \in T \Longrightarrow s \in T$,
- $\mathrm{Max}(T)$ is the set of all *endpoints* of a given tree T,
- a tree T is *well-founded*, if it has no *infinite branches*, *i.e.* there is no function $b : \omega \to \omega_1$ such that $b\restriction n \in T$ for all n.

Finally, let *a Borel code* (for the space $\mathbb{R}$), be any pair $\langle T, d\rangle$, where $\varnothing \neq T \subseteq \omega_1^{<\omega}$ *is a finite or countable* well-founded tree, and $d \subseteq T \times \mathbb{Q} \times \mathbb{Q}$. In this case, the well-foundedness of the tree T allows to uniquely define the Borel set $[T, d, s] \subseteq \mathbb{R}$ for each $s \in T$ so that:

b1 (I) $[T, d, s] = \mathbb{R} \setminus \bigcup_{\langle s,p,q\rangle\in d}(p, q)$, in case $s \in \mathrm{Max}(T)$, where $(p, q) = \{x : p < x < q\}$;

b2 (II) $[T, d, s] = \mathbb{R} \setminus \bigcup_{s^\frown\alpha\in T}[T, d, s^\frown\alpha]$, in case $s \in T \setminus \mathrm{Max}(T)$;

b3 (III) finally $[T, d] = [T, d, \langle\rangle]$.

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Thus the scheme (I)–(II)–(III) defines the set $[T,d]\subseteq\mathbb{R}$ from rational intervals, iterating the operation of the complement to the countable union, *i.e.* a Borel set. Conversely every Borel set $X\subseteq\mathbb{R}$ admits a Borel code $\langle T,d\rangle$ (with a countable tree T!) for which $X=[T,d]$.

We observe that this definition corresponds to the topology of the real line $\mathbb{R}$. For encoding Borel sets, *e.g.*, of *the Baire space* $\mathbb{I}=\omega^\omega$ we should take sets $d\subseteq T\times\omega^{<\omega}$, and also change (I) to the form $[T,d,s]=\mathbb{I}\smallsetminus\bigcup_{\langle s,\sigma\rangle\in d}I_\sigma$, where $I_\sigma=\{a\in\mathbb{I}:\sigma\subset a\}$.

As for the *planar* Borel sets, there is no need to consider the coding in full generality here, as in fact only planar sets $U\subseteq\mathbb{R}\times\mathbb{Q}$ occur in the proof of Lemma 4 and Theorem 1. Let a *Borel multicode* be any indexed system of Borel codes $c=\langle T_r,d_r\rangle_{r\in\mathbb{Q}}$. Then we define

$$[c]=\bigcup_{r\in\mathbb{Q}}[T_r,d_r]\times\{r\}\subseteq\mathbb{R}\times\mathbb{Q},$$

so that $[c]=U$ in case $[U]_r=[T_r,d_r]$ for all $r\in\mathbb{Q}$.

With this definition, Borel codes and multicodes are *hereditarily countable* sets, which is quite essential in some definability issues. If we allowed rational intervals per se instead of pairs of their endpoints in the conditions for d (which would seem to be a simpler and more natural solution), then this hereditary countability would disappear, of course.

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7. Getting codes to complete the proof of the main theorem

The goal of this section is **the proof of statements (B), (C), (D) of Theorem 1** for the Borel sets (21)/(23) constructed above in § 5 for the proof of Theorem 1 in part (A). The next intermediate result gives Borel codes for curves U_ξ and sets D_ξ as in (8); the codes will belong to the class $\mathbf{L}$ of *Gödel-constructible* sets.

U1 **Lemma 5.** (i) *There exist Borel multicodes $c_\xi=\langle T^\xi_r,d^\xi_r\rangle_{r\in\mathbb{Q}}\in\mathbf{L}$ for all curves U_ξ, and Borel codes $\langle T'_\xi,d'_\xi\rangle\in\mathbf{L}$ for all sets D_ξ, $\xi<\omega_1$, such that:*

U2 (ii) *the ω_1-sequences of these codes are constructible as well.*

Proof. We begin by defining a constructible multicode for the initial set G given by the proof of Lemma 1. Equations (9)–(11) make it possible to effectively define, by transfinite induction, Borel multicodes $c_\xi=\langle T^\xi_r,d^\xi_r\rangle_{r\in\mathbb{Q}}$ for sets U_ξ, *i.e.* $U_\xi=[c_\xi]$, and then, using (12), Borel codes $\langle T'_\xi,d'_\xi\rangle$ for sets $D_\xi=\mathbf{proj}\,U_\xi$ as well. We leave this as a simple exercise. Inductive construction of all these codes is absolute for the class $\mathbf{L}$ of all constructible sets, *i.e.* gives the same result in $\mathbf{L}$ and in the universe of all sets. □

Lemma 5 together with lemma 2 above give a complete proof of Theorem 1 for $\mu=\omega_1$. **The proof for the general case** $\mu<\omega_2$ is based on the next lemma, similar to Lemma 5(i), but relevant to sequences of Borel sets from the proof of Lemma 4.

cor2 **Lemma 6.** *Let $\omega_1<\mu<\omega_2$. In the context of the notation and assumptions of the inductive step in the proof of Lemma 4, suppose that*

2* $(*)$ *there exist Borel multicodes $\varphi^\xi_\alpha\in\mathbf{L}$ for the sets F^ξ_α as in (15).*

Then there exist Borel multicodes $\pi_\alpha\in\mathbf{L}$ for the resulting sets P_α in (23) in the proof of Lemma 4.

Proof. First, we define intermediate multicodes $\wp^\xi_\alpha\in\mathbf{L}$ for sets Q^ξ_α, from the codes given by $(*)$, using relations (16) in § 5. Then define multicodes $\sigma^\xi_\alpha\in\mathbf{L}$ for sets S^ξ_α using relations (20) in the same place. which, Along with the multicodes for the sets U_ξ, provided by Lemma 5, this allows to define the required multicodes $\pi_\alpha\in\mathbf{L}$ for sets P_α after the re-enumeration in the course of the transition from (21) to (23). Once again, the constructibility of all these multicodes follows from the obvious absoluteness. □

This completes **the proof of Theorem 1, part (B)**.

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Now prove **parts (C), (D) of the theorem**. Note that the constructions given in §§ 4 and 5 contain only one ineffective action of arbitrary choice, namely, the choice of a specific enumeration $\mu = \{\nu_\xi : \xi < \omega_1\}$ of all ordinals $\nu < \mu$ in the agreement (14) in the proof of Lemma 4. This does not allow to strengthen Lemma 6 by requiring the constructibility of the resulting μ-sequence of codes for sets (21)/(23) in the proof of Lemma 4.

Fortunately, this does not affect the constructibility of the codes for sets (21)/(23), since, as long as $\xi < \omega_1$ is fixed, the formulas (18) and (20) depend only on the value ν_ξ for this ξ, and do not depend on the entire sequence $\langle \nu_\xi \rangle_{\xi < \omega_1}$.

As for the case when $\mu < \Omega$, considered in statement (C) of Theorem 1, the inequality $\mu < \Omega$ allows makes it possible to select a *specific* enumeration $\mu = \{\nu_\xi : \xi < \omega_1\}$, namely, the *smallest one* of all such enumerations, in the sense of the canonical well-ordering $<_{\mathbf{L}}$ of the constructible universe $\mathbf{L}$. And then the construction of a sequence of sets (21)/(23), and the according codes π_α in the proof of Lemma 6 becomes fully effective and absolute for $\mathbf{L}$.

This completes **the proof of Theorem 1 in its part (C)**.

We finish this section with **the proof of Theorem 1 in its part (D)**.

Consider the set **BMC** of all Borel multicodes as defined in § 6. Suppose that $\omega_1 = \omega_\gamma^{\mathbf{L}}$; then by definition $\Omega = \omega_{\gamma+1}^{\mathbf{L}}$. On the other hand, the set $\mathbf{BMC} \cap \mathbf{L}$ of all *constructible* multicodes (this is different from $(\mathbf{BMC})^{\mathbf{L}}$!) has cardinality $\aleph_\gamma^{\mathbf{L}}$ in $\mathbf{L}$ since GCH holds in $\mathbf{L}$. Thus no *constructible* Ω-sequence of pairwise different Borel multicodes in $\mathbf{BMC} \cap \mathbf{L}$ exist.

This completes **the proof of Theorem 1 in its last part (D)**. □

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8. Discussion and conclusions

Our Theorem 1 closes the long-known classical problem of Petr Novikov and Luzin on the lengths of transfinite sequences of planar curves, and in the strongest form of *Borel* curves situated in the "plane" $\mathbb{R} \times \mathbb{Q}$ (*i.e.* with rational ordinates), and with constructive Borel codes. We expect that the results obtained will find applications in modern research in descriptive set theory.

We also expect that our methods of effective transfinite constructions will make a definite contribution to the modern theory of generalized computability on uncountable structures and the theory of information transmission between structures on ordinals and Borel structures associated with the real line, as in recent works [10–13].

We finish with a problem. Coming back to (C), (D) of Theorem 1, the ordinal Ω can be seen as a separator between the lengths of effectively existing and effectively non-existing $\lhd$-chains of uniform Borel sets, with the effectivity being understood in terms of the existence of a related constructible sequence of Borel codes. But there are other measures of effectivity, like *e.g.* the ordinal-definability. The following problem arises:

Problem 2. *Find a model in which* $\Omega < \omega_2$*, and there is no* ordinal-definable *(or even* real-ordinal-definable*) chains of Borel uniform sets of length* $\geq \Omega$*.*

We also hope that this research can be useful in creating algorithms or computational algorithmic models that represent the evolution of cell types and are related to the storage and processing of genomic information.

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References

1. Luzin, N. *On some new results of descriptive theory of functions*; USSR Academy of Science: Moscow – Leningrad, 1935; p. 86. (Russian).
2. Lusin, N.; Novikoff, P. Choix effectif d'un point dans un complémentaire analytique arbitraire, donné par un crible. *Fundamenta Math.* **1935**, *25*, 559–560.
3. Novikoff, P. Sur la séparabilité des ensembles projectifs de seconde classe. *Fundam. Math.* **1935**, *25*, 459–466.
4. Novikoff, P. Les projections des complémentaires analytiques uniformes. *Rec. Math. Moscou, n. Ser.* **1937**, *2*, 3–15.
5. Novikoff, P. Sur quelques relations entre les familles des ensembles projectifs de classe 2 et des projections des complémentaires analytiques uniformes. *Izv. Akad. Nauk SSSR, Ser. Mat.* **1937**, *1*, 231–252.
6. Jech, T. *Set theory*, The third millennium revised and expanded ed.; Springer-Verlag: Berlin-Heidelberg-New York, 2003. Pages xiii + 769, https://doi.org/10.1007/3-540-44761-X.
7. Kanovei, V.; Lyubetsky, V. On some classical problems in descriptive set theory. *Russ. Math. Surv.* **2003**, *58*, 839–927.
8. Kanovei, V.; Lyubetsky, V. *Modern set theory: absolutely undecidable classical problems*; "MCCME": Moscow, 2013. 384 p. (Russian).
9. Lebesgue, H. Sur les fonctions représentables analytiquement. *Journ. de Math. (6)* **1905**, *1*, 139–216.
10. Kanovei, V.; Lyubetsky, V. Jensen Δ^1_n reals by means of ZFC and second-order Peano arithmetic. *Axioms* **2024**, *13*. Article No 96, https://doi.org/10.3390/axioms13020096.
11. Kanovei, V.; Lyubetsky, V. Notes on the Equiconsistency of ZFC Without the Power Set Axiom and Second-Order Arithmetic. *Axioms* **2025**, *14*. https://doi.org/10.3390/axioms14120865.
12. Greenberg, N.; Hamkins, J.D.; Hirschfeldt, D.; Miller, R., Eds. *Effective mathematics of the uncountable*; Vol. 41, *Lect. Notes Log.*, Cambridge: Cambridge University Press; Ithaca, NY: Association for Symbolic Logic (ASL), 2013. https://doi.org/10.1017/CBO9781139028592.
13. Hamkins, J.D.; Lewis, A. Infinite time Turing machines. *Journal of Symbolic Logic* **2000**, *65*, 567–604. https://doi.org/10.2307/2586556.

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