By Jacobson's theorem [7], either $(\mathcal{L}_6^{(\xi)}, (\text{-stand})) \simeq (\mathcal{L}_6, \text{stand})$, or $(\mathcal{L}_6^{(\xi)}, (\text{-stand})) \simeq (\mathcal{D}_3, \text{symp})$, hence either $K^{(\xi)} \simeq H(\mathcal{L}_6, \text{stand})$ or $K^{(\xi)} \simeq H(\mathcal{D}_3, \text{symp})$. Comparing dimensions, we see that $K^{(\xi)} \simeq H(\mathcal{D}_3, \text{symp})$. Now, by Lemma 4,

$$G_1(K^{(\xi)}) \simeq G_1(H(\mathcal{D}_3, \text{symp})) = H(\mathcal{D}_3, \text{symp}).$$

But $G_1(K^{(\xi)}) \subseteq G_1(K)$ hence $G_1(K) = K$ and $G_1(K) \neq 0$. Contradiction. The theorem is proved.

In conclusion, the author thanks E. I. Zel'manov for posing this problem and his interest in this research.

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INTUITIONISTIC THEORY OF ALGEBRAIC SYSTEMS AND HEYTING-VALID ANALYSIS

V. A. Lyubetskii

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In this paper, given a ring f with a metric and a formula $\varphi \Rightarrow \psi$ in the language of rings, we shall construct a translation $\varphi^+ \Rightarrow \psi^+$ such that the (classical) truth of the inference $f \models (\varphi \Rightarrow \psi)$ (i.e., the inference $\varphi_f \Rightarrow \psi_f$) in the classical theory will imply, in a certain sense, the (intuitionistic) truth of the inference $f \models (\varphi^+ \Rightarrow \psi^+)$ [i.e., the inference $(\varphi^+)_f \Rightarrow (\psi^+)_f$] in intuitionistic set theory. Of course, the question of what one understands by "(intuitionistic) truth in intuitionistic set theory" is a difficult one and the answer is by no means unique. From the standpoint of the intuitionist it might be a derivation in Grayson's formal-axiomatic set theory ZF_I' (see [1]). As far as the classicist, i.e., the mathematician oriented toward using intuitionistic logic to ultimately obtain results pertaining to conventional mathematics, it might be Heyting validity. Let us recall the definition of the latter.

A formula φ in the language of ZF, with parameters $x_1, \ldots, x_n$ in the class of all sets V, is said to be Heyting-valid [notation: $cHa \models \varphi(x_1, \ldots, x_n)$] if, for any complete Heyting

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algebra (cHa) Ω , we have $[\![\varphi(\check{x}_1,\dots,\check{x}_n)]\!]_\Omega = 1$, where $[\![\cdot]\!]_\Omega$ is the usual valuation in ZF with parameter set V^Ω . The definitions and results referred to here and below may be found, e.g., in [2]. Essentially, the definition of Heyting validity appeared in [1]. Of course, $(ZF'_I \vdash \varphi) \Rightarrow cHa \models \varphi$, but the converse is false. [It is sometimes useful to specialize the predicate $cHa \models (\cdot)$, letting Ω range over only all topologies or only all zero-dimensional compacta, etc.] Even if one takes a single (non-Boolean) cHa Ω it is not true that $[\![\varphi \vee \neg\varphi]\!]_\Omega = 1$, where φ is "almost any" formula. We are going to operate with the predicate (semantics) $cHa \models (\cdot)$ in such a way that, whenever we assert that $cHa \models \varphi$, it will be true that $ZF'_I \vdash (cHa \models \varphi)$. In this sense our metamathematics may be considered strictly intuitionistic. The semantics $cHa \models (\cdot)$ also preserves some features of the theory ZF'_I . All this perhaps makes it possible to consider $cHa \models (\cdot)$ as an intuitionistic semantics in set theory. In the sequel this view of Heyting validity as a variety of intuitionistic truth in set theory will not be absolutely necessary, so it may be considered simply a convenient means of operating with intuitionistic logic in applications of nonstandard analysis.

If the algebra Ω in the definition of the predicate $cHa \models \varphi$ ranges only over all complete Boolean algebras (abbreviation: cBa), we obtain a new predicate, denoted by $cBa \models \varphi$. Of course, if $ZFC \vdash \varphi$ then $cBa \models \varphi$.

Thus, in Theorem 1 we shall establish a result of the following type. If $cBa \models (\varphi_f \Rightarrow \psi_f)$, then $cHa \models ((\varphi^+)_f \Rightarrow (\psi^+)_f)$. In more detail: certain restrictions will be imposed on the ring f or, more precisely, on its description by a formula $\varkappa$ in the language of ZF, and Theorem 1 will then state: "If $cBa \models \xi$, where $\xi \leftrightharpoons \forall f(\varkappa(f) \Rightarrow \forall x_1,\dots,x_n \in f[\varphi_f(x_1,\dots,x_n) \Rightarrow \psi_f(x_1,\dots, x_n)])$, then $cHa \models \xi^+$, where $\xi^+ \leftrightharpoons \forall f(\varkappa(f) \Rightarrow \forall x_1,\dots,x_n \in f\,[(\varphi^+)_f(x_1,\dots,x_n) \Rightarrow (\psi^+)_f(x_1,\dots,x_n)])$." These restrictions on $\varkappa$ will be contained in the concept of a "$\varkappa$-Dedekind formula," which will be substantially broader in this paper (see below) than in [2]. Thus Theorem 1 provides a tool which, when given any result of the form ξ (of course, such that $ZFC \vdash \xi$), will obtain $cBa \models \xi$, then $cHa \models \xi^+$, and hence $[\![\xi^+]\!]_\Omega = 1$ for the specific cHa Ω in which we are interested.

In the sense just proposed, results of the type $cHa \models \chi$ or, more precisely, $ZF'_I \vdash (cHa \models \chi)$, constitute an intuitionistic theory of algebraic systems.

Nonstandard analysis enables us to continue this chain of arguments (we have dwelled on the fact that $[\![\xi^+]\!]_\Omega = 1$). We choose any $f \in V^\Omega$ such that $[\![\varkappa(f)]\!]_\Omega = 1$, and obtain

$$[\![(\varphi^+)_f]\!]_\Omega \leqslant [\![(\psi^+)_f]\!]_\Omega .$$

At this point yet another "dramatis personae" appears:

$$\hat{f}^\Omega \leftrightharpoons \{g \in V^\Omega \mid [\![g \in f]\!]_\Omega = 1\}.$$

And, since $[\![f \text{ is a ring}]\!]_\Omega = 1$ it follows, putting $(t + r = s) \leftrightharpoons [\![t + r = s]\!]_\Omega = 1$ for any $t, r, s \in \hat{f}^\Omega$ and also for the operations $-$, $\cdot$ and constants 0, 1, that $\hat{f}^\Omega$ is also a ring. Suppose that some condition holds on $\hat{f}^\Omega$ ensuring that $[\![(\varphi^+)_f]\!]_\Omega = 1$ (often this is simply $\hat{f}^\Omega \models \varphi^+$). Then we get $[\![(\psi^+)_f]\!]_\Omega = 1$, and then $\hat{f}^\Omega \models \psi^+$ (the last step usually takes place when ψ is a Horn formula). Sometimes one gets not $\hat{f}^\Omega \models \psi^+$ but $\hat{f}^\Omega \models (\psi^+)'$, where

$(t)'$ is some transform of the formula t. Finally, in certain cases one can deduce from $\hat{f}^{\Omega} \models \psi^+$ that $\hat{f}^{\Omega} \models \psi$ via arguments that have nothing to do with this technique. Thus, our goal is, given some f such that $[\![\alpha(f)]\!]_{\Omega} = 1$, to establish, roughly speaking, that $\hat{f}^{\Omega} \models (\varphi^+ \Rightarrow \psi^+)$.

For the case of a complete Boolean algebra Ω and an ordinary Boolean-valued universe V^{Ω} (when translation and all other intuitionistic peculiarities are unnecessary), this program and some appropriate results were obtained by G. Takeuti, the author and E. I. Gordon in 1977-1980. Naturally, one bottleneck for this program is the question: just what is the ring $\hat{f}^{\Omega}$? It turns out that precisely in the case of non-Boolean complete Heyting algebra Ω (and therefore also the specifically intuitionistic situations) many important mathematical objects may be represented by a suitable $\hat{f}^{\Omega}$. For example, this is the case for the ring of all continuous functions $C(Z,\mathbb{R})$, in which case Ω must be the $\mathcal{T}(Z)$ -topology of the space Z. In that case f is chosen from V^{Ω} as the field of reals, understood as the completion of the field $\mathbb{Q}$ by Dedekind cuts or Cauchy filters. Of course, various interesting objects are expressible as $\hat{f}^{\Omega}$ for a Boolean algebra Ω as well. We proceed now to a systematic exposition.

Throughout, Y will denote a uniformly locally compact space, $\mathcal{T}$ its topology and Σ a base of symmetric open entourages. Let $\Omega^{\mathcal{T}}$ denote the set of all cHa -morphisms of $cHa\,\mathcal{T}$ into $cHa\,\Omega$. Let $\Omega^{\mathcal{T}}_{u}$ denote the set of all u -morphisms of the same form; for $\rho \in \mathcal{F}_Y \leftrightharpoons \bigcup\{\Omega^{\mathcal{T}}_{u} \mid u \in \Omega\}$ we define $\tilde{Y}(\rho) \leftrightharpoons \rho(Y)$ (see [2]). Since $\mathcal{F}_Y \subseteq V^{\Omega}$ (taking into consideration the identification of α in $\mathcal{T}$ with $\alpha^{\vee}$), it follows that $\tilde{Y} \in V^{\Omega}$. It was shown there that $(\tilde{Y})^{\wedge\Omega} \cong \Omega^{\mathcal{T}}$ and $[\![\tilde{Y}$ is the set of all bases of Cauchy filters in $\check{\mathcal{T}}$ for the space $\langle \check{Y}, \mathcal{T}^{\vee}, \Sigma^{\vee}\rangle]\!]_{\Omega} = 1$, i.e., $\tilde{Y}$ is the completion of $\check{Y}$ in V^{Ω}. It was shown in [2] that $\Omega^{\mathcal{T}}$ is isomorphic to $C_f(X, Y)$, where X is the Stone space of Ω; denote this isomorphism by ψ and the corresponding image of an equivalence class $[f]$ by ρ_f, where $\rho_f(\alpha) \leftrightharpoons j f^{-1}(\alpha)$, $\alpha \in \mathcal{T}$ (see [2]). The following predicates are defined in $\Omega^{\mathcal{T}}$:

$$(\rho \underset{\sigma}{=} q) \leftrightharpoons [\![\exists \alpha \in \check{\mathcal{T}}(\alpha^2 \subseteq \check{\sigma} \wedge \alpha \in \rho \wedge \alpha \in q)]\!]_{\Omega} = \bigvee_{\alpha^2 \subseteq \sigma},$$

$$\rho(\alpha) \wedge q(\alpha) = 1 \text{ and } (\rho \# q) \leftrightharpoons [\![\exists \alpha, \beta \in \check{\mathcal{T}}(\alpha \in \rho \wedge \beta \in q \wedge$$

$$\wedge \bar{\alpha} \cap \bar{\beta} = \emptyset)]\!]_{\Omega} = \bigvee\{\rho(\alpha) \wedge q(\beta) \mid \bar{\alpha}\, d\, \bar{\beta}\} = 1,$$

where d denotes disjunctivity. We call them, respectively, σ -equality (the analog of ε -equality) and separability. Clearly, $\rho_f \underset{\sigma}{=} q_g \Rightarrow \exists S$ which is Ω-dense in X $(\forall x \in S\, (\langle f(x), g(x)\rangle \in \sigma))$, and $\exists S$, which is Ω -dense in X $(\forall x \in S\, (\langle f(x), g(x)\rangle \in \sigma)) \Rightarrow \rho_f \underset{\sigma^2}{=} q_g$. In particular, $\forall \sigma (\rho_f \underset{\sigma}{=} q_g) \Leftrightarrow [f] = [g]$, i.e., σ -equality has the usual sense of ε -equality. Clearly, $\rho_f \# q_g \Leftrightarrow \exists S$ which is Ω -dense in $X(\forall x \in S\, (f(x) \neq g(x)))$. In particular, $\rho \# q \Rightarrow \rho \neq q$, i.e., intuitionistic separability is stronger than inequality.

It is known that any cHa Ω canonical determines a cBa $\mathcal{B}$ in which Ω can be cHa -embedded. We thus obtain an embedding $V^{\Omega} \subseteq V^{\mathcal{B}}$. This embedding is such that

$$[\![f=g]\!]_{\Omega} \leq [\![f=g]\!]_{B}, \quad [\![f\in g]\!]_{\Omega} \leq [\![f\in g]\!]_{B},$$

$$[\![p \underset{\varepsilon}{=} q]\!]_{\Omega} = [\![p \underset{\varepsilon}{=} q]\!]_{B}, \ [\![p \# q]\!]_{\Omega} = [\![p \# q]\!]_{B},$$

and $\Omega_u^{\mathcal{T}}$ is embedded in $B_u^{\mathcal{T}}$, including operations, for any $u \in \Omega$. All this follows at once from the fact that Ω is embeddable in B. It is important that

$$[\![\forall p, q \in \tilde{Y}\,(p \neq q \Longleftrightarrow p \# q)]\!]_{B} = 1$$

(this is false for the valuation $[\![\cdot]\!]_{\Omega}$). Here and below $\#$ and $\underset{\varepsilon}{=}$ also denote the formulas in the language of ZF indicated within the valuations. Of course, if $p = q$ and $p \# q$, we have a contradiction, so that intuitionistically $p \# q$ implies $p \neq q$. Let $[\![p \neq q]\!]_{B} = 1$, i.e., $[\![p = q]\!]_{B} = 0$, where $p, q \in B^{\mathcal{T}}$. The above-mentioned isomorphism gives a representation for elements p and q by functions f and g of the type $f, g : O \to Y$, O an open dense subset of $X(B)$, where $X(B)$ is the Stone space of B; we then have $[\![p = q]\!]_{B} = j\{x \mid f(x) = g(x)\}^{o}$ (see [2]). Therefore $F \rightleftharpoons \{x \mid f(x) = g(x)\}$ is nowhere dense, as is $\bar{F}$, and $(C\bar{F}) \cap O$ is an open and dense subset of $X(B)$ on which $f \neq g$. This set is also B-dense, whence it follows that $[\![p \# q]\!]_{B} = 1$.

The idea of the next definition is that σ-equality and separability in some abstract f may be defined as $\|k\| \underset{\varepsilon}{=} \|t\|$ or $\|k - t\| \underset{\varepsilon}{=} 0$, and similarly $\|k\| \# \|t\|$ or $\|k - t\| \# 0$, where $k, t \in f$ and we have a "metric" $\|\cdot\| : f \to \hat{Y}$, where $\hat{Y}$ is the completion of Y by Cauchy filters. Another "metric" $\|\cdot\| : f \to R^{d}$, where R^{d} are the Dedekind reals, was considered in [2]. In that case the roles of "$\underset{\varepsilon}{=}$" and "$\#$" are played by the predicates $\lambda^2 < \varepsilon$ and $\lambda^2 > 0$, where $\lambda \in R^{d}$.

A ring with metric is defined as a structure $\langle f, Y, +, -, \cdot, 0, 1, \|\cdot\| \rangle$, where $\langle f, +, -, \cdot, 0, 1\rangle$ is the ring structure, $\|\cdot\| : f \to \hat{Y}$, and $\hat{Y}$ is the completion of the uniform space Y by Cauchy filters. A formula $\varkappa$ is called a Dedekind formula if, for any fixed Y,

1) $CHa \models \forall f, +, -, \cdot, 0, 1, \|\cdot\|\, (\varkappa\,(f, \check{Y}, +, -, \cdot, 0, 1, \|\cdot\|) \Rightarrow \langle f, \check{Y}, +, -, \cdot, 0, 1, \|\cdot\| \rangle$ is a ring with metric);

2) $\forall \Omega\, ([\![\varkappa(f, \check{Y}, +, -, \cdot, 0, 1, \|\cdot\|)]\!]_{\Omega} \leq [\![\varkappa(f, \check{Y}, \ldots)]\!]_{B})$;

3) $\forall \Omega\, ([\![\varkappa(f, \check{Y}, \ldots)]\!]_{\Omega} \wedge f(k) \wedge f(t) \leq [\![k = t \Leftrightarrow \|k -_{B} t\| = \check{0}]\!]_{B})$,

where f is an extensional function in V^{Ω}, $\mathcal{D}(f)$ a set which is closed under all the operations and 0 a fixed element of Y;

4) $\forall \Omega\, ([\![\varkappa(f, \check{Y}, \ldots)]\!]_{\Omega} \wedge f(k) \wedge f(t) \leq ([\![\|k\|_{\Omega} = \|k\|_{B}]\!]_{B} \wedge [\![k +_{\Omega} t = k +_{B} t]\!]_{B} \wedge \ldots)$,

where ... means the same conditions for the operations $-, \cdot, 0, 1$. For example, if Y is a topological ring and $f \subseteq \tilde{Y}$, we can define $\|p\| \rightleftharpoons p, \forall p \in f$. An example of a Dedekind formula is: "f is a set of bases in $\check{\mathcal{T}}$ of Cauchy filters which is closed under the operations

$+, -, \cdot, 0, 1$ in $\check{Y}$." This will still be a Dedekind formula if one adds, say, conditions such as to be a field, to be an algebraically closed or real closed field, etc.

Now let φ, ψ be formulas in the language of rings with narrow negations, i.e., φ, ψ are without implications and with negations only on atomic formulas; in addition, ψ is an AE -formula. Define "translation in the premise" φ^+ by replacing each $k \neq t$ with $\|k-t\| \neq 0$ and "translation in the conclusion" ψ^+ by replacing each $k \neq t$ as in the premise and each $k=t$ by $\|k-t\| \underset{\sigma}{=} \check{0}$, the variable σ being bound in the form $\forall \sigma \in \check{\Sigma}$ in the universal quantifier block of ψ. Thus, given $\varphi \Rightarrow \psi$ we form $\varphi^+ \Rightarrow \psi^+$.

THEOREM 1. Let $\mathscr{X}$ be a Dedekind formula and ζ, ζ^+ formed as before on the basis of $\mathscr{X}$, Y, φ, and ψ. If $\mathcal{B}a \models \zeta$, then $\mathcal{H}a \models \zeta^+$.

Proof. Choose any $Y \in V$, where Y is a uniform space with bases $\mathcal{T}$ and Σ, and also any extensional $f \in V^{\Omega}$ with the appropriate domain of definition and any $k_1, \dots k_n \in \mathcal{D}(f)$ such that $0 < u \doteq [\![\mathscr{X}(f, \check{Y}, \dots)]\!]_{\Omega} \wedge f(k_1) \wedge \dots \wedge f(k_n) \wedge [\![(\varphi^+)_f(k_1, \dots, k_n)]\!]_{\Omega}$. We want to show that $u \leq [\![(\psi^+)_f(\bar{k})]\!]_{\Omega}$, whence the truth of the theorem will follow. It is easily verified by induction on the length of φ that

$$u \wedge [\![(\varphi^+)_f(\bar{k})]\!]_{\Omega} \leq [\![\varphi_f(\bar{k})]\!]_{\mathcal{B}} .$$

By assumption, we obtain $u \leq [\![\psi_f]\!]_{\mathcal{B}}$, so that $u \leq [\![\psi_f^+]\!]_{\mathcal{B}}$. It is easily verified by induction on the length of ψ (up to a $\forall$ quantifier) that

$$u \wedge [\![\psi_f^+]\!]_{\mathcal{B}} \leq [\![\psi_f^+]\!]_{\Omega} .$$

Remark. In order to prove that $[\![\zeta^+]\!]_{\Omega} = 1$, we need the condition $[\![\zeta]\!]_{\mathcal{B}} = 1$ only for one specific algebra $\mathcal{B}$, which is canonically determined by the algebra Ω. This also enables us to pass from a class of Boolean algebras to the corresponding class of Heyting algebras. Theorem 1 will remain valid if we additionally permit quantifiers over standard sets, e.g., over N, Q and so on; and also permit multibasis algebraic systems with supports $f_1, \dots, f_t$.

Let Y be a locally compact field. It is clear that $[\![\tilde{Y}$ is a ring$]\!]_{\mathcal{B}} = 1$ both in V^{Ω} and in $V^{\mathcal{B}}$. We shall denote the object $\tilde{Y}$ in V^{Ω} by $\tilde{Y}_{\Omega}$ and in $V^{\mathcal{B}}$ by $\tilde{Y}_{\mathcal{B}}$. The field is defined by the condition $\forall x \in f(x \neq 0 \Rightarrow \exists y \in f(x \cdot y = 1))$. It is easy to verify that $[\![f \subseteq \tilde{Y}$ is a field$]\!] = 1$ both in V^{Ω} and in $V^{\mathcal{B}}$. Indeed, if $p \in \Omega_u^{\mathcal{T}}$ and $u \doteq \tilde{Y}(p) \wedge [\![p \neq 0]\!]_{\Omega}$, there exists an Ω-dense $S \subseteq u$ on which f (corresponding to p) differs from 0. Define the function f^{-1} on S, which gives $q \in \Omega_u^{\mathcal{T}}$. This q is the desired inverse element.

Note that $[\![f \subseteq \tilde{Y} \wedge f$ is a field$]\!]_{\Omega} \leq [\![f \subseteq \tilde{Y}_{\Omega} \wedge f$ is a field$]\!]_{\mathcal{B}}$, since

$$\bigwedge_{x \in \mathcal{D}(f)}{}_{\Omega} f(x) \underset{\Omega}{\to} \bigvee_p \tilde{Y}(p) \wedge [\![x = p]\!]_{\Omega} \leq [\![f \subseteq \tilde{Y}_{\Omega}]\!]_{\mathcal{B}}$$

and

$$\bigwedge_{x \in \mathcal{D}(f)}{}_{\Omega} f(x) \underset{\Omega}{\to} \left([\![x \neq 0]\!]_{\Omega} \underset{\Omega}{\to} \bigvee_y f(y) \wedge [\![x \cdot y = 1]\!]_{\Omega}\right) \leq [\![f \text{ is a field}]\!]_{\mathcal{B}} .$$

In addition, $[\![\tilde{Y}_{\Omega}$ is a subfield of $\tilde{Y}_{\mathcal{B}}]\!]_{\mathcal{B}} = 1$, since $\Omega_u^{\mathcal{T}}$ is contained in $\mathcal{B}_u^{\mathcal{T}}$. Therefore $\mathscr{X}(f) \doteq$ "$f \subseteq \tilde{Y} \wedge f$ is a field" is a Dedekind formula which determines the family of all subfields of $\widetilde{Y}_{\Omega}$ in V^{Ω} and a certain family of subfields of $\widetilde{Y}_{\mathcal{B}}$ in $V^{\mathcal{B}}$. Hence properties of subfields of $\widetilde{Y}_{\mathcal{B}}$ or, more precisely, the properties of subfields of $\widetilde{Y}_{\Omega}$ in $V^{\mathcal{B}}$, transfer to subfields of $\widetilde{Y}_{\Omega}$ in V^{Ω}. We now consider Hilbert's Nullstellensatz in accordance with a somewhat different scheme: without the participation of $\mathscr{X}$ and φ.

Let Y be a locally compact field and Y^{a} any algebraically closed locally compact field containing Y. Then $[\![\widetilde{Y}_{\Omega}$ is a subfield of $\widetilde{Y}_{\mathcal{B}}$ and $\widetilde{Y}_{\mathcal{B}}$ is a subfield of $\widetilde{Y}^{a}_{\mathcal{B}}]\!]_{\mathcal{B}}=1$ and $[\![\widetilde{Y}^{a}_{\mathcal{B}}$ is an algebraically closed field $]\!]_{\mathcal{B}}=1$. We may therefore use $V^{\mathcal{B}}$ as the extension in the Nullstellensatz in $\widetilde{Y}^{a}_{\mathcal{B}}$. In that case

$$C_1(X,Y)\cong(\widetilde{Y}_{\Omega})^{\wedge\Omega}\subseteq(\widetilde{Y}_{\Omega})^{\wedge\mathcal{B}},$$

since

$$1=\bigvee_{p}\widetilde{Y}_{\Omega}(p)\wedge[\![x=p]\!]_{\Omega}\leq\bigvee_{p}\widetilde{Y}_{\Omega}(p)\wedge[\![x=p]\!]_{\mathcal{B}}.$$

Hence

$$C_1(X,Y)\subseteq(\widetilde{Y}^{a}_{\mathcal{B}})^{\wedge\mathcal{B}}\cong C_1(X_{\mathcal{B}},Y^{a}),$$

where $X_{\mathcal{B}}$ is the Stone space of $\mathcal{B}$. For polynomials $h_1,\dots,h_r,h$ over $C_1(X,Y)$ the premise $V^{\mathcal{B}}$ of the Nullstellensatz means

$$\forall x_1,\dots,x_n\in C_1(X_{\mathcal{B}},Y^{a})([\![h_1(\bar{x})=0]\!]_{\mathcal{B}}\wedge\dots\wedge[\![h_r(\bar{x})=0]\!]_{\mathcal{B}}\leq[\![h(\bar{x})=0]\!]_{\mathcal{B}}),$$

where $[\![f=g]\!]_{\mathcal{B}}=j\{x\in X_{\mathcal{B}}\mid f(x)=g(x)\}^{\circ}$, and $j^{\circ}\alpha$ is the interior of the closure of α (i.e., the computation of $[\![h_1(\bar{x})=0]\!]_{\mathcal{B}}$ and so on may be carried out with participation of the valuation in $V^{\mathcal{B}}$). If polynomials $h_1,\dots,h_r,h$ over $C_1(X,Y)$ satisfy this condition we shall say that they are compatible.

For purely illustrative purposes, we present the following analog of the Hilbert Nullstellensatz.

Example 1. Let Y be a locally compact field and Y^{0} an algebraically closed locally compact field containing Y. Let $h_1,\dots,h_r,h$ be any compatible polynomials over $C_1(X,Y)$ and Ω a Heyting algebra for the Stone space X. If Ω is compact zero-dimensional, then for any $\sigma\in\Sigma_Y$ there exist polynomials $g_1,\dots,g_r$ over $C_1(X,Y)$, such that $h^{\rho}=_{\sigma}g_1\cdot h_1+\dots+g_r\cdot h_r$. Equality of polynomials is understood here in the algebraic sense and ρ and the degrees of the polynomials $g_1,\dots,g_r$ are determined by the degrees of $h_1,\dots,h_r,h$; only the coefficients of the polynomials $g_1,\dots,g_r$ depend on σ. The proof follows at once from the foregoing remarks and Theorem 1.

The simplest example of such a compact zero-dimensional algebra Ω is $\Omega\leftrightharpoons\mathcal{T}(Z)$, where $\mathcal{T}(Z)$ is the topology of a topological space Z and Z a totally disconnected compact space. In that case V^{Ω} has the accessibility property: for any formula φ, if $[\![\exists x\varphi(x)]\!]_{\Omega}=1$, then $[\![\varphi(g)]\!]_{\Omega}=1$ for some $g\in V^{\Omega}$. In Example 1 Ω is required to satisfy this accessibility property only for an atomic formula φ. In the same example we may replace the ring $C_1(X,Y)$ by $C(Z,Y)$ for any totally disconnected compact space Z; for example, for $Y\leftrightharpoons\mathbb{R}$. This follows at once from the fact that the rings $C_1(X,\mathbb{R})$ and $C(Z,\mathbb{R})$ (and even the corresponding sheaves of functions) are isomorphic; see corollary, below.

THEOREM 2. Let Ω be an arbitrary complete Heyting algebra. Let R^d denote the ring of Dedekind cuts in $\mathbb{Q}$. Then the ring $(R^d_\Omega)^{\wedge\Omega}$ is isomorphic to $C_1(X,R)$ by the map $\varphi:[f]\mapsto \langle\lambda,\lambda_1\rangle$, where $\lambda(\tau)\leftrightharpoons jf^{-1\sigma}(\tau,\infty)$ and $\lambda_1(s)\leftrightharpoons jf^{-1\sigma}(-\infty,s)$, $\tau,s\in\mathbb{Q}$.

Let Z be any topological space and $\Omega\leftrightharpoons\mathcal{T}(Z)$. It is well known that in that case the ring $(R^d_\Omega)^{\wedge\Omega}$ is isomorphic to $C(Z,R)$ by the map $h\mapsto\langle\lambda,\lambda_1\rangle$, where $\lambda(\tau)\leftrightharpoons h^{-1\sigma}(\tau,\infty)$ and $\lambda_1(s)\leftrightharpoons h^{-1\sigma}(-\infty,s)$. The inverse map has the form $\langle\lambda,\lambda_1\rangle\mapsto h$, where $h(z)\leftrightharpoons\sup\{\tau\in\mathbb{Q}\mid z\in\lambda(\tau)\}$, the set in braces, taken for any z, being the lower class of a cut in $\mathbb{Q}$ and λ,λ_1 may always be considered to be defined on $\mathbb{Q}$. Let λ_h denote the cut corresponding to $h\in C(Z,R)$. Hence we obtain

COROLLARY. The rings $C_1(X,R)$ and $C(Z,R)$ (and the corresponding sheaves of functions) are isomorphic. The rings R^d_Ω and $\tilde{R}_\Omega$ are isomorphic in V^Ω for any Ω. If Z is a regular Baire space (and $\dot{Z}$ its absolute), this isomorphism has the form $h\mapsto(h\circ\varkappa)'$, where $h\in C(Z,R)$ and $\varkappa:\dot{Z}\to Z$ is the canonical map, while $(\cdot)'$ denotes the uniquely determined extension of $h\circ\varkappa$ to a continuous function g_h on an open neighborhood S_h of the absolute $\dot{Z}$, which is Ω-dense in X, and also the equivalence class $[g]$ of this function g in $C_1(X,R)$.

Proof of Theorem 2. Instead of $\varphi([f])$ we write λ_f. A Dedekind real (cut in $\mathbb{Q}$) is defined as a pair of subsets of $\mathbb{Q}$ which satisfies four conditions: 1) $\exists\tau,s\,(\tau\in\lambda\wedge s\in\lambda_1)$; 2) λ and λ_1 are disjoint; 3) $\tau\in\lambda\Leftrightarrow\exists s>\tau\,(s\in\lambda)$ and $s\in\lambda_1\Leftrightarrow\exists\tau<s\,(\tau\in\lambda_1)$; 4) $\tau<s\Rightarrow\tau\in\lambda\vee s\in\lambda_1$. A direct computation of valuations shows that $[\![\langle\lambda,\lambda_1\rangle$ is a section in $\check{\mathbb{Q}}]\!]_\Omega=1$. If g is a new representative of the equivalence class $[f]$, i.e., $f=g$ on $S_f\cap S_g$, then $(f^{-1\sigma}(\tau,\infty)\cap S_g=(g^{-1\sigma}(\tau,\infty))\cap S_f$ and $\varphi([f])=\varphi([g])$. Thus φ is well defined. This φ is injective: if $f\nsim g$, i.e., $f(x)\neq g(x)$ for $x\in S_f\cap S_g$, then there exists a neighborhood u of x (where $u\subseteq S_f\cap S_g$) such that $f(u)>\tau$ and $g(u)<\tau$. Hence $u\subseteq f^{-1\sigma}(\tau,\infty)$ and $u\cap g^{-1\sigma}(\tau,\infty)=\emptyset$, i.e., $[\![\lambda_f=\lambda_g]\!]_\Omega=1$. The proof that φ is surjective is more complicated.

Let $[\![\lambda\leftrightharpoons\langle\lambda,\lambda_1\rangle$ is a cut in $\check{\mathbb{Q}}]\!]_\Omega=1$. We may assume that λ,λ_1 are defined on $\mathbb{Q}$. Regarding λ as given, construct the function $\rho(\alpha)\leftrightharpoons[\![\exists\tau,s\in\check{\mathbb{Q}}\,(\tau\in\lambda\wedge s\in\lambda_1\wedge(\tau,s)\subseteq\check{\alpha})]\!]_\Omega$. Then $[\![\rho$ is a base of a Cauchy filter in $\mathcal{F}]\!]_\Omega=1$ and $\rho(\alpha)=[\![\check{\alpha}\in\rho]\!]_\Omega=\bigvee_{(\tau,s)\subseteq\alpha}\lambda(\tau)\wedge\lambda_1(s)$, and moreover $\rho\in\Omega^{\mathcal{T}}$. Using ψ^{-1} and considering ρ given, we form $f\in C_1(X,R)$, i.e., $\rho(\alpha)=j(f^{-1}(\alpha))$. Hence $\rho^\sigma(\tau,\infty)=j(f^{-1\sigma}(\tau,\infty))$, i.e., $\rho^\sigma(\tau,\infty)=\lambda_f(\tau)$, $\forall\tau\in\mathbb{Q}$. On the other hand, $[\![(\check{\tau},\infty)\in\rho\Longleftrightarrow\check{\tau}\in\lambda]\!]_\Omega=1$, i.e., $\rho^\sigma(\tau,\infty)=\lambda(\tau)$. Thus $[\![\lambda=\lambda_f]\!]_\Omega=1$, i.e., $\varphi([f])=\langle\lambda,\lambda_1\rangle$. The fact that φ is a homomorphism follows from the obvious formulas $(f+g)^{-1\sigma}(t,\infty)=\bigcup_{t<\tau+s}f^{-1\sigma}(\tau,\infty)\cap g^{-1\sigma}(s,\infty)$ and similarly for $f\cdot g$. We moreover observe that $[\![\lambda_h=\lambda_{h_1}]\!]_\Omega=\{z\in Z\mid h(z)=h_1(z)\}^\circ=[\![\lambda_f=\lambda_{f_1}]\!]_\Omega=j\{x\in X\mid f(x)=f_1(x)\}^\circ$, where h corresponds to f, and $[\![\lambda=\mu]\!]_\Omega=[\![\rho_\lambda=\rho_\mu]\!]_\Omega$.

Proof of the Corollary. We first add a few comments to complete Theorem 2. Let $\lambda\leftrightharpoons\lambda_f$ be a cut obtained from $[f]\in C_1(X,R)$. Let $S_f\leftrightharpoons\mathcal{D}(f)$. Set $S_g\leftrightharpoons\bigcup\lambda(\tau)\wedge\lambda_1(s)$ – this is an open

Ω-dense set, and $g(x) \leftrightharpoons \sup\{\overset{r,s}{r} \in Q \mid x \in \lambda(r)\}, x \in S_g$. It is clear that the set $t \leftrightharpoons \{r \in Q \mid x \in \lambda(r)\}$ is nonempty, bounded above and transitive downward; in particular, g is defined on S_g and we have $S_g \supseteq S_{f\text{-}lo}$ and $g = f$ on S_f. Indeed, if $x \in S_f$, then $x \in jf(r,\infty) = \lambda(r)$ and similarly $x \in \lambda_1(s)$ for some r and s, i.e., $x \in S_g$. Let $x \in S_f$. If $f(x) > r$, then $x \in \lambda(r)$. Hence $g(x) \geq f(x)$. If $x \in \lambda(r)$, then $f(x) \geq r$ (this implies equality), because otherwise $f(x) < r$, in which case $x \in jf^{-1\sigma}(-\infty, r)$, $x \in jf^{-1\sigma}(-\infty, r) \wedge jf^{-1\sigma}(r,\infty) = \emptyset$ – contradiction.

We do not claim that g is necessarily continuous on S_g. But if Ω is a topology, then $\dot{Z} \subseteq S_g$ and g is continuous on S_g. Indeed, S_g contains the absolute $\dot{Z}$, because S_g "in Z" coincides with the whole of Z. The set t has the property $r \in t \Leftrightarrow \exists s > r\,.\, s \in t$, $\forall r \in Q$, since the valid property $[\check{r} \in \lambda \Leftrightarrow \exists s > \check{r}(s \in \lambda)]_{\Omega} = 1$ implies that $\lambda(r) = \bigcup_{s>r} \lambda(s)$. Hence we have $g(x) > s \Leftrightarrow x \in (\lambda(s) \cap S_g)$, i.e., $g^{-1\sigma}(s,\infty)$ is open in S_g; similarly for $g^{-1}(-\infty, r)$. Thus g is continuous on S_g.

We recall that $\dot{Z}$ consists of all ultrafilters in $\mathcal{T}(Z)$ that converge to points of Z. From now on we shall assume that the points of the Stone space X of $\mathcal{T}(Z)$ are coprime filters and not prime ideals; then $\dot{Z} \subseteq X$. Both $\dot{Z}$ and X will be considered with the Zariski topology, in which case $\dot{Z}$ is a dense subspace of X. We define $\varkappa : \dot{Z} \to Z$, $\varkappa(x) \leftrightharpoons \lim x$ – this is an irreducible perfect map; $\dot{Z}$ is an extremally disconnected regular space. We have thus proved that in any equivalence class $[f]$ there are elements g such that $S_g \supseteq \dot{Z}$. Once again, let $\lambda \leftrightharpoons \lambda_f$ be the cut constructed from $[f]$, where $f \sim g$, g being as described above. Form $h_\lambda \in C(Z, R)$ and $h_\lambda \circ \varkappa : \dot{Z} \to R$. It turns out that g is not only an extension of f from S_f to S_g, but also an extension of $h_\lambda \circ \varkappa$ from $\dot{Z}$ to S_g, i.e., $g \upharpoonright \dot{Z} = h_\lambda \circ \varkappa$. Indeed, let $x \in \dot{Z}$ and $\lim x = z$. Then $h_\lambda \circ \varkappa(x) = h_\lambda(z) \leftrightharpoons \sup\{r \mid z \in \lambda(r)\}$, and $g(x) \leftrightharpoons \sup\{r \mid x \in \lambda(r)\}$. We shall find a dense subset of $\dot{Z}$, say D, on which $h_\lambda \circ \varkappa$ coincides with g; hence $h_\lambda \circ \varkappa$ will coincide with g on all of $\dot{Z}$. Denote $F_\sigma \leftrightharpoons \bigcup_r gr(\lambda(r)) \subseteq Z$, where $gr(\cdot)$ is the boundary of the set. The set F_σ is metager (i.e., a countable union of nowhere dense sets). Denote its complement by G_δ. Since by assumption Z is a Baire space, it follows that G_δ is dense in Z. Let $D \leftrightharpoons \varkappa^{-1}(G_\delta)$. If $\bar{D} \neq \dot{Z}$, then $\varkappa(\bar{D})$ is closed in Z and $\bar{G}_\delta \subseteq \varkappa(\bar{D})$, $\varkappa(\bar{D}) = Z$, contrary to the irreducibility of $\varkappa$. If $x \in D$, then $\varkappa \in \lambda(r) \Leftrightarrow x \in \lambda(r)$, $\forall r \in Q$. Hence, if g corresponds to h (in the sense that $g \upharpoonright \dot{Z} = h \circ \varkappa$), then $[\lambda_g = \lambda_h]_{\Omega} = 1$, where $[g] \in C_1(X, R)$ and $h \in C(Z, R)$.

Example 2. Let K be an arbitrary ring. Given its Pierce sheaf $\mathcal{F}(\cdot)$, we define an object $K' \in V^{\Omega}$, where Ω is the topology of the Stone space of the Boolean algebra of all central idempotents in K, such that $(K')^{\wedge\Omega} \cong K$. The sheaf $\mathcal{F}(\cdot)$ extends to B, which is canonically determined by Ω. This extension defines an object $K'' \in V^{B}$ and $[K' \subseteq K'']_{B} = 1$. By Theorem 1, the properties of K' in V^{B} carry over to K' in V^{Ω}, and then all of them (not only the Horn properties) carry over to K as well (in the form $\varphi \mapsto \varphi'$, see [2]). In other words, we consider K' in V^{B} and form $L \leftrightharpoons (K')^{\wedge B}$. This L is an orthogonally complete ring and has the properties of an "orthogonally complete closure" of K. The properties of K' in V^{Ω} and K' in V^{B} are closely linked by the above translation.

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Added in Proof (November 1990). Essentially, what we have proved in this paper is the following proposition. Let φ, ψ be formulas in the language of rings; in the premise of any implication in φ there is no quantifier $\forall$ and the quantifier $\exists$ does not occur in the scope of $\Longrightarrow$; let T be the theory of such formulas φ, and ψ an AE-formula, $\xi = \forall \bar{k}(T(\bar{k}) \Rightarrow \psi(\bar{k}))$ and $\xi' = \forall \bar{k}(T'(\bar{k}) = \psi'(\bar{k}))$; let K be a special variable which runs over all rings with countable support (which are normal in the conclusion). In the following statements the premise U is understood as $ZF \vdash U$, and the conclusion V is understood as $ZFI' \vdash V$ (if $Z \vdash U$ in the premise, then $ZFI \vdash V$ in the conclusion). Thus, 1) if $\forall K(\xi)_K$, then $\forall K$ (indecomposable $\Rightarrow \xi$) and $\forall K(\xi')_K$; 2) if $\forall K(i' \Rightarrow \xi)_K$, then $\forall K(i \Rightarrow \xi')_K$, where $i \leftrightarrow i'$ are the pairs of properties in [2] (it is always true that $i' \Longrightarrow$ indecomposable); 3) if (intuitionistically) $\forall K$ [indecomposable $\wedge T) \Rightarrow \varphi]_K$ then $\forall K\ [\quad T' \Longrightarrow \psi']_K$; finally, one can replace T by the set-theoretical condition $\text{æ}(K)$, where æ is an absolute formula.

ON THE LÖWENHEIM NUMBERS FOR THE SKELETONS OF VARIETIES OF BOOLEAN ALGEBRAS

A. G. Pinus

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The concepts of skeletons of varieties of algebras have been introduced by the author in [1, 2] and, afterwards, in a series of subsequent investigations one has studied in a sufficiently detailed manner the skeletons of congruence-distributive varieties. In particular, in [3, 4] one has proved the undecidability of the elementary theories of epimorphism skeletons and of the elementary theories of Cartesian skeletons for nontrivial congruence-distributive varieties, while in [5] one has proved the undecidability of the elementary theory of the imbeddability skeleton of an arbitrary variety, containing a non-one-element quasiminimal algebra. In [6] one has introduced the concept of the Löwenheim numbers for skeletons of varieties and one has proved the equality between the Löwenheim number of the Cartesian skeleton of any nontrivial, finitely based, congruence-distributive variety and the Löwenheim number of the full second-order logic. In [7] it is proved that the Löwenheim number of the full second-order logic coincides with the Löwenheim number of the so-called multiplicative epimorphism skeleton of Boolean algebras. This paper is devoted to the proof of an analogous result, with the replacement of epimorphism by imbeddability.

If $\mathcal{K}$ is an arbitrary class of algebras, then by $\mathcal{IK}$ we denote the collection of all isomorphism types of $\mathcal{K}$-algebras. On the collection $\mathcal{IK}$ we introduce the quasiorder relations $\ll$, $\leq$ and the operations, possibly partial, of products $\times, *$: for $a, b, c \in \mathcal{IK}$ $a \ll b$, $a \leq b$ if and only if an algebra of type a is the homomorphic image of an algebra of type b (an

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