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Grossone approach to Hutton and Euler transforms

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ABSTRACT

The aim of this paper is to demonstrate that several non–rigorous methods of mathematical reasoning in the field of divergent series, mostly related to the Euler and Hutton transforms, may be developed in a correct and consistent way by methods of the grossone analysis.

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1. Introduction

We begin with the following sequence of symbolic transformations with the shift operator, related to summability of divergent series.

Shift operator. An arbitrary series

$$A = a_0 + a_1 + a_2 + a_3 + \cdots \tag{1}$$

can be rewritten as

$$A = a_0 + \tau a_0 + \tau^2 a_0 + \tau^3 a_0 + \cdots = (1 + \tau + \tau^2 + \tau^3 + \cdots)\, a_0, \tag{2}$$

where τ is the *shift operator*, an operator of yet indefinite mathematical nature, but acting so that $\tau a_k = a_{k+1}$, — and then as

$$A \;=\; \frac{1}{1-\tau}\, a_0, \tag{3}$$

summing up $1 + \tau + \tau^2 + \tau^3 + \cdots$ according to the informal equality

$$1 + \tau + \tau^2 + \tau^3 + \cdots \;=\; \frac{1}{1-\tau}. \tag{4}$$

Hutton transform. Now, let $d \neq -1$ and $\sigma[d] = d + \tau$. Formally,

$$\left.\begin{aligned} \tfrac{1}{1-\tau} \;&=\; \tfrac{1-\tau+d+\tau}{(1+d)(1-\tau)} = \tfrac{1}{1+d} + \tfrac{1}{1-\tau}\,\tfrac{\sigma[d]}{1+d} &&= \\ &=\; \tfrac{1}{1+d} + (1 + \tau + \tau^2 + \cdots)\,\tfrac{\sigma[d]}{1+d} &&= \\ &=\; \tfrac{1}{1+d} + \tfrac{d+\tau}{1+d} + \tfrac{d\tau+\tau^2}{1+d} + \tfrac{d\tau^2+\tau^3}{1+d} + \cdots && \end{aligned}\right\} \tag{5}$$

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and we get *the Hutton transform* (H,d) (Hardy [1] for $d=1$) of the original series,

$$A=\frac{a_0}{1+d}+\frac{da_0+a_1}{1+d}+\frac{da_1+a_2}{1+d}+\frac{da_2+a_3}{1+d}+\cdots. \tag{6}$$

Iterated Hutton, or Euler–Jakimovski transform. Let $\{d_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers $d_n\neq-1$. Applying transformations $(H,d_1),\ (H,d_2),\ (H,d_3),\ldots$ – as in (5) and (6) – consecutively, so that the first term of every intermediate series is separated after each iteration, we obtain the series of separated (frameboxed) terms as the final result:

$$\left.\begin{aligned}
\frac{1}{1-\tau} &= \boxed{\frac{1}{1+d_1}}+\left(\frac{d_1+\tau}{1+d_1}\,\frac{1}{1-\tau}\right)=\\
&=\frac{1}{1+d_1}+\frac{d_1+\tau}{1+d_1}\left(\frac{1}{1+d_2}+\frac{d_2+\tau}{1+d_2}\,\frac{1}{1-\tau}\right)=\\
&=\boxed{\frac{1}{1+d_1}+\frac{d_1+\tau}{(1+d_1)(1+d_2)}}+\left(\frac{(d_1+\tau)(d_2+\tau)}{(1+d_1)(1+d_2)}\,\frac{1}{1-\tau}\right)=\\
&=\boxed{\frac{1}{1+d_1}+\frac{d_1+\tau}{(1+d_1)(1+d_2)}+\frac{(d_1+\tau)(d_2+\tau)}{(1+d_1)(1+d_2)(1+d_3)}}+\\
&\qquad\qquad+\left(\frac{(d_1+\tau)(d_2+\tau)(d_3+\tau)}{(1+d_1)(1+d_2)(1+d_3)}\,\frac{1}{1-\tau}\right)=\\
&\ \ \ldots\qquad\ldots\qquad\ldots\\
&=\sum_{k=0}^{\infty}\frac{(d_1+\tau)(d_2+\tau)\ldots(d_k+\tau)}{(1+d_1)(1+d_2)\ldots(1+d_k)(1+d_{k+1})}.
\end{aligned}\right\} \tag{7}$$

Remark 1. The final series is a formal Newton's interpolation of the function $\frac{1}{1-\tau}$ with the nodes $-d_1,-d_2,-d_3,\ldots$.

We conclude that, in the spirit of (3),

$$A=\frac{1}{1-\tau}\,a_0\ =\ \sum_{k=0}^{\infty}\frac{(d_1+\tau)(d_2+\tau)\ldots(d_k+\tau)}{(1+d_1)(1+d_2)\ldots(1+d_k)(1+d_{k+1})}\,a_0, \tag{8}$$

where each polynomial $P_k(\tau)=(d_1+\tau)(d_2+\tau)\ldots(d_k+\tau)$ formally acts on a_0 in accordance with the basic equalities $\tau^n a_0=a_n$.

Remark 2. Transformation (7) and (8) was explicitly introduced by Jakimovski [2] (as $[F,d_n]$) based on a series of earlier studies. Yet most notably, the whole idea of iterated transformation with separation of first terms of intermediate series belongs to Leonhard Euler, *Institutiones Calculi Differentialis*, Part II, Section 10 – see a discussion in Hardy [1, Section 2.6]. This is why we call it the *Euler–Jakimovski transformation* here. The summability method based on the Euler–Jakimovski transformation works, pending appropriate choice of d_n, for rapidly divergent oscillating series like $0!-1!+2!-3!+\cdots$. See [3] for further references. □

2. Regression: some linear transformations

The transformations considered above can be represented by the following infinite matrices:

$$H(d)=\frac{1}{1+d}\begin{pmatrix}1&0&0&0&\ldots\\ d&1&0&0&\ldots\\ 0&d&1&0&\ldots\\ 0&0&d&1&\ldots\\ \ldots&\ldots&\ldots&\ldots&\ldots\end{pmatrix},\quad S(d)\ =\begin{pmatrix}d&1&0&0&\ldots\\ 0&d&1&0&\ldots\\ 0&0&d&1&\ldots\\ 0&0&0&d&\ldots\\ \ldots&\ldots&\ldots&\ldots&\ldots\end{pmatrix},$$

$$E(\{d_n\}_{n=1}^{\infty})\ =\begin{pmatrix}\frac{1}{D_0}&0&0&0&0&\ldots\\ \frac{d_1}{D_1}&\frac{1}{D_1}&0&0&0&\ldots\\ \frac{d_1d_2}{D_2}&\frac{d_1+d_2}{D_2}&\frac{1}{D_2}&0&0&\ldots\\ \frac{d_1d_2d_3}{D_3}&\frac{d_1d_2+d_1d_3+d_2d_3}{D_3}&\frac{d_1+d_2+d_3}{D_3}&\frac{1}{D_3}&0&\ldots\\ \ldots&\ldots&\ldots&\ldots&\ldots&\ldots\end{pmatrix},$$

where $D_k=(1+d_1)(1+d_2)\ldots(1+d_{k+1})$ – so that

$$[E]_k\ =\frac{1}{1+d_{k+1}}\left[\frac{S(d_1)}{1+d_1}\cdot\frac{S(d_2)}{1+d_2}\cdot\ldots\cdot\frac{S(d_k)}{1+d_k}\right]_0,\quad k=0,1,2,\ldots, \tag{9}$$

where $[M]_k,\ k=0,1,2,\ldots$, is the kth row of any matrix M.

That is, let $\vec{a} = \langle a_0, a_1, a_2, \dots \rangle$, $\vec{h} = \left\langle \frac{a_0}{1+d}, \frac{da_0+a_1}{1+d}, \frac{da_1+a_2}{1+d}, \dots \right\rangle$, and $\vec{e} = \left\langle \frac{a_0}{1+d_1}, \frac{d_1 a_0 + a_1}{(1+d_1)(1+d_2)}, \dots \right\rangle$ be the infinite vectors representing the sums in the right-hand parts of equalities resp. (1), (6) and (8). Then formally

$$\vec{h} = H(d) \cdot \vec{a} \quad \text{and} \quad \vec{e} = E(\{d_n\}_{n=1}^{\infty}) \cdot \vec{a}. \tag{10}$$

3. The grossone approach

The transformations above and according equalities (2)–(8) obviously do not look rigorous in any way, in particular, since the nature of the "operator" $\boldsymbol{\tau}$ is not clear. Meanwhile, methods related to hyperreal fields (see [5,6] on modern trends in this direction) were applied in [4] to give a precise meaning to transformations (2)–(6). The goal of this paper is to extend this study to more complicated equalities (7) and (8) of the Euler–Jakimovski transformation, by methods of the grossone analysis. The reader is recommended to follow this paper in connection with [4].

Remark 3. Whilst the foundations of the "grossone" paradigm of Sergeyev [8] remain work in progress [7,9,10], we will stick to a version introduced in [11, Section 1.2]. It includes some key features of the paradigm, especially suited for dealing with infinite sequences and series. It sees **the grossone unit** ① is an infinite integer, divisible by any finite integer. $\square$

We replace series (1) by a sum of the form

$$\mathsf{A} \;=\; \mathsf{a}_0 + \mathsf{a}_1 + \mathsf{a}_2 + \cdots + \mathsf{a}_{①}. \tag{11}$$

Sums of this type, that is, with ① or another grossone-based infinite quantity, were earlier considered in [11].

Definition 4 (*shadows, Definition 1 in [4]*). A standard series $a_0 + a_1 + a_2 + \cdots$ is called the *shadow* of a hyperfinite sum $\mathsf{a}_0 + \mathsf{a}_1 + \cdots + \mathsf{a}_{①}$ iff $a_k \approx \mathsf{a}_k$ for all standard k, where

$$x \approx y \quad \text{iff} \quad |x - y| \text{ is infinitesimal.}$$

A standard infinite matrix X with terms x_{kl} is called the *shadow* of a hyperreal $(① + 1) \times (① + 1)$ matrix Y iff $x_{kl} \approx y_{kl}$ for all standard k, l. $\square$

In this paper, we will introduce certain linear transformations of infinite sums like (11) by matrices of dimension $(① + 1) \times (① + 1)$, which correspond to the Euler–Jakimovski transformation, and whose action does not change the sum value. We also show that, provided some conditions are satisfied, shadows of transformed infinite sums are equal to the results of standard transformations of their shadows.

It must be said that the Hutton and Jakimovski summation methods belong to a wide variety of linear summation methods (see [1]). Our choice of these methods to be considered in this paper is based not on their special position in this variety (on the contrary, the Hutton summability method is dominated by some other methods, see, e.g. [1], comments to Chapter 1), but rather because of their transparent connection with the shift operator, exploited in Section 1. It is a challenging problem to consider some other summability methods on the base of the same methodology, of course.

4. The shift matrix and the Hutton ①-transform

In this section, we present some definitions and results of [4], mainly related to the Hutton transform, which are necessary for understanding of the main content of the paper. We are going to consider (11) in the form

$$\mathsf{A} \;=\; \mathsf{a}_0 + \mathsf{a}_1 1^{(1)} + \mathsf{a}_2 1^{(2)} + \cdots + \mathsf{a}_{①} 1^{(①)}, \tag{12}$$

where

$$\xi^{(k)} \;=\; (\xi - x_0)(\xi - x_1)\dots(\xi - x_{k-1}), \quad \text{and separately, } \xi^{(0)} = 1, \tag{13}$$

which is implied by *Newton's interpolation theorem* with nodes $x_0, x_1, \dots, x_{①}$ (Proposition 2 in [4], called the nonstandard Taylor expansion there.)

Blanket Assumption 5. It will be assumed that:

1^* : an internal sequence of hyperreals x_k, $0 \leqslant k \leqslant ①$, is fixed;
2^* : $\xi^{(k)} \;=\; (\xi - x_0)(\xi - x_1)\dots(\xi - x_{k-1})$, in accordance with (13);
3^* : hyperreals $x_0, \dots, x_{①}$ are pairwise different and $x_{①} = 1$. $\square$

Working in these assumptions and making use of the matrices

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \mathsf{U} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

of dimension $(① + 1) \times (① + 1)$, and $(① + 1)$-vectors

$$\vec{\mathsf{a}}_k = \langle \mathsf{a}_k, \mathsf{a}_{k+1}, \dots, \mathsf{a}_①, \underbrace{0, 0, \dots, 0}_{k\,zeros} \rangle, \quad k = 0, 1, 2, \dots, ①, \tag{14}$$

we demonstrated in [4] that

$$[\mathbf{T}^k]_0 \, \vec{\mathsf{a}}_0 = [\mathbf{T}^k \vec{\mathsf{a}}_0]_0 = [\vec{\mathsf{a}}_k]_0 = \mathsf{a}_k. \tag{15}$$

(Recall that $[M]_n$ denotes the nth row of a matrix M and rows are enumerated from top to bottom starting with 0 as the index of the top row.) Then equality (12) takes the form

$$\mathbf{A} = \left[\mathbf{U} + \mathbf{T}\, 1^{(1)} + \mathbf{T}^2\, 1^{(2)} + \dots + \mathbf{T}^{①}\, 1^{(①)} \right]_0 \vec{\mathbf{a}}_0, \tag{16}$$

considered as the ①-counterpart of (2). We proved in [4] that moreover

$$\left[\mathbf{U} + \mathbf{T}\, 1^{(1)} + \mathbf{T}^2\, 1^{(2)} + \dots + \mathbf{T}^{①}\, 1^{(①)} \right]_0 = \left[\frac{1}{\mathbf{1} - \mathbf{T}} \right]_0, \tag{17}$$

where

$$\mathbf{1} = \mathbf{U} + \mathbf{XT} = \begin{pmatrix} 1 & x_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & x_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & x_2 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & x_{①-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

(the modified unit matrix of dimension $(① + 1) \times (① + 1)$, and

$$\mathbf{X} = \begin{pmatrix} x_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & x_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & x_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & x_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & x_① \end{pmatrix} \text{ of dimension } (① + 1) \times (① + 1).$$

so that the equality (17) is the ①-counterpart of (3). But (17) has the precise meaning of a sum and a complete proof which does not depend on any special assumption about the nature of the initial series (12).

Now we multiply both sides of (17) with $\vec{\mathsf{a}}_0$ and, using (15) and (16), obtain the following counterpart of (4):

$$\mathsf{a}_0 + \mathsf{a}_1\, 1^{(1)} + \mathsf{a}_2\, 1^{(2)} + \dots + \mathsf{a}_①\, 1^{(①)} = \mathbf{A} = \left[\frac{1}{\mathbf{1} - \mathbf{T}} \right]_0 \vec{\mathsf{a}}_0. \tag{18}$$

Thus the analogy between (1)–(4) on the one hand and (12), (16), (18), (17) on the other, is based on the identification of $\boldsymbol{\tau}$ with $\mathbf{T}$, 1 (in the denominator $1 - \boldsymbol{\tau}$) with $\mathbf{1}$, and on the adjoining of factors $1^{(k)}$ and taking the 0th row. Following this line, we developed the Hutton transform from this standpoint in [4]. Namely, given a parameter $d \neq -1$, we defined the *Hutton ①-matrix*

$$\mathbf{H}(d) = \frac{d\mathbf{T}^* + d\mathbf{X} + \mathbf{U}}{1 + d} = \frac{1}{1 + d} \begin{pmatrix} 1 + dx_0 & 0 & \dots & 0 & 0 \\ d & 1 + dx_1 & \dots & 0 & 0 \\ 0 & d & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d & 1 + dx_① \end{pmatrix}$$

of dimension $(① + 1) \times (① + 1)$, where

$$\mathbf{T}^* = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad \text{of dimension } (①+1)\times(①+1)$$

is the transpose of $\mathbf{T}$. Then, for any $(①+1)$-vector $\vec{a} = \langle a_0, a_1, \dots, a_① \rangle$, the product $\vec{h} = \mathbf{H}(d)\cdot\vec{a}$ is still a $(①+1)$-vector, defined by

$$h_0 = \frac{1+dx_0}{1+d}a_0, \quad \text{and} \quad h_k = \frac{da_{k-1} + (1+dx_k)a_k}{1+d} \quad \text{for} \quad 1 \leqslant k \leqslant ①.$$

Thus the action of $\mathbf{H}(d)$ on $\vec{a}$ converts the hyperfinite sum (12) to

$$\left.\begin{aligned} \sum_{k=0}^{①} h_k 1^{(k)} &= \tfrac{1+dx_0}{1+d}a_0 + \sum_{k=1}^{①} \tfrac{da_{k-1}+(1+dx_k)a_k}{1+d} 1^{(k)} = \\ &= \sum_{k=0}^{①} [\mathbf{H}(d)]_k \cdot 1^{(k)} \cdot \vec{a}. \end{aligned}\right\} \tag{19}$$

Theorem 6 (*proved in [4]*). *Assume that the series in (1) is the shadow of the sum in (12), and $x_k \approx 0$ for all standard k. Then the standard matrix $H(d)$ of Section 2 is the shadow of $\mathbf{H}(d)$ and the series in (6) is the shadow of the sum in the 2nd line of (19). Moreover, if $\vec{h} = \mathbf{H}(d)\cdot\vec{a}$ as above, then $\sum_{k=0}^{①} h_k 1^{(k)} = \sum_{k=0}^{①} a_k 1^{(k)}$, so that the Hutton ①-transform does not change the value of sums of the form (12).*

Proof (*sketch*). The first claim is rather obvious. To check the moreover claim note that the coefficient of each a_k in the right–hand side of the first line of (19) is, indeed, equal to $1^{(k)}$, in accordance with (12). □

5. A technical theorem

Here we prove a theorem (Theorem 7) which generalizes an earlier result in [4, 5.2]. We will need this result in Section 7. We first define the matrix

$$\mathbf{S}(d) = d\cdot\mathbf{1} + \mathbf{T} = \begin{pmatrix} d & 1+dx_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & d & 1+dx_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & d & 1+dx_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & d & 1+dx_{①-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & d \end{pmatrix}$$

of dimension $(①+1)\times(①+1)$. Recall that $\mathbf{T}^*$ is the transpose of $\mathbf{T}$. Put

$${}^k\mathbf{X} = \mathbf{T}^k\,\mathbf{X}\,\mathbf{T}^{*k} = \begin{pmatrix} x_k & 0 & 0 & 0 & \dots \\ 0 & x_{k+1} & 0 & 0 & \dots \\ 0 & 0 & x_{k+2} & 0 & \dots \\ 0 & 0 & 0 & x_{k+3} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

a diagonal matrix of dimension $(①+1)\times(①+1)$, with $x_k, x_{k+1}, \dots, x_①, [0]\,0, [0]\dots, 0$ (k zeros) on the diagonal. Then we define, for $k = 0, 1, \dots, ①$,

$${}^k\mathbf{1} = \mathbf{U} + {}^k\mathbf{X}\,\mathbf{T}, \quad {}^k\mathbf{S}(d) = d\cdot{}^k\mathbf{1} + \mathbf{T}, \quad \text{and} \quad {}^k\mathbf{F}(x) = \left({}^k\mathbf{1} - x\mathbf{T}\right)^{-1}.$$

In particular, ${}^0\mathbf{X} = \mathbf{X}$, ${}^0\mathbf{1} = \mathbf{1}$, ${}^0\mathbf{S}(d) = \mathbf{S}(d)$, and ${}^0\mathbf{F}(x) = (\mathbf{1} - x\mathbf{T})^{-1}$. Moreover, in the assumptions of Blanket Agreement 5, if $x_n \approx 0$ for all standard n, and d, k are limited, then the shadow of ${}^k\mathbf{S}(d)$ is equal to the matrix $S(d)$ in Section 2, the shadow of $\mathbf{X}$ is the infinite (in the usual sense) null matrix, and the shadow of ${}^k\mathbf{1}$ is the infinite unit matrix.

Theorem 7. *For any k, we have*

$$\left[\frac{1}{{}^k\mathbf{1} - \mathbf{T}}\right]_0 = \frac{1+dx_k}{1+d}[\mathbf{U}]_0 + \frac{1-x_0}{1+d}\left[{}^k\mathbf{S}(d)\cdot\frac{1}{{}^k\mathbf{1}-\mathbf{T}}\right]_0 = \frac{1+dx_k}{1+d}[\mathbf{U}]_0 + \frac{1-x_k}{1+d}\left[\frac{1}{{}^{k+1}\mathbf{1} - \mathbf{T}}\right]_0 {}^{k+1}\mathbf{S}(d).$$

Proof. We first observe that $[\mathbf{U}]_0 = \frac{1+dx_k}{1+d}\,[{}^k\mathbf{1} - \mathbf{T}]_0 + \frac{1-x_k}{1+d}\,[{}^k\mathbf{S}(d)]_0$. Multiplying both sides by $({}^k\mathbf{1} - \mathbf{T})^{-1}$, we obtain

$$\left[\frac{1}{{}^k\mathbf{1} - \mathbf{T}}\right]_0 = [\mathbf{U}]_0\,\frac{1}{{}^k\mathbf{1} - \mathbf{T}} = \frac{1+dx_0}{1+d}\,[\mathbf{U}]_0 + \frac{1-x_0}{1+d}\left[{}^k\mathbf{S}(d)\cdot\frac{1}{{}^k\mathbf{1} - \mathbf{T}}\right]_0,$$

which is the first equality of the theorem.

To establish the second equality, it is enough to show that

$$({}^{k+1}\mathbf{1} - \mathbf{T})\;{}^k\mathbf{S}(d) = {}^{k+1}\mathbf{S}(d)\;({}^k\mathbf{1} - \mathbf{T}),$$

or equivalently

$$({}^{k+1}\mathbf{1} - \mathbf{T})\;(d\cdot{}^k\mathbf{1} + \mathbf{T}) = (d\cdot{}^{k+1}\mathbf{1} + \mathbf{T})\;({}^k\mathbf{1} - \mathbf{T}),$$

or equivalently, after opening the brackets and simplifying,

$${}^{k+1}\mathbf{1}\,\mathbf{T} - d\cdot\mathbf{T}\,{}^k\mathbf{1} = \mathbf{T}\,{}^k\mathbf{1} - d\cdot{}^{k+1}\mathbf{1}\,\mathbf{T},$$

that is, $(1+d)\cdot{}^{k+1}\mathbf{1}\,\mathbf{T} = (1+d)\cdot\mathbf{T}\,{}^k\mathbf{1}$, or ${}^{k+1}\mathbf{1}\,\mathbf{T} = \mathbf{T}\,{}^k\mathbf{1}$, or

$$(\mathbf{U} + {}^{k+1}\mathbf{X}\,\mathbf{T})\,\mathbf{T} = \mathbf{T}\,(\mathbf{U} + {}^k\mathbf{X}\,\mathbf{T}),$$

or equivalently (as $\mathbf{U}\mathbf{T} = \mathbf{T}\mathbf{U}$): ${}^{k+1}\mathbf{X}\mathbf{T} = \mathbf{T}\,{}^k\mathbf{X}$. Yet by definition

$${}^{k+1}\mathbf{X}\mathbf{T} = \mathbf{T}\,{}^k\mathbf{X} = \begin{pmatrix} 0 & x_{k+1} & 0 & 0 & 0 & \dots \\ 0 & 0 & x_{k+2} & 0 & 0 & \dots \\ 0 & 0 & 0 & x_{k+3} & 0 & \dots \\ 0 & 0 & 0 & 0 & x_{k+4} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

as required. $\square$

6. Euler–Jakimovski ①-transform

In this section we concentrate on the ①-interpretation of the Euler–Jakimovski transform. The plan is roughly the same as in the previous section. We define a ①-version $\mathbf{E}$ of the Euler–Jakimovski matrix $E = E(\{d_n\})$ of Section 2, prove that the latter is the shadow of the former (under certain conditions) and that its action commutes with the shadow operation (this will require a bit more efforts than in the case of the Hutton transform), prove that $\mathbf{E}$ can be obtained by a certain simulation of Eqs. (7) and (8), and then show, in Section 7, that a suitable reiteration of the Hutton ①-transform of Section 4 leads to $\mathbf{E}$ as well.

Blanket Assumption 8. A sequence of hyperreals $x_k,\ k \leqslant ①$, satisfying conditions of Blanket Assumption 5, a hyperfinite sum (12), and the corresponding vector $\vec{a} = \vec{a}_0$ continue to be fixed. Also fix an internal sequence $\{d_n\}_{n=1}^{①}$ of hyperreals $d_n \neq -1$. $\square$

Definition 9 (*Euler–Jakimovski ①-transform*). We set

$${}^{(k)}\mathbf{S} = {}^k\mathbf{S}(d_k)\cdot{}^{k-1}\mathbf{S}(d_{k-1})\cdot\ \dots\ \cdot{}^1\mathbf{S}(d_1)$$

for all $k \leqslant ①$. Define the *Euler–Jakimovski matrix* $\mathbf{E}$ by the equalities

$$[\mathbf{E}]_k = \frac{1 + d_{k+1}x_k}{(1+d_1)(1+d_2)\dots(1+d_k)(1+d_{k+1})}\left[{}^{(k)}\mathbf{S}\right]_0 \tag{20}$$

— for all $k, 0 \leqslant k \leqslant ①$. (Compare with (9).) The *Euler–Jakimovski ①-transform* $\vec{e}$ of $\vec{a}$ ($\vec{e}, \vec{a}$ are $① + 1$-vectors) is defined by $\vec{e} = \mathbf{E}\,\vec{a}$. $\square$

We are able now to define the ①-version

$$\left.\begin{aligned} &\sum_{k=0}^{①} e_k\,1^{(k)}, \quad \text{where} \quad \langle e_0, \dots, e_{①}\rangle = \vec{e} = \mathbf{E}\,\vec{a} = \mathbf{E}\,\vec{a}_0, \\ &\text{so that} \quad e_k = \tfrac{(1+d_{k+1}x_k)\,\left[{}^{(k)}\mathbf{S}\right]_0}{(1+d_1)(1+d_2)\dots(1+d_{k+1})}\ \vec{a} \quad \text{for all } k \end{aligned}\right\} \tag{21}$$

of (8), similar to (19) in Section 4. The difference between (21) and (19) is that it is not evident that $\sum_{k=0}^{①} e_k\,1^{(k)} = \sum_{k=0}^{①} a_k\,1^{(k)}$ — compare with Theorem 6. We will address this question in the next section.To illustrate the definition, we present the first few values of e_k:

$$\begin{aligned}
\mathsf{e}_0 &= K_0\,\mathsf{a}_0;\\
\mathsf{e}_1 &= K_1\,(d_1\mathsf{a}_0 + (1+d_1x_1)\,\mathsf{a}_1);\\
\mathsf{e}_2 &= K_2\,(d_1d_2\,\mathsf{a}_0 + (d_1+d_2+d_1d_2(x_1+x_2))\,\mathsf{a}_1 + (1+d_1x_2)(1+d_2x_2)\,\mathsf{a}_2);\\
\mathsf{e}_3 &= K_3\,[d_1d_2d_3\,\mathsf{a}_0 + (d_1d_2+d_2d_3+d_1d_3+d_1d_2d_3(x_1+x_2+x_3))\,\mathsf{a}_1 + (d_1+d_2+d_3+(d_1d_2+d_2d_3+d_1d_3)(x_2+x_3)\\
&\quad + d_1d_2d_3(x_2^2+x_2x_3+x_3^2))\,\mathsf{a}_2 + (1+d_1x_3)(1+d_2x_3)(1+d_3x_3)\,\mathsf{a}_3];
\end{aligned}$$

and so on, where $K_n = \frac{1+d_{n+1}x_n}{(1+d_1)(1+d_2)\dots(1+d_{n+1})}$, so that in particular $K_0 = \frac{1+d_1x_0}{1+d_1}$, $K_1 = \frac{1+d_2x_1}{(1+d_1)(1+d_2)}$, $K_2 = \frac{1+d_3x_2}{(1+d_1)(1+d_2)(1+d_3)}$, et cetera.

Proposition 10. *Assume that the series in (1) is the shadow of the sum in (12), and $x_k \approx 0$ for limited k. Then the series in (8) is the shadow of the sum $\sum_{k=0}^{①} \mathsf{e}_k\,1^{(k)}$ in (21).*

Proof. Say that a matrix M is *almost triangular* iff there is a finite natural number n_0 such that $l \leqslant k + n_0$ whenever $M_{kl} \neq 0$. (M_{kl} is the $\langle k, l\rangle$th element of M.) Such a matrix can contain nonzero elements above the diagonal only in finitely many consecutive diagonals.

The matrices $\mathbf{T}$, $\mathbf{T}^*$, $\mathbf{U}$, $\mathbf{E}$, $\mathbf{H}$, and ${}^k\mathbf{X}$, ${}^k\mathbf{1}$, ${}^k\mathbf{S}(d_k)$, ${}^{(k)}\mathbf{S}$ for all *standard* indices k are almost triangular; the product of a standard number of almost triangular matrices is almost triangular as well.

It is also clear that if a standard double-infinite matrix Z is the shadow (see Definition 4) of an almost triangular $(①+1)\times(①+1)$ matrix $\mathbf{Z}$, and a standard infinite vector $\vec{v}$ is the shadow of a $(①+1)$-dimensional vector $\vec{\mathbf{v}}$, then $Z\,\vec{v}$ is the shadow of $\mathbf{Z}\,\vec{\mathbf{v}}$.

Note that, in the assumptions of Proposition 10, the shadow of every matrix ${}^k\mathbf{X}$ (k standard) is the zero matrix. Since the property of being the shadow is preserved under addition and multiplication (for almost triangular matrices), the matrix $S(d_k)$ introduced in Section 2 is the shadow of every matrix ${}^k\mathbf{S}(d_k)$, and accordingly the product $S(d_k)\cdot S(d_k)\cdot\ldots\cdot S(d_1)$ is the shadow of ${}^{(k)}\mathbf{S}$ (provided k is standard). Therefore $E = E(\{d_n\})$ is the shadow of $\mathbf{E}$. This completes the proof of Proposition 10. □

7. Euler–Jakimovski matrix via reiteration of Hutton transform

We are going to demonstrate here that the hyperfinite sum $\sum_{k=0}^{①} \mathsf{e}_k\,1^{(k)}$ in (21) can be obtained by reiteration of the Hutton transform of Section 4, similarly to Eqs. (7) and (8).

Theorem 11 (*under Blanket Assumption 8*). *We have*

$$\left[\frac{1}{\mathbf{1}-\mathbf{T}}\right]_0 = \sum_{k=0}^{①} \frac{1+d_{k+1}x_k}{(1+d_1)(1+d_2)\dots(1+d_{k+1})}\,\left[{}^{(k)}\mathbf{S}\right]_0\,1^{(k)}.$$

Proof. Let us apply the equality of Theorem 7 consecutively, starting with $\frac{1}{1-\mathbf{T}}$, for the indices $k = 0, 1, \dots, ① - 1$ and accordingly values $d = d_1, d_2, \dots, d_{①}$. The first two steps will be as follows:

$$\begin{aligned}
\left[\frac{1}{\mathbf{1}-\mathbf{T}}\right]_0 &= \frac{1+d_1x_0}{1+d_1}\,[\mathbf{U}]_0 + \frac{1-x_0}{1+d_1}\left[\frac{1}{{}^1\mathbf{1}-\mathbf{T}}\right]_0 {}^1\mathbf{S}(d_1)\\
&= \frac{1+d_1x_0}{1+d_1}\,[\mathbf{U}]_0 + \frac{1-x_0}{1+d_1}\left\{\frac{1+d_2x_1}{1+d_2}\,[\mathbf{U}]_0 + \frac{1-x_1}{1+d_2}\left[\frac{1}{{}^2\mathbf{1}-\mathbf{T}}\right]_0 {}^2\mathbf{S}(d_2)\right\}{}^1\mathbf{S}(d_1)\\
&= \frac{1+d_1x_0}{1+d_1}\,[\mathbf{U}]_0 + \frac{(1-x_0)(1+d_2x_1)}{(1+d_1)(1+d_2)}\,[\mathbf{U}]_0\,{}^1\mathbf{S}(d_1) + \frac{(1-x_0)(1-x_1)}{(1+d_1)(1+d_2)}\left[\frac{1}{{}^2\mathbf{1}-\mathbf{T}}\right]_0 {}^{(2)}\mathbf{S}.
\end{aligned}$$

After ① steps, the rightmost term turns out to be

$$N_{①+1}\left[\frac{1}{{}^{①+1}\mathbf{1}-\mathbf{T}}\right]_0\cdot{}^{(①+1)}\mathbf{S}\cdot 1^{(①+1)} = N_{①+1}\cdot[\mathbf{U}]_0\cdot{}^{(①+1)}\mathbf{S}\cdot 1^{(①+1)} = 0,$$

where

$$N_k = \frac{1}{(1+d_1)(1+d_2)\dots(1+d_k)}$$

and $1^{(k)} = (1-x_0)(1-x_1)\dots(1-x_{k-1})$ — so that $1^{(①+1)} = 0$ because $x_{①} = 1$. Therefore, we obtain the equality of the theorem after exactly the number ① of "Hutton" steps. □

Corollary 12. *In the notation of (12) of Section 4 and (21) of Section 6*,

$$\sum_{k=0}^{①} \mathsf{e}_k \, 1^{(k)} = \sum_{k=0}^{①} \frac{(1+d_{k+1}x_k)\ \left[{}^{(k)}\mathbf{S}\right]_0}{(1+d_1)(1+d_2)\dots(1+d_{k+1})} \cdot \vec{\mathsf{a}} = \sum_{k=0}^{①} \mathsf{a}_k \, 1^{(k)} = \mathsf{A}.$$

Proof. The left and right equalities hold by definition. To prove the middle one, recall that $\left[\frac{1}{1-\mathbf{T}}\right]_0 \cdot \vec{\mathsf{a}} = \sum_{k=0}^{①} \mathsf{a}_k \, 1^{(k)}$ by (18) in Section 4. It remains to multiply the equality of the theorem by $\vec{\mathsf{a}}$. $\square$

It follows that the Euler–Jakimovski ①-transform does not change the value of sums of the form (12).

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