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An infinity which depends on the axiom of choice

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ABSTRACT

In the early years of set theory, Du Bois Reymond introduced a vague notion of *infinitary pantachie* meant to symbolize an infinity bigger than the infinity of real numbers. Hausdorff reformulated this concept rigorously as a maximal *chain* (a linearly ordered subset) in a partially ordered set of certain type, for instance, the set $\mathbb{N}^{\mathbb{N}}$ under eventual domination. Hausdorff proved the existence of a pantachy in any partially ordered set, using the axiom of choice **AC**. We show in this note that the pantachy existence theorem fails in the absense of **AC**, and moreover, even if **AC** is assumed, hence pantachies do exist, one may not be able to come up with an individual, effectively defined example of a pantachy.

1. Introduction

Linear order relations, which typically appear in conventional mathematics, are *countably cofinal*, that is, they admit countable strictly increasing cofinal subsequences. In fact *every* Borel (as a set of pairs) linear order on a subset of a Polish space is countably cofinal: see, e.g. [11]. Uncountably cofinal orders were introduced in mathematics, in the form of *partial* rather than linear orders, by Du Bois Reymond.

The *rate of growth* partial order $\leqslant_{\text{RG}}$ is defined on positive real functions so that $f \leqslant_{\text{RG}} g$ iff the limit $\mathtt{lim}_{\,x\to+\infty} \frac{g(x)}{f(x)}$ exists and is >0. This ordering of functions was known long before Du Bois Reymond, but he was the first who considered $\leqslant_{\text{RG}}$ in [1] as a relation on the whole totality of positive real functions. He also proved in [1] that the ordering $\leqslant_{\text{RG}}$ is great deal non-separable: in particular, for any countable collection $\{f_n\}_{n\in\mathbb{N}}$ of positive real functions there is a function f satisfying $f_n <_{\text{RG}} f$ strictly, that is, $f_n \leqslant_{\text{RG}} f$ but $f \not\leqslant_{\text{RG}} f_n$ for all n. (Indeed, let $f(x) = x\ \mathtt{sup}_{n\leqslant x} f_n(x)$ for all $x > 0$. This was the first application of the diagonal method in mathematics.)

Somewhat later, Du Bois Reymond published a monograph [2], with a mixed mathematical and philosophical content, where he stipulated that the totality of all real functions ordered by $\leqslant_{\text{RG}}$, which he called *the infinitary pantachy*, might serve as an extension of the continuum of real numbers, where infinitesimal and infinitely large quantities coexist with usual reals (corresponding to constant functions), thus manifesting a sort of infinity which exceeds the infinity of the real continuum. This concept was met with mixed reception among contemporary mathematicians. In particular, Hausdorff [7,8] noted that obvious existence of $\leqslant_{\text{RG}}$-incomparable functions makes *the infinitary pantachy* rather useless in the role of an extended analytic domain (see more on controversies around Du Bois Reymond's approach in [5]). Instead, Hausdorff suggested to consider *maximal linearly ordered* sets of functions (or infinite real sequences, that can be ordered the same way), in the sense of $\leqslant_{\text{RG}}$ or any other similar order based on the comparison of behaviour of functions or sequences at infinity. He called such maximal linearly ordered sets *pantachies*.

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Hausdorff [7,8] proved the existence of a pantachy in any partially ordered set. This result was one of the earliest explicit applications of the axiom of choice **AC**. The method of one of two Hausdorff's pantachy existence proofs is known nowadays as *the maximality principle*.

Typically for the **AC**-based existence proofs, Hausdorff's pantachy existence proof did not produce anything near a concrete, individual, effectively defined example of a pantachy in the $\leqslant_{RG}$ ordered set of real functions or in any partial order of the same kind. Haudorff writes in [7,p. 110]:

> Since the attempt to actually legitimately construct a pantachy seems completely hopeless, it would now be a matter of gathering information ... about the order type of any pantachy ...[3]

Working in this direction, Hausdorff proved, in particular, that any pantachy is uncountably cofinal, uncountably coinitial, and has no (ω,ω^*)-gaps – hence, is extremely nonseparable, a type of infinity rather uncommon for mathematics of the early 1900s. Yet those studies left open the major problem of *effective existence* of pantachies. One may ask:

(1) whether the pantachy existence can be established not assuming the axiom of choice **AC**,
(2) whether, even assuming **AC**, one can actually define an individual example of a pantachy.

Advances in modern set theory allow us to answer both questions **in the negative**, both for the $\leqslant_{RG}$-ordering of positive functions and for a variety of similar partial orderings. This is the main result of this paper, and it supports Haudorff's observation cited above. The result is not unexpected. The unexpected feature is that we will have to apply two difficult special results in set theory related to Solovay's models (Propositions 12 and 13), since the basic technique of Solovay's models does not seem to be sufficient in this case.

The negative answer we obtain is a motivation for the title of the paper: pantachies in the $\leqslant_{RG}$-ordering of positive real functions is the type of infinity which depends on the axiom of choice!

2. Preliminaries

We precede the formulation of our main results with several definitions and notational comments. First of all, we adjust to modern terminology related to partial and linear orderings.

Definition 1. A *partial quasi-order*, PQO for brevity, is a binary relation $\leqslant$ satisfying $x \leqslant y \wedge y \leqslant z \Rightarrow x \leqslant z$ (transitivity) and $x \leqslant x$ (reflexivity) on its domain. In this case, an *associated equivalence relation* $\equiv$ and an *associated strict partial order* $<$ are defined, on the same domain, so that

$$x \equiv y \text{ iff } x \leqslant y \wedge y \leqslant x \quad \text{and} \quad x < y \text{ iff } x \leqslant y \wedge y \nleqslant x.$$

If a PQO $\leqslant$ also satisfies the antisymmetry condition $x \leqslant y \wedge y \leqslant x \Rightarrow x = y$ (which is not assumed, generally speaking) then it is called a *partial order*, PO for brevity. Thus, a PQO is a PO iff the associated equivalence relation is the equality.

A PQO is *linear*, LQO for brevity, if we have $x \leqslant y \vee y \leqslant x$ for all x, y in its domain. A *linear order*, or LO, is any LQO which satisfies the same antisymmetry condition $x \leqslant y \wedge y \leqslant x \Rightarrow x = y$.

An LQO $\langle X; \leqslant\rangle$ (meaning: X is the domain of $\leqslant$) is *of countable cofinality* iff there is a set $Y \subseteq X$, at most countable and *cofinal* in X, that is, if x belong to X then there exists an element $y \in Y$ such that $x \leqslant y$. In this case, we also say that X is countably $\leqslant$-cofinal. □

For instance, if X has a $\leqslant$-largest element x then X is countably cofinal: indeed, take $Y = \{x\}$.

The set $2^{<\omega_1} = \bigcup_{\xi<\omega_1} 2^\xi$ consists of all transfinite binary sequences of length $<\omega_1$, and if $\xi < \omega_1$ then 2^ξ is the set of all binary sequences of length exactly ξ. By $<_{\mathtt{lex}}$ we denote the strict lexicographical order on $2^{<\omega_1}$, that is, if $s,t \in 2^{<\omega_1}$ then $s <_{\mathtt{lex}} t$ iff $s \not\subset t$, $t \not\subset s$, and the least ordinal $\xi < \mathtt{dom}\, s$, $\mathtt{dom}\, t$ such that $s(\xi) \neq t(\xi)$ satisfies $s(\xi) < t(\xi)$. Here $s \not\subset t$ means that t is a proper extension of s. We define $s \leqslant_{\mathtt{lex}} t$ iff either $s = t$ or $s <_{\mathtt{lex}} t$. Note that $\leqslant_{\mathtt{lex}}$ is a *partial* order on $2^{<\omega_1}$, but if $\xi < \omega_1$ then $\leqslant_{\mathtt{lex}}$ *linearly* orders 2^ξ.

The next lemma will be used below.

Lemma 2. *If $\xi < \omega_1$ then any set $C \subseteq 2^\xi$ is countably $\leqslant_{\mathtt{lex}}$-cofinal.*

Proof. We argue by transfinite induction on ξ.

If $\xi = 0$ then there is nothing to prove.

To carry out the step $\xi \to \xi + 1$, suppose that $C \subseteq 2^{\xi+1}$. If $s \in 2^{\xi+1}$ then $x{\restriction}\xi \in 2^\xi$ is the restriction of s to ξ. Consider the set $C' = \{s{\restriction}\xi : s \in C\}$. By the inductive hypothesis, there is a set $Y' \subseteq C'$, cofinal in C' and at most countable. Then the set $Y = \{s \in C : s{\restriction}\xi \in C'\}$ is cofinal in C' and still at most countable, as required.

[3] English translation taken from [9].

To carry out the limit step, let $\lambda < \omega_1$ be a limit ordinal, and $C \subseteq 2^\lambda$. By the inductive hypothesis, for any $\xi < \lambda$ there is a set $Y'_\xi \subseteq C_\xi = \{s \restriction \xi : s \in C\}$, cofinal in C_ξ and at most countable, and then there is a set $Y_\xi \subseteq C$, at most countable and such that $Y'_\xi = \{s \restriction \xi : s \in Y_\xi\}$. We claim that the set $Y = \bigcup_{\xi<\lambda} Y_\xi$ (still at most countable) is cofinal in C, or else C has a $\leqslant_{\mathtt{lex}}$-largest element. Indeed suppose that $t \in C$. Then by construction for each ξ there exist elements $s'_\xi \in Y'_\xi$ and $s_\xi \in Y_\xi$ such that $s'_\xi = s_\xi \restriction \xi,\ t \restriction \xi \leqslant_{\mathtt{lex}} s'_\xi$, and moreover, either $t \restriction \xi <_{\mathtt{lex}} s'_\xi$ or $t \restriction \xi$ is a $\leqslant_{\mathtt{lex}}$-largest element of Y'_ξ.

If $t \restriction \xi <_{\mathtt{lex}} s'_\xi$ for at least one ξ then clearly $t <_{\mathtt{lex}} s_\xi$, as required.

It remains to consider the case when $t \restriction \xi$ is a $\leqslant_{\mathtt{lex}}$-largest element of Y'_ξ, hence, of C_ξ as well, for each ordinal $\xi < \lambda$. But then t itself is obviously a $\leqslant_{\mathtt{lex}}$-largest element of C, as required. □

Definition 3. A PQO $\langle X; \leqslant \rangle$ is *Borel* iff the set X is a Borel set in a suitable Polish space $\mathbb{X}$, and the relation $\leqslant$ (as a set of pairs) is a Borel subset of $\mathbb{X} \times \mathbb{X}$. □

Corollary 4. *Every Borel LQO $\leqslant$ is countably cofinal, and moreover, there is no strictly increasing ω_1 sequences.*

Proof. It was established in [6] (see also [11]) that if $\langle X; \leqslant \rangle$ is a Borel LQO then there is an ordinal $\xi < \omega_1$ and a Borel map $\vartheta : X \to 2^\xi$ such that we have $x \leqslant y$ iff $\vartheta(x) \leqslant_{\mathtt{lex}} \vartheta(y)$ for all $x, y \in X$. In other words, any Borel linear quasi-order is Borel-isomorphic to a suborder [4] of $\langle 2^\xi; \leqslant_{\mathtt{lex}} \rangle$ for a suitable countable ordinal ξ. It remains to apply Lemma 2. □

Informally, a *pantachy* is a maximal linearly ordered subset of a PQO set [7,8]. This can be rigorously defined in two versions.

Definition 5. A *pantachy* in a PQO $\langle X; \leqslant \rangle$ is any set $P \subseteq X$ such that $\leqslant \restriction P$ is an LO and (the maximality!) if $x \in X \setminus P$ then $\leqslant \restriction (P \cup \{x\})$ is **not** an LO.

A *quasi-pantachy* in a PQO $\langle X; \leqslant \rangle$ is any set $P \subseteq X$ such that $\leqslant \restriction P$ is an LQO and if $x \in X \setminus P$ then $\leqslant \restriction (P \cup \{x\})$ is **not** an LQO.

To get a pantachy from a quasi-pantachy $Q \subseteq X$ in a PQO $\langle X; \leqslant \rangle$, it suffices to pick an element in each $\equiv$ class which intersects Q – so generally speaking this needs the axiom of choice **AC**. On the other hand, if $P \subseteq X$ is a pantachy then $Q = [P]_\equiv = \{x \in X : \exists p \in P\ (p \equiv x)\}$ is a quasi-pantachy, and here there is no need in **AC**.

3. The main technical theorem

As usual, **ZFC** and **ZF** are Zermelo – Fraenkel set theories resp. with and without the axiom of choice **AC**. The *principle of dependent choices* **DC** is the strongest possible form of the *countable* **AC**; it allows countable sequences of choices even in the case when the set $X_n \neq \varnothing$, in which the next choice x_n is to be made, itself depends not only on the index $n \in \mathbb{N}$, but also on the results x_k, $k < n$, of all previous choices.

The next theorem (Theorem 7), our main technical result, deals with **ROD** linearly ordered subsets in Borel partial orderings. □

Definition 6. **ROD** is the class of *real-ordinal definable* sets, that is, those definable by a set theoretic formula with reals and ordinals as parameters. □

ROD is the widest class known in modern mathematics, which consists of sets that can be considered as "effectively defined". (It contains all Borel, Souslin, and projective sets of Polish spaces, by the way.) Therefore any nonexistence result for the **ROD** domain (as, for instance, the first claim of Theorem 7) is usually treated in the sense that there is no individual, effectively defined examples of sets of the type considered.

Let **WIC** be the sentence "there is a weakly inaccessible cardinal".

Recall that *weakly inaccessible cardinals* are uncountable regular limit cardinal numbers, see, e.g. [10] for background on inaccessible cardinals.

Theorem 7. *Suppose that* **WIC** *is consistent with the axioms of* **ZFC**.

Then, first, the following sentence is consistent with **ZFC** *as well:*

(i) if $\leqslant$ is a Borel PQO on a (Borel) set $D \subseteq \mathbb{N}^\mathbb{N}, X \subseteq D$ is a **ROD** *set, and $\leqslant \restriction X$ is a LQO, then $\leqslant \restriction X$ is of countable cofinality.*

And second, the following sentence is consistent with **ZF + DC**:

(ii) if $\leqslant$ is a Borel PQO on a (Borel) set $D \subseteq \mathbb{N}^\mathbb{N}, X \subseteq D$ is any set, and $\leqslant \restriction X$ is a LQO, then $\leqslant \restriction X$ is of countable cofinality.

[4] By *suborder* we mean the restriction of a given partial order to a subset of its domain.

Thus (assuming the consistency of **ZFC + WIC**) it is consistent with **ZFC** that all **ROD** linear suborders of Borel PQOs are countably cofinal, and it is consistent with **ZF + DC** that all in general linear suborders of Borel PQOs are countably cofinal. We proceed with a few remarks.

A. It is known that **WIC** cannot be proved in **ZFC** (unless **ZFC** is inconsistent), neither the consistency of the extended theory **ZFC + WIC** can be proved assuming the consistency of **ZFC** alone. Nevertheless **ZFC + WIC** is considered as a legitimate extension of **ZFC** itself, and accordingly consistency proofs carried out in the assumption of the consistency of **ZFC + WIC** are considered as legitimate consistency proofs.

B. On the other hand, it turns out that, conversely, if statement (i) of Theorem 7 is consistent with **ZFC**, or statement (ii) of Theorem 7 is consistent with **ZF + DC**, then **WIC** is consistent with **ZFC**. In other words, the three theories **ZFC + WIC**, **ZFC** + (i), **ZF + DC** + (ii) are *equiconsistent*.

Indeed, suppose that, say, (i) holds. We claim that then $\omega_1^{\mathrm{L}[x]}$, the first uncountable cardinal in the class L[x] of sets constructible from x, satisfies $\omega_1^{\mathrm{L}[x]} < \omega_1$ strictly for every $x \in \mathbb{N}^{\mathbb{N}}$. It is known that the "true" ω_1 is a weakly (even strongly) inaccessible cardinal in L[x] for any $x \in \mathbb{N}^{\mathbb{N}}$ in the assumption $\forall x \in \mathbb{N}^{\mathbb{N}}\ (\omega_1^{\mathrm{L}[x]} < \omega_1)$ (see, e.g. [10,15]), and hence we get an inaccessible cardinal. Thus it remains to prove the claim.

Suppose, towards the contrary, that $x \in \mathbb{N}^{\mathbb{N}}$ and $\omega_1^{\mathrm{L}[x]} = \omega_1$. Consider the *eventual domination* PQO $\leqslant^*$ on $\mathbb{N}^{\mathbb{N}}$, defined so that $x \leqslant^* y$ iff there is n_0 such that $x(n) \leqslant y(n)$ for all $n \geqslant n_0$. Clearly $\leqslant^*$ is a Borel relation. Let $<^*$ be the corresponding strict PQO, so that $x <^* y$ iff $x \leqslant^* y$ but $y \nleqslant^* x$. A simple diagonal argument by Du Bois Reymond [1] shows that for any countable set $X \subseteq \mathbb{N}^{\mathbb{N}}$ there is an element $y \in \mathbb{N}^{\mathbb{N}}$ such that $x <^* y$ for all $x \in X$. Therefore, arguing in L[x] and using the canonical Gödel **ROD** wellordering of $\mathbb{N}^{\mathbb{N}} \cap \mathrm{L}[x]$ (this wellordering is **ROD** with parameter x), we can define a **ROD** strictly $<^*$ increasing sequence $\{x_\xi\}_{\xi<\omega_1^{\mathrm{L}[x]}}$ of length $\omega_1^{\mathrm{L}[x]} = \omega_1$. The sequence is not countably $\leqslant^*$ cofinal, of course, which contradicts to (ii) of Theorem 7.

C. Typically, properties of **ROD** sets consistent with **ZFC** tend to hold provably in **ZFC** for sets of low projective classes, say, for $\mathbf{\Sigma}_1^1$ sets. This turns out to be the case for Theorem 7 as well. We claim that.

if $\leqslant$ is a Borel PQO on a (Borel) set $D \subseteq \mathbb{N}^{\mathbb{N}}$, $X \subseteq D$ is a $\mathbf{\Sigma}_1^1$ set, and $\leqslant\restriction X$ is a linear quasi-order, then $\leqslant\restriction X$ is countably cofinal.

Indeed the set Y of all elements $y \in D$ $\leqslant$-comparable with *every* element $x \in X$ is a $\mathbf{\Pi}_1^1$ set, and $X \subseteq Y$ (as $\leqslant$ is linear on X). By Luzin's separability theorem, there is a Borel set Z such that $X \subseteq Z \subseteq Y$. The set U of all elements $z \in Z$ $\leqslant$-comparable with *every* element $y \in Y$ still is a $\mathbf{\Pi}_1^1$ set, and $X \subseteq U$ by the definition of Y. Once again, there is a Borel set W such that $X \subseteq W \subseteq U$. And by definition still $\leqslant$ is linear on W. It follows that W does not have increasing ω_1 sequences by Corollary 4, and hence neither does X.

We do not know whether the displayed claim *holds for all $\mathbf{\Pi}_1^1$ sets* X. We cannot go much higher though. Indeed, the axiom of constructibility (consistent with **ZFC**) implies the existence of a $<^*$ monotone ω_1 sequence of class Δ_1^2, where $<^*$ is the eventual domination order, see item B above.

D. We may note that Borel *linear* quasi-orderings themselves are countably cofinal in any case by Corollary 4. As for **ROD** LQOs in general (not necessarily linear **ROD** suborders of Borel PQOs, as in the theorem), one can define an uncountably-cofinal order of this class on a subset of $\mathbb{N}^{\mathbb{N}}$, and hence the theorem cannot be extended to **ROD** LQOs of subsets of Polish spaces. The following are two examples in the lowest possible definability classes Σ_1^1 and Π_1^1.

D1. Fix any recursive enumeration $\mathbb{Q} = \{q_k : k \in \mathbb{N}\}$ of the rationals. For any ordinal $\xi < \omega_1$, let $\mathbf{WO}_\xi$ be the set of all points $x \in \mathbb{N}^{\mathbb{N}}$ such that the set $Q_x = \{q_k : x(k) = 0\}$ is well-ordered, in the sense of the usual order of the rationals, and has the order type ξ. Let $\mathbf{WO} = \bigcup_{\xi<\omega_1} \mathbf{WO}_\xi$, all codes of ordinals. For $x, y \in \mathbf{WO}$ define $x \preccurlyeq y$ iff $x \in \mathbf{WO}_\xi$, $y \in \mathbf{WO}_\eta$, and $\xi \leqslant \eta$. Clearly $\preccurlyeq$ is a LQO of cofinality ω_1 on **WO**, and one can show that $\preccurlyeq$ is a lightface Π_1^1 relation, hence **OD**.

Strengthening this example, we let, for $x, y \in \mathbf{WO}$, $x \preccurlyeq' y$ iff either $x \in \mathbf{WO}_\xi$, $y \in \mathbf{WO}_\eta$, and $\xi < \eta$, or $x, y \in \mathbf{WO}_\xi$ for one and the same ξ and $x \leqslant_{\mathtt{lex}} y$ in the sense of the lexicographic linear order $\leqslant_{\mathtt{lex}}$ on $\mathbb{N}^{\mathbb{N}}$. Then $\preccurlyeq'$ is a true LO of cofinality ω_1 on **WO**, and still $\preccurlyeq'$ is Π_1^1 and **OD**.

D2. Modifying D1, we let, for any $\xi < \omega_1$, X_ξ be the set of all points $x \in \mathbb{N}^{\mathbb{N}}$ such that the maximal well-ordered initial segment of the set Q_x has the order type ξ. Thus $\mathbb{N}^{\mathbb{N}} = \bigcup_{\xi<\omega_1} X_\xi$. For $x, y \in \mathbb{N}^{\mathbb{N}}$ define $x \preccurlyeq y$ iff $x \in X_\xi$, $y \in X_\eta$, and $\xi \leqslant \eta$. Thus $\preccurlyeq$ is a LQO of cofinality ω_1, and now $\preccurlyeq$ belongs to Σ_1^1.

The same strengthening as in D1, that is, $x \preccurlyeq' y$ iff either $x \in X_\xi$, $y \in X_\eta$, and $\xi < \eta$, or $x, y \in X_\xi$ for one and the same ξ and $x \leqslant_{\mathtt{lex}} y$ lexicographically, yields a true LO of cofinality ω_1 on $\mathbb{N}^{\mathbb{N}}$, but it cannot be even boldface $\mathbf{\Sigma}_1^1$. (Indeed, any $\mathbf{\Sigma}_1^1$ LO $<$ on $\mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Pi}_1^1$ because $x < y$ is equivalent to $x \neq y \wedge y \nless x$, hence, it is Borel and countably cofinal by Corollary 4.)

4. Applications to the pantachy existence problem

Here we explain how Theorem 7 leads to the negative answers to questions 1 and 2 in the end of Section 1.

Definition 8. Let a *DBR-order* (from Du Bois Reymond) be any PQO $\langle X; \leqslant\rangle$ such that, first, $\langle X; \leqslant\rangle$ is a Borel in the sense of Definition 3, and second, for any countable set $Y \subseteq X$ there is an element $x \in X$ such that $y < x$ (that is, $y \leqslant x$ but $x \nleqslant y$) for all $y \in Y$. □

Corollary 9 (*Of Theorem 7*). *First, it is consistent with* **ZFC** *that no DBR-order contains a* **ROD** *pantachy. Second, it is consistent with* **ZF + DC** *that no DBR-order contains a pantachy of any kind.*

Proof. It suffices to note that a pantachy in a DBR-order cannot be countably cofinal because of the second requirement in Definition 8. □

There exist many notable orders of this type with the domains X being Borel sets in the Polish space $\mathbb{R}^{\mathbb{N}}$ of all infinite real sequences, see, e.g. [13]. For instance let $X = \mathbb{N}^{\mathbb{N}}$ (sequences of natural numbers). For $x,\ y \in \mathbb{N}^{\mathbb{N}}$ we define $x \leqslant_{RG} y$ (essentially the same rate of growth order as in Section 1) iff the limit $\mathtt{lim}_{\,n\to\infty} \frac{y(n)}{x(n)}$ exists and is >0 (including the limit value $+\infty$).

Lemma 10. $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{RG}\rangle$ *is a DBR-order. Therefore, by Corollary 9, it is consistent with* **ZF + DC** *that there is no pantachy in the structure* $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{RG}\rangle$, *and it is consistent with* **ZFC** *that there is no* **ROD** *pantachy in* $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{RG}\rangle$.

Thus *questions 1 and 2 in the end of Section 1 answer in the negative* for the ordering $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{RG}\rangle$, and hence for the ordering $\langle (\mathbb{R}^{+})^{\mathbb{N}}; \leqslant_{RG}\rangle$, in which $\mathbb{N}^{\mathbb{N}}$ is a cofinal subset. (See a note after Definition 6.)

Proof. The borelness of both the domain $X = \mathbb{N}^{\mathbb{N}}$ and the order $\leqslant_{RG}$ in the Polish space $\mathbb{R}^{\mathbb{N}}$ is rather clear. To check the second requirement in Definition 8, suppose that $x_0, x_1, x_2, \ldots \in \mathbb{N}^{\mathbb{N}}$. Put $x(k) = k\,\mathtt{max}_{n\leqslant k} x_n(k)$ for every k. Then $x_n <_{RG} x$ for all n. □

The actual Du Bois Reymond's domain $\mathscr{F}^{+}$ of all real positive functions is **not** a set in any Polish space since its cardinality $2^{2^{\aleph_0}} > 2^{\aleph_0}$ is too big. Thus formally $\langle \mathscr{F}^{+}; \leqslant_{RG}\rangle$ is **not** a DBR-order. Nevertheless the result of Lemma 10 easily extends to $\langle \mathscr{F}^{+}; \leqslant_{RG}\rangle$.

Indeed if $P \subseteq \mathscr{F}^{+}$ is a pantachy in $\langle \mathscr{F}^{+}; \leqslant_{RG}\rangle$ then $P{\restriction}\mathbb{N} = \{f{\restriction}\mathbb{N} : f \in P\}$ is a pantachy in the structure $\langle (\mathbb{R}^{+})^{\mathbb{N}}; \leqslant_{RG}\rangle$, and if P is **ROD** then so is $P{\restriction}\mathbb{N}$. (If f is a real function then $f{\restriction}\mathbb{N}$ is the infinite sequence of values $f(n),\ n \in \mathbb{N}$.) Therefore any pantachy-nonexistence result for $\langle (\mathbb{R}^{+})^{\mathbb{N}}; \leqslant_{RG}\rangle$ implies a corresponding pantachy-nonexistence result for $\langle \mathscr{F}^{+}; \leqslant_{RG}\rangle$. We conclude that *questions 1 and 2 in the end of Section 1 answer in the negative* for Du Bois Reymond's ordered domain $\langle \mathscr{F}^{+}; \leqslant_{RG}\rangle$ as well.

5. The Solovay model

The proof of Theorem 7 involves *the Solovay model*, a model of set theory introduced in [15] and applied for various purposes in many other papers. Basically, there are **two** Solovay models, that is,

(I) a model of **ZFC** in which all **ROD** sets of reals have some basic regularity properties, for instance, are Lebesgue measurable, have the Baire property, and the perfect subset property;[5]
(II) a model of **ZF + DC** in which all sets of reals are Lebesgue measurable, have the Baire property, and the perfect subset property.

The models are defined in the assumption that the sentence **WIC** ("there exists a weakly inaccessible cardinal") is consistent with **ZFC**. They are connected as follows:

Proposition 11 (*Solovay [15]*). *The Solovay model (II) is equal to the class* **HROD** *of all hereditarily* **ROD** *sets in the Solovay model (I). Both models have the same reals and ordinals.* □

By definition a set x is *hereditarily* **ROD** if x itself is **ROD**, all elements of x are **ROD**, all elements of x are **ROD**, *et cetera*. **HROD** is a transitive class containing all reals and all points of $\mathbb{N}^{\mathbb{N}}$.

We are not going to take much space for description of the construction of the Solovay model, since it can be found in detail in [15,16] and elsewhere. Moreover, our applications of the model are based on the following two results. The proofs of both of them are long and complicated, and involve a wide spectrum of methods of modern set theory.

Proposition 12 (*Stern [16]*). *It is true in the Solovay model (I) that if $\rho < \omega_1$ then there is no* **ROD** *ω_1 sequence of pairwise different sets in the class* $\mathbf{\Sigma}^{\rho}_{0}$. □

See [14] in matters of Borel classes $\mathbf{\Sigma}^{\rho}_{0}$ and Borel hierarchy in general.

The next claim was established in [12]. It can be viewed as a Solovay's model version of the first sentence of the Proof of Corollary 4.

Proposition 13. *It is true in the Solovay model (I) that if $\leqslant$ is a* **ROD** *LQO on a* **ROD** *set $D \subseteq \mathbb{N}^{\mathbb{N}}$ then there exist an antichain $A \subseteq 2^{<\omega_1}$ and a* **ROD** *map $\vartheta : D \overset{\text{onto}}{\to} A$ such that $x \leqslant y \Leftrightarrow \vartheta(x) \leqslant_{\mathtt{lex}} \vartheta(y)$ for all $x, y \in D$.* □

[5] A set X has *the perfect subset property* iff either X is at most countable or X contains a perfect subset.

See some basic definitions in Section 2. A set $A \subseteq 2^{<\omega_1}$ is an *antichain* if $s \not\subset t$ holds for every pair of $s \neq t$ in A. It is clear that the lexicographic order $\leqslant_{\mathtt{lex}}$ linearly orders any antichain $A \subseteq 2^{<\omega_1}$.

Using Propositions 12 and 13, we'll prove the following result below:

Proposition 14. *Sentence (i) of Theorem 7 is true in the Solovay model (I). Therefore sentence (ii) of Theorem 7 is true in the Solovay model (II).*

The "therefore" claim here is a consequence of the first claim by Proposition 11. On the other hand Proposition 14 implies Theorem 7 since a sentence true in a model is consistent.

6. The proof

Here we prove Proposition 14. *We argue in the Solovay model (I).*

According to (i) of Theorem 7, suppose that:

- $\leqslant$ is a Borel PQO on a Borel set $D \subseteq \mathbb{N}^{\mathbb{N}}$, while $\equiv$ and $<$ are resp. the associated equivalence relation and the associated strict order, and in addition $X \subseteq D$ is a **ROD** set, and $\leqslant \restriction X$ is a LQO.

Our goal will be to show that X is countably $\leqslant$ cofinal.

The restricted order $\leqslant \restriction X$ is **ROD**, of course, and hence, by Proposition 13, there is a **ROD** map $\vartheta : X \overset{\text{onto}}{\to} A$ onto an antichain $A \subseteq 2^{<\omega_1}$ (also obviously a **ROD** set) such that $x \leqslant y \Leftrightarrow \vartheta(x) \leqslant_{\mathtt{lex}} \vartheta(y)$ for all $x, y \in X$.

If $\xi < \omega_1$ then let $A_\xi = A \cap 2^\xi$ and $X_\xi = \{x \in X : \vartheta(x) \in A_\xi\}$.

Case 1: There is an ordinal $\eta < \omega_1$ such that the set A_η is $\leqslant_{\mathtt{lex}}$-cofinal in A. However, by Lemma 2, there is a set $A' \subseteq A_\eta$, countable and $\leqslant_{\mathtt{lex}}$-cofinal in A_η, and hence $\leqslant_{\mathtt{lex}}$-cofinal in A by the choice of η. If $s \in A'$ then pick an element $x_s \in X$ such that $\vartheta(x_s) = s$. Then $Y = \{x_s : s \in A'\}$ is a countable subset of X, $\leqslant$ cofinal in X. This ends the proof of (i) of Theorem 7.

Case 2: Not Case 1. That is, for any $\eta < \omega_1$ there is an ordinal $\xi < \omega_1$ and an element $s \in A_\xi$ such that $\eta < \xi$ and $t <_{\mathtt{lex}} s$ for all $t \in A_\eta$. Let

$$D_\xi = \{z \in D : \exists x \in X_\xi (z \leqslant x)\}$$

for each $\xi < \omega_1$, thus $X_\xi \subseteq D_\xi$. The sequence of sets D_ξ is **ROD**, of course.

We are going to get a contradiction. The first step is the following lemma.

Lemma 15. *The sequence $\{D_\xi\}_{\xi<\omega_1}$ has uncountably many different terms.*

Proof. As the sequence is $\subseteq$ increasing by obvious reasons, it suffices to prove that for any $\eta < \omega_1$ there exists an ordinal ξ, $\eta < \xi < \omega_1$, such that $D_\eta \subsetneqq D_\xi$ strictly. So let $\eta < \omega_1$. Then (see above) there exist: an ordinal ξ, $\eta < \xi < \omega_1$ and an element $s \in A_\xi$ such that $t <_{\mathtt{lex}} s$ for all $t \in A_\eta$. Take an element $z \in X_\xi$ such that $\vartheta(z) = s$. It remains to prove that $x \notin D_\eta$.

Indeed otherwise we have $z \leqslant x$ for some $x \in X_\eta$. By definition $t = \vartheta(x) \in A_\eta$, therefore $t <_{\mathtt{lex}} s$ by the choice of s. But on the other hand $s = \vartheta(z) \leqslant_{\mathtt{lex}} \vartheta(x) = t$ by the choice of ϑ, and this is a contradiction. □

Recall that $\leqslant$ is a Borel relation, hence there is an ordinal $1 \leqslant \rho < \omega_1$ such that $\leqslant$ (as a set of pairs) belongs to the Borel class $\mathbf{\Sigma}^\rho_0$.

Lemma 16. *If $\xi < \omega_1$ then the set D_ξ belong to $\mathbf{\Sigma}^\rho_0$.*

Proof. By Lemma 2 there exists a countable set $A' = \{s_n : n < \omega\} \subseteq A_\xi$, $\leqslant_{\mathtt{lex}}$-cofinal in A_ξ. If $n < \omega$ then pick an element $x_n \in X_\xi$ such that $\vartheta(x_n) = s_n$. Then by the choice of ϑ any element $x \in X$ with $\vartheta(x) = s_n$ satisfies $x \equiv x_n$, where $\equiv$ is the equivalence relation on D associated with $\leqslant$. It follows that

$$D_\xi = \bigcup_n Z_n, \quad \text{where} \quad Z_n = \{z \in D : z \leqslant x_n\},$$

so each Z_n is a $\mathbf{\Sigma}^\rho_0$ set together with $\leqslant$. We conclude that D_ξ is a $\mathbf{\Sigma}^\rho_0$ set as a countable union of sets in $\mathbf{\Sigma}^\rho_0$. □

The two lemmas contradict to Proposition 12, and the contradiction accomplishes the Proof of Proposition 14.

□ (Proposition 14 and Theorem 7)

7. Open problems

One may ask if linear **ROD** subsets X of Borel PQOs have some other special properties in the Solovay model besides the countable cofinality. For instance *is it true that such a set X has no monotone ω_1 sequences?*.

It will be interesting to figure out whether the results, of the type considered in this note but for quotient structures of the form $\mathbb{N}^{\mathbb{N}}/I$, where I is an ideal over $\mathbb{N}$, depend on the choice of I. (See Farah [3] for a comprehensive survey of various aspects of quotient structures, and [4,17] on interesting examples.)

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