Annals of Pure and Applied Logic 167 (2016) 262–283

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Counterexamples to countable-section Π^1_2 uniformization and Π^1_3 separation ☆

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ARTICLE INFO

Article history:
Received 27 March 2015
Accepted 11 November 2015
Available online 28 December 2015

MSC:
03E15
03E35
03E20
28A05

Keywords:
Uniformization
Separation
Jensen's forcing
Finite support

ABSTRACT

We make use of a finite support product of the Jensen minimal Π^1_2 singleton forcing to define a model in which Π^1_2 uniformization fails for a set with countable cross-sections. We also define appropriate submodels of the same model in which separation fails for Π^1_3.

1. Introduction

The uniformization problem, introduced by Luzin [17,18], is well known in modern set theory. (See Moschovakis [19], Kechris [16], Hauser and Schindler [6] for both older and more recent studies.) In particular, it is known that every $\mathbf{\Sigma}^1_2$ set can be uniformized by a set of the same class $\mathbf{\Sigma}^1_2$, but on the other hand, there is a $\mathbf{\Pi}^1_2$ set (in fact, a lightface Π^1_2 set), not uniformizable by any set in $\mathbf{\Pi}^1_2$. The negative part of this result cannot be strengthened much further in the direction of the absence of more complicated uniformizing sets since any $\mathbf{\Pi}^1_2$ set admits a $\mathbf{\Delta}^1_3$-uniformization assuming $\mathbf{V} = \mathbf{L}$ and admits a $\mathbf{\Pi}^1_3$-uniformization assuming the existence of sharps (the Martin–Solovay–Mansfield theorem, [19, 8H.10]).

☆ This document is a collaborative effort.

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1 Partial support of RFFI grant 13-01-00006 acknowledged.
2 Partial support of the Russian Science Foundation grant 14-50-00150 acknowledged.

http://dx.doi.org/10.1016/j.apal.2015.12.002

However, the mentioned $\mathbf{\Pi}^1_2$-non-uniformization theorem can be strengthened in the context of consistency. For instance, the Π^1_2 set

$$P = \{\langle x, y\rangle \,:\, x, y \in 2^\omega \wedge y \notin \mathbf{L}[x]\}$$

is not uniformizable by any ROD (real-ordinal definable) set in the Solovay model and many other models of **ZFC** in which it is not true that $\mathbf{V} = \mathbf{L}[x]$ for a real x, and then the cross-sections of P can be considered as "large", in particular, they are definitely uncountable. Therefore one may ask:

Question 1. Is it consistent that there is a ROD-non-uniformizable Π^1_2 set P such that all cross-sections $P_x = \{y \,:\, \langle x, y\rangle \in P\}$ are at most countable?

This question is obviously connected with another question, initiated and briefly discussed at the Mathoverflow exchange desk[3] and at FOM[4]:

Question 2. Is it consistent with **ZFC** that there is a *countable* definable set of reals $X \neq \varnothing$ which has no OD (ordinal definable) elements.

Ali Enayat (footnote 4) conjectured that Question 2 can be solved in the positive by the finite-support product $\mathbb{P}^{<\omega}$ of countably many copies of the Jensen "minimal Π^1_2 real singleton forcing" $\mathbb{P}$ defined in [9].[5] Enayat demonstrated in [2] that a symmetric part of the $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$, the constructible universe, definitely yields a model of **ZF** (not a model of **ZFC**!) in which there is a Dedekind-finite infinite OD set of reals with no OD elements.

Following the mentioned conjecture, we proved in [14] that indeed it is true in a $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$ that the set of $\mathbb{P}$-generic reals is a countable non-empty Π^1_2 set with no OD elements.[6] Using a finite-support product $\prod_{\xi<\omega_1} \mathbb{P}_\xi{}^{<\omega}$, where the forcing notions $\mathbb{P}_\xi$ are pairwise different clones of Jensen's forcing $\mathbb{P}$, we answer Question 1 in the positive.

Theorem 1.1. *In a suitable generic extension of* $\mathbf{L}$*, it is true that there is a lightface* Π^1_2 *set* $P \subseteq 2^\omega \times 2^\omega$ *whose all cross-sections* $P_x = \{y \,:\, \langle x, y\rangle \in P\}$ *are at most countable, but* P *is not uniformizable by a ROD set.*

Using an appropriate generic extension of a submodel of the same model, similar, to some extent, to models considered in Harrington's unpublished notes [5], we also prove

Theorem 1.2. *In a suitable generic extension of* $\mathbf{L}$*, it is true that there is a pair of disjoint lightface* Π^1_3 *sets* $X, Y \subseteq 2^\omega$*, not separable by disjoint* $\mathbf{\Sigma}^1_3$ *sets, and hence* $\mathbf{\Pi}^1_3$ *Separation and* Π^1_3 *Separation fail.*

This result was first proved by Harrington in [5] on the basis of almost disjoint forcing of Jensen–Solovay [10], and in this form has never been published, but was mentioned in [19, 5B.3] and [7, page 230]. A complicated alternative proof of Theorem 1.2 can be obtained with the help of *countable-support* products and iterations of Jensen's forcing studied earlier in [1,11,12]. The *finite-support* approach which we pursue here

[3] A question about ordinal definable real numbers. Mathoverflow, March 09, 2010. http://mathoverflow.net/questions/17608.

[4] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html.

[5] *Jensen's forcing* below, for the sake of brevity—on this forcing, see also 28A in [8].

[6] We also proved in [15] that the existence of a Π^1_2 E_0-class with no OD elements is consistent with **ZFC**, using a E_0-invariant version of the Jensen forcing. A related consistency result on countable Groszek–Laver pairs, established by similar methods, will appear in [3].

yields a significantly more compact proof, which still uses some basic constructions from [5]. As far as Theorem 1.1 is concerned, countable-support products and iterations hardly can lead to the countable-section non-uniformization results.

We recall that $\mathbf{\Pi}^1_3$ Separation *holds* in $\mathbf{L}$. Thus Theorem 1.2 in fact shows that the $\mathbf{\Pi}^1_3$ Separation principle is destroyed in an appropriate generic extension of $\mathbf{L}$. It would be interesting to find a generic extension in which, the other way around, the $\mathbf{\Sigma}^1_3$ Separation (false in $\mathbf{L}$) holds. This can be a difficult problem. At least, the model used to prove Theorem 1.2 does not help: we prove (Theorem 14.1 below) that any pair of disjoint $\mathbf{\Sigma}^1_3$ sets, non-separable by disjoint $\mathbf{\Pi}^1_3$ sets in $\mathbf{L}$, remains $\mathbf{\Sigma}^1_3$ and non-separable by disjoint $\mathbf{\Pi}^1_3$ sets in the extension.

2. Trees and splitting

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers $0, 1$. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\wedge i$ is the extension of t by i. If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s, while $s \subset t$ means proper extension. If $s \in 2^{<\omega}$ then $\mathtt{lh}(s)$ is the length of s, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length n).

A set $T \subseteq 2^{<\omega}$ is a *tree* iff for any strings $s \subset t$ in $2^{<\omega}$, if $t \in T$ then $s \in T$. Thus every non-empty tree $T \subseteq 2^{<\omega}$ contains the *empty string* Λ.

If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T{\restriction}_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let $\mathbf{PT}$ be the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$. Thus a non-empty tree $T \subseteq 2^{<\omega}$ belongs to $\mathbf{PT}$ iff it has no endpoints and no isolated branches. Then there is a largest string $s \in T$ such that $T = T{\restriction}_s$; it is denoted by $s = \mathtt{stem}(T)$ (the *stem* of T); we have $s^\wedge 1 \in T$ and $s^\wedge 0 \in T$ in this case.

Definition 2.1 *(Perfect sets).* If $T \in \mathbf{PT}$ then $[T] = \{a \in 2^\omega : \forall n\, (a{\restriction}n \in T)\}$ is the set of all *paths through* T, a perfect set in 2^ω. $\square$

The *simple splitting* of a tree $T \in \mathbf{PT}$ consists of smaller trees

$$T(\to 0) = T{\restriction}_{\mathtt{stem}(T)^\wedge 0} \quad \text{and} \quad T(\to 1) = T{\restriction}_{\mathtt{stem}(T)^\wedge 1},$$

so that $[T(\to i)] = \{x \in [T] : x(h) = i\}$, where $h = \mathtt{lh}(\mathtt{stem}(T))$. Clearly $T(\to i) \in \mathbf{PT}$. The splitting can be iterated, so that if $s \in 2^n$ then

$$T(\to s) = T(\to s(0))(\to s(1))(\to s(2)) \dots (\to s(n-1)).$$

We separately define $T(\to \Lambda) = T$, for the empty string Λ.

Lemma 2.2. *Suppose that $T \in \mathbf{PT}$. If $u \in T$ then there is a unique string $s \in 2^{<\omega}$ such that $T(\to s) = T{\restriction}_u$. Conversely, if $s \in 2^{<\omega}$ then the string $u[s] = \mathtt{stem}(T(\to s))$ belongs to T and we have $T(\to s) = T{\restriction}_{u[s]}$.* $\square$

If $T, S \in \mathbf{PT}$ and $n \in \omega$ then let $S \subseteq_n T$ (S *n-refines* T) mean that $S(\to s) \subseteq T(\to s)$ for all strings $s \in 2^{\leqslant n}$. In particular, $S \subseteq_0 T$ iff simply $S \subseteq T$. By definition if $S \subseteq_{n+1} T$ then we have $S \subseteq_n T$ (and $S \subseteq T$), too.

Lemma 2.3. *Suppose that $T \in \mathbf{PT}$ and $n < \omega$. Then $T = \bigcup_{s \in 2^n} T(\to s)$ and $[T(\to s)] \cap [T(\to t)] = \varnothing$ for all $s \neq t$ in 2^n.*

In addition if $Y_s \in \mathbf{PT}$ and $Y_s \subseteq T(\to s)$ for every $s \in 2^n$, then $T' = \bigcup_{s \in 2^n} Y_s \in \mathbf{PT}$ and $T'(\to s) = Y_s$ for all $s \in 2^n$, and hence $T' \subseteq_n T$. $\square$

Lemma 2.4 *(Fusion). Let* $\ldots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ *be an infinite decreasing sequence of trees in* $\mathbf{PT}$*. Then* $T = \bigcap_n T_n \in \mathbf{PT}$*, and if* $n < \omega$ *and* $\mathtt{lh}(s) \le n+1$*, then* $T(\to s) = T \cap T_n(\to s) = \bigcap_{m \ge n} T_m(\to s)$*.* □

3. Perfect-tree forcing notions

Let a **perfect-tree forcing notion** (sometimes called *arboreal forcing*) be any set $\mathbb{P} \subseteq \mathbf{PT}$ such that if $u \in T \in \mathbb{P}$ then $T{\restriction}_u \in \mathbb{P}$. Let $\mathbf{PTF}$ be the set of all such sets $\mathbb{P}$. A perfect-tree forcing notion $\mathbb{P} \in \mathbf{PTF}$ is *regular* if $2^{<\omega} \in \mathbb{P}$.

Any set $\mathbb{P} \in \mathbf{PTF}$ can be considered as a forcing notion (if $T \subseteq T'$ then T is a stronger condition); such a forcing $\mathbb{P}$ obviously adds a real in 2^ω.

Example 3.1. If $s \in 2^{<\omega}$ then the tree $T[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subseteq s\}$ belongs to $\mathbf{PT}$ and $T[s] = (2^{<\omega})(\to s) = (2^{<\omega}){\restriction}_s$, $\forall s$. The set $\mathbb{P}_{\rm coh} = \{T[s] : s \in 2^{<\omega}\}$ (the Cohen forcing) is a regular perfect-tree forcing notion. □

If $\mathbb{P} \subseteq \mathbf{PT}$, $T \in \mathbf{PT}$, $n < \omega$, and all split trees $T(\to s)$, $s \in 2^n$, belong to $\mathbb{P}$, then we say that T is a *n-collage tree over* $\mathbb{P}$. Let $\mathbf{CT}_n(\mathbb{P})$ be the set of all trees $T \in \mathbf{PT}$ which are n-collage trees over $\mathbb{P}$, and let $\mathbf{CT}(\mathbb{P}) = \bigcup_n \mathbf{CT}_n(\mathbb{P})$.

Lemma 3.2.

(i) *If* $T \in \mathbb{P} \in \mathbf{PTF}$ *and* $s \in 2^{<\omega}$ *then* $T(\to s) \in \mathbb{P}$*.*
(ii) *If* $\mathbb{P} \in \mathbf{PTF}$ *and* $n < \omega$ *then* $\mathbb{P} = \mathbf{CT}_0(\mathbb{P}) \subseteq \mathbf{CT}_n(\mathbb{P}) \subseteq \mathbf{CT}_{n+1}(\mathbb{P})$*.*
(iii) *If* $\mathbb{P} \in \mathbf{PTF}$*,* $n < \omega$*, and, in Lemma 2.3, every tree* Y_s *belongs to* $\mathbb{P}$*, then the resulting tree* T' *belongs to* $\mathbf{CT}_n(\mathbb{P})$*.*

Proof. To prove (i) use Lemma 2.2. To prove (ii) use (i). □

Lemma 3.3 *(Disjoint splitting). Let* $\mathbb{P}, \mathbb{P}'$ *be perfect-tree forcings. Then*

(i) *if* $T \in \mathbb{P}$ *and* $T' \in \mathbb{P}'$*, then there are trees* $S \in \mathbb{P}$*,* $S' \in \mathbb{P}'$ *such that* $S \subseteq T$*,* $S' \subseteq T'$*, and* $[S] \cap [S'] = \varnothing$*.*
(ii) *if* $n < \omega$ *and* $T \in \mathbf{CT}_n(\mathbb{P})$*,* $T' \in \mathbf{CT}_n(\mathbb{P}')$*, then there exist trees* $S \in \mathbf{CT}_n(\mathbb{P})$*,* $S' \in \mathbf{CT}_n(\mathbb{P}')$ *such that* $S \subseteq_n T$*,* $S' \subseteq_n T'$*, and* $[S] \cap [S'] = \varnothing$*.*

Proof. (i) If $T = T'$ then let $s = \mathtt{stem}(T)$ and $S = T{\restriction}_{s^\wedge 0}$, $S' = T'{\restriction}_{s^\wedge 1}$. If say $T \not\subseteq T'$ then let $u \in T \smallsetminus T'$, $S = T{\restriction}_u$, and simply $S' = T'$. To prove (ii) iterate (i) and make use of Lemma 3.2(iii). □

4. Multitrees and splitting systems

Suppose in this Section that $\vartheta \in \mathbf{Ord}$ and $\mathbb{p} = \langle \mathbb{P}_\xi \rangle_{\xi<\vartheta}$ is a sequence of sets $\mathbb{P}_\xi \in \mathbf{PTF}$; we'll call such a $\mathbb{p}$ a $\mathbf{PTF}$*-sequence (of length* ϑ*)*. Sequences of this type will be systematically considered below, and if $\mathbb{q} = \langle \mathbb{Q}_\xi \rangle_{\xi<\vartheta}$ is another such a sequence of the same length then we let $\mathbb{p} \vee \mathbb{q} = \langle \mathbb{P}_\xi \cup \mathbb{Q}_\xi \rangle_{\xi<\vartheta}$.

Definition 4.1. A *multitree* is a "matrix" of the form $\boldsymbol{p} = \langle T^{\boldsymbol{p}}_{\xi k} \rangle^{\xi<\vartheta}_{k<\omega}$, where each $T^{\boldsymbol{p}}_{\xi k}$ belongs to $\mathbf{PT}$ and the *support* $|\boldsymbol{p}| = \{\langle \xi, k \rangle : T^{\boldsymbol{p}}_{\xi k} \neq 2^{<\omega}\}$ is finite. Let

$$[\boldsymbol{p}] = \{x \in 2^{\vartheta\times\omega} : \forall \langle\xi,k\rangle \in |\boldsymbol{p}|\,(x(\xi,k) \in [T^{\boldsymbol{p}}_{\xi k}])\} =$$
$$= \{x \in 2^{\vartheta\times\omega} : \forall \langle\xi,k\rangle \in |\boldsymbol{p}|\,\forall m\,(x(\xi,k){\restriction}m \in T^{\boldsymbol{p}}_{\xi k})\}$$

in this case; this is a cofinite-dimensional perfect cube in $2^{\vartheta\times\omega}$.

If $\mathbb{p} = \langle \mathbb{P}_\xi\rangle_{\xi<\vartheta}$ is a **PTF**-sequence then a $\mathbb{p}$-*multitree* is any multitree $\boldsymbol{p}$ with $T^{\boldsymbol{p}}_{\xi k} \in \mathbb{P}_\xi \cup \{2^{<\omega}\}$ for all ξ, k. Let $\mathbf{MT}(\mathbb{p})$ be the set of all $\mathbb{p}$-multitrees. □

The set $\mathbf{MT}(\mathbb{p})$ is equal to the finite support product $\prod_{\xi<\vartheta}(\mathbb{P}_\xi \cup \{2^{<\omega}\})^\omega$ of $(\vartheta\times\omega)$-many factors, with each factor $\mathbb{P}_\xi$ in ω-many copies. We order $\mathbf{MT}(\mathbb{p})$ componentwise: $\boldsymbol{q} \le \boldsymbol{p}$ ($\boldsymbol{q}$ is stronger) iff $T^{\boldsymbol{q}}_{\xi k} \subseteq T^{\boldsymbol{p}}_{\xi k}$ for all ξ, k. The forcing $\mathbf{MT}(\mathbb{p})$ adds a "matrix" $\langle x_{\xi k}\rangle^{\xi<\vartheta}_{k<\omega}$, where each $x_{\xi k} \in 2^\omega$ is a $\mathbb{P}_\xi$-generic real. The multitree $\boldsymbol{\Lambda}$ defined by $T^{\boldsymbol{\Lambda}}_{\xi k} = 2^{<\omega}$ for all ξ, k, belongs to $\mathbf{MT}(\mathbb{p})$, satisfies $|\boldsymbol{\Lambda}| = \varnothing$ and $[\boldsymbol{\Lambda}] = 2^{\vartheta\times\omega}$, and is the weakest condition.

The intention of the next definition is to formalize the construction of "generic" multitrees by means of Lemma 2.4 in the next section.

Definition 4.2. A $\mathbb{p}$-*system* is a "matrix" $\varphi = \langle\langle h^\varphi_{\xi m}, \tau^\varphi_{\xi m}\rangle\rangle^{\xi<\vartheta}_{m<\omega}$, where

(1) if $\xi < \vartheta$ and $m < \omega$ then $h^\varphi_{\xi m} \in \omega \cup \{-1\}$, and $|\varphi| = \{\langle\xi,m\rangle : h^\varphi_{\xi m} \ne -1\}$ (the *support* of φ) is a finite set;

(2) if $\langle\xi,m\rangle \in |\varphi|$ then $\tau^\varphi_{\xi m} = \langle T^\varphi_{\xi m}(0), T^\varphi_{\xi m}(1), \dots, T^\varphi_{\xi m}(h^\varphi_{\xi m})\rangle$, where each $T^\varphi_{\xi m}(h)$ is a tree in $\mathbf{CT}_h(\mathbb{P}_\xi)$ and $T^\varphi_{\xi m}(h) \subseteq_h T^\varphi_{\xi m}(h-1)$ whenever $1 \le h \le h^\varphi_{\xi m}$, while if $h^\varphi_{\xi m} = -1$ then simply $\tau^\varphi_{\xi m} = \Lambda$ (the empty sequence).

In this case, if $h \le h^\varphi_{\xi m}$ and $s \in 2^h$ then let $T^\varphi_{\xi m}(s) = T^\varphi_{\xi m}(h)(\to s)$; then the tree $T^\varphi_{\xi m}(s)$ belongs to $\mathbb{P}_\xi$ since $T^\varphi_{\xi m}(h) \in \mathbf{CT}_h(\mathbb{P}_\xi)$.[7]

Let $\mathbf{MS}(\mathbb{p})$ be the set of all $\mathbb{p}$-systems. □

Say that a system $\varphi \in \mathbf{MS}(\mathbb{p})$ is *pairwise disjoint* if $T^\varphi_{\xi m}(h^\varphi_{\xi m}) \cap T^\varphi_{\eta n}(h^\varphi_{\eta n}) = \varnothing$ for all pairs $\langle\xi,m\rangle \ne \langle\eta,n\rangle$ in $|\varphi|$.

Let $\varphi, \psi \in \mathbf{MS}(\mathbb{p})$. Say that φ *extends* $\psi \in \mathbf{MS}(\mathbb{p})$, symbolically $\psi \preccurlyeq \varphi$, if $|\psi| \subseteq |\varphi|$, and, for every $\langle\xi,m\rangle \in |\psi|$, we have $h^\varphi_{\xi m} \ge h^\psi_{\xi m}$ and $\tau^\varphi_{\xi m}$ extends $\tau^\psi_{\xi m}$, so that simply $T^\varphi_{\xi m}(h) = T^\psi_{\xi m}(h)$ for all $h \le h^\psi_{\xi m}$.

Say that a multitree $\boldsymbol{p}$ *occurs* in a system $\varphi \in \mathbf{MS}(\mathbb{p})$ if for each pair $\langle\xi,k\rangle \in |\boldsymbol{p}|$ there is a number $m = m_{k\xi} < \omega$ and a string $s = s_{k\xi} \in 2^{<\omega}$ with $\mathtt{lh}(s) \le h^\varphi_{\xi m}$ such that $T^{\boldsymbol{p}}_{\xi k} = T^\varphi_{\xi m}(s)$—then $\boldsymbol{p} \in \mathbf{MT}(\mathbb{p})$, of course.

Lemma 4.3. *Let $\mathbb{p} = \langle\mathbb{P}_\xi\rangle_{\xi<\vartheta}$ be a* **PTF**-*sequence and $\varphi \in \mathbf{MS}(\mathbb{p})$.*

(i) *If $\langle\xi,m\rangle \in |\varphi|$ and $h = h^\varphi_{\xi m}$ then the extension φ' of φ by $h^{\varphi'}_{\xi m} = h+1$ and $T^{\varphi'}_{\xi m}(h+1) = T^\varphi_{\xi m}(h)$ belongs to $\mathbf{MS}(\mathbb{p})$ and $\varphi \preccurlyeq \varphi'$.*

(ii) *If $\langle\xi,m\rangle \notin |\varphi|$ then the extension φ' of φ by $|\varphi'| = |\varphi| \cup \{\langle\xi,m\rangle\}$, $h^{\varphi'}_{\xi m} = 0$ and $T^{\varphi'}_{\xi m}(0) = T$, where $T \in \mathbb{P}_\xi$, belongs to $\mathbf{MS}(\mathbb{p})$ and $\varphi \preccurlyeq \varphi'$.*

(iii) *There is a pairwise disjoint system $\varphi' \in \mathbf{MS}(\mathbb{p})$ such that $|\varphi'| = |\varphi|$ and $\varphi \preccurlyeq \varphi'$.*

Proof. (i) Use Lemma 3.2(ii) and the fact that $T \subseteq_n T$ for all n, T.

To prove (iii) use Lemma 3.3(ii). □

[7] Note that the split trees $T^\varphi_{\xi m}(s) = T^\varphi_{\xi m}(h)(\to s)$ ($h \le h^\varphi_{\xi m}$, $s \in 2^h$) belong to $\mathbb{P}_\xi$, while the trees $T^\varphi_{\xi m}(h)$, which actually participate in φ, are assumed to only belong to $\mathbf{CT}_h(\mathbb{P}_\xi)$.

5. Jensen's extension of a perfect tree forcing

Let $\mathbf{ZFC}'$ be the subtheory of $\mathbf{ZFC}$ including all axioms except for the power set axiom, plus the axiom saying that $\mathscr{P}(\omega)$ exists. (Then ω_1 and some typical sets related to the continuum, like $\mathbf{PT}$, exist, too.) Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$.

Suppose in this Section that $\mathbb{p} = \langle \mathbb{P}_\xi \rangle_{\xi<\theta} \in \mathfrak{M}$ is a $\mathbf{PTF}$-sequence of (countable) sets $\mathbb{P}_\xi \in \mathbf{PTF}$, of a fixed length $\theta < \omega_1^{\mathfrak{M}}$. Then the sets $\mathbb{P}_\xi$ for all $\xi < \theta$, as well as the sets $\mathbf{MT}(\mathbb{p})$ and $\mathbf{MS}(\mathbb{p})$, belong to $\mathfrak{M}$, too.

Definition 5.1. (i) Let us fix any $\preccurlyeq$-increasing sequence $\Phi = \langle \varphi(j) \rangle_{j<\omega}$ of systems $\varphi(j) \in \mathbf{MS}(\mathbb{p})$, *generic over* $\mathfrak{M}$ in the sense that it intersects every set $D \in \mathfrak{M}$, $D \subseteq \mathbf{MS}(\mathbb{p})$, dense in $\mathbf{MS}(\mathbb{p})$.[8]

(ii) Suppose that $\xi < \theta$ and $m < \omega$. In particular, the sequence Φ intersects every (dense by Lemma 4.3(i), (ii)) set of the form

$$D_{\xi m h} = \{\varphi \in \mathbf{MS}(\mathbb{p}) \,:\, h^{\varphi}_{\xi m} \geq h\}, \quad \text{where } h < \omega\,.$$

Therefore if $\xi < \theta$ and $m < \omega$ then by definition there is an infinite sequence

$$\ldots \subseteq_5 \boldsymbol{T}^{\Phi}_{\xi m}(4) \subseteq_4 \boldsymbol{T}^{\Phi}_{\xi m}(3) \subseteq_3 \boldsymbol{T}^{\Phi}_{\xi m}(2) \subseteq_2 \boldsymbol{T}^{\Phi}_{\xi m}(1) \subseteq_1 \boldsymbol{T}^{\Phi}_{\xi m}(0)$$

of trees $\boldsymbol{T}^{\Phi}_{\xi m}(h) \in \mathbf{CT}_h(\mathbb{P})$, such that, for any j, if $\langle \xi, m\rangle \in |\varphi(j)|$ and $h \leq h^{\varphi(j)}_{\xi m}$ then $T^{\varphi(j)}_{\xi m}(h) = \boldsymbol{T}^{\Phi}_{\xi m}(h)$. If $h < \omega$ and $s \in 2^h$ then we let $\boldsymbol{T}^{\Phi}_{\xi m}(s) = \boldsymbol{T}^{\Phi}_{\xi m}(h)(\to s)$; then $\boldsymbol{T}^{\Phi}_{\xi m}(s) \in \mathbb{P}_\xi$ since $\boldsymbol{T}^{\Phi}_{\xi m}(h) \in \mathbf{CT}_h(\mathbb{P}_\xi)$.

(iii) Then it follows from Lemma 2.4 that each set

$$\boldsymbol{U}^{\Phi}_{\xi m} = \textstyle\bigcap_h \boldsymbol{T}^{\Phi}_{\xi m}(h) = \bigcap_h \bigcup_{s \in 2^h} \boldsymbol{T}^{\Phi}_{\xi m}(s)$$

is a tree in $\mathbf{PT}$ (not necessarily in $\mathbb{P}_\xi$), as well as the trees $\boldsymbol{U}^{\Phi}_{\xi m}(s) := \boldsymbol{U}^{\Phi}_{\xi m}(\to s)$, and still by Lemma 2.4,

$$\boldsymbol{U}^{\Phi}_{\xi m}(s) = \boldsymbol{U}^{\Phi}_{\xi m} \cap \boldsymbol{T}^{\Phi}_{\xi m}(s) = \textstyle\bigcap_{h \geq \mathtt{lh}(s)} \boldsymbol{T}^{\Phi}_{\xi m}(h)(\to s)\,,$$

and obviously $\boldsymbol{U}^{\Phi}_{\xi m} = \boldsymbol{U}^{\Phi}_{\xi m}(\Lambda)$.

(iv) If $\xi < \theta$ then let $\mathbb{U}_\xi = \{\boldsymbol{U}^{\Phi}_{\xi m}(s) \,:\, m < \omega \wedge s \in 2^{<\omega}\}$.

Let $\mathbb{u} = \langle \mathbb{U}_\xi \rangle_{\xi<\theta}$ and $\mathbb{p} \vee \mathbb{u} = \langle \mathbb{P}_\xi \cup \mathbb{U}_\xi \rangle_{\xi<\theta}$. $\square$

Lemma 5.2.

(i) *If $\xi < \theta$ then the sets $\mathbb{U}_\xi$ and $\mathbb{P}_\xi \cup \mathbb{U}_\xi$ belong to* $\mathbf{PTF}$*;*

(ii) *if $\xi < \theta$, $m < \omega$, and strings $s \subseteq t$ belong to $2^{<\omega}$ then $[\boldsymbol{T}^{\Phi}_{\xi m}(s)] \subseteq [\boldsymbol{T}^{\Phi}_{\xi m}(t)]$ and $[\boldsymbol{U}^{\Phi}_{\xi m}(s)] \subseteq [\boldsymbol{U}^{\Phi}_{\xi m}(t)]$;*

(iii) *if $\xi < \theta$, $m < \omega$, and strings $t' \neq t$ in $2^{<\omega}$ are $\subseteq$-incomparable then $[\boldsymbol{U}^{\Phi}_{\xi m}(t')] \cap [\boldsymbol{U}^{\Phi}_{\xi m}(t)] = [\boldsymbol{T}^{\Phi}_{\xi m}(t')] \cap [\boldsymbol{T}^{\Phi}_{\xi m}(t)] = \varnothing$.*

Proof. To prove (iii) note that $[\boldsymbol{T}^{\Phi}_{\xi m}(s^\wedge 0)] \cap [\boldsymbol{T}^{\Phi}_{\xi m}(s^\wedge 1)] = \varnothing$. $\square$

The following two lemmas present rather simple consequences of genericity of the background sequence of systems $\Phi = \langle \varphi(j) \rangle_{j<\omega}$ in Definition 5.1.

[8] Meaning that for any $\psi \in \mathbf{MS}(\mathbb{p})$ there is $\varphi \in D$ with $\psi \preccurlyeq \varphi$.

Lemma 5.3. *If* $\langle \xi, m\rangle \neq \langle \eta, n\rangle$ *then* $[\boldsymbol{U}^{\Phi}_{\xi m}] \cap [\boldsymbol{U}^{\Phi}_{\eta n}] = \varnothing$.

Therefore if $U \in \bigcup_{\xi<\theta} \mathbb{U}_\xi$ *then there is a unique triple of* $\xi < \theta$, $m < \omega$, *and* $s \in 2^{<\omega}$ *such that* $U = \boldsymbol{U}^{\Phi}_{\xi m}(s)$.

Proof. By Lemma 4.3(iii), the set D of all pairwise disjoint systems is dense. □

Lemma 5.4. *Let* $\xi < \theta$. *The set* $\mathbb{U}_\xi$ *is dense in* $\mathbb{U}_\xi \cup \mathbb{P}_\xi$.

Proof. If $T \in \mathbb{P}_\xi$ then the set $D(T)$ of all systems $\varphi \in \mathbf{MS}(\mathbb{p})$, such that $T^{\varphi}_{\xi m}(0) = T$ for some m, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{MS}(\mathbb{p})$. It follows that $\varphi(J) \in D(T)$ for some $J < \omega$, by the choice of Φ. Then $\boldsymbol{T}^{\Phi}_{\xi m}(\Lambda) = T$ for some $m < \omega$. However $\boldsymbol{U}^{\Phi}_{\xi m}(\Lambda) \subseteq \boldsymbol{T}^{\Phi}_{\xi m}(\Lambda)$. □

6. Preservation of density

This Section contains several key results related to pre-dense sets in the frameworks of Jensen's extension construction. We still suppose that $\mathfrak{M}$ is a countable transitive model of $\mathbf{ZFC}'$, $\mathbb{p} = \langle \mathbb{P}_\xi \rangle_{\xi<\theta} \in \mathfrak{M}$ is a **PTF**-sequence of (countable) sets $\mathbb{P}_\xi \in \mathbf{PTF}$, of a fixed length $\theta < \omega_1^{\mathfrak{M}}$, and we argue in terms of Definition 5.1.

Lemma 6.1. *If* $\xi < \theta$ *and a set* $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}_\xi$ *is pre-dense in* $\mathbb{P}_\xi$, *and* $U \in \mathbb{U}_\xi$, *then* $U \subseteq^{\mathtt{fin}} \bigcup D$, *that is, there is a finite set* $D' \subseteq D$ *with* $U \subseteq \bigcup D'$.

Proof. Suppose that $U = \boldsymbol{U}^{\Phi}_{\xi M}$, $M < \omega$. The set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{MS}(\mathbb{p})$ such that $\langle \xi, M\rangle \in |\varphi|$, and for each $t \in 2^h$, where $h = h^{\varphi}_{\xi M}$, there is a tree $S_t \in D$ with $T^{\varphi}_{\xi M}(t) \subseteq S_t$, is dense in $\mathbf{MS}(\mathbb{p})$ by the pre-density of D. Therefore there is an index j such that $\varphi(j) \in \Delta$. Let this be witnessed by trees $S_t \in D$, $t \in 2^h$, where $h = h^{\varphi(j)}_{\xi M}$, so that $T^{\varphi(j)}_{\xi M}(t) \subseteq S_t$, $\forall t$. Then

$$U \;=\; \boldsymbol{U}^{\Phi}_{\xi M}(s) \subseteq \boldsymbol{U}^{\Phi}_{\xi M}(\Lambda) \;\subseteq\; \bigcup_{t\in 2^h} T^{\varphi(j)}_{\xi M}(t) \;\subseteq\; \bigcup_{t\in 2^h} S_t \;=\; \bigcup D'$$

by construction, where $D' = \{S_t \,:\, t \in 2^h\} \subseteq D$ is finite. □

Corollary 6.2. *If* $\xi < \theta$ *and trees* $T, T' \in \mathbb{P}_\xi$ *are incompatible in* $\mathbb{P}_\xi$ *then* T, T' *remain incompatible in* $\mathbb{U}_\xi \cup \mathbb{P}_\xi$.

Proof. By the incompatibility assumption, if $S \in \mathbb{P}_\xi$ then either $S \not\subseteq T$ or $S \not\subseteq T'$. In both cases, there is a smaller tree $S' \in \mathbb{P}_\xi$, $S' \subseteq S$, such that $[S'] \cap [T] \cap [T'] = \varnothing$. It follows that the set D of all trees $S \in \mathbb{P}_\xi$ satisfying $[S] \cap [T] \cap [T'] = \varnothing$ is dense in $\mathbb{P}_\xi$. It remains to apply Lemma 6.1. □

Theorem 6.3. *In the assumptions above, if a set* $D \in \mathfrak{M}$, $D \subseteq \mathbf{MT}(\mathbb{p})$ *is pre-dense in* $\mathbf{MT}(\mathbb{p})$ *then it remains pre-dense in* $\mathbf{MT}(\mathbb{p} \vee \mathbb{u})$.

Proof. Given a multitree $\boldsymbol{p} \in \mathbf{MT}(\mathbb{p}\vee\mathbb{u})$, let us prove that $\boldsymbol{p}$ is compatible in $\mathbf{MT}(\mathbb{p}\vee\mathbb{u})$ with a multitree $\boldsymbol{q} \in D$. By Lemma 5.4, assume that $\boldsymbol{p} \in \mathbf{MT}(\mathbb{u})$. Then each term $T^{\boldsymbol{p}}_{\xi k}$ of $\boldsymbol{p}$ ($\langle \xi, k\rangle \in |\boldsymbol{p}|$) is equal to some $\boldsymbol{U}^{\Phi}_{\xi, m_{\xi k}}(s'_{\xi k})$, where $m_{\xi k} < \omega$ and $s'_{\xi k} \in 2^{<\omega}$. Choose a number $h > \mathtt{max}\{\mathtt{lh}(s'_{\xi k}) \,:\, \langle \xi, k\rangle \in |\boldsymbol{p}|\}$ big enough for there to exist strings $s_{\xi k} \in 2^h$ such that $s'_{\xi k} \subset s_{\xi k}$ and $s_{\xi k} \neq s_{\eta \ell}$ whenever pairs $\langle \xi, k\rangle \neq \langle \eta, \ell\rangle$ belong to $|\boldsymbol{p}|$. Define a multitree $\boldsymbol{u} \in \mathbf{MT}(\mathbb{u})$ so that $|\boldsymbol{u}| = |\boldsymbol{p}|$ and $T^{\boldsymbol{u}}_{\xi k} = \boldsymbol{U}^{\Phi}_{\xi, m_{\xi k}}(s_{\xi k})$ for all $\langle \xi, k\rangle \in |\boldsymbol{u}|$. Obviously $\boldsymbol{u} \leq \boldsymbol{p}$.

Consider the set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{MS}(\mathbb{p})$ such that there is a number $H > h$ and multitrees $\boldsymbol{q} \in D$ and $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$, satisfying $\boldsymbol{r} \leq \boldsymbol{q}$ and

(1) $|\boldsymbol{u}| \subseteq |\boldsymbol{r}|$ and $\boldsymbol{r}$ occurs in φ;
(2) if $\langle\xi,k\rangle \in |\boldsymbol{u}|$ then $\langle\xi,m_{\xi k}\rangle \in |\varphi|$, $h^{\varphi}_{\xi,m_{\xi k}} = H$, and $T^{\boldsymbol{r}}_{\xi k} = T^{\varphi}_{\xi,m_{\xi k}}(t_{\xi k})$, where $t_{\xi k} \in 2^H$ and $s_{\xi k} \subset t_{\xi k}$.

Lemma 6.4. *The set Δ is dense in $\mathbf{MS}(\mathbb{p})$.*

Proof. [Lemma] Suppose that $\psi \in \mathbf{MS}(\mathbb{p})$; we have to find a system $\varphi \in \mathbf{MS}(\mathbb{p})$ with $\psi \preccurlyeq \varphi$. First of all, by Lemma 4.3(i) we can assume that there is a number $g \geq h$ such that $h^{\psi}_{\xi m} = g$ for all $\langle\xi,m\rangle \in |\psi|$. We can also assume that if $\langle\xi,k\rangle \in |\boldsymbol{u}|$ then $\langle\xi,m_{\xi k}\rangle \in |\psi|$, for if not then just add $\langle\xi,m_{\xi k}\rangle$ to $|\psi|$ and define $h^{\psi}_{\xi m_{\xi k}} = g$ and $T^{\psi}_{\xi m_{\xi k}}(n) = S$ for all $n \leq g$, where $S \in \mathbb{P}_\eta$ is any tree, one and the same for all n.

Let $H = g+1$. Define a system $\chi \in \mathbf{MS}(\mathbb{p})$ extending ψ so that $|\chi| = |\psi|$, and $h^{\chi}_{\xi m} = H$, $T^{\chi}_{\xi m}(H) = T^{\psi}_{\xi m}(g)$ for all $\langle\xi,m\rangle \in |\psi|$; then $\psi \preccurlyeq \chi$. Pick strings $t_{\xi k} \in 2^H$ with $s_{\xi k} \subset t_{\xi k}$ for all $\langle\xi,k\rangle \in |\boldsymbol{u}|$. Then we have $t_{\xi k} \neq t_{\eta\ell}$ whenever pairs $\langle\xi,k\rangle \neq \langle\eta,\ell\rangle$ belong to $|\boldsymbol{u}|$, by the choice of $s_{\xi k}$.

Define a multitree $\boldsymbol{\pi} \in \mathbf{MT}(\mathbb{p})$ by $|\boldsymbol{\pi}| = |\boldsymbol{u}|$ and $T^{\boldsymbol{\pi}}_{\xi k} = T^{\chi}_{\xi,m_{\xi k}}(t_{\xi k})$ for all $\langle\xi,k\rangle \in |\boldsymbol{u}|$. By the pre-density of D there exist multitrees $\boldsymbol{q} \in D$ and $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$, such that $\boldsymbol{r} \leq \boldsymbol{q}$ and $\boldsymbol{r} \leq \boldsymbol{\pi}$; then $|\boldsymbol{u}| = |\boldsymbol{\pi}| \subseteq |\boldsymbol{r}|$.

Now define a system $\varphi \in \mathbf{MS}(\mathbb{p})$ so that $|\chi| \subseteq |\varphi|$ and $h^{\varphi}_{\xi m} = h^{\chi}_{\xi m} = H$, $T^{\varphi}_{\xi m}(n) = T^{\chi}_{\xi m}(n)$ for all $\langle\xi,m\rangle \in |\chi|$ and $n < H$. As for the values $T^{\varphi}_{\xi m}(H)$ and possible additional pairs in $|\varphi| \smallsetminus |\chi|$, proceed as follows.

(I) If some pair $\langle\xi,m\rangle \in |\chi|$ is not of the form $\langle\xi,m_{\xi k}\rangle$, where $\langle\xi,k\rangle \in |\boldsymbol{u}| = |\boldsymbol{\pi}|$, then simply keep $T^{\varphi}_{\xi m}(H) = T^{\chi}_{\xi m}(H)$.
(II) Now suppose that $\langle\xi,k\rangle \in |\boldsymbol{\pi}| = |\boldsymbol{u}|$, so that $\langle\xi,m_{\xi k}\rangle \in |\chi|$. Then $T^{\boldsymbol{r}}_{\xi k} = R \subseteq T = T^{\boldsymbol{\pi}}_{\xi k} = T^{\chi}_{\xi,m_{\xi k}}(t_{\xi k})$ since $\boldsymbol{r} \leq \boldsymbol{\pi}$. We let $T^{\varphi}_{\xi,m_{\xi k}}(t_{\xi k}) = R$.

Note that all trees R involved in (I) belong to $\mathbb{P}_\xi$ since $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$. Therefore, by Lemma 3.2(iii), the definition of the values of $T^{\varphi}_{\xi m}(t)$ for different $m = m_{\xi k}$ and $t = t_{\xi k} \in 2^H$ by (I) still results in trees $T^{\varphi}_{\xi m}(H)$ in $\mathbf{CT}_H(\mathbb{P}_\xi)$.

(III) Finally suppose that $\langle\xi,k\rangle \in |\boldsymbol{r}| \smallsetminus |\boldsymbol{u}|$. Then pick a number $m'_{\xi k} < \omega$ such that $\langle\xi,m'_{\xi k}\rangle \notin |\varphi|$ (and take care that all $m'_{\xi k}$ are pairwise different), add $\langle\xi,m'_{\xi k}\rangle$ to $|\varphi|$, and put $h^{\varphi}_{\xi,m'_{\xi k}} = 0$ and $T^{\varphi}_{\xi,m'_{\xi k}}(0) = T^{\boldsymbol{r}}_{\xi k}$.

The system φ still belongs to $\mathbf{MS}(\mathbb{p})$ (since $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$) and satisfies $\psi \preccurlyeq \varphi$ (as we only change the Hth level of χ absent in ψ), and $\boldsymbol{r}$ occurs in φ and satisfies $\boldsymbol{r} \leq \boldsymbol{q}$ and (1), (2) by construction. $\square$ (Lemma)

By the lemma, there is an index j such that the system $\varphi(j)$ belongs to Δ. Let this be witnessed by a number $H > h$, multitrees $\boldsymbol{q} \in D$ and $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$, and strings $t_{\xi k} \in 2^H$, satisfying $\boldsymbol{r} \leq \boldsymbol{q}$ and (1), (2) for $\varphi(j)$ instead of φ. Define a multitree $\boldsymbol{v} \in \mathbf{MT}(\mathbb{u})$ by $|\boldsymbol{v}| = |\boldsymbol{r}|$, $T^{\boldsymbol{v}}_{\xi k} = \boldsymbol{U}^{\Phi}_{\xi,m_{\xi k}}(t_{\xi k})$ for all $\langle\xi,k\rangle \in |\boldsymbol{u}|$, and

- if $\langle\xi,k\rangle \in |\boldsymbol{r}| \smallsetminus |\boldsymbol{u}|$ then, as $\boldsymbol{r}$ occurs in $\varphi(j)$ by (1), there is a number $m < \omega$ and a string $t \in 2^{<\omega}$ such that $\langle\xi,m\rangle \in |\varphi(j)|$, $\mathtt{lh}(t) \leq h^{\varphi(j)}_{\xi m}$, and $T^{\boldsymbol{r}}_{\xi k} = T^{\varphi(j)}_{\xi m}(t)$—in this case put $T^{\boldsymbol{v}}_{\xi k} = \boldsymbol{U}^{\Phi}_{\xi m}(t)$.

Then $\boldsymbol{v} \leq \boldsymbol{u}$ (since $s_{\xi k} \subset t_{\xi k}$), therefore $\boldsymbol{v} \leq \boldsymbol{u} \leq \boldsymbol{p}$. Moreover $\boldsymbol{v} \leq \boldsymbol{r}$. Indeed if $\langle\xi,k\rangle \in |\boldsymbol{u}|$ then $T^{\boldsymbol{v}}_{\xi k} = \boldsymbol{U}^{\Phi}_{\xi,m_{\xi k}}(t_{\xi k}) \subseteq \boldsymbol{T}^{\Phi}_{\xi,m_{\xi k}}(t_{\xi k}) = T^{\varphi(j)}_{\xi,m_{\xi k}}(t_{\xi k}) = T^{\boldsymbol{r}}_{\xi k}$. Similarly if $\langle\xi,k\rangle \in |\boldsymbol{r}| \smallsetminus |\boldsymbol{u}|$ then still $T^{\boldsymbol{v}}_{\xi k} \subseteq T^{\boldsymbol{r}}_{\xi k}$ by the same argument. Thus $\boldsymbol{v}$ witnesses that $\boldsymbol{p}$ is compatible with $\boldsymbol{q} \in D$. $\square$

7. Real names and direct forcing

Let $\mathfrak{M}$ be still a countable transitive model of $\mathbf{ZFC}'$ and $\mathbb{p} = \langle\mathbb{P}_\xi\rangle_{\xi<\theta} \in \mathfrak{M}$ be a *regular* $\mathbf{PTF}$-sequence of length $\theta < \omega_1^{\mathfrak{M}}$; those are fixed in this section. Our goal will be to introduce a suitable notation related to names of reals in 2^ω in the context of forcing notions of the form $\mathbf{MT}(\mathbb{p})$.

Definition 7.1. A $\mathbf{MT}(\mathbb{p})$-*real name* is a system $\mathbf{c} = \langle C_{ni}\rangle_{n<\omega,\, i<2}$ of sets $C_{ni} \subseteq \mathbf{MT}(\mathbb{p})$ such that each set $C_n = C_{n0} \cup C_{n1}$ is dense or at least pre-dense in $\mathbf{MT}(\mathbb{p})$ and if $\boldsymbol{p} \in C_{n0}$ and $\boldsymbol{q} \in C_{n1}$ then $\boldsymbol{p}, \boldsymbol{q}$ are incompatible in $\mathbf{MT}(\mathbb{p})$.

If a set $G \subseteq \mathbf{MT}(\mathbb{p})$ is $\mathbf{MT}(\mathbb{p})$-generic at least over the collection of all sets C_n then we define $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_{ni} \neq \varnothing$. □

Thus any $\mathbf{MT}(\mathbb{p})$-real name $\mathbf{c} = \langle C_{ni}\rangle$ is a $\mathbf{MT}(\mathbb{p})$-name for a real in 2^ω.

Recall that $\mathbf{MT}(\mathbb{p})$ adds a generic sequence $\langle x_{\xi k}\rangle_{\xi<\theta, k<\omega}$ of reals $x_{\xi k} \in 2^\omega$.

Example 7.2. If $\xi < \theta$ and $k < \omega$ then define a $\mathbf{MT}(\mathbb{p})$-real name $\dot{\boldsymbol{x}}_{\xi k} = \langle C^{\xi k}_{ni}\rangle_{n<\omega,\, i<2}$ such that each set $C^{\xi k}_{ni}$ contains all (finitely many) multitrees $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$, such that $|\boldsymbol{r}| = \{\langle \xi, k\rangle\}$ and $T^{\boldsymbol{r}}_{\xi k} = [s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subseteq s\}$, where $s \in 2^{k+1}$ (a string of length $k+1$) and $s(k) = i$.

Note that every multitree $\boldsymbol{r} = \boldsymbol{r}_{\xi k s}$ of this form belongs to $\mathbf{MT}(\mathbb{p})$. Indeed since the $\mathbf{PTF}$-sequence $\mathbb{p}$ considered is assumed to be regular, we have $2^{<\omega} \in \mathbb{P}_\xi$. It follows that $[s] \in \mathbb{P}_\xi$ as well for any ξ and any string $s \in 2^{<\omega}$, and hence $\boldsymbol{r}_{\xi k s} \in \mathbf{MT}(\mathbb{p})$. Therefore the name $\dot{\boldsymbol{x}}_{\xi k}$ defined this way is a $\mathbf{MT}(\mathbb{p})$-real name of the real $x_{\xi k}$, the (ξ, k)th term of a $\mathbf{MT}(\mathbb{p})$-generic sequence $\langle x_{\xi k}\rangle_{\xi<\theta,\, k<\omega}$. □

Let $\mathbf{c} = \langle C_{ni}\rangle$ and $\mathbf{d} = \langle D_{ni}\rangle$ be $\mathbf{MT}(\mathbb{p})$-real names. Let us say that a multitree $\boldsymbol{p}$ (not necessarily $\boldsymbol{p} \in \mathbf{MT}(\mathbb{p})$):

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$ and $i = 0, 1$, iff there is a multitree $\boldsymbol{q} \in C_{ni}$ such that $\boldsymbol{p} \leq \boldsymbol{q}$;
- *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \mathtt{lh}(s)$, $\boldsymbol{p}$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{d} \neq \mathbf{c}$, iff there are strings $s, t \in 2^{<\omega}$, incomparable in $2^{<\omega}$ and such that $\boldsymbol{p}$ directly forces $s \subset \mathbf{c}$ and $t \subset \mathbf{d}$;
- *directly forces* $\mathbf{c} \notin [T]$, where $T \in \mathbf{PT}$, iff there is a string $s \in 2^{<\omega} \smallsetminus T$ such that $\boldsymbol{p}$ directly forces $s \subset \mathbf{c}$.

The definition of direct forcing is not explicitly associated with any concrete forcing notion, but in fact the direct forcing relation (in all four instances) is compatible with any perfect tree forcing notion $\mathbb{P} \in \mathbf{PTF}$.

8. Forcing a real away of a pre-dense set

The goal of the following Theorem 8.1 is to prove that, under the conditions and notation of Definition 5.1, if $\xi < \theta$ and $\mathbf{c}$ is a $\mathbf{MT}(\mathbb{p})$-name of a real in 2^ω then the extended forcing $\mathbf{MT}(\mathbb{p} \vee \mathbb{u})$ forces that $\mathbf{c}$ does not belong to sets $[U]$ where U is a tree in $\mathbb{U}_\xi$—unless $\mathbf{c}$ is the name $\dot{\boldsymbol{x}}_{\xi k}$ of one of generic reals $x_{\xi k}$ themselves.

Theorem 8.1. *In the assumptions of Definition 5.1, let* $\mathbf{c} = \langle C^i_n\rangle_{n<\omega,\, i<2} \in \mathfrak{M}$ *be a* $\mathbf{MT}(\mathbb{p})$*-real name,* $\boldsymbol{\zeta} < \theta$ *is fixed, and for all* k *the set*

$$D(k) = \{\boldsymbol{p} \in \mathbf{MT}(\mathbb{p}) \;:\; \boldsymbol{p} \textit{ directly forces } \mathbf{c} \neq \dot{\boldsymbol{x}}_{\boldsymbol{\zeta} k}\}$$

is dense in $\mathbf{MT}(\mathbb{p})$*. Let* $\boldsymbol{u} \in \mathbf{MT}(\mathbb{p} \vee \mathbb{u})$*, and* $U \in \mathbb{U}_{\boldsymbol{\zeta}}$*. Then there is a stronger multitree* $\boldsymbol{v} \in \mathbf{MT}(\mathbb{u})$*,* $\boldsymbol{v} \leq \boldsymbol{u}$*, which directly forces* $\mathbf{c} \notin [U]$*.*

Proof. By construction $U \subseteq \boldsymbol{U}^{\Phi}_{\boldsymbol{\zeta} M}$ for some $M < \omega$; thus we can assume that simply $U = \boldsymbol{U}^{\Phi}_{\boldsymbol{\zeta} M}$. The indices $\boldsymbol{\zeta}$ and M are fixed in the proof. As in the proof of Theorem 6.3, we can assume that $\boldsymbol{u} \in \mathbf{MT}(\mathbb{u})$, and there is a number h and, for each $\langle \xi, k\rangle \in |\boldsymbol{u}|$, a number $m_{\xi k} < \omega$ and a string $s_{\xi k} \in 2^h$, such that $T^{\boldsymbol{u}}_{\xi k} = \boldsymbol{U}^{\Phi}_{\xi, m_{\xi k}}(s_{\xi k})$, and $s_{\xi k} \neq s_{\eta \ell}$ whenever $\langle \xi, k\rangle \neq \langle \eta, \ell\rangle$.

Consider the set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{MS}(\mathbb{p})$ such that there is a number $H > h$ and a multitree $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$, satisfying

(1) $|\boldsymbol{u}| \subseteq |\boldsymbol{r}|$ and $\boldsymbol{r}$ occurs in φ,
(2) if $\langle \xi, k\rangle \in |\boldsymbol{u}|$ then $\langle \xi, m_{\xi k}\rangle \in |\varphi|$, $h^{\varphi}_{\xi,m_{\xi k}} = H$, and $T^{\boldsymbol{r}}_{\xi k} = T^{\varphi}_{\xi,m_{\xi k}}(t_{\xi k})$, where $t_{\xi k} \in 2^H$ and $s_{\xi k} \subset t_{\xi k}$,
(3) $\boldsymbol{r}$ directly forces $\mathbf{c} \notin [T]$, where $T = T^{\varphi}_{\boldsymbol{\zeta} M}(H)$.

Lemma 8.2. *$\mathscr{D}$ is dense in* $\mathbf{MS}(\mathbb{p})$.

Proof. Suppose that $\psi \in \mathbf{MS}(\mathbb{p})$; we have to find a system $\varphi \in \mathscr{D}$ such that $\psi \preccurlyeq \varphi$. As in the proof of Theorem 6.3, we can assume that there is a number $g > h$ such that $h^{\psi}_{\xi m} = g$ for all $\langle \xi, m\rangle \in |\psi|$, $\langle \boldsymbol{\zeta}, M\rangle \in |\psi|$, and if $\langle \xi, k\rangle \in |\boldsymbol{u}|$ then $\langle \xi, m_{\xi k}\rangle \in |\psi|$.

Let $H = g + 1$. Define a system $\chi \in \mathbf{MS}(\mathbb{p})$ extending ψ so that $|\chi| = |\psi|$, and $h^{\chi}_{\xi m} = H$, $T^{\chi}_{\xi m}(H) = T^{\psi}_{\xi m}(g)$ for all $\langle \xi, m\rangle \in |\psi|$; then $\psi \preccurlyeq \chi$. Pick strings $t_{\xi k} \in 2^H$ with $s_{\xi k} \subset t_{\xi k}$ for all $\langle \xi, k\rangle \in |\boldsymbol{u}|$. Then we have $t_{\xi k} \neq t_{\eta \ell}$ whenever $\langle \xi, k\rangle \neq \langle \xi, \ell\rangle$ belong to $|\boldsymbol{u}|$, by the choice of $s_{\xi k}$.

Define a multitree $\boldsymbol{\pi} \in \mathbf{MT}(\mathbb{p})$ by $|\boldsymbol{\pi}| = |\boldsymbol{u}|$ and $T^{\boldsymbol{\pi}}_{\xi k} = T^{\chi}_{\xi,m_{\xi k}}(t_{\xi k})$ for all $\langle \xi, k\rangle \in |\boldsymbol{u}|$; then $\boldsymbol{\pi}$ occurs in χ. We can also assume that

(a) if $t \in 2^H$ then there exists a pair $\langle \boldsymbol{\zeta}, k(t)\rangle \in |\boldsymbol{\pi}|$ such that $m_{\boldsymbol{\zeta},k(t)} = M$ and $t_{\boldsymbol{\zeta},k(t)} = t$.

Indeed if this fails then for any such string t pick a number $k(t) < \omega$ satisfying $\langle \boldsymbol{\zeta}, k(t)\rangle \notin |\boldsymbol{\pi}|$ (taking care that $k(t) \neq k(t')$ whenever $t \neq t'$), add the pair $\langle \boldsymbol{\zeta}, k(t)\rangle$ to $|\boldsymbol{\pi}|$, and let $T^{\boldsymbol{\pi}}_{\boldsymbol{\zeta},k(t)} = T^{\chi}_{\boldsymbol{\zeta} M}(t)$. Then define $m_{\boldsymbol{\zeta},k(t)} = M$ and $t_{\boldsymbol{\zeta},k(t)} = t$. The extended multitree $\boldsymbol{\pi}$ satisfies $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$ (and $\subseteq$ can be $\subsetneqq$ here) and (a), and by construction satisfies

(b) $\langle \boldsymbol{\zeta}, M\rangle \in |\chi|$, and if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ then $\langle \xi, m_{\xi k}\rangle \in |\chi|$, and
(c) $t_{\xi k} \neq t_{\eta \ell}$ whenever pairs $\langle \xi, k\rangle \neq \langle \eta, \ell\rangle$ belong to $|\boldsymbol{\pi}|$.

By the density of sets $D(k)$, there exists a multitree $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$, $\boldsymbol{r} \leq \boldsymbol{\pi}$, which directly forces $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\boldsymbol{\zeta} k}$ whenever $k \in K = \{k : \langle \boldsymbol{\zeta}, k\rangle \in |\boldsymbol{\pi}|\}$. Then there are strings u and $\{v_k : k \in K\}$ in $2^{<\omega}$ such that u is incompatible in $2^{<\omega}$ with each v_k and $\boldsymbol{r}$ directly forces each of the formulas $u \subset \mathbf{c}$ and $v_k \subset \dot{\boldsymbol{x}}_{\boldsymbol{\zeta} k}$—for all $k \in K$. Yet $\boldsymbol{r}$ directly forces $v_k \subset \dot{\boldsymbol{x}}_{\boldsymbol{\zeta} k}$ iff $v_k \subseteq \mathtt{stem}(T^{\boldsymbol{r}}_{\boldsymbol{\zeta} k})$. Thus $\boldsymbol{r}$ directly forces $\mathbf{c} \notin [T^*]$, where $T^* = \bigcup_{k \in K} T^{\boldsymbol{r}}_{\boldsymbol{\zeta} k}$.

Define a system $\varphi \in \mathbf{MS}(\mathbb{p})$ so that $|\chi| \subseteq |\varphi|$ and $h^{\varphi}_{\xi m} = h^{\chi}_{\xi m} = H$, $T^{\varphi}_{\xi m}(n) = T^{\chi}_{\xi m}(n)$ for all $\langle \xi, m\rangle \in |\chi|$, $n < H$. As for the values $T^{\varphi}_{\xi m}(H)$ and possible pairs in $\langle \xi, m\rangle \in |\varphi| \smallsetminus |\chi|$, proceed as follows.

(I) If $\langle \xi, m\rangle \in |\chi|$ is not of the form $\langle \xi, m_{\xi k}\rangle$, where $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, then put $T^{\varphi}_{\xi m}(H) = T^{\chi}_{\xi m}(H)$.
(II) Suppose that $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, so that $\langle \xi, m_{\xi k}\rangle \in |\chi|$. Then $T^{\boldsymbol{r}}_{\xi k} = R \subseteq T = T^{\boldsymbol{\pi}}_{\xi k} = T^{\chi}_{\xi,m_{\xi k}}(t_{\xi k})$ since $\boldsymbol{r} \leq \boldsymbol{\pi}$. We let $T^{\varphi}_{\xi,m_{\xi k}}(t_{\xi k}) = R$.
(III) Finally suppose that $\langle \xi, k\rangle \in |\boldsymbol{r}| \smallsetminus |\boldsymbol{\pi}|$. Then pick a number $m'_{\xi k} < \omega$ such that $\langle \xi, m'_{\xi k}\rangle \notin |\varphi|$ (and we assume that all $m'_{\xi k}$ are pairwise different), add $\langle \xi, m'_{\xi k}\rangle$ to $|\varphi|$, and put $h^{\varphi}_{\xi,m'_{\xi k}} = 0$ and $T^{\varphi}_{\xi,m'_{\xi k}}(0) = T^{\boldsymbol{r}}_{\xi k}$.

As in the proof of Lemma 6.4, the extended system φ still belongs to $\mathbf{MS}(\mathbb{p})$ and satisfies $\psi \preccurlyeq \varphi$, the multitree $\boldsymbol{r}$ occurs in φ, and we have $T^{\boldsymbol{r}}_{\xi k} = T^{\varphi}_{\xi,m_{\xi k}}(t_{\xi k})$ whenever $\langle \xi, k\rangle \in |\boldsymbol{r}|$.

To complete the proof of the lemma, suppose that $t \in 2^H$. Then by (a) there is a number $k < \omega$ such that $\langle \boldsymbol{\zeta}, k\rangle \in |\boldsymbol{\pi}|$—hence, $k \in K$,—and $m_{\boldsymbol{\zeta} k} = M$, $t_{\boldsymbol{\zeta} k} = t$. Then by construction $T^{\varphi}_{\boldsymbol{\zeta} M}(t) = T^{\boldsymbol{r}}_{\boldsymbol{\zeta} k}$, therefore

$T^{\varphi}_{\boldsymbol{\zeta} M}(t) \subseteq T^*$. As $t \in 2^H$ is arbitrary in this argument, we conclude that $T^{\varphi}_{\boldsymbol{\zeta} M}(H) = \bigcup_{t \in 2^H} T^{\varphi}_{\boldsymbol{\zeta} M}(t) \subseteq T^*$. It follows that $\boldsymbol{r}$ directly forces $\mathbf{c} \notin [T^{\varphi}_{\boldsymbol{\zeta} M}(H)]$, as required. □ (Lemma)

We now return to the proof of the theorem. It follows from the lemma that there is an index j such that the system $\varphi(j)$ belongs to $\mathscr{D}$. Let this be witnessed by a number $H > h$, a multitree $\boldsymbol{r} \in \mathbf{MT}(\mathbb{p})$, and a collection of strings $t_{\xi k} \in 2^H$ ($\langle \xi, k \rangle \in |\boldsymbol{u}|$), such that conditions (1), (2), (3) are satisfied for $\varphi = \varphi(j)$.

Define a multitree $\boldsymbol{v} \in \mathbf{MT}(\mathbb{u})$ so that $|\boldsymbol{v}| = |\boldsymbol{r}|$, if $\langle \xi, k \rangle \in |\boldsymbol{u}|$ then $T^{\boldsymbol{v}}_{\xi k} = \boldsymbol{U}^{\Phi}_{\xi, m_{\xi k}}(t_{\xi k}) = T^{\varphi(j)}_{\xi, m_{\xi k}}(t_{\xi k})$, and if $\langle \xi, k \rangle \in |\boldsymbol{r}| \smallsetminus |\boldsymbol{u}|$ then $T^{\boldsymbol{v}}_{\xi k} = \boldsymbol{U}^{\Phi}_{\xi m}(t) = T^{\varphi(j)}_{\xi m}(t)$, for any $m < \omega$ and $t \in 2^{<\omega}$ such that $T^{\boldsymbol{r}}_{\xi k} = T^{\varphi(j)}_{\xi m}(t) = \boldsymbol{U}^{\Phi}_{\xi m}(t)$. Then $\boldsymbol{v} \leq \boldsymbol{u}$ and $\boldsymbol{v} \leq \boldsymbol{r}$ (see the end of the proof of Theorem 6.3). Finally, by (3), $\boldsymbol{r}$ directly forces $\mathbf{c} \notin [T]$, where $T = T^{\varphi(j)}_{\boldsymbol{\zeta} M}(H)$. However $U = \boldsymbol{U}^{\Phi}_{\boldsymbol{\zeta} M} \subseteq T^{\varphi(j)}_{\boldsymbol{\zeta} M}(H)$. □

9. The basic product forcing

In this section, **we argue in $\mathbf{L}$, the constructible universe.** Let $\leq_{\mathbf{L}}$ be the canonical wellordering of $\mathbf{L}$.

Definition 9.1. (In $\mathbf{L}$.) We define, by induction on $\alpha < \omega_1$, a **PTF**-sequence $\mathbb{u}^{\alpha} = \langle \mathbb{U}^{\alpha}_{\xi} \rangle_{\xi < \alpha}$, and a regular **PTF**-sequence $\mathbb{p}^{\alpha} = \langle \mathbb{P}^{\alpha}_{\xi} \rangle_{\xi < \alpha}$, of countable sets of trees $\mathbb{U}^{\alpha}_{\xi}$, $\mathbb{P}^{\alpha}_{\xi}$ in **PTF**, as follows.

First of all, we let $\mathbb{P}^{\alpha}_{\alpha} = \varnothing$ and $\mathbb{U}^{\alpha}_{\alpha} = \mathbb{P}_{\text{coh}}$ (see Example 3.1) for all α; note that the terms $\mathbb{P}^{\alpha}_{\alpha}, \mathbb{U}^{\alpha}_{\alpha}$ do not participate in the sequences $\mathbb{p}^{\alpha}$ and $\mathbb{u}^{\alpha}$.

The case $\boldsymbol{\alpha = 0}$. Let $\mathbb{p}^0 = \mathbb{u}^0 = \Lambda$ (the empty sequence).

The step. Suppose that $0 < \lambda < \omega_1$, and $\mathbb{u}^{\alpha}$, $\mathbb{p}^{\alpha}$ as above are already defined for every $\alpha < \lambda$. Let $\mathfrak{M}_{\lambda}$ be the least model $\mathfrak{M}$ of $\mathbf{ZFC}'$ of the form $\mathbf{L}_{\mu}$, $\mu < \omega_1$, containing $\langle \mathbb{u}^{\alpha} \rangle_{\alpha < \lambda}$ and $\langle \mathbb{p}^{\alpha} \rangle_{\alpha < \lambda}$, and such that $\lambda < \omega_1^{\mathfrak{M}}$ and $\mathbb{U}^{\alpha}_{\xi}$, $\mathbb{P}^{\alpha}_{\xi}$ are countable in $\mathfrak{M}$ for all $\xi < \alpha < \lambda$.

We define a sequence $\mathbb{p}^{\lambda} = \langle \mathbb{P}^{\lambda}_{\xi} \rangle_{\xi < \lambda}$ so that $\mathbb{P}^{\lambda}_{\xi} = \bigcup_{\xi \leq \alpha < \lambda} \mathbb{U}^{\alpha}_{\xi}$ for all $\xi < \lambda$. Thus if $\lambda = \alpha + 1$ then $\mathbb{P}^{\alpha+1}_{\xi} = \mathbb{P}^{\alpha}_{\xi} \cup \mathbb{U}^{\alpha}_{\xi}$ for all $\xi \leq \alpha$ (since $\mathbb{P}^{\alpha}_{\xi} = \bigcup_{\xi \leq \alpha' < \alpha} \mathbb{U}^{\alpha'}_{\xi}$ at the previous step). In particular, for $\xi = \alpha$, $\mathbb{P}^{\alpha+1}_{\alpha} = \mathbb{P}^{\alpha}_{\alpha} \cup \mathbb{U}^{\alpha}_{\alpha} = \mathbb{P}_{\text{coh}}$ (see above). Thus $\mathbb{p}^{\alpha+1}$ is the extension of $\mathbb{p}^{\alpha} \vee \mathbb{u}^{\alpha}$ (see Section 4) by the default assignment $\mathbb{P}^{\alpha+1}_{\alpha} = \mathbb{P}_{\text{coh}}$. For instance, $\mathbb{p}^1 = \langle \mathbb{P}^1_0 \rangle$, where $\mathbb{P}^1_0 = \mathbb{P}_{\text{coh}}$.

To define $\mathbb{u}^{\lambda}$ and accomplish the step, let $\Phi = \langle \varphi^j \rangle_{j < \omega}$ be the $\leq_{\mathbf{L}}$-least sequence of systems $\varphi^j \in \mathbf{MS}(\mathbb{p}^{\lambda})$, $\preccurlyeq$-increasing and generic over $\mathfrak{M}_{\lambda}$, and let $\mathbb{u}^{\lambda} = \langle \mathbb{U}^{\lambda}_{\xi} \rangle_{\xi < \lambda}$ be defined, on the base of this sequence, as in Definition 5.1.

Final. After the sequences $\mathbb{u}^{\alpha} = \langle \mathbb{U}^{\alpha}_{\xi} \rangle_{\xi < \alpha}$, $\mathbb{p}^{\alpha} = \langle \mathbb{P}^{\alpha}_{\xi} \rangle_{\xi < \alpha}$ and models $\mathfrak{M}_{\alpha}$ have been defined for all $\alpha < \omega_1$, we let $\mathbb{P}_{\xi} = \bigcup_{\xi \leq \alpha < \omega_1} \mathbb{U}^{\alpha}_{\xi}$ for all $\xi < \omega_1$, and $\mathbb{p} = \mathbb{p}^{\omega_1} = \langle \mathbb{P}_{\xi} \rangle_{\xi < \omega_1}$, which is a regular **PTF**-sequence of length $\omega_1^{\mathbf{L}}$ in $\mathbf{L}$. Let $\mathbf{P} = \mathbf{MT}(\mathbb{p})$. If $\alpha < \omega_1^{\mathbf{L}}$ then let $\mathbf{P}^{\alpha} = \mathbf{MT}(\mathbb{p}^{\alpha})$. □

The next result (a routine proof is omitted) accounts for the definability class of the constructions introduced by Definition 9.1. Recall that HC is the set of all *hereditarily countable* sets.

Proposition 9.2. *In* $\mathbf{L}$, *all three sequences* $\langle \mathbb{u}^{\alpha} \rangle_{\alpha < \omega_1}$, $\langle \mathbb{p}^{\alpha} \rangle_{\alpha < \omega_1}$, $\langle \mathfrak{M}_{\alpha} \rangle_{\alpha < \omega_1}$ *belong to the definability class* Δ^{HC}_1. □

The set $\mathbf{P} = \mathbf{MT}(\mathbb{p}) = \prod_{\xi < \omega_1^{\mathbf{L}}} \mathbb{P}_{\xi}^{<\omega}$ of all $\mathbb{p}$-multitrees (see Definition 4.1) will be our principal forcing notion; $\mathbf{P}$ belongs to $\mathbf{L}$ as so does $\mathbb{p}$. The forcing $\mathbf{P}$ can be identified with the finite-support product $\prod_{\xi < \omega_1^{\mathbf{L}}} \prod_{k < \omega} \mathbb{P}_{\xi k}$, where each factor $\mathbb{P}_{\xi k}$ is equal to the set $\mathbb{P}_{\xi} = \bigcup_{\xi \leq \alpha < \omega_1^{\mathbf{L}}} \mathbb{U}^{\alpha}_{\xi}$ of Definition 9.1.

Remark 9.3. If $\alpha < \gamma \leq \omega_1^{\mathbf{L}}$ then the sets $\mathbf{P}^{\alpha} = \mathbf{MT}(\mathbb{p}^{\alpha})$ and $\mathbf{P}^{\gamma} = \mathbf{MT}(\mathbb{p}^{\gamma})$ of multitrees are formally disjoint. However we can naturally embed the former in the latter. Indeed each multitree $\boldsymbol{p} = \langle T^{\boldsymbol{p}}_{\xi k} \rangle^{\xi < \alpha}_{k < \omega} \in \mathbf{P}^{\alpha}$ can be identified as an element of $\mathbf{P}^{\gamma}$ by the default extension $T^{\boldsymbol{p}}_{\xi k} = 2^{<\omega}$ whenever $\alpha \leq \xi < \gamma$, $k < \omega$.

With such an identification, we can assume that $\mathbb{P}^\alpha \subseteq \mathbb{P}^\gamma \subseteq \mathbb{P}$, and similarly $\mathbb{P}^\lambda = \bigcup_{\alpha<\lambda} \mathbb{P}^\alpha$ for all limit λ, and the like. □

10. Preservation of density revisited

Here we establish some corollaries of results in Section 6, as well as some close results, including the CCC property. We argue in terms of Definition 9.1.

Lemma 10.1.

(i) *If $\alpha < \omega_1^{\mathbf{L}}$ and a set $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}^\alpha$ is pre-dense in $\mathbb{P}^\alpha$ then it remains pre-dense in $\mathbb{P}$.*
(ii) *In particular the set $\mathbf{MT}(\mathbb{u}^\alpha)$ itself is pre-dense in $\mathbb{P}$.*

Proof. (i) By induction on γ, $\xi \le \gamma < \omega_1^{\mathbf{L}}$, if D is pre-dense in $\mathbb{P}^\gamma = \mathbf{MT}(\mathbb{p}^\gamma)$ then it remains pre-dense in $\mathbf{MT}(\mathbb{p}^\gamma \vee \mathbb{u}^\gamma)$ by Theorem 6.3, hence in $\mathbb{P}^{\gamma+1} = \mathbf{MT}(\mathbb{p}^{\gamma+1})$ too by construction. Limit steps including the step $\omega_1^{\mathbf{L}}$ are obvious.

(ii) Note that $\mathbf{MT}(\mathbb{u}^\alpha)$ is dense in $\mathbf{MT}(\mathbb{p}^\alpha \vee \mathbb{u}^\alpha)$ by Lemma 5.4, therefore, pre-dense in $\mathbb{P}^{\alpha+1} = \mathbf{MT}(\mathbb{p}^{\alpha+1})$, and $\mathbf{MT}(\mathbb{u}^\alpha) \in \mathfrak{M}_{\alpha+1}$. Apply (i). □

Corollary 10.2. *If $\xi < \alpha < \omega_1^{\mathbf{L}}$ then the set $\mathbb{U}_\xi^\alpha$ is pre-dense in $\mathbb{P}_\xi$.*

Proof. Let $T \in \mathbb{P}_\xi$. Consider a multitree $\boldsymbol{p} \in \mathbb{P} = \mathbf{MT}(\mathbb{p})$ defined so that $T_{\xi 0}^{\boldsymbol{p}} = T$ and $T_{\eta k}^{\boldsymbol{p}} = 2^{<\omega}$ whenever $\langle \eta, k\rangle \ne \langle \xi, 0\rangle$. By Lemma 10.1 $\boldsymbol{p}$ is compatible in $\mathbb{P}$ with a multitree $\boldsymbol{u} \in \mathbf{MT}(\mathbb{u}^\alpha)$. We conclude that T is compatible in $\mathbb{P}_\xi$ with the tree $U = T_{\xi 0}^{\boldsymbol{u}} \in \mathbb{U}_\xi^\alpha$. □

Corollary 10.3. *If $\xi < \alpha < \omega_1^{\mathbf{L}}$ and trees $T, T' \in \mathbb{P}_\xi^\alpha$ are incompatible in $\mathbb{P}_\xi^\alpha$ then T, T' remain incompatible in $\mathbb{P}_\xi$. Therefore if multitrees $\boldsymbol{p}, \boldsymbol{p}' \in \mathbb{P}^\alpha = \mathbf{MT}(\mathbb{p}^\alpha)$ are incompatible in $\mathbf{MT}(\mathbb{p}^\alpha)$ then $\boldsymbol{p}, \boldsymbol{p}'$ remain incompatible in $\mathbb{P}$.*

Proof. Let $T, T' \in \mathbb{P}_\xi^\alpha$ be incompatible in $\mathbb{P}_\xi^\alpha$. Use Corollary 6.2 at successor steps to prove by induction on γ that if $\alpha < \gamma \le \omega_1^{\mathbf{L}}$ that the trees T, T' remain incompatible in $\mathbb{P}_\xi^\gamma$. □

Corollary 10.4. *If $\alpha < \omega_1^{\mathbf{L}}$ and a set (filter) $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$ then the set $G' = G \cap \mathbb{P}^\alpha$ is $\mathbb{P}^\alpha$-generic over $\mathfrak{M}_\alpha$.*

Proof. Elements of G' are still pairwise compatible in $\mathbb{P}^\alpha = \mathbf{MT}(\mathbb{p}^\alpha)$ by Corollary 10.3. Furthermore if a set $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}^\alpha$, is dense in $\mathbb{P}^\alpha$ then it is pre-dense in $\mathbb{P}$ by Lemma 10.1, so that $G \cap D \ne \varnothing$ and $G' \cap D \ne \varnothing$. □

To prove the CCC property, we'll need the following reflection-type result.

Lemma 10.5. *If $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1^{\mathbf{L}}}$ then the set $\mathscr{O}_X$ of all ordinals $\alpha < \omega_1^{\mathbf{L}}$, such that $\langle \mathbf{L}_\alpha\,; X \cap \mathbf{L}_\alpha\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1^{\mathbf{L}}}\,; X\rangle$ and $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$, is unbounded in $\omega_1^{\mathbf{L}}$. More generally, if $X_n \subseteq \mathrm{HC}$ for all n then the set $\mathscr{O}$ of all ordinals $\alpha < \omega_1^{\mathbf{L}}$, such that $\langle \mathbf{L}_\alpha\,; \langle X_n \cap \mathbf{L}_\alpha\rangle_{n<\omega}\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1^{\mathbf{L}}}\,; \langle X_n\rangle_{n<\omega}\rangle$ and $\langle X_n \cap \mathbf{L}_\alpha\rangle_{n<\omega} \in \mathfrak{M}_\alpha$, is unbounded in $\omega_1^{\mathbf{L}}$.*

Proof. Let $\alpha_0 < \omega_1^{\mathbf{L}}$. Let M be a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing $\alpha_0, \omega_1^{\mathbf{L}}, X$, and such that $M \cap \mathbf{L}_{\omega_1}$ is transitive. Let $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and let $\alpha = \phi(\omega_1^{\mathbf{L}})$. Then $\alpha_0 < \alpha < \lambda < \omega_1^{\mathbf{L}}$ and $\phi(X) = X \cap \mathbf{L}_\alpha$ by the choice of M. It follows that $\langle \mathbf{L}_\alpha\,; X \cap \mathbf{L}_\alpha\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1^{\mathbf{L}}} ; X\rangle$. Moreover, α is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\alpha$. We conclude that $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$ since $X \cap \mathbf{L}_\alpha \in \mathbf{L}_\lambda$ by construction.

The second, more general claim does not differ much. □

Corollary 10.6. *The forcing $\mathbb{P}$ satisfies CCC, therefore $\mathbb{P}$-generic extensions of $\mathbf{L}$ preserve cardinals.*

Proof. Suppose that $A \subseteq \mathbb{P} = \mathbf{MT}(\mathbb{p})$ is a maximal antichain. By Lemma 10.5, there is an ordinal α such that $A' = A \cap \mathbb{P}^\alpha$ is a maximal antichain in $\mathbb{P}^\alpha = \mathbf{MT}(\mathbb{p}^\alpha)$ and $A' \in \mathfrak{M}_\alpha$. But then A' remains pre-dense, therefore, maximal, in the whole set $\mathbb{P}$ by Lemma 10.1. It follows that $A = A'$ is countable. □

11. The basic extension: product structure and generic reals

Working in terms of Definition 9.1, we let $\mathbb{P}\restriction\Delta = \{\boldsymbol{p} \in \mathbb{P} : |\boldsymbol{p}| \subseteq \Delta\}$ for any set $\Delta \subseteq \omega_1^{\mathbf{L}} \times \omega$. The forcing $\mathbb{P}$ has an obvious product structure:

Lemma 11.1. *Suppose that $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}} \times \omega$. Then $\mathbb{P}$ is equal to the product $(\mathbb{P}\restriction\Delta) \times (\mathbb{P}\restriction\Delta')$, where $\Delta' = (\omega_1^{\mathbf{L}} \times \omega) \smallsetminus \Delta$. If $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$, then the set $G\restriction\Delta = \{\boldsymbol{p} \in G : |\boldsymbol{p}| \subseteq \Delta\}$ is accordingly $(\mathbb{P}\restriction\Delta)$-generic over $\mathbf{L}$.* □

Assume that $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}} \times \omega$. Similarly to Definition 7.1, let a $(\mathbb{P}\restriction\Delta)$-*real name* be a system $\mathbf{c} = \langle C_{ni}\rangle_{n<\omega,\, i<2}$ of sets $C_{ni} \subseteq \mathbb{P}\restriction\Delta$ such that each set $C_n = C_{n0} \cup C_{n1}$ is pre-dense in $\mathbb{P}\restriction\Delta$ and if $\boldsymbol{p} \in C_{n0}$, $\boldsymbol{q} \in C_{n1}$ then $\boldsymbol{p}, \boldsymbol{q}$ are incompatible in $\mathbb{P}\restriction\Delta$. A name is *countable* if such are all sets C_{ni}.

If a set $G \subseteq \mathbb{P}\restriction\Delta$ is at least pairwise compatible then we define $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_{ni} \neq \varnothing$.

Lemma 11.2. *Suppose that $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}} \times \omega$. If a set $G' \subseteq \mathbb{P}\restriction\Delta$ is generic over $\mathbf{L}$ and $x \in 2^\omega \cap \mathbf{L}[G']$ then there is a $(\mathbb{P}\restriction\Delta)$-real name $\mathbf{c} \in \mathbf{L}$, countable in $\mathbf{L}$ and such that $x = \mathbf{c}[G']$.*

Proof. To reduce an arbitrary name to a countable one, note that $\mathbb{P}\restriction\Delta$ is CCC in $\mathbf{L}$ as a factor of the CCC (by Corollary 10.6) forcing $\mathbb{P} = \mathbf{MT}(\mathbb{p})$. □

Definition 11.3 *(Generic reals).* Let $G \subseteq \mathbb{P}$ be a set (filter) $\mathbb{P}$-generic over $\mathbf{L}$. Note that $\omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}}$ by Corollary 10.6.

If $\xi < \omega_1^{\mathbf{L}}$ and $k < \omega$ then let $G_{\xi k} = \{T_{\xi k}^{\boldsymbol{p}} : \boldsymbol{p} \in G\}$, so that each set $G_{\xi k}$ is $\mathbb{P}_\xi$-generic over $\mathbf{L}$, and $X_{\xi k} = \bigcap_{T \in G_{\xi k}} [T]$ is a singleton $X_{\xi k} = \{x_{\xi k}\}$, whose only element $x_{\xi k} = x_{\xi k}[G] \in 2^\omega$ is a real $\mathbb{P}_\xi$-generic over $\mathbf{L}$. □

The product structure of $\mathbb{P}$ further reflects in the following lemma.

Lemma 11.4. *(In the notation of Definition 11.3.) If $\xi < \omega_1^{\mathbf{L}}$ and $k < \omega$ then*

(i) $x_{\xi k}[G] \notin \mathbf{L}[G\restriction\Delta_{\xi k}]$, *where* $\Delta_{\xi k} = (\omega_1^{\mathbf{L}} \times \omega) \smallsetminus \{\langle \xi, k\rangle\}$,
(ii) $x_{\xi k}[G]$ *is not* $\mathrm{OD}(G\restriction\Delta_\xi)$ *in* $\mathbf{L}[G]$, *where* $\Delta_\xi = (\omega_1^{\mathbf{L}} \smallsetminus \{\xi\}) \times \omega$. □

Proof. To prove (ii) make use of the fact that by construction the ξ-part of the forcing is itself a finite-support product of countably many copies of $\mathbb{P}_\xi$. □

12. Definability of generic reals and non-uniformization model

We continue to argue in terms of Definitions 9.1 and 11.3. The next lemma is similar to Lemma 7 in [9].

Lemma 12.1. *Let $\xi < \omega_1^{\mathbf{L}}$. A real $x \in 2^\omega$ is $\mathbb{P}_\xi$-generic over $\mathbf{L}$ iff $x \in Z_\xi = \bigcap_{\xi<\alpha<\omega_1^{\mathbf{L}}} \bigcup_{U \in \mathbb{U}_\xi^\alpha} [U]$.*

Proof. All sets $\mathbb{U}_\xi^\alpha$ are pre-dense in $\mathbb{P}_\xi$ by Corollary 10.2, therefore any $\mathbb{P}_\xi$-generic real belongs to Z_ξ. On the other hand, if $A \in \mathbf{L}$, $A \subseteq \mathbb{P}_\xi$ is a maximal antichain in $\mathbb{P}_\xi$, then A is countable by Corollary 10.6, and hence easily $A \subseteq \mathbb{P}_\xi^\alpha$ and $A \in \mathfrak{M}_\alpha$ for some α, $\xi < \alpha < \omega_1^{\mathbf{L}}$. But then every tree $U \in \mathbb{U}_\xi^\alpha$ satisfies $U \subseteq^{\mathtt{fin}} \bigcup A$ by Lemma 6.1, and we conclude that $\bigcup_{U \in \mathbb{U}_\xi^\alpha} [U] \subseteq \bigcup_{T \in A} [T]$. $\square$

Corollary 12.2. *In any generic extension of $\mathbf{L}$ with the same ω_1, the set*

$$\mathbb{W} = \{\langle \xi, x\rangle \,:\, \xi < \omega_1^{\mathbf{L}} \wedge x \in 2^\omega \text{ is } \mathbb{P}_\xi\text{-generic over } \mathbf{L}\} \subseteq \omega_1^{\mathbf{L}} \times 2^\omega$$

is Π_1^{HC}, and Π_2^1 in terms of a usual coding system of ordinals $< \omega_1$ by reals.

Proof. Use Lemma 12.1 and Proposition 9.2. $\square$

Now prove that $\mathbf{L}[G]$ contains no $\mathbb{P}_\xi$-generic reals except for the reals $x_{\xi k}[G]$. This is the key property of the forcing extensions considered.

Lemma 12.3. *Let a set $G \subseteq \mathbf{P}$ be $\mathbf{P}$-generic over $\mathbf{L}$. If $\xi < \omega_1^{\mathbf{L}}$ and $x \in \mathbf{L}[G] \cap 2^\omega$ then x is a $\mathbb{P}_\xi$-generic real over $\mathbf{L}$ iff $x \in \{x_{\xi k}[G] \,:\, k < \omega\}$.*

Proof. Otherwise there is a $\mathbf{P}$-real name $\mathbf{c} = \langle C_{ni}\rangle_{n<\omega,\, i=0,1} \in \mathbf{L}$ and a multitree $\boldsymbol{p} \in \mathbf{P} = \mathbf{MT}(\mathbb{p})$ which $\mathbf{P}$-forces that $\mathbf{c}$ is $\mathbb{P}_\xi$-generic over $\mathbf{L}$ while $\mathbf{P}$ itself forces $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\xi k}$, $\forall k$. (Recall that $\dot{\boldsymbol{x}}_{\xi k}$ is a name for $x_{\xi k}[G]$.) We can assume that $\mathbf{c}$ is a name countable in $\mathbf{L}$, by Lemma 11.2. Then there is an ordinal λ, $\xi < \lambda < \omega_1$, such that $\mathbf{c} \in \mathfrak{M}_\lambda$ and each set C_{ni} satisfies $C_{ni} \subseteq \mathbf{P}^\lambda = \mathbf{MT}(\mathbb{p}^\lambda)$ for all n, i.

Further, if $k < \omega$ then, as $\mathbf{P}$ forces that $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\xi k}$, the set D_k of all multitrees $\boldsymbol{p} \in \mathbf{P}$ which directly force $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\xi k}$, is dense in $\mathbf{P}$. Therefore, by Lemma 10.5, we may assume that the same ordinal λ as above satisfies the following: each set $D'_k = D_k \cap \mathbf{P}^\lambda$ is dense in $\mathbf{P}^\lambda$.

Applying Theorem 8.1 with $\mathbb{p} = \mathbb{p}^\lambda$, $\mathbb{u} = \mathbb{u}^\lambda$, $\vartheta = \lambda$, $\boldsymbol{\zeta} = \xi$, we conclude that for each $U \in \mathbb{U}_\xi^\lambda$ the set Q_U of all multitrees $\boldsymbol{v} \in \mathbf{P}^\lambda = \mathbf{MT}(\mathbb{u}^\lambda)$ which directly force $\mathbf{c} \notin [U]$, is dense in $\mathbf{MT}(\mathbb{u}^\lambda \vee \mathbb{p}^\lambda)$, therefore, pre-dense in $\mathbf{P}^{\lambda+1} = \mathbf{MT}(\mathbb{p}^{\lambda+1})$. As obviously $Q_U \in \mathfrak{M}_{\lambda+1}$, we further conclude that Q_U is pre-dense in $\mathbf{P}$ by Lemma 10.1. Therefore $\mathbf{P}$ forces $\mathbf{c} \notin \bigcup_{U \in \mathbb{U}_\xi^\lambda} [U]$, hence, forces that $\mathbf{c}$ is not $\mathbb{P}_\xi$-generic, by Lemma 12.1. But this contradicts to the choice of $\boldsymbol{p}$. $\square$

The results obtained allow us to easily prove Theorem 1.1.

Example 12.4 *(Non-uniformizable Π_1^{HC} set).* Let a set $G \subseteq \mathbf{P}$ be $\mathbf{P}$-generic over $\mathbf{L}$. Consider the set $\mathbb{W}$ of Corollary 12.2 in the model $\mathbf{L}[G]$. First of all $\mathbb{W}$ is Π_1^{HC} in $\mathbf{L}[G]$ by Corollary 12.2. Further we have

$$\mathbb{W} = \{\langle \xi, x_{\xi k}[G]\rangle \,:\, \xi < \omega_1^{\mathbf{L}} \wedge k < \omega\}$$

by Lemma 12.3, and hence all vertical cross-sections of $\mathbb{W}$ are countable. And the set $\mathbb{W}$ is not ROD uniformizable by Lemma 11.4, since by Corollary 10.6 any real in $\mathbf{L}[G]$ belongs to a submodel of the form $\mathbf{L}[G{\restriction}(\zeta \times \omega)]$, where $\zeta < \omega_1^{\mathbf{L}}$. $\square$

Example 12.5 *(Non-uniformizable Π_2^1 set).* Let $\mathbf{WO} \subseteq 2^\omega$ be the Π_1^1 set of codes of countable ordinals, and for $w \in \mathbf{WO}$ let $|w| < \omega_1$ be the ordinal coded by w. In continuation of Example 12.4, we consider

$$\mathbb{W}' = \{\langle w, x\rangle \in \mathbf{WO} \times 2^\omega \,:\, \langle |w|, x\rangle \in \mathbb{W}\},$$

a Π^1_2 set in $\mathbf{L}[G]$. Suppose towards the contrary that, in $\mathbf{L}[G]$, $\mathbb{W}'$ is uniformizable by a ROD set $Q' \subseteq \mathbb{W}'$. As $\omega_1^{\mathbf{L}} = \omega_1$ by Corollary 10.6, for any $\xi < \omega_1$ there is a code $w \in \mathbf{WO} \cap \mathbf{L}$ with $|w| = \xi$. Let w_ξ be the $\le_{\mathbf{L}}$-least of those. Then

$$Q = \{\langle \xi, x\rangle \in \mathbb{W} : \langle w_\xi, x\rangle \in Q'\}$$

is a ROD subset of $\mathbb{W}$ which uniformizes $\mathbb{W}$, contrary to Example 12.4. $\square$ $\square$ (Theorem 1.1)

13. Non-separation model

Here we prove Theorem 1.2. The model we use will be defined on the base of a $\mathbb{P}$-generic extension $\mathbf{L}[G]$ of $\mathbf{L}$. More exactly, it will have the form $\mathbf{L}[G{\restriction}\Delta]$, where $\Delta \subseteq \omega_1^{\mathbf{L}} \times \{0\}$ will itself be a generic set over $\mathbf{L}$.[9]

Let $\mathbb{Q} = \{1, 2, 12\}^{\omega_1^{\mathbf{L}}} \cap \mathbf{L}$ with *countable* support, so that a typical element of $\mathbb{Q}$ is a partial map $q \in \mathbf{L}$ from $\omega_1^{\mathbf{L}}$ to the 3-element set $\{1, 2, 12\}$, with a domain $\mathtt{dom}\, q \subseteq \omega_1^{\mathbf{L}}$ countable in $\mathbf{L}$, that is, just bounded in $\omega_1^{\mathbf{L}}$. (The choice of the 3-element set $\{1, 2, 12\}$ is explained by later considerations, see Definition 13.3.) We order $\mathbb{Q}$ opposite to extension, that is, let $q \le q'$ (meaning: q is stronger) iff $q' \subseteq q$. Thus $\mathbb{Q} \in \mathbf{L}$, and, inside $\mathbf{L}$, $\mathbb{Q}$ is equal to the product $\{1, 2, 12\}^{\omega_1}$ with countable support. Accordingly a $\mathbb{Q}$-generic object is a full $\mathbb{Q}$-generic map $H : \omega_1^{\mathbf{L}} \to \{1, 2, 12\}$.

Recall that $\mathbb{P}$ is a CCC forcing in $\mathbf{L}$ by Corollary 10.6.

Lemma 13.1. *$\mathbb{P}$ remains CCC in any $\mathbb{Q}$-generic extension $\mathbf{L}[H]$ of $\mathbf{L}$, therefore $\mathbb{P} \times \mathbb{Q}$ preserves cardinals over $\mathbf{L}$.*

Proof. Suppose towards the contrary that some $q' \in \mathbb{Q}$ forces the opposite, that is, forces that C is an uncountable antichain in $\mathbb{P}$, where C is a $\mathbb{Q}$-name. Note that, in $\mathbf{L}$, $\mathbb{Q}$ is *countably complete*: if $q_0 \ge q_1 \ge q_2 \ge \dots$ is a sequence of conditions in $\mathbb{Q}$ then there is a condition $q = \bigcup_k q_k \in \mathbb{Q}$ satisfying $q \le q_k$, $\forall k$. Therefore, *arguing in* $\mathbf{L}$, we can define by induction a decreasing sequence $\langle q_\xi\rangle_{\xi<\omega_1}$ in $\mathbb{Q}$ and a sequence of pairwise incompatible conditions $p_\xi \in \mathbb{P}$, such that $q_0 \le q'$ and each q_ξ forces that $p_\xi \in C$. But then $A = \{p_\xi : \xi < \omega_1\} \in \mathbf{L}$ is an uncountable antichain in $\mathbb{P}$, a contradiction. $\square$

Lemma 13.2. *Assume that a set $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over $\mathbf{L}$. Then*

(i) *all reals in $\mathbf{L}[G, H]$ belong to $\mathbf{L}[G]$;*
(ii) *if $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}} \times \omega$ then all reals in $\mathbf{L}[G{\restriction}\Delta, H]$ belong to $\mathbf{L}[G{\restriction}\Delta]$;*
(iii) *if $\Delta \in \mathbf{L}[H]$, $\Delta \subseteq \omega_1^{\mathbf{L}} \times \omega$, and $\langle \xi, k\rangle \in \omega_1^{\mathbf{L}} \times \omega$ then $x_{\xi k}[G] \in \mathbf{L}[G{\restriction}\Delta]$ iff $\langle \xi, k\rangle \in \Delta$.*

Proof. (i) Note that $\mathbb{Q}$ may not be countably complete in $\mathbf{L}[G]$ any more, so that the most elementary way to prove (i) does not work. However consider $\mathbf{L}[G, H]$ as a $\mathbb{P}$-generic extension $\mathbf{L}[H][G]$ of $\mathbf{L}[H]$. Let $x = \dot{\boldsymbol{x}}[G]$ be a real in $\mathbf{L}[H][G]$, where $\dot{\boldsymbol{x}} \in \mathbf{L}[H]$ is a $\mathbb{P}$-real name as in Definition 7.1. But $\mathbb{P}$ is CCC in $\mathbf{L}[H]$ by Lemma 13.1. Therefore we may assume that $\dot{\boldsymbol{x}}$ is hereditarily countable in $\mathbf{L}[H]$, that is, essentially a real. Yet $\mathbf{L}[H]$ has just the same reals as $\mathbf{L}$, so we conclude that $\dot{\boldsymbol{x}} \in \mathbf{L}$ and $x = \dot{\boldsymbol{x}}[G] \in \mathbf{L}[G]$.

The proof of (ii) is similar.

[9] The countable number of instances of each factor $\mathbb{P}_\xi$ in the product $\mathbb{P} = \prod_{\xi<\omega_1^{\mathbf{L}}} \mathbb{P}_\xi{}^{<\omega}$, crucial in the definition of the non-uniformization model above, is irrelevant to the non-separation model. In fact we'll need just one copy of each $\mathbb{P}_\xi$, and the background model $\mathbf{L}[G{\restriction}(\omega_1^{\mathbf{L}} \times \{0\})]$, a submodel $\mathbf{L}[G{\restriction}\Delta]$ $(\Delta \subseteq \omega_1^{\mathbf{L}} \times \{0\})$ of which we'll use to prove Theorem 1.2, is a $(\prod_{\xi<\omega_1^{\mathbf{L}}} \mathbb{P}_\xi)$-generic extension (one copy of each $\mathbb{P}_\xi$) of $\mathbf{L}$ by Lemma 11.1.

(iii) In the nontrivial direction, suppose that $\langle\xi,k\rangle \notin \Delta$. Consider the set $\Delta' = (\omega_1^{\mathbf{L}} \times \omega) \smallsetminus \{\langle\xi,k\rangle\} \in \mathbf{L}$. As obviously $G{\restriction}\Delta \in \mathbf{L}[G{\restriction}\Delta', H]$, any real in $\mathbf{L}[G{\restriction}\Delta]$ belongs to $\mathbf{L}[G{\restriction}\Delta']$ by (ii). But $x_{\xi k}[G] \notin \mathbf{L}[G{\restriction}\Delta']$ by Lemma 11.4. □

Recall that if $\nu \in \mathbf{Ord}$ then the ordinal product 2ν is considered as the ordered sum of ν copies of $2 = \{0,1\}$. Thus if $\nu = \lambda + m$, where λ is a limit ordinal or 0 and $m < \omega$, then $2\nu = \lambda + 2m$ and $2\nu + 1 = \lambda + 2m + 1$.

Definition 13.3. If $H : \omega_1^{\mathbf{L}} \to \{1,2,12\}$ then let

$$\begin{aligned} \mathbb{1}_H &= \{\nu < \omega_1^{\mathbf{L}} : H(\nu) = 1\}, \qquad \mathbb{2}_H = \{\nu < \omega_1^{\mathbf{L}} : H(\nu) = 2\}, \\ \mathbb{12}_H &= \{\nu < \omega_1^{\mathbf{L}} : H(\nu) = 12\}, \\ \Xi_H &= \{2\nu : \nu \in \mathbb{1}_H \cup \mathbb{12}_H\} \cup \{2\nu+1 : \nu \in \mathbb{2}_H \cup \mathbb{12}_H\}, \\ \Delta_H &= \Xi_H \times \{0\} = \{\langle\xi,0\rangle : \xi \in \Xi_H\}. \end{aligned}$$

If a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$ then consider the model $\mathbf{L}[G{\restriction}\Delta_H]$ and let $\mathrm{HC}(G,H) = (\mathrm{HC})^{\mathbf{L}[G\restriction\Delta_H]}$. □

Note that $\mathbf{L}[G{\restriction}\Delta_H]$ is not necessarily a submodel of $\mathbf{L}[G]$ since the set Δ_H does not necessarily belong to $\mathbf{L}[G]$ (unless $H \in \mathbf{L}[G]$); but we have $\mathbf{L}[G{\restriction}\Delta_H] \subseteq \mathbf{L}[G][H]$, of course.

Theorem 13.4. *Let a set $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $\mathbf{L}$ and $H : \omega_1^{\mathbf{L}} \to \{1,2,12\}$ be a map $\mathbb{Q}$-generic over $\mathbf{L}[G]$. Then it is true in $\mathbf{L}[G{\restriction}\Delta_H]$ that $\mathbb{1}_H$ and $\mathbb{2}_H$ are disjoint Π_2^{HC} sets not separable by disjoint $\mathbf{\Sigma}_2^{\mathrm{HC}}$ sets.*

By boldface $\mathbf{\Sigma}_2^{\mathrm{HC}}$ we always mean Σ_2 definability in HC with any reals as parameters.

Proof. To see that, say, $\mathbb{1}_H$ is Π_2^{HC} in $\mathbf{L}[G{\restriction}\Delta_H]$, prove that the equality

$$\mathbb{1}_H = \{\nu < \omega_1 : \neg\,\exists\, x\,(\langle 2\nu+1, x\rangle \in \mathbb{W})\}$$

holds in $\mathbf{L}[G{\restriction}\Delta_H]$, where $\mathbb{W}$ is the Π_1^{HC} set of Corollary 12.2. (For $\mathbb{2}_H$ it would be $\langle 2\nu, x\rangle \in \mathbb{W}$ in the displayed formula.)

First suppose that $\nu < \omega_1^{\mathbf{L}}$, $\xi = 2\nu+1$, $x \in \mathbf{L}[G{\restriction}\Delta_H] \cap 2^\omega$, and $\mathbb{W}(\xi,x)$ holds in $\mathbf{L}[G{\restriction}\Delta_H]$; prove that $\nu \notin \mathbb{1}_H$. Note that $x \in \mathbf{L}[G]$ by Lemma 13.2(i). Further, by definition x is $\mathbb{P}_\xi$-generic over $\mathbf{L}$, therefore $x = x_{\xi k}[G]$ for some k by Lemma 12.3, and we have $\langle\xi,k\rangle \in \Delta_H$ by Lemma 13.2(iii). Therefore $\xi \in \Xi_H$ and $k = 0$. But then $\nu \in \mathbb{2}_H \cup \mathbb{12}_H$, so $\nu \notin \mathbb{1}_H$, as required.

To prove the converse, suppose that $\nu \notin \mathbb{1}_H$, so that $\nu \in \mathbb{2}_H \cup \mathbb{12}_H$. Then $\xi = 2\nu+1 \in \Xi_H$, and hence $x = x_{\xi 0} \in \mathbf{L}[G{\restriction}\Delta_H]$. It follows that $\langle\xi,x\rangle = \langle 2\nu+1, x\rangle \in \mathbb{W}$ by Lemma 12.3, as required.

To prove the non-separability claim, suppose towards the contrary that, in $\mathbf{L}[G{\restriction}\Delta_H]$, the sets $\mathbb{1}_H, \mathbb{2}_H$ are separated by disjoint $\mathbf{\Sigma}_2^{\mathrm{HC}}$ sets $A, B \subseteq \omega_1 = \omega_1^{\mathbf{L}}$. The sets A, B are defined, in the set $\mathrm{HC}(G,H) = (\mathrm{HC})^{\mathbf{L}[G\restriction\Delta_H]}$, by Σ_2 formulas, resp., $\varphi(a,\xi), \psi(a,\xi)$, with a real parameter $a \in \mathbf{L}[G{\restriction}\Delta_H] \cap 2^\omega$; hence, $a \in \mathbf{L}[G]$ by Lemma 13.2. Let $\lambda < \omega_1^{\mathbf{L}}$ be a limit ordinal such that $a \in \mathbf{L}[G{\restriction}\Delta_{H\lambda}]$, where $\Delta_{H\lambda} = \Delta_H \cap (\lambda \times \{0\}) \in \mathbf{L}$.

If $K : \omega_1^{\mathbf{L}} \to \{1,2,12\}$ (for instance, $K = H$), then let

$$A_K^* = \{\xi < \omega_1^{\mathbf{L}} : \varphi(a,\xi)^{\mathrm{HC}(G,K)}\}, \quad B_K^* = \{\xi < \omega_1^{\mathbf{L}} : \psi(a,\xi)^{\mathrm{HC}(G,K)}\}. \tag{$*$}$$

Then by definition $\mathbb{1}_H \subseteq A = A_H^*$, $\mathbb{2}_H \subseteq B = B_H^*$, and $A_H^* \cap B_H^* = \varnothing$. Fix a condition $q_0 \in \mathbb{Q}$ compatible with H (here meaning that simply $q_0 \subset H$), which forces the choice of A, B, so that,

(†) if $K : \omega_1^{\mathbf{L}} \to \{1, 2, 12\}$ is a map $\mathbb{Q}$-generic over $\mathbf{L}[G]$ and compatible with q_0, then $\mathbb{1}_K \subseteq A^*_K$, $\mathbb{2}_K \subseteq B^*_K$, and $A^*_K \cap B^*_K = \varnothing$.

We may assume that $\mathtt{dom}\, q_0 \subseteq \lambda$, otherwise just increase λ.

Let ν_0 be any ordinal, $\lambda \le \nu_0 < \omega_1$. Consider the maps $H_1, H_2, H_{12} : \omega_1^{\mathbf{L}} \to \{1, 2, 12\}$, generic over $\mathbf{L}[G]$, compatible with q_0, and satisfying $H_i(\nu_0) = i$, $i = 1, 2, 12$, and $H_1(\nu) = H_2(\nu) = H_{12}(\nu)$ for all $\nu \ne \nu_0$. Then $\Xi_{H_{12}} = \Xi_{H_1} \cup \{2\nu_0 + 1\}$ by Definition 13.3, hence, $\mathbf{L}[G{\restriction}\Delta_{H_1}] \subseteq \mathbf{L}[G{\restriction}\Delta_{H_{12}}]$. It follows by Shoenfield that $A^*_{H_1} \subseteq A^*_{H_{12}}$ (since φ is an essentially Σ^1_3 formula), therefore $\mathbb{1}_{H_1} \subseteq A^*_{H_1} \subseteq A^*_{H_{12}}$ by (†). We conclude that $\nu_0 \in A^*_{H_{12}}$, just because $\nu_0 \in \mathbb{1}_{H_1}$ by the choice of H_1. And we have $\nu_0 \in B^*_{H_{12}}$ by a similar argument (with H_2). Thus $A^*_{H_{12}} \cap B^*_{H_{12}} \ne \varnothing$, contrary to (†). The contradiction ends the proof. □

Example 13.5 *(Non-separable Π^1_3 sets).* In the notation of Example 12.5, let

$$X = \{w_\xi \,:\, \xi \in \mathbb{1}_H\} \quad \text{and} \quad Y = \{w_\xi \,:\, \xi \in \mathbb{2}_H\}.$$

The sets $X, Y \subseteq \mathbf{WO} \cap \mathbf{L}$ are $\Pi_2^{\mathrm{HC}(G,H)}$ together with $\mathbb{1}_H$ and $\mathbb{2}_H$, and hence Π^1_3, and $X \cap Y = \varnothing$. (Recall that $\mathrm{HC}(G, H) = (\mathrm{HC})^{\mathbf{L}[G{\restriction}\Delta_H]}$, Definition 13.3.) Suppose towards the contrary that $X', Y' \subseteq 2^\omega$ are disjoint sets in $\mathbf{\Sigma}^1_3$, hence in $\mathbf{\Sigma}_2^{\mathrm{HC}(G,H)}$, such that $X \subseteq X'$ and $Y \subseteq Y'$. Then

$$A = \{\xi < \omega_1^{\mathbf{L}} \,:\, w_\xi \in X'\} \quad \text{and} \quad B = \{\xi < \omega_1^{\mathbf{L}} \,:\, w_\xi \in Y'\}$$

are disjoint sets in $\mathbf{\Sigma}_2^{\mathrm{HC}(G,H)}$, and we have $\mathbb{1}_H \subseteq A$ and $\mathbb{2}_H \subseteq B$ by construction, contrary to Theorem 13.4. □ □ (Theorem 1.2)

14. The failure of $\mathbf{\Sigma}^1_3$ separation persists

As the $\mathbf{\Pi}^1_3$ Separation, known to be true in $\mathbf{L}$, fails in a certain generic extension of $\mathbf{L}$ by Theorem 1.2, one may ask what happens with the $\mathbf{\Sigma}^1_3$ Separation, known to fail already in $\mathbf{L}$, in that same or similar extension.

Here, first of all, we can easily manufacture a version of the model of Section 13, where the $\mathbf{\Sigma}^1_3$ Separation fails for very similar reasons. Namely, coming back to Definition 13.3, we make use of the sets

$$\Xi'_H = \{2\nu \,:\, \nu \in \mathbb{1}_H\} \cup \{2\nu + 1 \,:\, \nu \in \mathbb{2}_H\}$$

instead of Ξ_H, and $\Delta'_H = \Xi'_H \times \{0\}$. Then, similarly to Theorem 13.4, it is true in the model $\mathbf{L}[G{\restriction}\Delta'_H]$ that $\mathbb{1}_H$ and $\mathbb{2}_H$ are disjoint Σ_2^{HC} sets not separable by disjoint $\mathbf{\Pi}_2^{\mathrm{HC}}$ sets.

Moreover, it is possible to maintain both constructions in the same model, so that Separation fails in the model for both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$, see [5](A).

Yet it is perhaps not less interesting to prove that a counterexample to the $\mathbf{\Sigma}^1_3$ Separation in $\mathbf{L}$ survives in the extension say of the type considered in Section 13.

Theorem 14.1. *Let a set $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $\mathbf{L}$ and $H : \omega_1^{\mathbf{L}} \to \{1, 2, 12\}$ be a map $\mathbb{Q}$-generic over $\mathbf{L}[G]$. Suppose that, in $\mathbf{L}$, $X, Y \subseteq 2^\omega$ are disjoint Σ^1_3 sets not separable by disjoint $\mathbf{\Pi}^1_3$ sets. Then it holds in $\mathbf{L}[G{\restriction}\Delta_H]$ that X, Y are still Σ^1_3 sets not separable by disjoint $\mathbf{\Pi}^1_3$ sets, and hence $\mathbf{\Sigma}^1_3$ Separation fails.*

Proof. That X, Y are still Σ^1_3 sets in $\mathbf{L}[G{\restriction}\Delta_H]$ holds by standard arguments, therefore we can focus on the non-separability claim.

14.1. Contrary assumption and notation

Suppose to the contrary that, in $\mathbf{L}[G{\restriction}\Delta_H]$, the sets X, Y are separable by disjoint $\mathbf{\Pi}^1_3$ sets $A, B \subseteq 2^\omega \cap \mathbf{L}$, so that $X \subseteq A$, $Y \subseteq B$, $A \cap B = \varnothing$. These sets A, B are defined, in $\mathbf{L}[G{\restriction}\Delta_H]$, by $\mathit{\Pi}^1_3$ formulas, resp., $\varphi(a, \cdot)$, $\psi(a, \cdot)$, with a real parameter $a \in \mathbf{L}[G{\restriction}\Delta_H] \cap 2^\omega$. We let, for any map $K : \omega_1^{\mathbf{L}} \to \{1, 2, 12\}$,

$$\left.\begin{aligned} A^*_K &= \{x \in 2^\omega \cap \mathbf{L} \,:\, \varphi(a, x)^{\mathbf{L}[G{\restriction}\Delta_K]}\}\,, \\ B^*_K &= \{x \in 2^\omega \cap \mathbf{L} \,:\, \psi(a, x)^{\mathbf{L}[G{\restriction}\Delta_K]}\}\,, \end{aligned}\right\} \tag{**}$$

so that A^*_K and B^*_K are $\mathbf{\Pi}^1_3$ sets in $\mathbf{L}[G{\restriction}\Delta_K]$, and, with $K = H$, we have $X \subseteq A = A^*_H$, $Y \subseteq B = B^*_H$, $A^*_H \cap B^*_H = \varnothing$.

Let $\lambda < \omega_1^{\mathbf{L}}$ be a limit ordinal such that $a \in \mathbf{L}[G{\restriction}\Delta_{H\lambda}]$, where $\Delta_{H\lambda} = \Delta_H \cap (\lambda \times \{0\}) \in \mathbf{L}$ (since $\mathbf{L}[H]$ does not add new reals to $\mathbf{L}$), and let $\dot{a} \in \mathbf{L}$ be a $(\mathbb{P}{\restriction}\Delta_{H\lambda})$-real name, countable in $\mathbf{L}$ and such that $a = \dot{a}[G{\restriction}\Delta_{H\lambda}]$ (Lemma 11.2).

14.2. Reduction to a constructible map

We are going to define a map $J : \omega_1^{\mathbf{L}} \to \{1, 2, 12\}$, which, unlike H above, belongs to $\mathbf{L}$, but still the sets X, Y are separable by disjoint $\mathbf{\Pi}^1_3$ sets in $\mathbf{L}[G{\restriction}\Delta_J]$. To get such a map, let us fix a condition $q_0 \in \mathbb{Q}$ compatible with H which $\mathbb{Q}$-forces, over $\mathbf{L}[G]$, the choice of A, B, so that

(‡) if $K : \omega_1^{\mathbf{L}} \to \{1, 2, 12\}$ is a map $\mathbb{Q}$-generic over $\mathbf{L}[G]$ and $q_0 \subset K$ then $X \subseteq A^*_K$, $Y \subseteq B^*_K$, $A^*_K \cap B^*_K = \varnothing$.

We may assume that $\mathtt{dom}\, q_0 \subseteq \lambda$, otherwise just increase λ. Then $q_1 = H{\restriction}\lambda$ is a condition in $\mathbb{Q}$ stronger than q_0 and compatible with H. Recall that $\mathbb{Q} \in \mathbf{L}$.

If $\lambda < \vartheta \le \omega_1^{\mathbf{L}}$ then let a map H_ϑ be defined so that still $q_1 \subset H_\vartheta$—hence $H_\vartheta{\restriction}\lambda = H{\restriction}\lambda = q_1$, and also $H_\vartheta{\restriction}(\omega_1^{\mathbf{L}} \smallsetminus \vartheta) = H{\restriction}(\omega_1^{\mathbf{L}} \smallsetminus \vartheta)$, but $H_\vartheta(\nu) = 2$ whenever $\lambda \le \nu < \vartheta$. For instance $H_\lambda = H$, and if $\lambda < \vartheta < \omega_1^{\mathbf{L}}$ strictly then H_ϑ is still $\mathbb{Q}$-generic over $\mathbf{L}[G]$. Let $J = H_{\omega_1^{\mathbf{L}}}$; J is a map $\omega_1^{\mathbf{L}} \to \{1, 2, 12\}$, $J{\restriction}\lambda = q_1$, and $J(\nu) = 2$ for all $\nu \ge \lambda$. (J is not a $\mathbb{Q}$-generic map, of course.)

Lemma 14.2. *$J \in \mathbf{L}$, $X \subseteq A^*_J$, $Y \subseteq B^*_J$, and $A^*_J \cap B^*_J = \varnothing$.*

Proof. If $\vartheta \le \gamma < \omega_1^{\mathbf{L}}$ then obviously $\Xi_{H_\vartheta} \subseteq \Xi_{H_\gamma}$ and $\Delta_{H_\vartheta} \subseteq \Delta_{H_\gamma}$, therefore, $X \subseteq A^*_{H_\gamma} \subseteq A^*_{H_\vartheta}$ and $Y \subseteq B^*_{H_\gamma} \subseteq B^*_{H_\vartheta}$ by Shoenfield and (‡). Sending ϑ to $\omega_1^{\mathbf{L}}$, we easily obtain the results required. $\square$

Let's look closer at the map $J = H_{\omega_1^{\mathbf{L}}}$. The set $\Delta_J = \Delta_{H\lambda} \cup \big((\omega_1^{\mathbf{L}} \smallsetminus \lambda) \times \{0\}\big)$ belongs to $\mathbf{L}$, where, we recall, $\Delta_{H\lambda} = \Delta_H \cap (\lambda \times \{0\}) \in \mathbf{L}$. It follows that $\mathbf{L}[G{\restriction}\Delta_J] \subseteq \mathbf{L}[G]$. The parameter a in (**) belongs to $\mathbf{L}[G{\restriction}\Delta_J]$ and the sets A^*_J and B^*_J of (**) are disjoint $\mathbf{\Pi}^1_3$ sets in $\mathbf{L}[G{\restriction}\Delta_J]$ which separate X and Y by Lemma 14.2.

14.3. Evaluation of forcing

By Lemma 11.1, the model $\mathbf{L}[G{\restriction}\Delta_J]$ is a $(\mathbb{P}{\restriction}\Delta_J)$-generic extension of $\mathbf{L}$, where $\mathbb{P}{\restriction}\Delta_J = \{\boldsymbol{p} \in \mathbb{P} : |\boldsymbol{p}| \subseteq \Delta_J\} \in \mathbf{L}$ is a subforcing of $\mathbb{P}$. To estimate the complexity of the $(\mathbb{P}{\restriction}\Delta_J)$-forcing relation in $\mathbf{L}$, we introduce an auxiliary forcing relation $\boldsymbol{p}\, \mathtt{forc}\, \varphi$, where $\boldsymbol{p} \in \mathbb{P}{\restriction}\Delta_J$ while φ is a formula of certain type.

Let's define some classes of formulas.

Let $\widetilde{\Sigma}^1_1$ consist of all Σ^1_1 formulas of the language of the 2nd order PA, with variables of the real type over 2^ω, and with $(\mathbb{P}{\restriction}\Delta_J)$-real names $\mathbf{c}_i \in \mathbf{L}$, countable in $\mathbf{L}$, as parameters. The collection $\widetilde{\Pi}^1_1$ is defined similarly. Let $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1$ be the closure of $\widetilde{\Sigma}^1_1 \cup \widetilde{\Pi}^1_1$ under $\neg, \wedge, \vee$ and both quantifiers over ω.

By induction, we define $\widetilde{\Sigma}^1_{n+1}$, resp., $\widetilde{\Pi}^1_{n+1}$ $(n \ge 1)$ to consist of all formulas of the form $\exists x\, \varphi(x)$, resp., $\forall x\, \varphi(x)$, where x is a variable over 2^ω and φ is $\widetilde{\Pi}^1_n$, resp., $\widetilde{\Sigma}^1_n$ (or φ is $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1$ whenever $n = 1$, in both cases). If φ belongs to $\widetilde{\Sigma}^1_n$, $n \ge 2$, then let φ^- be the result of canonical transformation of $\neg\varphi$ to $\widetilde{\Pi}^1_n$, and similarly for $\varphi \in \widetilde{\Pi}^1_n$. Separately, if $\varphi \in (\widetilde{\Sigma}+\widetilde{\Pi})^1_1$ then φ^- is just $\neg\varphi$.

The definition of the relation $\boldsymbol{p}\,\mathtt{forc}\,\varphi$ in (A), (B), (C) goes on by induction on the complexity of formulas φ in $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1 \cup \widetilde{\Sigma}^1_2 \cup \widetilde{\Pi}^1_2 \cup \widetilde{\Sigma}^1_3 \cup \widetilde{\Pi}^1_3 \cup \dots$.

(A) Let $\varphi = \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ belong to $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1$. We define $\boldsymbol{p}\,\mathtt{forc}\,\varphi$ iff $\boldsymbol{p}$ $(\mathbb{P}^\alpha{\restriction}\Delta_J)$-forces φ in the usual sense over $\mathfrak{M}_\alpha$, where, we recall, $\mathbb{P}^\alpha = \mathbf{MT}(\mathbb{p}^\alpha)$, and α is any ordinal such that $\boldsymbol{p} \in \mathbb{P}^\alpha{\restriction}\Delta_J$, the condition $q_1 = H{\restriction}\lambda$ belongs to $\mathfrak{M}_\alpha$, and each name $\mathbf{c}_i$ in φ is a $(\mathbb{P}^\alpha{\restriction}\Delta_J)$-real name.

Lemma 14.3. *The definition in* (A) *does not depend on the choice of* α.

Proof. [Lemma] It suffices to prove that if α is as indicated then $\boldsymbol{p}$ $(\mathbb{P}^\alpha{\restriction}\Delta_J)$-forces φ over $\mathfrak{M}_\alpha$ iff $\boldsymbol{p}$ $(\mathbb{P}{\restriction}\Delta_J)$-forces φ over $\mathbf{L}$ in the usual sense.

To prove $\Longrightarrow$, suppose that $\boldsymbol{p}$ does not $(\mathbb{P}{\restriction}\Delta_J)$-force φ over $\mathbf{L}$, so that (in a bigger universe) there is a set $\mathbf{g} \subseteq \mathbb{P}{\restriction}\Delta_J$, $(\mathbb{P}{\restriction}\Delta_J)$-generic over $\mathbf{L}$, containing $\boldsymbol{p}$, and $\varphi[\mathbf{g}]$ is false in $\mathbf{L}[\mathbf{g}]$. Then $\mathbf{g}' = \mathbf{g} \cap (\mathbb{P}^\alpha{\restriction}\Delta_J)$ is $(\mathbb{P}^\alpha{\restriction}\Delta_J)$-generic over $\mathfrak{M}_\alpha$ by Corollary 10.4, and the formula $\varphi[\mathbf{g}] = \varphi[\mathbf{g}'] = \varphi(\mathbf{c}_1[\mathbf{g}'], \dots, \mathbf{c}_n[\mathbf{g}'])$ is false in $\mathfrak{M}_\alpha[\mathbf{g}']$ by Shoenfield. Thus $\boldsymbol{p}$ does not $(\mathbb{P}^\alpha{\restriction}\Delta_J)$-force φ over $\mathfrak{M}_\alpha$.

Conversely if $\boldsymbol{p}$ does not $(\mathbb{P}^\alpha{\restriction}\Delta_J)$-force φ over $\mathfrak{M}_\alpha$ then there is a stronger condition $\boldsymbol{q} \in \mathbb{P}^\alpha{\restriction}\Delta_J$ which forces $\neg\varphi$ over $\mathfrak{M}_\alpha$. Then $\boldsymbol{q}$ $(\mathbb{P}{\restriction}\Delta_J)$-forces $\neg\varphi$ over $\mathbf{L}$ by the above, hence $\boldsymbol{p}$ does not $(\mathbb{P}{\restriction}\Delta_J)$-force φ over $\mathbf{L}$. $\square$

(B) We define that $\boldsymbol{p}\,\mathtt{forc}\,\exists x\, \varphi(x)$ (x being a variable over 2^ω), where φ is a formula in $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1$ or $\widetilde{\Pi}^1_n$, $n \ge 2$, iff there is a $(\mathbb{P}{\restriction}\Delta_J)$-real name $\mathbf{c} \in \mathbf{L}$, countable in $\mathbf{L}$ and such that $\boldsymbol{p}\,\mathtt{forc}\,\varphi(\mathbf{c})$.

(C) We define that $\boldsymbol{p}\,\mathtt{forc}\,\varphi$, where φ belongs to $\widetilde{\Pi}^1_n$, $n \ge 2$, iff no condition $\boldsymbol{q} \in \mathbb{P}{\restriction}\Delta_J$ stronger than $\boldsymbol{p}$ satisfies $\boldsymbol{p}\,\mathtt{forc}\,\varphi^-$.

The following lemma shows that $\mathtt{forc}$ is an adequate approximation of the true $(\mathbb{P}{\restriction}\Delta_J)$-forcing over $\mathbf{L}$ as the ground model.

Lemma 14.4. *Suppose that* $\boldsymbol{p} \in \mathbb{P}{\restriction}\Delta_J$, φ *is a closed formula in* $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1 \cup \widetilde{\Sigma}^1_2 \cup \widetilde{\Pi}^1_2 \cup \widetilde{\Sigma}^1_3 \cup \widetilde{\Pi}^1_3 \cup \dots$, *and* $\mathbf{g} \subseteq \mathbb{P}{\restriction}\Delta_J$ *is a set* $(\mathbb{P}{\restriction}\Delta_J)$*-generic over* $\mathbf{L}$. *Then* $\varphi[\mathbf{g}]$ *is true in* $\mathbf{L}[\mathbf{g}]$ *iff there is a multitree* $\boldsymbol{p} \in \mathbf{g}$ *satisfying* $\boldsymbol{p}\,\mathtt{forc}\,\varphi$.

Proof. We argue by induction. If φ is a formula in $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1$ then $\boldsymbol{p}\,\mathtt{forc}\,\varphi$ iff $\boldsymbol{p}$ $(\mathbb{P}{\restriction}\Delta_J)$-forces φ over $\mathbf{L}$ in the usual sense (see the proof of Lemma 14.3), so the result immediately follows. The steps $\Pi^1_n \to \Sigma^1_{n+1}$ are justified by Lemma 11.2, on the base of (B). The steps $\Sigma^1_n \to \Pi^1_n$ $(n \ge 2)$ are justified, on the base of (C), the same way as in the case of usual forcing. $\square$ (Lemma)

The next lemma evaluates the complexity of the relation $\mathtt{forc}$.

Lemma 14.5 *(in* $\mathbf{L}$*). Let* $\varphi(v_1, \dots, v_m)$ *be a formula in* $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1$ *or in* $\widetilde{\Sigma}^1_n \cup \widetilde{\Pi}^1_n$, $n \ge 2$, *with exactly* m *free variables* $v_1, \dots, v_m$, *all of them over* 2^ω. *Then*

(i) *if* φ *belong to* $(\widetilde{\Sigma}+\widetilde{\Pi})^1_1$ *then the set*

$$W_\varphi = \{\langle \boldsymbol{p}, \mathbf{c}_1, \dots, \mathbf{c}_m\rangle \,:\, \boldsymbol{p} \in \mathbb{P}{\restriction}\Delta_J \wedge \boldsymbol{p}\,\mathtt{forc}\,\varphi(\mathbf{c}_1, \dots, \mathbf{c}_m)\}$$

belongs to $\Delta^{\mathrm{HC}}_1(q_1)$*;*

(ii) *if* φ *is a* $\widetilde{\Sigma}^1_n$ *formula,* $n \ge 2$*, then the set* W_φ *belongs to* $\Sigma^{\mathrm{HC}}_{n-1}(q_1)$*;*

(iii) *if* φ *is a* $\widetilde{\Pi}^1_n$ *formula,* $n \ge 2$*, then the set* W_φ *belongs to* $\Pi^{\mathrm{HC}}_{n-1}(q_1)$*.*

Proof. (i) Any forcing relation over a countable transitive model $\mathfrak{M}$ is known to be Δ^{HC}_1 uniformly on $\mathfrak{M}$ and the forcing notion $P \in \mathfrak{M}$ involved, while $q_1 = H{\restriction}\gamma$ is a parameter to naturally define J and Δ_J. On the other hand, $\alpha \mapsto \mathfrak{M}_\alpha$ and $\alpha \mapsto \mathbf{MT}(\mathbb{p}^\alpha)$ are Δ^{HC}_1 maps by Proposition 9.2. And finally the choice of α itself, given a multitree and a finite set of names, can be made in both Σ^{HC}_1 and Π^{HC}_1 way, see (A) above.

To prove (ii) and (iii) use induction based on (B), (C) above. □ (Lemma)

14.4. Constant names

Now we introduce a subset of *constant* names among the $(\mathbb{P}{\restriction}\Delta)$-real names (see Section 11). Suppose that $x \in 2^\omega \cap \mathbf{L}$. Define a $(\mathbb{P}{\restriction}\Delta_J)$-real name $\underline{x} = \langle C_{ni}(x)\rangle_{n<\omega,\, i<2}$ (a canonical name for x), where

$$C_{ni}(x) = \begin{cases} \varnothing, \text{ whenever } x(n) \ne i\,, \\ \{\mathbf{\Lambda}\}, \text{ whenever } x(n) = i\,, \end{cases}$$

and $\mathbf{\Lambda}$ is the default multitree with $|\mathbf{\Lambda}| = \varnothing$, see Section 4.

Then $\underline{x} \in \mathbf{L}$ and $\underline{x}[G] = x$ for any non-empty set $G \subseteq \mathbb{P}{\restriction}\Delta_J$.

Recall that $\dot{a} \in \mathbf{L}$ is a $(\mathbb{P}{\restriction}\Delta_{H\lambda})$-real name, and by construction it is a $(\mathbb{P}{\restriction}\Delta_J)$-real name as well (since $q_1 = H{\restriction}\lambda = J{\restriction}\lambda$), and we have

$$a \,=\, \dot{a}[G{\restriction}\Delta_{H\lambda}] \,=\, \dot{a}[G{\restriction}\Delta_J] \,\in\, \mathbf{L}[G{\restriction}\Delta_J]\,,$$

see Step 1 above. The next corollary deals with Π^1_3 formulas φ, ψ of Step 1. Their adjusted negations φ^- and ψ^- are Σ^1_3, of course.

Corollary 14.6. *(Of Lemma 14.5.) It is true in* $\mathbf{L}$ *that the sets*

$$\Phi \,=\, \{\langle \boldsymbol{r}, x\rangle \,:\, \boldsymbol{r} \in \mathbb{P}{\restriction}\Delta_J \wedge x \in 2^\omega \wedge \boldsymbol{r}\,\mathtt{forc}\,\varphi^-(\dot{a}, \underline{x})\}\,,$$
$$\Psi \,=\, \{\langle \boldsymbol{r}, y\rangle \,:\, \boldsymbol{r} \in \mathbb{P}{\restriction}\Delta_J \wedge y \in 2^\omega \wedge \boldsymbol{r}\,\mathtt{forc}\,\psi^-(\dot{a}, \underline{y})\}$$

belong to $\Sigma^{\mathrm{HC}}_2(q_1, \dot{a})$*, hence, to* $\mathbf{\Sigma}^1_3$*.* □

14.5. Final argument

Here we accomplish the proof of Theorem 14.1. Recall that $J \in \mathbf{L}$ is a map $\omega_1^{\mathbf{L}} \to \{1, 2, 12\}$, $G \subseteq \mathbb{P}$ is a set $\mathbb{P}$-generic over $\mathbf{L}$, and we deal with the model $\mathbf{L}[G{\restriction}\Delta_J]$ (see Definition 13.3 on Δ_J), which is a $G{\restriction}\Delta_J$-generic extension of $\mathbf{L}$. Moreover by Lemma 14.2 the following is true in $\mathbf{L}[G{\restriction}\Delta_J]$:

$$\forall\, x \in X\, \varphi(a, x)\,, \quad \forall\, y \in Y\, \psi(b, y)\,, \quad \forall\, z \in 2^\omega\, \neg\,(\varphi(a, z) \wedge \psi(b, z))\,.$$

This is $\mathbb{P}{\restriction}\Delta_J$-forced by a multitree $\boldsymbol{p} \in G{\restriction}\Delta_J$, or, to be more pedantic, the multitree $\boldsymbol{p}$ $(\mathbb{P}{\restriction}\Delta_J)$-forces $\varphi(\dot{a}, \underline{x}) \wedge \psi(\dot{a}, \underline{y})$ over $\mathbf{L}$, whenever $x \in X$ and $y \in Y$, and also forces $\neg\,(\varphi(\dot{a}, \underline{z}) \wedge \psi(\dot{a}, \underline{z}))$, whenever $z \in 2^\omega$.

Corollary 14.7. *(In* $\mathbf{L}$*.) Assume that* $\boldsymbol{q} \in \mathbb{P}{\restriction}\Delta_J$, $\boldsymbol{q} \le \boldsymbol{p}$, *and* $z \in 2^\omega$. *Then*

(i) *there is a multitree* $\boldsymbol{r} \in \mathbb{P}{\restriction}\Delta_J$, $\boldsymbol{r} \le \boldsymbol{q}$, *such that* $\langle \boldsymbol{r}, z\rangle \in \Phi \cup \Psi$;
(ii) *if* $z \in X$ *then* $\langle \boldsymbol{q}, z\rangle \notin \Phi$, *and similarly if* $z \in Y$ *then* $\langle \boldsymbol{q}, z\rangle \notin \Psi$.

Proof. (i) As $\boldsymbol{p}$ forces $\neg(\varphi(\dot a, \underline{z}) \wedge \psi(\dot a, \underline{z}))$, there is a condition $\boldsymbol{r} \in \mathbb{P}{\restriction}\Delta_J$, $\boldsymbol{r} \le \boldsymbol{q}$, which $(\mathbb{P}{\restriction}\Delta_J)$-forces $\neg\varphi(\dot a, \underline{z})$ or forces $\neg\psi(\dot a, \underline{z})$. By Lemma 14.4 we can assume that $\boldsymbol{r}\,\mathtt{forc}\,\varphi^-(\dot a, \underline{z})$ or $\boldsymbol{r}\,\mathtt{forc}\,\psi^-(\dot a, \underline{z})$, that is, $\langle \boldsymbol{r}, z\rangle \in \Phi \cup \Psi$.

(ii) Suppose that $\langle \boldsymbol{q}, z\rangle \in \Phi$. Then by definition $\boldsymbol{q}\,\mathtt{forc}\,\varphi^-(\dot a, \underline{z})$, hence $\boldsymbol{q}$ $(\mathbb{P}{\restriction}\Delta_J)$-forces $\neg\varphi(\dot a, \underline{z})$ by Lemma 14.4. Then $z \notin X$ by the choice of $\boldsymbol{p}$. □

Corollary 14.7(i) allows to define, in $\mathbf{L}$, a transfinite sequence of pairs $\langle \boldsymbol{q}_\xi, z_\xi\rangle$, $\xi < \omega_1^{\mathbf{L}}$, such that $\boldsymbol{q}_\xi \in \mathbb{P}{\restriction}\Delta_J$, $\boldsymbol{q}_\xi \le \boldsymbol{p}$, $\langle \boldsymbol{q}_\xi, z_\xi\rangle \in \Phi \cup \Psi$ for all ξ, and $\{z_\xi : \xi < \omega_1^{\mathbf{L}}\} = 2^\omega \cap \mathbf{L}$. In addition, by Corollary 14.6, we can maintain the construction so that the sequence belongs to $\Sigma_2^{\mathrm{HC}}(q_1, \dot a)$ together with the sets Φ, Ψ, hence in fact to $\Delta_2^{\mathrm{HC}}(q_1, \dot a)$ as the domain ω_1 is a Δ_1^{HC} set. It follows that the sets

$$A' = \{z \in 2^\omega : \exists \xi < \omega_1^{\mathbf{L}}\,(\langle \boldsymbol{q}_\xi, z_\xi\rangle \in \Psi \wedge z = z_\xi)\},$$
$$B' = \{z \in 2^\omega : \exists \xi < \omega_1^{\mathbf{L}}\,(\langle \boldsymbol{q}_\xi, z_\xi\rangle \in \Phi \wedge z = z_\xi)\}$$

belong to $\Sigma_2^{\mathrm{HC}}(q_1, \dot a)$, hence, to $\mathbf{\Sigma}_3^1$, and satisfy $A' \cup B' = 2^\omega\ \mathbf{L}$. Therefore, by the Reduction theorem of Addison, it is true in $\mathbf{L}$ that there exist *disjoint* $\mathbf{\Sigma}_3^1$ sets $A \subseteq A'$ and $B \subseteq B'$ such that $A \cup B = A' \cup B' = 2^\omega$, so that both A and B in fact belong to $\mathbf{\Delta}_3^1$.

Now to prove Theorem 14.1 by getting a contradiction, it remains to check that $X \subseteq A$ and $Y \subseteq B$—so X, Y are separable by a $\mathbf{\Delta}_3^1$ set in $\mathbf{L}$. By construction it suffices to verify that $X \cap B' = Y \cap A' = \varnothing$. Suppose that say $z \in X \cap B'$. By definition there is a multitree $\boldsymbol{q}$ such that $\boldsymbol{q} \le \boldsymbol{p}$ and $\langle \boldsymbol{q}, z\rangle \in \Phi$. But then $z \notin X$ by Corollary 14.7(ii), as required. □ (Theorem 14.1)

Acknowledgements

The authors thank Ali Enayat and Mohammad Golshani for fruitful discussions, and an anonymous reviewer for valuable remarks and comments.

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