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The full basis theorem does not imply analytic wellordering [☆]

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A finite support product of ω_1 clones of Jensen's minimal Π^1_2 singleton forcing is used to define a model in which any non-empty analytically definable set of reals contains an analytically definable real (the full basis theorem), but there is no analytically definable wellordering of the reals.

Contents

☆ This document is a collaborative effort.

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1. Introduction

A set of reals B is a *basis* for a family $\mathbf{F}$ of sets of reals, if any non-empty set $X \in \mathbf{F}$ contains a real in B. Typically both B and $\mathbf{F}$ here are connected with this or another type of definability or mathematical constructivity, so the question then can be understood as whether a non-empty definable set contains a definable element. Questions related to the definability of mathematical objects, appeared in the focus of attention of discussions on mathematical foundations immediately after the publication of the famous Zermelo's paper on the axiom of choice and its application to the problem of wellorderability in 1905, and also, to some extent, in connection with a simultaneous publication of the Richard paradox. For instance, Hadamard, Borel, Baire, and Lebesgue, participants of the discussion published in [13], in spite of significant differences in their positions regarding problems of mathematical foundations, emphasized that a proof of nonemptiness, that is, a proof of pure existence of an element in a given set, and a direct definition (or an effective construction) of such an element are different mathematical results, and the second of them does not follow from the first. In particular, Lebesgue, in his part of [13], pointed at the difficulties in the problem of effective choice, that is, a selection of a definable element in a definable (nonempty) set.[3]

For the sake of convenience, let us represent Lebesgue's remark as follows.

Does every non-empty definable set of reals contain a definable element?

The answer depends on the type of definability considered. In the context of the classes Σ^1_n, Π^1_n, Δ^1_n (see [39]) of the *analytical hierarchy*, the most profound basis result, a corollary of the Novikov–Kondo–Addison

[3] "Ainsi je vois déjà une difficulté dans ceci dans un M' déterminé je puis choisir un m' déterminé", in the original. Thus I already see a difficulty with the assertion that "in a determinate M' I can choose a determinate m'", in the English translation.

uniformization theorem, claims that every non-empty (lightface) Σ^1_2 set of reals has a Δ^1_2 element, see [39, 4E.4 and 4E.5].[4] As for the dual class Π^1_2, and subsequently all higher classes, no similar basis result is possible. Indeed, Levy [36] defined a model of **ZFC**, in which it is true that a certain Π^1_2 set of reals (the set of all non-constructible reals), is non-empty but contains no *analytically definable* reals of any class Δ^1_n, and even more, no *ordinal-definable*[5] elements. This non-basis result has been recently sharpened to the extent that a Π^1_2 set $X \neq \varnothing$ of reals, containing no OD elements, can be countable [24], and even can be equal to a Vitali class (a shift of the rationals)[6] [22] in appropriate models. See related discussions at the Mathoverflow[7] and FOM[8] exchange boards.

The basis problem, as well as the *uniformization problem*, introduced by Luzin [37,38], and the related *wellordering problem*, are well known in modern set theory. (See Moschovakis [39, Section 6C and 8H.10], Kechris [35], Hauser and Schindler [15], Ressayre [41], Woodin [42], Caicedo and Schindler [3], Vera Fischer *e.a.* [7,6] among other, both older and more recent studies on different aspects of basis, uniformization, and wellorderings.)

Anyway, it seems that the most transparent way to get a basis result is to make use of an analytically definable wellordering $<$ of the reals, which enables one to pick the $<$-least real in each non-empty set of reals. This leads to the question: is the existence of an analytically definable wellordering $<$ of the reals necessary for the basis theorem. In principle, **the negative answer** is well-known: indeed, the axiom of projective determinacy **PD**: 1) is incompatible with the existence of a projective (let alone analytically definable) wellordering of the reals, but 2) implies that any non-empty Σ^1_{2n} set of reals contains a Δ^1_{2n} real, for each $n \geq 1$, see [39, 6C.6]. However **PD** is an extraordinarily strong axiom, whose consistency strength crucially exceeds **ZFC**. Therefore, following the approach outlined in [8], it can be interesting to get **the negative answer** by pure means of a **set-generic extension of the constructible universe L**, in particular, with no reference to any extra axioms. This is the main content of this paper.

Theorem 1.1. *In a suitable set-generic extension of* $\mathbf{L}$*, it is true that every non-empty lightface analytically definable set of reals contains a lightface analytically definable real* ***(the full basis theorem)****, but there is no lightface analytically definable wellordering of the continuum.*

More precisely, there is a cardinal-preserving generic extension $\mathbf{L}[X]$ *of* $\mathbf{L}$*, such that* $X = \langle x_{\xi k}\rangle_{\xi<\omega_1^{\mathbf{L}} \wedge k<\omega}$*, where each* $x_{\xi k}$ *is a real in* 2^ω*, and in addition*

(I) *if* $m < \omega$ *then it is true in the submodel* $\mathbf{L}[X_m]$ *that there is a* Δ^1_{m+3} *wellordering of the reals of length* ω_1*, where* $X_m = \langle x_{\xi k}\rangle_{\xi<\omega_1^{\mathbf{L}} \wedge k<m}$*;*

(II) *if* $m < \omega$ *then* $\omega^\omega \cap \mathbf{L}[X_m]$ *is a* Σ^1_{m+3} *set in* $\mathbf{L}[X]$*;*

(III) *if* $m < \omega$ *then* $\mathbf{L}[X_m]$ *is an elementary submodel of* $\mathbf{L}[X]$ *w.r.t. all* Σ^1_{m+2} *formulas with reals in* $\mathbf{L}[X_m]$ *as parameters;*

(IV) *it is true in* $\mathbf{L}[X]$ *that there is no lightface analytically definable wellordering of the reals.*

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[4] Some important and difficult basis results related to classes lower than Σ^1_2, for instance the Kleene and Gandy basis theorems, A.1.3 and A.1.4 in [10], are out of the scope of this paper.

[5] The class OD of ordinal-definable sets contains all sets definable by a set theoretic formula with only ordinals as parameters. This is perhaps the most broad version of effective definability admitted in modern set theory. Unlike the pure (parameterfree) definability, the ordinal definability admits a set theoretic formula, which adequately expresses the property of a set x to be ordinal definable, see [40]. The class Ord of all ordinals is an extension of the natural numbers, unique and determined enough for not to insist on definability of the ordinals themselves.

[6] Or a E_0-equivalence class, if we consider reals in 2^ω or ω^ω. Recall that E_0 is an equivalence relation on ω^ω defined so that $x \mathrel{\mathsf{E}_0} y$ iff $x(n) = y(n)$ for all but finite n.

[7] A question about ordinal definable real numbers. Mathoverflow, March 09, 2010. http://mathoverflow.net/questions/17608.

[8] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html.

To see that the additional claims imply the main claim (the full basis theorem), let, in $\mathbf{L}[X]$, $Z \subseteq \omega^\omega$ be a non-empty Σ^1_{m+2} set of reals. Then $Z' = Z \cap \mathbf{L}[X_m]$ is a Σ^1_{m+3} set by (II), and $Z' \neq \varnothing$ by (III). It remains to pick the least real in Z' in the sense of the lightface Δ^1_{m+3} wellordering given by (I).

2. Comments

To prove the theorem, we make use of a generic extension of $\mathbf{L}$ via a finite-support product of forcing notions that resemble Jensen's minimal Π^1_2 singleton forcing. The history of this approach goes down to Jensen [17], where a subforcing $\mathbb{J}$ of the Sacks forcing is defined in $\mathbf{L}$, the constructible universe, such that the canonical $\mathbb{J}$-generic real in 2^ω is the only $\mathbb{J}$-generic real in the extension, and 'being a $\mathbb{J}$-generic real' is a Π^1_2 property. Thus any $\mathbb{J}$-generic extension of $\mathbf{L}$ contains a Π^1_2 nonconstructible singleton. See 28A in [16] for an up-to-date exposition of Jensen's forcing.[9]

Later Jensen's forcing construction was extended by Abraham. This included a model [2] containing a definable minimal collapsing real, and a minimal model [1] for the negation of CH, based on iterations of Jensen's forcing constructed with some semblance of the Sacks forcing iterations as in [12] or [21]. Another modification of Jensen's forcing construction by Jensen and Johnsbraten [19] yields such a forcing notion in $\mathbf{L}$ that any extension of $\mathbf{L}$, containing two generic reals $x \neq y$, necessarily satisfies $\omega_1^{\mathbf{L}} < \omega_1$.

Somewhat later, Ali Enayat (Footnote 8) conjectured that some definability questions can be solved by finite-support products of Jensen's [17] forcing $\mathbb{J}$. Enayat demonstrated in [4] that a symmetric part of the $\mathbb{J}^\omega$-generic extension of $\mathbf{L}$ definitely yields a model of **ZF** (not a model of **ZFC**!) in which there is a Dedekind-finite infinite Π^1_2 set of reals with no OD elements.

Following the conjecture, we proved in [24] that indeed it is true in a $\mathbb{J}^\omega$-generic extension of $\mathbf{L}$ that the set of $\mathbb{J}$-generic reals is a countable non-empty Π^1_2 set with no OD elements. We also proved in [22] that the existence of a Π^1_2 E_0-equivalence class with no OD elements is consistent with **ZFC**, using a E_0-invariant version of Jensen's forcing.[10] A similar technique was used in [11] to define a generic extension $\mathbf{L}[x,y]$ of $\mathbf{L}$ by reals x, y, in which the union of E_0-classes of x and y is a lightface Π^1_2 set, but neither of these two E_0-classes is an ordinal-definable set.

A suitable finite-support product of Jensen's forcing was employed in [23] to define a generic extension of $\mathbf{L}$ where there is a Π^1_2 set $P \subseteq \omega^\omega \times \omega^\omega$ which is non-uniformizable by a projective set and has countable cross-sections $P_x = \{y : \langle x, y\rangle \in P\}$, and, more specifically, whose all non-empty cross-sections are Vitali classes [26]. A combination of finite-support product and iteration of Jensen's forcing was used in [9] to define a model of **ZF** in which the countable **AC** holds but the principle of dependent choices **DC** fails for a Π^1_2 relation on the reals.

A different modification of Jensen's forcing construction was engineered in [20] in order to define an extension of $\mathbf{L}$ in which, for a given $n \geq 2$, there is a nonconstructible Π^1_n singleton while all Σ^1_n reals are still constructible. The idea is to complicate the inductive construction of Jensen's forcing $\mathbb{J} = \bigcup_{\alpha<\omega_1} \mathbb{J}_\alpha$ in $\mathbf{L}$ by the requirement that the sequence $\langle \mathbb{J}_\alpha \rangle_{\alpha<\omega_1}$ intersects any set, of a certain definability level, dense in the collection of all possible countable initial steps of the construction — *inner definable genericity*.[11] Using this tool, generic modes have been recently defined, in which, for a given $n \geq 3$:

[9] Another nonconstructible Π^1_2 singleton was defined in [18] via the almost-disjoint forcing, yet the construction in [17] has the advantage of *minimality* of $\mathbb{J}$-generic reals and some other advantages as a coding system (as well as some disadvantages).

[10] On the contrary, it is true in some "more elementary" generic extensions of $\mathbf{L}$, like Cohen, Solovay-random, Sacks, that every countable OD set contains only constructible elements [25] (also true for dominating real extensions), and OD Borel sets admit constructible Borel codes of the same rank [28,29]. The elementarity of such extensions is somewhat illusory though. See *e.g.* [34] on Cohen reals, or [5] on a theorem, originally by Solovay, that it is true in a generic extension $\mathbf{L}[a]$ of $\mathbf{L}$ by a Sacks real that there is an ordinal-definable equivalence relation on the reals, which has exactly two equivalence classes, and these classes are not OD.

[11] This method of *inner genericity* was originally introduced and developed in Harrington's handwritten notes [14] on the base of the Jensen–Solovay almost-disjoint forcing [18], and applied towards some great results in set theory that unfortunately have never been published in a mathematical journal. We have recently reproved some results of [14] in [32,33].

(1) there is a Π^1_n E_0-equivalence class with no OD elements, and in the same time every countable Σ^1_n set consists of OD elements, [27];
(2) there is a Δ^1_n real that codes a collapse of $\aleph_1^{\mathbf{L}}$, and in the same time every Σ^1_{n-1} set $x \subseteq \omega$ is constructible, [30];
(3) there is a planar Π^1_{n-1} set with countable vertical cross-sections, not uniformizable by a real-ordinal-definable set, and in the same time all planar boldface $\mathbf{\Sigma}^1_{n-1}$ sets with countable cross-sections are $\mathbf{\Delta}^1_n$-uniformizable, [31].

3. The structure of the paper

To prove Theorem 1.1, we define, in $\mathbf{L}$, a system of forcing notions $\mathbb{P}_{\xi k}$, $\xi < \omega_1$ and $k < \omega$, whose finite-support product $\mathbb{P} = \prod_{\xi,k} \mathbb{P}_{\xi k}$ adds a generic array $X = \langle x_{\xi k} \rangle_{\xi<\omega_1, k<\omega}$ of reals $x_{\xi k}$ to $\mathbf{L}$, such that conditions (I), (II), (III), (IV) of Theorem 1.1 hold in $\mathbf{L}[X]$.

We employ the inner definable genericity idea here in such a way that if $m < \omega$ then the m-tail $\langle \mathbb{P}_{\xi k} \rangle_{\xi<\omega_1 \wedge k\ge m}$ of the forcing construction, bears an amount of inner definable genericity which strictly depends on m. (See Definition 21.1, where a key concept is introduced.) This obscures the coding construction to the extent that the partial analytical wellorderings mentioned in (I) of Theorem 1.1 cannot be glued into a common analytically definable wellordering.

Chapter I contains a general formalism related to forcing by perfect trees and finite-support products, convenient for our goals. Following Jensen [17], we consider forcing notions of the form $\mathbb{P} = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha$, where $\lambda < \omega_1$ and each $\mathbb{P}_\alpha$ is a countable set of perfect trees in $2^{<\omega}$. Each term $\mathbb{P}_\alpha$ has to satisfy some routine conditions of *refinement* with respect to the previous terms, in particular, to make sure that each $\mathbb{P}_\alpha$ remains pre-dense at further steps. Also, each $\mathbb{P}_\alpha$ has to *seal* some dense sets in $\bigcup_{\xi<\alpha} \mathbb{P}_\xi$ so that they remain pre-dense at further steps as well. And this procedure has to be extended from single forcing notions to their finite-support products. These issues are dealt with in **Chapter II**.

Then we consider real names with respect to finite-support products of perfect-tree forcing notions in **Chapter III**. Here the key issue is to make sure that if $\mathbb{P}$ is a factor in a product forcing considered then there is no other $\mathbb{P}$-generic real in the whole product extension except for the obvious one.

In **Chapter IV** we define the forcing notion $\mathbf{P} = \prod_{\xi<\omega_1, k<\omega} \mathbb{P}_{\xi k}$ to prove the main theorem, in the form of a limit of a certain increasing sequence of countable products of countable perfect-tree forcing notions. Quite a complicated construction of this sequence in $\mathbf{L}$ involves ideas related to diamond-style constructions, as well as to some sort of definable genericity, as explained above.

The forcing $\mathbb{P}$ as a whole is not analytically definable; however each k-th layer $\langle \mathbb{P}_{\xi k} \rangle_{\xi<\omega_1}$ belongs to Δ^1_{k+4}. But it is a key property that the $\mathbf{P}$-forcing relation restricted to Σ^1_n formulas is essentially Σ^1_n. We prove this in **Chapter V**, with the help of an auxiliary forcing notion $\mathtt{forc}$. We also establish the invariance of $\mathtt{forc}$ with respect to countable-support permutations of $\omega_1 \times \omega$.

We finally prove Theorem 1.1 in **Chapter VI**, on the base of the results obtained in two previous chapters.

I. Basic constructions

We begin with some basic things: perfect trees in the Cantor space 2^ω, perfect tree forcing notions (those which consist of perfect trees), their finite-support products, and a splitting construction of perfect trees.

4. Perfect trees

Let $2^{<\omega}$ be the set of all *strings* (finite sequences) of numbers $0, 1$. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\frown k$ is the extension of t by k. If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s, while $s \subset t$ means proper extension. If $s \in 2^{<\omega}$ then $\mathtt{lh}(s)$ is the length of s, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length n).

A set $T \subseteq 2^{<\omega}$ is a *tree* iff for any strings $s \subset t$ in $2^{<\omega}$, if $t \in T$ then $s \in T$. Every non-empty tree $T \subseteq 2^{<\omega}$ contains the empty string Λ. If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T{\restriction}_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let **PT** be the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$. Thus a non-empty tree $T \subseteq 2^{<\omega}$ belongs to **PT** iff it has no endpoints and no isolated branches. Then there is a largest string $s \in T$ such that $T = T{\restriction}_s$; it is denoted by $s = \mathtt{stem}(T)$ (the *stem* of T); we have $s^\frown 1 \in T$ and $s^\frown 0 \in T$ in this case.

Definition 4.1 *(perfect sets).* If $T \in \mathbf{PT}$ then $[T] = \{a \in 2^\omega : \forall n\,(a{\restriction}n \in T)\}$ is the set of all *paths through* T, a perfect set in 2^ω. Conversely if $X \subseteq 2^\omega$ is a perfect set then $\mathtt{tree}(X) = \{a{\restriction}n : a \in X \wedge n < \omega\} \in \mathbf{PT}$ and $[\mathtt{tree}(X)] = X$.

Trees $T, S \in \mathbf{PT}$ are *almost disjoint*, AD for brevity, iff the intersection $S \cap T$ is finite; this is equivalent to just $[S] \cap [T] = \varnothing$. □

The *simple splitting* of a tree $T \in \mathbf{PT}$ consists of smaller trees

$$T(\to 0) = T{\restriction}_{\mathtt{stem}(T)^\frown 0} \quad \text{and} \quad T(\to 1) = T{\restriction}_{\mathtt{stem}(T)^\frown 1}$$

in **PT**, so that $[T(\to i)] = \{x \in [T] : x(h) = i\}$, where $h = \mathtt{lh}(\mathtt{stem}(T))$. We let

$$T(\to u) = T(\to u(0))(\to u(1))(\to u(2))\dots(\to u(n-1))$$

for each string $u \in 2^{<\omega}$, $\mathtt{lh}(u) = n$; and separately $T(\to \Lambda) = T$.

Lemma 4.2. *Suppose that $T \in \mathbf{PT}$. Then:*

(i) *if $u \in 2^{<\omega}$ then there is a string $s \in 2^{<\omega}$ such that $T(\to u) = T{\restriction}_s$;*
(ii) *if $s \in 2^{<\omega}$ then there is a string $u \in 2^{<\omega}$ such that $T{\restriction}_s = T(\to u)$;*
(iii) *if $\varnothing \neq U \subseteq [T]$ is a (relatively) open subset of $[T]$, or at least U has a non-empty interior in $[T]$, then there is a string $s \in T$ such that $T{\restriction}_s \subseteq U$.* □

If $T \in \mathbf{PT}$ and $a \in 2^\omega$ then the intersection $T(\to a) = \bigcap_{n<\omega} T(\to a{\restriction}n) = \{\boldsymbol{h}_T(a)\}$ is a singleton, and the map $\boldsymbol{h}_T$ is a *canonical homeomorphism* from 2^ω onto $[T]$. Accordingly if $S, T \in \mathbf{PT}$ then the map $\boldsymbol{h}_{ST}(x) = \boldsymbol{h}_T({\boldsymbol{h}_S}^{-1}(x))$ is a *canonical homeomorphism* from $[S]$ onto $[T]$.

5. Perfect tree forcing notions

Let a **perfect-tree forcing notion** be any non-empty set $\mathbb{P} \subseteq \mathbf{PT}$ such that if $s \in T \in \mathbb{P}$ then $T{\restriction}_s \in \mathbb{P}$, or equivalently, by Lemma 4.2, if $u \in 2^{<\omega}$ then $T(\to u) \in \mathbb{P}$. Let **PTF** be the set of all such forcing notions $\mathbb{P} \subseteq \mathbf{PT}$.

Example 5.1. If $s \in 2^{<\omega}$ then the tree $[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subseteq s\}$ belongs to **PT**. The set $\mathbb{P}_{\rm coh} = \{[s] : s \in 2^{<\omega}\}$ of all such trees (the Cohen forcing) is a perfect-tree forcing notion, special and regular in the sense of Definition 5.4. □

Lemma 5.2. *Let $\mathbb{P} \in \mathbf{PTF}$. If $T \in \mathbb{P}$ and a set $X \subseteq [T]$ is (relatively) open (resp., clopen) in $[T]$, then there is a countable (resp., finite) set $\mathscr{S}$ of pairwise* AD *trees $S \in \mathbb{P}$, satisfying $\bigcup_{S \in \mathscr{S}} [S] = X$.* □

Lemma 5.3. (i) *If $s \in T \in \mathbb{P} \in \mathbf{PTF}$ then $T{\restriction}_s \in \mathbb{P}$.*

(ii) *If $\mathbb{P}, \mathbb{P}' \in \mathbf{PTF}$, $T \in \mathbb{P}$, $T' \in \mathbb{P}'$, then there are trees $S \in \mathbb{P}$, $S' \in \mathbb{P}'$ such that $S \subseteq T$, $S' \subseteq T'$, and $[S] \cap [S'] = \varnothing$.*

Proof. (i) use Lemma 4.2. (ii) If $T = T'$ then let $S = T(\to 0)$, $S' = T(\to 1)$. If say $T \not\subseteq T'$ then let $s \in T \smallsetminus T'$, $S = T{\restriction}_s$, and simply $S' = T'$. □

Definition 5.4. A set $\mathbb{A} \subseteq \mathbf{PT}$ is an *antichain* iff any trees $T \neq T'$ in $\mathbb{A}$ are AD, that is, $[T] \cap [T'] = \varnothing$. A forcing notion $\mathbb{P} \in \mathbf{PTF}$ is:

small, if it is countable;
special, if there is an antichain $\mathbb{A} \subseteq \mathbb{P}$ such that $\mathbb{P} = \{T{\restriction}_s : s \in T \in \mathbb{A}\}$ — note that $\mathbb{A}$ is unique if exists; we write $\mathbb{A} = \mathtt{base}(\mathbb{P})$ (the *base* of $\mathbb{P}$);
regular, if for any $S, T \in \mathbb{P}$, the intersection $[S] \cap [T]$ is clopen in $[S]$ or clopen in $[T]$ (or clopen in both $[S]$ and $[T]$). □

Lemma 5.5. *Let $\mathbb{P} \in \mathbf{PTF}$. If $\mathbb{P}$ is special and $S, T \in \mathbb{P}$ are not* AD*, then they are comparable: $S \subseteq T$ or $T \subseteq S$.*

If $\mathbb{P}$ is special then $\mathbb{P}$ is regular. If $\mathbb{P}$ is regular, then

(i) *if $S, T \in \mathbb{P}$ are not* AD*, then they are compatible in $\mathbb{P}$, that is, there is a tree $R \in \mathbb{P}$ such that $R \subseteq S \cap T$;*
(ii) *if $S_1, \dots, S_k \in \mathbb{P}$ then there is a finite set of pairwise* AD *trees $R_1, \dots, R_n \in \mathbb{P}$ such that $[S_1] \cap \ldots \cap [S_k] = [R_1] \cup \ldots \cup [R_n]$;*
(iii) *if $\mathscr{S}_1, \dots, \mathscr{S}_k$ are finite collections of trees in $\mathbb{P}$ then there is a finite set of trees $R_1, \dots, R_n \in \mathbb{P}$ such that $\bigcup_{S \in \mathscr{S}_1} [S] \cap \ldots \cap \bigcup_{S \in \mathscr{S}_k} [S] = [R_1] \cup \ldots \cup [R_n]$, and for any $\mathscr{S}_i$ and R_j, there is $S \in \mathscr{S}_i$ such that $R_j \subseteq S$.*

Proof. (iii) Apply (ii) to every set of the form $[S_1] \cap \ldots \cap [S_k]$, where $S_i \in \mathscr{S}_i$, $\forall i$, then gather all trees R_i obtained in one finite set. □

Remark 5.6. Any set $\mathbb{P} \in \mathbf{PTF}$ can be considered as a forcing notion (if $T \subseteq T'$ then T is a stronger condition); then $\mathbb{P}$ adds a real $x \in 2^\omega$. □

Lemma 5.7. *If a set $G \subseteq \mathbb{P}$ is generic over a ground set universe* $\mathbf{V}$ (*resp., over a transitive model, e.g.* $\mathbf{L}$) *then*

(i) *the intersection $\bigcap_{T \in G} [T]$ contains a single real $x = x[G] \in 2^\omega$, and*
(ii) *this real x is $\mathbb{P}$-**generic**, in the sense that if $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ and belongs to* $\mathbf{V}$ (*resp., to the ground model*) *then $x \in \bigcup_{T \in D} [T]$.* □

As usual, a set $D \subseteq \mathbb{P}$ is:

– *open* in $\mathbb{P}$, if for any trees $T \subseteq S$ in $\mathbb{P}$, $S \in D \Longrightarrow T \in D$;
– *dense* in $\mathbb{P}$, if for any $T \in \mathbb{P}$ there is $S \in D$, $S \subseteq T$;
– *pre-dense* in $\mathbb{P}$, if the set $D' = \{T \in \mathbb{P} : \exists\, S \in D\, (T \subseteq S)\}$ is dense in $\mathbb{P}$.

6. Splitting construction

We proceed with an important splitting/fusion construction of perfect trees by means of infinite splitting systems of such trees.

Definition 6.1. Let **FSS** be the set of all *finite splitting systems*, that is, systems of the form $\varphi = \langle T_s \rangle_{s\in 2^{\le n}}$, where $n = \mathtt{hgt}(\varphi) < \omega$ (the height of φ), each value $T_s = T_s^\varphi = \varphi(s)$ is a tree in **PT**, and

($*$) if $s \in 2^{<n}$ and $i = 0, 1$ (so $s^\frown i \in 2^{\le n}$) then $T_{s^\frown i} \subseteq T_s(\to i)$ — it easily follows that $[T_{s^\frown 0}] \cap [T_{s^\frown 1}] = \varnothing$.

We add the *empty system* $\mathbf{\Lambda}$ to **FSS**, with $\mathtt{hgt}(\mathbf{\Lambda}) = -1$. □

A tree T *occurs in* $\varphi \in$ **FSS** if $T = \varphi(s)$ for some $s \in 2^{\le \mathtt{hgt}(\varphi)}$. If all trees occurring in φ belong to some $\mathbb{P} \in$ **PTF** then say that φ is a finite splitting system *over* $\mathbb{P}$, symbolically $\varphi \in \mathbf{FSS}(\mathbb{P})$.

Let φ, ψ be systems in **FSS**. Say that φ *extends* ψ, symbolically $\psi \preccurlyeq \varphi$, if $n = \mathtt{hgt}(\psi) \le \mathtt{hgt}(\varphi)$ and $\psi(s) = \varphi(s)$ for all $s \in 2^{\le n}$, and *properly extends*, $\psi \prec \varphi$, if in fact $\mathtt{hgt}(\psi) < \mathtt{hgt}(\varphi)$ strictly.

Each system $\varphi \in \mathbf{FSS}(\mathbb{P})$ with $\mathtt{hgt}(\varphi) = 0$ consists essentially of a single tree $T_\Lambda^\varphi \in \mathbb{P}$. The next lemma provides systems of arbitrary height.

Lemma 6.2. *Assume that* $\mathbb{P} \in$ **PTF**. *If* $n \ge 1$ *and* $\psi = \langle T_s \rangle_{s\in 2^{\le n}} \in \mathbf{FSS}(\mathbb{P})$ *then there is a system* $\varphi = \langle T_s \rangle_{s\in 2^{\le n+1}} \in \mathbf{FSS}(\mathbb{P})$ *which properly extends* ψ.

Proof. If $s \in 2^n$ and $i = 0, 1$ then let $T_{s^\frown i} = T_s(\to i)$. □

The next well-known lemma belongs to the type of *splitting/fusion* lemmas widely used in connection with the perfect set forcing and some similar forcings.

Lemma 6.3. *Let* $\mathbb{P} \in$ **PTF**. *Then there is an* $\prec$*-increasing sequence* $\langle \varphi_n \rangle_{n<\omega}$ *of systems in* $\mathbf{FSS}(\mathbb{P})$. *And if* $\langle \varphi_n \rangle_{n<\omega}$ *is such then:*

(i) *the limit system* $\varphi = \bigcup_n \varphi_n = \langle T_s \rangle_{s\in 2^{<\omega}}$ *satisfies* ($*$) *of Definition 6.1 on the whole domain of strings* $s \in 2^{<\omega}$*;*
(ii) $T = \bigcap_n \bigcup_{s\in 2^n} T_s$ *is a perfect tree in* **PT** *and* $[T] = \bigcap_n \bigcup_{s\in 2^n} [T_s]$*;*
(iii) *if* $u \in 2^{<\omega}$ *then* $T(\to u) = T \cap T_u = \bigcap_{n \ge \mathtt{lh}(u)} \bigcup_{s\in 2^n, u\subseteq s} T_s$. □

7. Multiforcings and multitrees

We'll systematically make use of finite support products of perfect tree forcings in this paper. The following definitions introduce suitable notation.

Call a **multiforcing** any map $\boldsymbol{\pi} : |\boldsymbol{\pi}| \to$ **PTF**, where $|\boldsymbol{\pi}| = \mathtt{dom}\,\boldsymbol{\pi} \subseteq \omega_1 \times \omega$. Thus each set $\boldsymbol{\pi}(\xi, k)$, $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$, is a perfect tree forcing notion. Such a $\boldsymbol{\pi}$ is:

- *small*, if both $|\boldsymbol{\pi}|$ and each forcing $\boldsymbol{\pi}(\xi, k)$, $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$, are countable;
- *special*, if each $\boldsymbol{\pi}(\xi, k)$ is special in the sense of Definition 5.4;
- *regular*, if each $\boldsymbol{\pi}(\xi, k)$ is regular, in the sense of Definition 5.4.

Let **MF** be the set of all multiforcings.

Let a **multitree** be any map $\boldsymbol{p} : |\boldsymbol{p}| \to$ **PT**, such that $|\boldsymbol{p}| = \mathtt{dom}\,\boldsymbol{p} \subseteq \omega_1 \times \omega$ is finite and each value $T_{\xi k}^{\boldsymbol{p}} = \boldsymbol{p}(\xi, k)$ is a tree in **PT**. In this case we define a cofinite-dimensional perfect cube in $(2^\omega)^{\omega_1 \times \omega}$

$$\begin{aligned}[\boldsymbol{p}] &= \{x \in (2^\omega)^{\omega_1\times\omega} : \forall \langle \xi, k \rangle \in |\boldsymbol{p}|\, (x(\xi, k) \in [T_{\xi k}^{\boldsymbol{p}}])\} = \\ &= \{x \in (2^\omega)^{\omega_1\times\omega} : \forall \langle \xi, k \rangle \in |\boldsymbol{p}|\, \forall m\, (x(\xi, k){\restriction}m \in T_{\xi k}^{\boldsymbol{p}})\}\,.\end{aligned}$$

Let **MT** be the set of all multitrees. We order **MT** componentwise: $\boldsymbol{q} \leqslant \boldsymbol{p}$ ($\boldsymbol{q}$ is stronger) iff $|\boldsymbol{p}| \subseteq |\boldsymbol{q}|$ and $T^{\boldsymbol{q}}_{\xi k} \subseteq T^{\boldsymbol{p}}_{\xi k}$ for all $\langle \xi, k\rangle \in |\boldsymbol{p}|$; this is equivalent to $[\boldsymbol{q}] \subseteq [\boldsymbol{p}]$, so that stronger multitrees correspond to smaller cubes. The weakest multitree $\boldsymbol{\Lambda} \in \mathbf{MT}$ is just the empty map; $|\boldsymbol{\Lambda}| = \varnothing$ and $[\boldsymbol{\Lambda}] = 2^{\omega_1 \times \omega}$.

Definition 7.1. Multitrees $\boldsymbol{p}, \boldsymbol{q}$ are *somewhere almost disjoint*, or SAD, if, for at least one pair of indices $\langle \xi, k\rangle \in |\boldsymbol{p}| \cap |\boldsymbol{q}|$, the trees $T^{\boldsymbol{p}}_{\xi k}$, $T^{\boldsymbol{q}}_{\xi k}$ are AD, that is, $[T^{\boldsymbol{p}}_{\xi k}] \cap [T^{\boldsymbol{q}}_{\xi k}] = \varnothing$, or equivalently, $T^{\boldsymbol{p}}_{\xi k} \cap T^{\boldsymbol{q}}_{\xi k}$ is finite. □

If $\boldsymbol{\pi}$ is a multiforcing then a $\boldsymbol{\pi}$*-multitree* is any multitree $\boldsymbol{p}$ with $|\boldsymbol{p}| \subseteq |\boldsymbol{\pi}|$ and $T^{\boldsymbol{p}}_{\xi k} \in \boldsymbol{\pi}(\xi, k)$ for all $\langle \xi, k\rangle \in |\boldsymbol{p}|$. Let $\mathbf{MT}(\boldsymbol{\pi})$ be the set of all $\boldsymbol{\pi}$-multitrees; it is equal to the finite support product $\prod_{\langle \xi,k\rangle \in |\boldsymbol{\pi}|} \boldsymbol{\pi}(\xi, k)$.

Corollary 7.2 *(of Lemma 5.5(i)). If $\boldsymbol{\pi}$ is a regular multiforcing and multitrees $\boldsymbol{p}, \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ are not* SAD, *then $\boldsymbol{p}, \boldsymbol{q}$ are compatible in $\mathbf{MT}(\boldsymbol{\pi})$, so that there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ with $\boldsymbol{r} \leqslant \boldsymbol{p}$, $\boldsymbol{r} \leqslant \boldsymbol{q}$.* □

The following is similar to Lemma 5.5(iii).

Lemma 7.3. *If a multiforcing $\boldsymbol{\pi}$ is regular, $\xi \subseteq |\boldsymbol{\pi}|$ is finite, and $U_1, \dots, U_k$ are finite collections of multitrees in $\mathbf{MT}(\boldsymbol{\pi})$ with $|\boldsymbol{p}| = \xi$ for all $\boldsymbol{p} \in \bigcup_i U_i$, then there is a finite set of multitrees $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{u}_j| = \xi$ for all j,*

$$\bigcup\nolimits_{\boldsymbol{p} \in U_1} [\boldsymbol{p}] \cap \dots \cap \bigcup\nolimits_{\boldsymbol{p} \in U_k} [\boldsymbol{p}] = [\boldsymbol{u}_1] \cup \dots \cup [\boldsymbol{u}_n],$$

and for any U_i and $\boldsymbol{u}_j$, there is $\boldsymbol{p} \in U_i$ such that $[\boldsymbol{u}_j] \subseteq [\boldsymbol{p}]$. □

We consider sets of the form $\mathbf{MT}(\boldsymbol{\pi})$ in the role of **product forcing notions**. A set $D \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is:

- *open* in $\mathbf{MT}(\boldsymbol{\pi})$, if for any $\boldsymbol{p} \leqslant \boldsymbol{q}$ in $\mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{q} \in D \Longrightarrow \boldsymbol{p} \in D$;
- *dense* in $\mathbf{MT}(\boldsymbol{\pi})$, if for any $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, there is $\boldsymbol{q} \in D$, $\boldsymbol{q} \leqslant \boldsymbol{p}$;
- *pre-dense* in $\mathbf{MT}(\boldsymbol{\pi})$, if the set $D' = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \exists\, \boldsymbol{q} \in D\, (\boldsymbol{p} \leqslant \boldsymbol{q})\}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$.

Remark 7.4. As a forcing notion, each $\mathbf{MT}(\boldsymbol{\pi})$ adds an array $\langle x_{\xi k}\rangle_{\langle \xi,k\rangle \in |\boldsymbol{\pi}|}$ of reals, where each real $x_{\xi k} \in 2^\omega$ is a $\boldsymbol{\pi}(\xi, k)$-generic real. Namely if a set $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is generic over the ground set universe $\mathbf{V}$ then each factor

$$G(\xi, k) = \{T^{\boldsymbol{p}}_{\xi k} : \boldsymbol{p} \in G \wedge \langle \xi, k\rangle \in |\boldsymbol{p}|\} \subseteq \boldsymbol{\pi}(\xi, k)$$

(where $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$) is accordingly a set $\boldsymbol{\pi}(\xi, k)$-generic over $\mathbf{V}$, the real $x_{\xi k} = x_{\xi k}[G] = x[G(\xi, k)] \in 2^\omega$ is the only real satisfying $x_{\xi k} \in \bigcap_{T \in G(\xi,k)} [T]$, and $x_{\xi k}$ is $\boldsymbol{\pi}(\xi, k)$-generic over $\mathbf{V}$ as in Lemma 5.7. □

The reals of the form $x_{\xi k}[G]$ will be called *principal generic reals* in $\mathbf{V}[G]$.

Definition 7.5. A *componentwise union* of multiforcings $\boldsymbol{\pi}, \boldsymbol{♀}$ is a multiforcing $\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{♀}$ satisfying $|(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{♀})| = |\boldsymbol{\pi}| \cup |\boldsymbol{♀}|$ and

$$(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{♀})(\xi, k) = \begin{cases} \boldsymbol{\pi}(\xi, k), & \text{whenever } \langle \xi, k\rangle \in |\boldsymbol{\pi}| \smallsetminus |\boldsymbol{♀}| \\ \boldsymbol{♀}(\xi, k), & \text{whenever } \langle \xi, k\rangle \in |\boldsymbol{♀}| \smallsetminus |\boldsymbol{\pi}| \\ \boldsymbol{\pi}(\xi, k) \cup \boldsymbol{♀}(\xi, k), & \text{whenever } \langle \xi, k\rangle \in |\boldsymbol{\pi}| \cap |\boldsymbol{♀}| \end{cases}$$

Similarly, if $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha\rangle_{\alpha<\lambda}$ is a sequence of multiforcings then define a multiforcing $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathsf{cw}}_{\alpha<\lambda} \boldsymbol{\pi}_\alpha$ so that $|\boldsymbol{\pi}| = \bigcup_{\alpha<\lambda} |\boldsymbol{\pi}_\alpha|$ and if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ then $\boldsymbol{\pi}(\xi, k) = \bigcup_{\alpha<\lambda,\, \langle \xi,k\rangle \in |\boldsymbol{\pi}_\alpha|} \boldsymbol{\pi}_\alpha(\xi, k)$. □

8. Multisystems

The next definition introduces *multisystems*, a multi version of the splitting/fusion technique of Section 6, whose intention is to define suitable multiforcings, as will be shown in Section 11 below.

Definition 8.1. A **multisystem** is any map $\boldsymbol{\varphi} : |\boldsymbol{\varphi}| \to \mathbf{FSS}$, such that $|\boldsymbol{\varphi}| \subseteq \omega_1 \times \omega \times \omega$ is finite.[12] This amounts to

(1) the map $h^{\boldsymbol{\varphi}}(\xi, k, m) = \mathtt{hgt}(\boldsymbol{\varphi}(\xi, k, m)) : |\boldsymbol{\varphi}| \to \omega$, and
(2) the finite collection of trees $T^{\boldsymbol{\varphi}}_{\xi k,m}(s) = \boldsymbol{\varphi}(\xi, k, m)(s)$, where $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}|$ and $s \in 2^{\le h^{\boldsymbol{\varphi}}(\xi,k,m)}$, such that if $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}|$ then $\boldsymbol{\varphi}(\xi, k, m) = \langle T^{\boldsymbol{\varphi}}_{\xi k,m}(s)\rangle_{s\in 2^{\le h^{\boldsymbol{\varphi}}(\xi,k,m)}}$ is a finite splitting system in **FSS**.

If $\boldsymbol{\pi}$ is a multiforcing, $|\boldsymbol{\varphi}| \subseteq (|\boldsymbol{\pi}|)\times\omega$, and $\boldsymbol{\varphi}(\xi, k, m) \in \mathbf{FSS}(\boldsymbol{\pi}(\xi, k))$ for all $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}|$ (or equivalently, $T^{\boldsymbol{\varphi}}_{\xi k,m}(s) \in \boldsymbol{\pi}(\xi, k)$ whenever $\langle \xi, k, m\rangle \in \boldsymbol{\varphi}$ and $s \in 2^{\le h^{\boldsymbol{\varphi}}(\xi,k,m)}$), then say that $\boldsymbol{\varphi}$ is a **$\boldsymbol{\pi}$-multisystem**, $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$. □

Let $\boldsymbol{\varphi}, \boldsymbol{\psi}$ be multisystems. Say that $\boldsymbol{\varphi}$ *extends* $\boldsymbol{\psi}$, symbolically $\boldsymbol{\psi} \preccurlyeq \boldsymbol{\varphi}$, if $|\boldsymbol{\psi}| \subseteq |\boldsymbol{\varphi}|$, and, for every $\langle \xi, k, m\rangle \in |\boldsymbol{\psi}|$, $\boldsymbol{\varphi}(\xi, k, m)$ extends $\boldsymbol{\psi}(\xi, k, m)$, that is, $h^{\boldsymbol{\varphi}}(\xi, k, m) \ge h^{\boldsymbol{\psi}}(\xi, k, m)$ and $T^{\boldsymbol{\varphi}}_{\xi k,m}(s) = T^{\boldsymbol{\psi}}_{\xi k,m}(s)$ for all $s \in 2^{\le h^{\boldsymbol{\psi}}(\xi,k,m)}$.

It will be demonstrated in Section 11 that a suitably increasing infinite sequence $\boldsymbol{\varphi}_0 \preccurlyeq \boldsymbol{\varphi}_1 \preccurlyeq \boldsymbol{\varphi}_2 \preccurlyeq \dots$ of multisystems in some $\mathbf{MS}(\boldsymbol{\pi})$ leads to a "limit" multiforcing φ with $|\varphi| = \bigcup_n |\boldsymbol{\varphi}_n|$, such that each factor $\varphi(\xi, k)$, $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, is filled in by trees $Q_{\xi k,m}$, $m < \omega$, in such a way, that the (ξ, k, m)-components of the systems $\boldsymbol{\varphi}_n$ are responsible for the construction of the tree $Q_{\xi k,m}$.

The next lemma introduces different ways to extend a given multisystem.

Say that a multisystem φ is *2wise disjoint* if $[T^{\varphi}_{\xi k,m}(s)] \cap [T^{\varphi}_{\eta\ell,n}(t)] = \varnothing$ for all triples $\langle \xi, k, m\rangle \ne \langle \eta, \ell, n\rangle$ in $|\varphi|$ and all $s \in 2^{h^{\boldsymbol{\varphi}}(\xi,k,m)}$ and $t \in 2^{h^{\boldsymbol{\varphi}}(\eta,\ell,n)}$.

Lemma 8.2. *Let $\boldsymbol{\pi}$ be a multiforcing and $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$.*

(i) *If $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}|$ and $h = h^{\boldsymbol{\varphi}}(\xi, k, m)$ then the extension $\boldsymbol{\psi}$ of $\boldsymbol{\varphi}$ by $h^{\boldsymbol{\psi}}(\xi, k, m) = h + 1$ and $T^{\boldsymbol{\psi}}_{\xi k,m}(s^\frown i) = T^{\boldsymbol{\varphi}}_{\xi k,m}(s)(\to i)$ for all $s \in 2^h$ and $i = 0, 1$, belongs to $\mathbf{MS}(\boldsymbol{\pi})$ and $\boldsymbol{\varphi} \preccurlyeq \boldsymbol{\psi}$.*
(ii) *If $\langle \xi, k, m\rangle \notin |\boldsymbol{\varphi}|$ then the extension $\boldsymbol{\psi}$ of $\boldsymbol{\varphi}$ by $|\boldsymbol{\psi}| = |\boldsymbol{\varphi}| \cup \{\langle \xi, k, m\rangle\}$, $h^{\boldsymbol{\psi}}(\xi, k, m) = 0$ and $T^{\boldsymbol{\psi}}_{\xi k,m}(\Lambda) = T$, where $T \in \boldsymbol{\pi}(\xi, k)$ and Λ is the empty string, belongs to $\mathbf{MS}(\boldsymbol{\pi})$ and $\boldsymbol{\varphi} \preccurlyeq \boldsymbol{\psi}$.*
(iii) *If $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}|$ and a set $D \subseteq \boldsymbol{\pi}(\xi, k)$ is open dense in $\boldsymbol{\pi}(\xi, k)$ then there is a multisystem $\boldsymbol{\psi} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{\psi}| = |\boldsymbol{\varphi}|$, $\boldsymbol{\varphi} \preccurlyeq \boldsymbol{\psi}$, and $T^{\boldsymbol{\psi}}_{\xi k,m}(s) \in D$ whenever $s \in 2^{h^{\boldsymbol{\psi}}(\xi,k,m)}$.*
(iv) *There is a 2wise disjoint $\boldsymbol{\psi} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{\psi}| = |\boldsymbol{\varphi}|$ and $\boldsymbol{\varphi} \preccurlyeq \boldsymbol{\psi}$.*

Proof. To prove (iii) first use (i) to get a multisystem $\boldsymbol{\psi} \in \mathbf{MS}(\boldsymbol{\pi})$ with $\boldsymbol{\varphi} \preccurlyeq \boldsymbol{\psi}$ and $h^{\boldsymbol{\psi}}(\xi, k, m) = h + 1$, where $h = h^{\boldsymbol{\varphi}}(\xi, k, m)$. Then replace each tree $T^{\boldsymbol{\psi}}_{\xi k,m}(s) = \boldsymbol{\psi}(\xi, k, m)(s)$, $s \in 2^{h+1}$, with a suitable tree $T' \in D$, $T' \subseteq T^{\boldsymbol{\psi}}_{\xi k,m}(s)$.

To prove (iv) first apply (i) to get a multisystem $\boldsymbol{\psi} \in \mathbf{MS}(\boldsymbol{\pi})$ with $\boldsymbol{\varphi} \preccurlyeq \boldsymbol{\psi}$, $|\boldsymbol{\psi}| = |\boldsymbol{\varphi}|$, and $h^{\boldsymbol{\psi}}(\xi, k, m) = h^{\boldsymbol{\varphi}}(\xi, k, m) + 1$ for all $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}|$. Now if $\langle \xi, k, m\rangle \ne \langle \eta, \ell, n\rangle$ are triples in $|\boldsymbol{\varphi}|$ and $s \in 2^{h^{\boldsymbol{\varphi}}(\xi,k,m)+1}$, $t \in 2^{h^{\boldsymbol{\varphi}}(\eta,\ell,n)+1}$, then, by Lemma 5.3(ii), there are trees $S \in \boldsymbol{\pi}(\xi, k)$ and $T \in \boldsymbol{\pi}(\eta, \ell)$ satisfying $[S] \cap [T] = \varnothing$ and $S \subseteq T^{\boldsymbol{\psi}}_{\xi k,m}(s)$, $T \subseteq T^{\boldsymbol{\psi}}_{\eta\ell,n}(t)$. Replace the trees $T^{\boldsymbol{\psi}}_{\xi k,m}(s)$, $T \subseteq T^{\boldsymbol{\psi}}_{\eta\ell,n}(t)$ with resp. S, T. Iterate this shrinking construction for all triples $\langle \xi, k, m\rangle \ne \langle \eta, \ell, n\rangle$ and strings s, t as above. □

[12] To explain the third dimension ω in $|\boldsymbol{\varphi}|$, we note that the goal of the multisystem technique will be to define a *refinement* of a given small multiforcing $\boldsymbol{\pi}$. This preassumes the construction of countably many AD and rather independent trees for each $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$. The natural number parameter m below will enumerate these new trees, and the extra dimension ω is its domain.

II. Refinements

Here we consider *refinements* of perfect tree forcings and multiforcings, the key technical tool of definition of various forcing notions in this paper.

9. Refining perfect tree forcings

If $T \in \mathbf{PT}$ (a perfect tree) and $D \subseteq \mathbf{PT}$ then $T \subseteq^{\mathtt{fin}} \bigcup D$ will mean that there is a finite set $D' \subseteq D$ such that $T \subseteq \bigcup D'$, or equivalently $[T] \subseteq \bigcup_{S \in D'} [S]$.

Definition 9.1. Let $\mathbb{P}, \mathbb{Q} \in \mathbf{PTF}$ be perfect tree forcing notions. Say that $\mathbb{Q}$ is a *refinement* of $\mathbb{P}$ (symbolically $\mathbb{P} \sqsubset \mathbb{Q}$) if

(1) the set $\mathbb{Q}$ is dense in $\mathbb{P} \cup \mathbb{Q}$: if $T \in \mathbb{P}$ then $\exists Q \in \mathbb{Q}\,(Q \subseteq T)$;
(2) if $Q \in \mathbb{Q}$ then $Q \subseteq^{\mathtt{fin}} \bigcup \mathbb{P}$;
(3) if $Q \in \mathbb{Q}$ and $T \in \mathbb{P}$ then $[Q] \cap [T]$ is clopen in $[Q]$ and $T \not\subseteq Q$. □

Lemma 9.2.

(i) *If $\mathbb{P} \sqsubset \mathbb{Q}$ and $S \in \mathbb{P}$, $T \in \mathbb{Q}$, then $[S] \cap [T]$ is meager in $[S]$, therefore $\mathbb{P} \cap \mathbb{Q} = \varnothing$ and $\mathbb{Q}$ is open dense in $\mathbb{P} \cup \mathbb{Q}$;*
(ii) *if $\mathbb{P} \sqsubset \mathbb{Q} \sqsubset \mathbb{R}$ then $\mathbb{P} \sqsubset \mathbb{R}$, thus $\sqsubset$ is a strict partial order;*
(iii) *if $\langle \mathbb{P}_\alpha \rangle_{\alpha<\lambda}$ is a $\sqsubset$-increasing sequence in $\mathbf{PTF}$ and $0 < \mu < \lambda$ then $\mathbb{P} = \bigcup_{\alpha<\mu} \mathbb{P}_\alpha \sqsubset \mathbb{Q} = \bigcup_{\mu \le \alpha < \lambda} \mathbb{P}_\alpha$;*
(iv) *if $\langle \mathbb{P}_\alpha \rangle_{\alpha<\lambda}$ is a $\sqsubset$-increasing sequence in $\mathbf{PTF}$ and each $\mathbb{P}_\alpha$ is special then $\mathbb{P} = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha$ is a regular forcing in $\mathbf{PTF}$;*
(v) *in (iv), each $\mathbb{P}_\gamma$ is pre-dense in $\mathbb{P} = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha$.*

Proof. (i) Otherwise there is a string $u \in S$ such that $S{\restriction}_u \subseteq [T] \cap [S]$. But $S{\restriction}_u \in \mathbb{P}$, which contradicts to Definition 9.1(3).

(ii), (iii) Make use of (i) to establish Definition 9.1(3).

(iv) To check the regularity let $S \in \mathbb{P}_\alpha$, $T \in \mathbb{P}_\beta$, $\alpha \le \beta$. If $\alpha = \beta$ then, as $\mathbb{P}_\alpha$ is special, the trees S, T are either AD or $\subseteq$-comparable by Lemma 5.5. If $\alpha < \beta$ then $[S] \cap [T]$ is clopen in $[T]$ by Definition 9.1(3).

(v) Let $S \in \mathbb{P}_\alpha$, $\alpha \ne \gamma$. If $\alpha < \gamma$ then by 9.1(1) there is a tree $T \in \mathbb{P}_\gamma$, $T \subseteq S$. Now let $\gamma < \alpha$. Then $S \subseteq^{\mathtt{fin}} \bigcup \mathbb{P}_\gamma$ by Definition 9.1(2), in particular, there is a tree $T \in \mathbb{P}_\gamma$ such that $[S] \cap [T] \ne \varnothing$. However $[S] \cap [T]$ is clopen in $[S]$ by Definition 9.1(3). Therefore $S{\restriction}_u \subseteq T$ for a string $u \in S$. Finally $S{\restriction}_u \in \mathbb{P}_\alpha$ since $\mathbb{P}_\alpha \in \mathbf{PTF}$. □

Note that if $\mathbb{P}, \mathbb{Q} \in \mathbf{PTF}$ and $\mathbb{P} \sqsubset \mathbb{Q}$ then a dense set $D \subseteq \mathbb{P}$ is not necessarily dense or even pre-dense in $\mathbb{P} \cup \mathbb{Q}$. Yet there is a special type of refinement which preserves at least pre-density. We modify the relation $\sqsubset$ as follows.

Definition 9.3. Let $\mathbb{P}, \mathbb{Q} \in \mathbf{PTF}$ and $D \subseteq \mathbb{P}$. Say that $\mathbb{Q}$ *seals D over* $\mathbb{P}$, symbolically $\mathbb{P} \sqsubset_D \mathbb{Q}$, if $\mathbb{P} \sqsubset \mathbb{Q}$ holds and every tree $S \in \mathbb{Q}$ satisfies $S \subseteq^{\mathtt{fin}} \bigcup D$. Then simply $\mathbb{P} \sqsubset \mathbb{Q}$ is equivalent to $\mathbb{P} \sqsubset_{\mathbb{P}} \mathbb{Q}$. □

As we'll see now, a sealed set has to be pre-dense both before and after the refinement. The additional importance of sealing refinements lies in fact that, once established, it preserves under further simple refinements, that is, $\sqsubset_D$ is transitive in a combination with $\sqsubset$ in the sense of (ii) of the following lemma:

Lemma 9.4. (i) *If* $\mathbb{P} \sqsubset_D \mathbb{Q}$ *then* D *is pre-dense in* $\mathbb{P} \cup \mathbb{Q}$*, and if in addition* $\mathbb{P}$ *is regular then* D *is pre-dense in* $\mathbb{P}$ *as well;*

(ii) *if* $\mathbb{P} \sqsubset_D \mathbb{Q} \sqsubset \mathbb{R}$ (note: the second $\sqsubset$ is not $\sqsubset_D$!) *then* $\mathbb{P} \sqsubset_D \mathbb{R}$*;*

(iii) *if* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* **PTF**, $0 < \mu < \lambda$*, and* $\mathbb{P} = \bigcup_{\alpha<\mu} \mathbb{P}_\alpha \sqsubset_D \mathbb{P}_\mu$*, then* $\mathbb{P} \sqsubset_D \mathbb{Q} = \bigcup_{\mu\le\alpha<\lambda} \mathbb{P}_\alpha$.

Proof. (i) To see that D is pre-dense in $\mathbb{P} \cup \mathbb{Q}$, let $T_0 \in \mathbb{P} \cup \mathbb{Q}$. By 9.1(1), there is a tree $T \in \mathbb{Q}$, $T \subseteq T_0$. Then $T \subseteq^{\mathtt{fin}} \bigcup D$, in particular, there is a tree $S \in D$ with $X = [S] \cap [T] \ne \varnothing$. However X is clopen in $[T]$ by 9.1(3). Therefore, by Lemma 5.2, there is a tree $T' \in \mathbb{Q}$ with $[T'] \subseteq X$, thus $T' \subseteq S \in D$ and $T' \subseteq T \subseteq T_0$. We conclude that T_0 is compatible with $S \in D$ in $\mathbb{P} \cup \mathbb{Q}$.

To see that D is pre-dense in $\mathbb{P}$ (assuming $\mathbb{P}$ is regular), let $S_0 \in \mathbb{P}$. It follows from the above that S_0 is compatible with some $S \in D$, hence, S and S_0 are not absolutely incompatible. It remains to use Lemma 5.5(i).

To prove (ii) on the top of Lemma 9.2(ii), let $R \in \mathbb{R}$. Then $R \subseteq^{\mathtt{fin}} \bigcup \mathbb{Q}$, but each $T \in \mathbb{Q}$ satisfies $T \subseteq^{\mathtt{fin}} \bigcup D$. The same for (iii). □

The existence of $\sqsubset_D$-refinements will be established below.

10. Refining multiforcings

Let $\boldsymbol{\pi}, \boldsymbol{\varrho}$ be multiforcings. Say that $\boldsymbol{\varrho}$ is an *refinement* of $\boldsymbol{\pi}$, symbolically $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$, if $|\boldsymbol{\pi}| \subseteq |\boldsymbol{\varrho}|$ and $\boldsymbol{\pi}(\xi,k) \sqsubset \boldsymbol{\varrho}(\xi,k)$ whenever $\langle \xi,k \rangle \in |\boldsymbol{\pi}|$.

Corollary 10.1 *(of Lemma 9.2). If* $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho} \sqsubset \boldsymbol{\rho}$ *then* $\boldsymbol{\pi} \sqsubset \boldsymbol{\rho}$.

If $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$ *then the multiforcing* $\mathbf{MT}(\boldsymbol{\varrho})$ *is open dense in* $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varrho})$. □

Our next goal is to introduce a version of Definition 9.3 suitable for multiforcings; we expect an appropriate version of Lemma 9.4 to hold.

First of all, we accommodate the definition of the relation $\subseteq^{\mathtt{fin}}$ in Section 9 for multitrees. Namely if $\boldsymbol{u}$ is a multitree and $\boldsymbol{D}$ a collection of multitrees, then $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}$ will mean that there is a finite set $\boldsymbol{D}' \subseteq \boldsymbol{D}$ satisfying 1) $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in \boldsymbol{D}'$, and 2) $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v}\in\boldsymbol{D}'} [\boldsymbol{v}]$.

Definition 10.2. Let $\boldsymbol{\pi}, \boldsymbol{\varrho}$ be multiforcings, and $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$. Say that $\boldsymbol{\varrho}$ *seals a set* $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$ *over* $\boldsymbol{\pi}$, symbolically $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$ if the following condition holds:

($*$) if $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varrho})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, then there is $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, still $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$, where

$$\boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|} = \{\boldsymbol{u}' \in \mathbf{MT}(\boldsymbol{\pi}) : |\boldsymbol{u}'| = |\boldsymbol{u}| \text{ and } \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}\}. \quad \square$$

Note that if $\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{D}, \boldsymbol{q}$ are as indicated then still $\boldsymbol{u} \cup \boldsymbol{q} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}$ holds via the finite set $\boldsymbol{D}' = \{\boldsymbol{u}' \cup \boldsymbol{q} : \boldsymbol{u}' \in \boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}\} \subseteq \boldsymbol{D}$. Anyway the definition of $\sqsubset_{\boldsymbol{D}}$ in 10.2 looks somewhat different and more complex than the definition of $\sqsubset_D$ in 9.3, which reflects the fact that finite-support products of forcing notions in **PTF** behave differently (and in more complex way) than single perfect-tree forcings. Accordingly, the next lemma, similar to Lemma 9.4, is way harder to prove.

Lemma 10.3. *Let* $\boldsymbol{\pi}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ *be multiforcings and* $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$*. Then:*

(i) *if* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$ *then* $\boldsymbol{D}$ *is dense in* $\mathbf{MT}(\boldsymbol{\pi})$ *and pre-dense in* $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varrho})$*;*

(ii) *if* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$ *and* $\boldsymbol{D} \subseteq \boldsymbol{D}' \subseteq \mathbf{MT}(\boldsymbol{\pi})$ *then* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}'} \boldsymbol{\varphi}$*;*
(iii) *if* $\boldsymbol{\pi}$ *is regular,* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}_i} \boldsymbol{\varphi}$ *for* $i = 1, \dots, n$*, all sets* $\boldsymbol{D}_i \subseteq \mathbf{MT}(\boldsymbol{\pi})$ *are open dense in* $\mathbf{MT}(\boldsymbol{\pi})$*, and* $\boldsymbol{D} = \bigcap_i \boldsymbol{D}_i$*, then* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$*;*
(iv) *if* $\boldsymbol{D}$ *is open dense in* $\mathbf{MT}(\boldsymbol{\pi})$ *and* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi} \sqsubset \boldsymbol{\sigma}$ *then* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\sigma}$*;*
(v) *if* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{MF}$*,* $0 < \mu < \lambda$*,* $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$*,* $\boldsymbol{D}$ *is open dense in* $\mathbf{MT}(\boldsymbol{\pi})$*, and* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\pi}_\mu$*, then* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi} = \bigcup^{\mathsf{cw}}_{\mu\le\alpha<\lambda} \boldsymbol{\pi}_\alpha$*.*

Proof. (i) To check that $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi})$, let $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi})$. Due to the product character of $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi})$, we can assume that $|\boldsymbol{r}| \subseteq |\boldsymbol{\pi}|$. Let

$$X = \{\langle \xi, k\rangle \in |\boldsymbol{r}| : T^{\boldsymbol{r}}_{\xi k} \in \boldsymbol{\varphi}(\xi, k)\}, \quad Y = \{\langle \xi, k\rangle \in |\boldsymbol{r}| : T^{\boldsymbol{r}}_{\xi k} \in \boldsymbol{\pi}(\xi, k)\}.$$

Then $\boldsymbol{r} = \boldsymbol{u} \cup \boldsymbol{p}$, where $\boldsymbol{u} = \boldsymbol{r} \restriction X \in \mathbf{MT}(\boldsymbol{\varphi})$, $\boldsymbol{p} = \boldsymbol{r} \restriction Y \in \mathbf{MT}(\boldsymbol{\pi})$. As $\boldsymbol{\varphi}$ seals $\boldsymbol{D}$, there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigcup \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. We assert that

(*) there is a multitree $\boldsymbol{u}' \in \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ compatible with $\boldsymbol{u}$ in $\mathbf{MT}(\boldsymbol{\varphi})$.

Indeed by definition there is a finite set $U \subseteq \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ such that 1) $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in U$, and 2) $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v}\in U} [\boldsymbol{v}]$. Then obviously there is a multitree $\boldsymbol{u}' \in U$ such that $[\boldsymbol{u}'] \cap [\boldsymbol{u}] \ne \varnothing$. This implies $[T^{\boldsymbol{u}}_{\xi k}] \cap [T^{\boldsymbol{u}'}_{\xi k}] \ne \varnothing$ for all $\langle \xi, k\rangle \in |\boldsymbol{u}'| = |\boldsymbol{u}|$. Here $T^{\boldsymbol{u}}_{\xi k} \in \boldsymbol{\varphi}(\xi, k)$ (as $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$) and $T^{\boldsymbol{u}'}_{\xi k} \in \boldsymbol{\pi}(\xi, k)$ (as $\boldsymbol{u}' \in \mathbf{MT}(\boldsymbol{\pi})$). However we have $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$, hence in particular $\boldsymbol{\pi}(\xi, k) \sqsubset \boldsymbol{\varphi}(\xi, k)$ for all $\langle \xi, k\rangle \in |\boldsymbol{u}'| = |\boldsymbol{u}|$. It follows by definition that if $\langle \xi, k\rangle \in |\boldsymbol{u}'| = |\boldsymbol{u}|$ then $[T^{\boldsymbol{u}}_{\xi k}] \cap [T^{\boldsymbol{u}'}_{\xi k}]$ is a non-empty clopen subset of $[T^{\boldsymbol{u}}_{\xi k}]$. Therefore by Lemma 5.2 there is a tree $T_{\xi k} \in \boldsymbol{\varphi}(\xi, k)$ with $[T_{\xi k}] \subseteq [T^{\boldsymbol{u}}_{\xi k}] \cap [T^{\boldsymbol{u}'}_{\xi k}]$. The multitree $\boldsymbol{w}$ defined by $|\boldsymbol{w}| = |\boldsymbol{u}'| = |\boldsymbol{u}|$ and $T^{\boldsymbol{w}}_{\xi k} = T_{\xi k}$ for all $\langle \xi, k\rangle \in |\boldsymbol{w}|$, belongs to $\mathbf{MT}(\boldsymbol{\varphi})$ and satisfies $\boldsymbol{w} \leqslant \boldsymbol{u}$ and $\boldsymbol{w} \leqslant \boldsymbol{u}'$. This completes the proof of $(*)$.

By (*), let $\boldsymbol{w} \in \mathbf{MT}(\boldsymbol{\varphi})$, $\boldsymbol{w} \leqslant \boldsymbol{u}$, $\boldsymbol{w} \leqslant \boldsymbol{u}'$, $|\boldsymbol{w}| = |\boldsymbol{u}'| = |\boldsymbol{u}|$. Then the multitree $\boldsymbol{r}' = \boldsymbol{w} \cup \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi} \vee \boldsymbol{\varphi})$ satisfies $\boldsymbol{r}' \leqslant \boldsymbol{r}$ and $\boldsymbol{r}' \leqslant \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}$.

To check that $\boldsymbol{D}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$, suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$. Let $\boldsymbol{u} = \boldsymbol{\Lambda}$ (the empty multitree) in $(*)$ of Definition 10.2, so that $|\boldsymbol{u}| = \varnothing$ and $\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}} = \boldsymbol{D}$.

(ii) is obvious. To prove (iii), let $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$. Iterating $(*)$ for $\boldsymbol{D}_i$, $i = 1, \dots, n$, we find a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee (\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ for all i, where

$$(\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}} = \{\boldsymbol{u}' \in \mathbf{MT}(\boldsymbol{\pi}) : |\boldsymbol{u}'| = |\boldsymbol{u}| \text{ and } \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}_i\}.$$

Thus there are finite sets $U_i \subseteq (\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ such that $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v}\in U_i} [\boldsymbol{v}]$ for all i. Using the regularity assumption and Lemma 7.3, we refine multitrees in $\bigcup_i U_i$, getting a finite set $W \subseteq \mathbf{MT}(\boldsymbol{\pi})$ such that still $|\boldsymbol{w}| = |\boldsymbol{u}|$ for all $\boldsymbol{w} \in W$, $\bigcap_i \bigcup_{\boldsymbol{v}\in U_i} [\boldsymbol{v}] = \bigcup_{\boldsymbol{w}\in W} [\boldsymbol{w}]$, and if $i = 1, \dots, n$ and $\boldsymbol{w} \in W$ then $[\boldsymbol{w}] \subseteq [\boldsymbol{v}]$ for some $\boldsymbol{v} \in U_i$ — therefore $\boldsymbol{w} \cup \boldsymbol{q} \in \boldsymbol{D}_i$. We conclude that if $\boldsymbol{w} \in W$ then $\boldsymbol{w} \cup \boldsymbol{q} \in \boldsymbol{D}$, hence $\boldsymbol{w} \in \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. Thus $W \subseteq \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. However $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{w}\in W} [\boldsymbol{w}]$ by the choice of W. We conclude that $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, as required.

(iv) It follows from Corollary 10.1 that $\boldsymbol{\pi} \sqsubset \boldsymbol{\sigma}$, hence it remains to check that $\boldsymbol{\sigma}$ seals $\boldsymbol{D}$ over $\boldsymbol{\pi}$. Assume that $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\sigma})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$. As $\boldsymbol{\varphi} \sqsubset \boldsymbol{\sigma}$, there is a finite set $U \subseteq \mathbf{MT}(\boldsymbol{\varphi})$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in U$, and $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v}\in U} [\boldsymbol{v}]$. As $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$, by iterated application of $(*)$ of Definition 10.2, we get a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, still $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and if $\boldsymbol{v} \in U$ then $\boldsymbol{v} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, where

$$\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}} = \{\boldsymbol{v}' \in \mathbf{MT}(\boldsymbol{\pi}) : |\boldsymbol{v}'| = |\boldsymbol{v}| = |\boldsymbol{u}| \wedge \boldsymbol{v}' \cup \boldsymbol{q} \in \boldsymbol{D}\}.$$

Note finally that $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee U$ by construction, hence $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ as well.

(v) We have to check that $\mathfrak{P}$ seals $\boldsymbol{D}$ over $\boldsymbol{\pi}$. Let $\boldsymbol{u} \in \mathbf{MT}(\mathfrak{P})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$. As above, there is a finite set $U \subseteq \mathbf{MT}(\boldsymbol{\pi}_\mu)$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in U$ and $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v}\in U}[\boldsymbol{v}]$. And so on as in the proof of (iv). □

11. Generic refinement of a multiforcing

Here we introduce a construction, due to Jensen in its original form, which implies the existence of refinements of forcings and multiforcings, of types $\sqsubset_D$ and $\sqsubset_{\boldsymbol{D}}$.

Definition 11.1. 1. Suppose that $\boldsymbol{\pi}$ is a small multiforcing, and $\mathfrak{M} \in \mathrm{HC}$ is any set. (Recall that HC = all hereditarily countable sets.) This is the input.

2. The set $\mathfrak{M}^+$ of all sets $X \in \mathrm{HC}$, $\in$-definable in HC by formulas with sets in $\mathfrak{M}$ as parameters, is still countable. Therefore there exists a $\preccurlyeq$-increasing sequence $\langle \boldsymbol{\varphi}(j)\rangle_{j<\omega}$ of multisystems $\boldsymbol{\varphi}(j) \in \mathbf{MS}(\boldsymbol{\pi})$, $\mathfrak{M}^+$-*generic* in the sense that it intersects any set $\Delta \subseteq \mathbf{MS}(\boldsymbol{\pi})$, $\Delta \in \mathfrak{M}^+$, dense in $\mathbf{MS}(\boldsymbol{\pi})$. (The density means that for any $\boldsymbol{\psi} \in \mathbf{MS}(\boldsymbol{\pi})$ there is a multisystem $\boldsymbol{\varphi} \in \Delta$ with $\boldsymbol{\psi} \preccurlyeq \boldsymbol{\varphi}$.)

Let us fix any such a $\mathfrak{M}^+$-generic sequence $\Phi = \langle \boldsymbol{\varphi}(j)\rangle_{j<\omega}$.

3. Suppose that $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ and $m < \omega$. In particular, the sequence Φ intersects every (dense by Lemma 8.2(i),(ii)) set of the form

$$\Delta_{\xi kmh} = \{\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi}) : h^{\boldsymbol{\varphi}}(\xi, k, m) \ge h\} \in \mathfrak{M}^+ , \quad \text{where } h < \omega .$$

Hence a tree $\boldsymbol{T}^{\Phi}_{\xi k,m}(s) \in \boldsymbol{\pi}(\xi, k)$ can be associated to any $s \in 2^{<\omega}$, such that, for all j, if $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}(j)|$ and $\mathtt{lh}(s) \le h^{\boldsymbol{\varphi}(j)}(\xi, k, m)$ then $T^{\boldsymbol{\varphi}(j)}_{\xi k,m}(s) = \boldsymbol{T}^{\Phi}_{\xi k,m}(s)$.

4. Then it follows from Lemma 6.3 that each set $\boldsymbol{Q}^{\Phi}_{\xi k,m} = \bigcap_h \bigcup_{s\in 2^h} \boldsymbol{T}^{\Phi}_{\xi k,m}(s)$ is a tree in $\mathbf{PT}$ (not necessarily in $\boldsymbol{\pi}(\xi, k)$), as well as the trees

$$\boldsymbol{Q}^{\Phi}_{\xi k,m}(s) = \bigcap_{n\ge \mathtt{lh}(s)} \bigcup_{t\in 2^n,\, s\subseteq t} \boldsymbol{T}^{\Phi}_{\xi k,m}(t) ,$$

and obviously $\boldsymbol{Q}^{\Phi}_{\xi k,m} = \boldsymbol{Q}^{\Phi}_{\xi k,m}(\Lambda)$. Let $\mathbb{Q}^{\Phi}_{\xi k} = \{\boldsymbol{Q}^{\Phi}_{\xi k,m}(s) : m < \omega \wedge s \in 2^{<\omega}\}$.

5. If $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ then let $\mathfrak{P}(\xi, k) = \mathbb{Q}^{\Phi}_{\xi k} = \{\boldsymbol{Q}^{\Phi}_{\xi k,m}(s) : m < \omega \wedge s \in 2^{<\omega}\}$.

6. Finally if $\mathfrak{P} = \mathfrak{P}[\Phi]$ is obtained this way from an $\mathfrak{M}^+$-generic sequence Φ of multisystems in $\mathbf{MS}(\boldsymbol{\pi})$, then $\mathfrak{P}$ is called an $\mathfrak{M}$-*generic refinement of* $\boldsymbol{\pi}$. □

Proposition 11.2 *(by the countability of $\mathfrak{M}^+$). If $\boldsymbol{\pi}$ is a small multiforcing and $\mathfrak{M} \in \mathrm{HC}$ then there is an $\mathfrak{M}$-generic refinement $\mathfrak{P}$ of $\boldsymbol{\pi}$.* □

Theorem 11.3. *If $\boldsymbol{\pi}$ is a small multiforcing, a set $\mathfrak{M} \in \mathrm{HC}$ contains $\boldsymbol{\pi}$, $|\boldsymbol{\pi}| \subseteq \mathfrak{M}$, and $\mathfrak{P}$ is an $\mathfrak{M}$-generic refinement of $\boldsymbol{\pi}$, then:*

(i) *$\mathfrak{P}$ is a small special multiforcing, $|\mathfrak{P}| = |\boldsymbol{\pi}|$, and $\boldsymbol{\pi} \sqsubset \mathfrak{P}$;*
(ii) *if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ and a set $D \in \mathfrak{M}$, $D \subseteq \boldsymbol{\pi}(\xi, k)$ is pre-dense in $\boldsymbol{\pi}(\xi, k)$ then $\boldsymbol{\pi}(\xi, k) \sqsubset_D \mathfrak{P}(\xi, k)$;*

(a) *if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, $m < \omega$, and $s \in 2^{<\omega}$ then $\boldsymbol{Q}^{\Phi}_{\xi k,m}(s) = \boldsymbol{Q}^{\Phi}_{\xi k,m}(\to s)$;*
(b) *if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, $m < \omega$, and $s \in 2^{<\omega}$ then $\boldsymbol{Q}^{\Phi}_{\xi k,m}(s) \subseteq \boldsymbol{T}^{\Phi}_{\xi k,m}(s)$;*
(c) *if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, $m < \omega$, and strings $t' \ne t$ in $2^{<\omega}$ are $\subseteq$-incomparable then $[\boldsymbol{Q}^{\Phi}_{\xi k,m}(t')] \cap [\boldsymbol{Q}^{\Phi}_{\xi k,m}(t)] = [\boldsymbol{T}^{\Phi}_{\xi k,m}(t')] \cap [\boldsymbol{T}^{\Phi}_{\xi k,m}(t)] = \varnothing$;*
(d) *if $\langle \xi, k, m\rangle \ne \langle \eta, \ell, n\rangle$ then $[\boldsymbol{Q}^{\Phi}_{\xi k,m}] \cap [\boldsymbol{Q}^{\Phi}_{\eta \ell,n}] = \varnothing$;*

(e) *if* $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, $S \in \varphi(\xi, k)$ *and* $T \in \boldsymbol{\pi}(\xi, k)$ *then* $[S] \cap [T]$ *is clopen in* $[S]$ *and* $T \not\subseteq S$, *in particular,* $\boldsymbol{\pi}(\xi, k) \cap \varphi(\xi, k) = \varnothing$;

(f) *if* $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ *then the set* $\varphi(\xi, k)$ *is open dense in* $\varphi(\xi, k) \cup \boldsymbol{\pi}(\xi, k)$.

If in addition $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\lambda} \boldsymbol{\pi}_\alpha$, *where* $\lambda < \omega_1$, $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence of small special multiforcings, and* $\mathfrak{M}$ *contains* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *and all* $\alpha < \lambda$, *then*

(iii) *if* $\alpha < \lambda$ *then* $\boldsymbol{\pi}_\alpha \sqsubset \varphi$.

Proof. Let $\varphi = \varphi[\Phi]$ be obtained from an $\mathfrak{M}^+$-generic sequence Φ of multisystems in $\mathbf{MS}(\boldsymbol{\pi})$, as above. We argue in the notation of Definition 11.1.

If $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ and $m < \omega$ then by construction the system of trees $\boldsymbol{T}^{\Phi}_{\xi k,m}(s) \in \boldsymbol{\pi}(\xi, k)$, $s \in 2^{<\omega}$, satisfies 6.1(∗) on the whole domain $s \in 2^{<\omega}$. This leads to (a), (b) (essentially corollaries of Lemma 6.3) and (c).

To prove (d) note that the set Δ of all 2wise disjoint multisystems $\boldsymbol{\varphi}$ such that $|\boldsymbol{\varphi}|$ contains both $\langle \xi, k, m\rangle$ and $\langle \eta, \ell, n\rangle$, is dense in $\mathbf{MS}(\boldsymbol{\pi})$ by Lemma 8.2, and obviously $\Delta \in \mathfrak{M}^+$. Therefore there is $j < \omega$ such that $\boldsymbol{\varphi}(j) \in \Delta$. Let $h = h^{\boldsymbol{\varphi}(j)}(\xi, k, m)$ and $h' = h^{\boldsymbol{\varphi}(j)}(\eta, \ell, n)$. Then the sets

$$A = \bigcup_{s\in 2^h} [T^{\boldsymbol{\varphi}(j)}_{\xi k,m}(s)] = \bigcup_{s\in 2^h} [\boldsymbol{T}^{\Phi}_{\xi k,m}(s)],\ B = \bigcup_{t\in 2^{h'}} [T^{\boldsymbol{\varphi}(j)}_{\xi \ell,n}(t)] = \bigcup_{t\in 2^{h'}} [\boldsymbol{T}^{\Phi}_{\xi \ell,n}(t)]$$

are disjoint as $\boldsymbol{\varphi}(j) \in \Delta$. However $[\boldsymbol{Q}^{\Phi}_{\xi k,m}] \subseteq A$ and $[\boldsymbol{Q}^{\Phi}_{\eta \ell,n}] \subseteq B$.

(i) Thus the sets $\varphi(\xi, k) = \mathbb{Q}^{\Phi}_{\xi k}$ are special **PTF**s (Definition 5.4), and hence φ is a small special multiforcing, and $|\varphi| = |\boldsymbol{\pi}|$. See a continuation below.

(e) To prove the clopenness claim, note that the set Δ of all multisystems $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$ such that $\langle \xi, k, m\rangle \in |\varphi|$ and if $s \in 2^h$, where $h = h^{\boldsymbol{\varphi}}(\xi, k, m)$, then either $T^{\boldsymbol{\varphi}}_{\xi k,m}(s) \subseteq T$ or $[T^{\boldsymbol{\varphi}}_{\xi k,m}(s)] \cap [T] = \varnothing$, is dense. To prove $T \not\subseteq S$, the set Δ' of all multisystems $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$ such that $\langle \xi, k, m\rangle \in |\varphi|$ and $T \not\subseteq \bigcup_{s\in 2^h} T^{\boldsymbol{\varphi}}_{\xi k,m}(s)$, where $h = h^{\boldsymbol{\varphi}}(\xi, k, m)$, is dense. Note that $\Delta, \Delta' \in \mathfrak{M}^+$ and argue as above.

(f) *Density.* If $T \in \boldsymbol{\pi}(\xi, k)$ then the set $\Delta(T)$ of all multisystems $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$, such that $T^{\boldsymbol{\varphi}}_{\xi k,m}(\Lambda) = T$ for some m, is dense in $\mathbf{MS}(\boldsymbol{\pi})$ by Lemma 8.2(ii), therefore $\varphi(j) \in \Delta(T)$ for some j. Then $\boldsymbol{T}^{\Phi}_{\xi k,m}(\Lambda) = T$ for some $m < \omega$. However $\boldsymbol{Q}^{\Phi}_{\xi m,k}(\Lambda) \subseteq \boldsymbol{T}^{\Phi}_{\xi k,m}(\Lambda)$. *Openness.* Suppose that $S \in \varphi(\xi, k)$, $T \in \varphi(\xi, k) \cup \boldsymbol{\pi}(\xi, k)$, $T \subseteq S$. Then $T \notin \boldsymbol{\pi}(\xi, k)$ by (e). Therefore $T \in \varphi(\xi, k)$.

(i), continuation. To establish $\boldsymbol{\pi} \sqsubset \varphi$, let $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$. We have to prove that $\boldsymbol{\pi}(\xi, k) \sqsubset \varphi(\xi, k)$. This comes down to conditions (1), (2), (3) of Definition 9.1, of which (1) follows from (f) and (3) from (e), and (2) is obvious since $\boldsymbol{Q}^{\Phi}_{\xi k,m}(s) \subseteq \boldsymbol{T}^{\Phi}_{\xi k,m}(s) \in \boldsymbol{\pi}(\xi, k)$ for all m.

(ii) As $\boldsymbol{\pi} \sqsubset \varphi$ has been checked, it remains to prove $\boldsymbol{Q}^{\Phi}_{\xi k,m} \subseteq^{\mathtt{fin}} \bigcup D$ for all m. It follows from the pre-density of D that the set

$$D' = \{T \in \boldsymbol{\pi}(\xi, k) : \exists S \in D\,(T \subseteq S)\} \qquad (\bullet)$$

is open dense in $\boldsymbol{\pi}(\xi, k)$, and still $D' \in \mathfrak{M}^+$. Then the set $\Delta \in \mathfrak{M}^+$ of all multisystems $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$ such that $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}|$ and $T^{\boldsymbol{\varphi}}_{\xi k,m}(s) \in D'$ for all $s \in h^{\boldsymbol{\varphi}}(\xi, k, m)$, is dense in $\mathbf{MS}(\boldsymbol{\pi})$ by Lemma 8.2(iii). Thus $\boldsymbol{\varphi}(j) \in \Delta$ for some j, which witnesses $\boldsymbol{Q}^{\Phi}_{\xi k,m} \subseteq^{\mathtt{fin}} \bigcup D'$, and hence $\subseteq^{\mathtt{fin}} \bigcup D$ as well by (•).

(iii) We have to prove that $\boldsymbol{\pi}_\alpha(\xi, k) \sqsubset \varphi(\xi, k)$ whenever $\langle \xi, k\rangle \in |\boldsymbol{\pi}_\alpha|$. And as $\boldsymbol{\pi}(\xi, k) \sqsubset \varphi(\xi, k)$ has been checked, it suffices to prove that $\boldsymbol{Q}^{\Phi}_{\xi k,m} \subseteq^{\mathtt{fin}} \bigcup \boldsymbol{\pi}_\alpha(\xi, k)$ for all m. However $D = \boldsymbol{\pi}_\alpha(\xi, k)$ is pre-dense in $\boldsymbol{\pi}(\xi, k)$ by Lemma 9.2(v), and still $D \in \mathfrak{M}^+$, hence we can refer to (ii). □

Corollary 11.4. *In the assumptions of Proposition 11.2, if* $|\boldsymbol{\pi}| \subseteq Z \subseteq \omega_1 \times \omega$ *and* Z *is at most countable then there is a small special multiforcing* φ *such that* $|\varphi| = Z$ *and* $\boldsymbol{\pi} \sqsubset \varphi$*.*

Proof. If $|\boldsymbol{\pi}| = Z$ then let $\mathfrak{M}$ be any countable set containing $\boldsymbol{\pi}$, pick φ by Proposition 11.2, and apply Theorem 11.3. If $|\boldsymbol{\pi}| \subsetneqq Z$ then we trivially extend the construction by $\varphi(\xi, k) = \mathbb{P}_{\text{coh}}$ (see Example 5.1) for all $\langle \xi, k \rangle \in Z \smallsetminus |\boldsymbol{\pi}|$. $\square$

Corollary 11.5. *Suppose that* $\lambda < \omega_1$*, and* $\langle \mathbb{P}_\alpha \rangle_{\alpha<\lambda}$ *is an* $\sqsubset$*-increasing sequence of countable special forcings in* **PTF***. Then there is a countable special forcing* $\mathbb{Q} \in \mathbf{PTF}$ *such that* $\mathbb{P}_\alpha \sqsubset \mathbb{Q}$ *for each* $\alpha < \lambda$*.*

Proof. If $\alpha < \lambda$ then let a multiforcing $\boldsymbol{\pi}_\alpha$ be defined by $|\boldsymbol{\pi}_\alpha| = \{\langle 0, 0 \rangle\}$ and by $\boldsymbol{\pi}_\alpha(0, 0) = \mathbb{P}_\alpha$. By Proposition 11.2 and Theorem 11.3 there is a multiforcing φ satisfying $|\varphi| = \{\langle 0, 0 \rangle\}$ and $\boldsymbol{\pi}_\alpha \sqsubset \varphi$, $\forall \alpha$. Let $\mathbb{Q} = \varphi(0, 0)$. $\square$

12. Sealing dense sets

This Section proves a special consequence of $\mathfrak{M}^+$-genericity of multiforcing refinements, the relation $\sqsubset$ of Definition 10.2 between a multiforcing and its refinement.

Theorem 12.1. *In the assumptions of Theorem 11.3, if* $\boldsymbol{D} \in \mathfrak{M}^+$*,* $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$*, and* $\boldsymbol{D}$ *is open dense in* $\mathbf{MT}(\boldsymbol{\pi})$*, then* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \varphi$*.*

Proof. We suppose that $\varphi = \varphi[\Phi]$ is obtained from an increasing $\mathfrak{M}^+$-generic sequence Φ of multisystems in $\mathbf{MS}(\boldsymbol{\pi})$, as in Definition 11.1, and argue in the notation of 11.1. Suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\varphi)$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, as in $(*)$ of Definition 10.2; the extra condition $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$ holds automatically as we still have $|\varphi| = |\boldsymbol{\pi}|$. Let $X = |\boldsymbol{u}|$, $Y = |\boldsymbol{\pi}| \smallsetminus X$. If $\langle \xi, k \rangle \in X$ then $T^{\boldsymbol{u}}_{\xi k} = \boldsymbol{Q}^{\Phi}_{\xi k, m_{\xi k}}(s_{\xi k})$, where $m_{\xi k} < \omega$ and $s_{\xi k} \in 2^{<\omega}$. By obvious reasons we can assume that $s_{\xi k} = \Lambda$, hence $T^{\boldsymbol{u}}_{\xi k} = \boldsymbol{Q}^{\Phi}_{\xi k, m_{\xi k}}$, for all $\langle \xi, k \rangle \in X$.

Consider the set Δ of all multisystems $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$ such that there is a number $H > 0$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying (1), (2), (3), (4) below.

(1) $|\boldsymbol{q}| \cap X = \varnothing$ and $\boldsymbol{q} \leqslant \boldsymbol{p}$;
(2) if $\langle \xi, k \rangle \in X$ then $\langle \xi, k, m_{\xi k} \rangle \in |\boldsymbol{\varphi}|$;
(3) if $\langle \xi, k, m \rangle \in |\boldsymbol{\varphi}|$ then $h^{\boldsymbol{\varphi}}(\xi, k, m) = H$.

To formulate the last requirement, we need one more definition. Suppose that $\tau = \langle t_{\xi k} \rangle_{\langle \xi, k \rangle \in X}$ is a system of strings $\tau(\xi, k) = t_{\xi k} \in 2^H$, symbolically $\tau \in (2^H)^X$. Define a multitree $\mathbf{s}(\boldsymbol{\varphi}, \tau) \in \mathbf{MT}(\boldsymbol{\pi})$ so that $|\mathbf{s}(\boldsymbol{\varphi}, \tau)| = X$ and $T^{\mathbf{s}(\boldsymbol{\varphi},\tau)}_{\xi k} = T^{\boldsymbol{\varphi}}_{\xi k, m_{\xi k}}(t_{\xi k})$ for all $\langle \xi, k \rangle \in X$. Note that $|\mathbf{s}(\boldsymbol{\varphi}, \tau)| = |\boldsymbol{u}|$, and hence the multitree $\mathbf{s}(\boldsymbol{\varphi}, \tau) \cup \boldsymbol{q}$ belongs to $\mathbf{MT}(\boldsymbol{\pi})$ as well.[13] Now goes the last condition.

(4) If $\tau \in (2^H)^X$ then $\mathbf{s}(\boldsymbol{\varphi}, \tau) \cup \boldsymbol{q} \in \boldsymbol{D}$.

Lemma 12.2. *The set* Δ *is dense in* $\mathbf{MS}(\boldsymbol{\pi})$*.*

Proof *(Lemma).* Suppose that $\boldsymbol{\psi} \in \mathbf{MS}(\boldsymbol{\pi})$; we have to find a multisystem $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$ with $\boldsymbol{\psi} \preccurlyeq \boldsymbol{\varphi}$. First of all, by Lemma 8.2(i)(ii) we can assume that

[13] Here, if $\boldsymbol{p}, \boldsymbol{q}$ are multitrees satisfying $|\boldsymbol{p}| \cap |\boldsymbol{q}| = \varnothing$ (disjoint domains), then $\boldsymbol{p} \cup \boldsymbol{q}$, a *disjoint union*, is a multitree such that $|\boldsymbol{p} \cup \boldsymbol{q}| = |\boldsymbol{p}| \cup |\boldsymbol{q}|$ and $T^{\boldsymbol{p} \cup \boldsymbol{q}}_{\xi k} = T^{\boldsymbol{p}}_{\xi k}$ whenever $\langle \xi, k \rangle \in |\boldsymbol{p}|$ but $T^{\boldsymbol{p} \cup \boldsymbol{q}}_{\xi k} = T^{\boldsymbol{q}}_{\xi k}$ whenever $\langle \xi, k \rangle \in |\boldsymbol{q}|$.

(a) if $\langle \xi, k \rangle \in X$ then $\langle \xi, k, m_{\xi k} \rangle \in |\boldsymbol{\psi}|$;
(b) there is a number $g > 0$ such that $h^{\boldsymbol{\psi}}(\xi, k, m) = g$ for all $\langle \xi, k, m \rangle \in |\boldsymbol{\psi}|$.

Let $H = g + 1$. Define $\boldsymbol{\chi} \in \mathbf{MS}(\boldsymbol{\pi})$ so that $|\boldsymbol{\chi}| = |\boldsymbol{\psi}|$, and $h^{\boldsymbol{\chi}}(\xi, k, m) = H$, $T^{\boldsymbol{\chi}}_{\xi k,m}(s^\frown i) = T^{\boldsymbol{\psi}}_{\xi k,m}(s)(\to i)$ for all $\langle \xi, k, m \rangle \in |\boldsymbol{\psi}|$ and $s^\frown i \in 2^H$; then $\boldsymbol{\psi} \preccurlyeq \boldsymbol{\chi}$.

It follows from the open density of $\boldsymbol{D}$ that there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying (1), and a multisystem $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$ satisfying (4) and such that still $|\boldsymbol{\varphi}| = |\boldsymbol{\psi}|$ and $h^{\boldsymbol{\varphi}}(\xi, k, m) = H$ for all $\langle \xi, k, m \rangle \in |\boldsymbol{\psi}|$, and in addition

(c) if $\langle \xi, k \rangle \in X$ and $s \in 2^H$ then $T^{\boldsymbol{\varphi}}_{\xi k,m_{\xi k}}(s) \subseteq T^{\boldsymbol{\chi}}_{\xi k,m_{\xi k}}(s)$;
(d) $T^{\boldsymbol{\varphi}}_{\xi k,m}(s) = T^{\boldsymbol{\chi}}_{\xi k,m}(s)$ for all applicable ξ, k, m, s not covered by (c).

Namely to achieve (4) for one particular $\tau \in (2^H)^X$, consider the multitree $\boldsymbol{r} = \mathbf{s}(\boldsymbol{\chi}, \tau) \cup \boldsymbol{p}$. There is a multitree $\boldsymbol{r}' \in \boldsymbol{D}$, $\boldsymbol{r}' \leqslant \boldsymbol{r}$. Let a new multisystem $\boldsymbol{\chi}'$ be obtained from $\boldsymbol{\chi}$ by the reassignment $T^{\boldsymbol{\chi}'}_{\xi k,m_{\xi k}}(\tau(\xi, k)) = T^{\boldsymbol{r}'}_{\xi k}$ for all $\langle \xi, k \rangle \in X$. To get the input for the next step, let $\boldsymbol{p}' = \boldsymbol{r}' \restriction Y$,[14] so that $\boldsymbol{r}' = \mathbf{s}(\boldsymbol{\chi}', \tau) \cup \boldsymbol{p}' \in \boldsymbol{D}$.

Now consider another $\tau' \in (2^H)^X$ and the multitree $\boldsymbol{r}' = \mathbf{s}(\boldsymbol{\chi}', \tau') \cup \boldsymbol{p}'$. There is $\boldsymbol{r}'' \in \boldsymbol{D}$, $\boldsymbol{r}'' \leqslant \boldsymbol{r}'$. Define $\boldsymbol{\chi}''$ from $\boldsymbol{\chi}'$ by the reassignment $T^{\boldsymbol{\chi}''}_{\xi k,m_{\xi k}}(\tau'(\xi, k)) = T^{\boldsymbol{r}''}_{\xi k}$ for all $\langle \xi, k \rangle \in X$. Let $\boldsymbol{p}'' = \boldsymbol{r}'' \restriction Y$, so that $\boldsymbol{r}'' = \mathbf{s}(\boldsymbol{\chi}'', \tau') \cup \boldsymbol{p}'' \in \boldsymbol{D}$.

And so on. The final multisystem and multitree of this construction will be $\boldsymbol{\varphi}$ and $\boldsymbol{q}$ satisfying (1), (2), (3), (4). Note that $\boldsymbol{\psi} \preccurlyeq \boldsymbol{\varphi}$, as we only amend the H-th level of $\boldsymbol{\chi}$ absent in $\boldsymbol{\psi}$. □(Lemma)

Note that Δ is defined in HC using sets $\boldsymbol{D}$, $\boldsymbol{\pi}$, $\boldsymbol{p}$, X, and the map $\langle \xi, k \rangle \to m_{\xi k} : X \to \omega$ as parameters. Now, $\boldsymbol{D}$, $\boldsymbol{\pi}$ belong to $\mathfrak{M}^+$ straightforwardly, X belongs to $\mathfrak{M}^+$ since it is a finite subset of a set $|\boldsymbol{\pi}| \subseteq \mathfrak{M}$, and $\boldsymbol{p}$ belongs to $\mathfrak{M}^+$ by similar reasons. It follows that Δ belongs to $\mathfrak{M}^+$ as well.

Therefore, by the lemma and the choice of Φ, there is an index j such that the multisystem $\boldsymbol{\varphi}(j)$ belongs to Δ, which is witnessed by a number $H > 0$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying (1), (2), (3), (4) for $\boldsymbol{\varphi}(j)$ instead of $\boldsymbol{\varphi}$. To prove that $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, note that the multitrees $\mathbf{s}(\boldsymbol{\varphi}(j), \tau) \cup \boldsymbol{q}$, $\tau \in (2^H)^X$, belong to $\boldsymbol{D}$ by (4), and easily $[\boldsymbol{u}] \subseteq \bigcup_{\tau \in (2^H)^X} [\mathbf{s}(\boldsymbol{\varphi}(j), \tau)]$. □

III. Structure of real names

Here we discuss the structure of reals in models of $\mathbf{MT}(\boldsymbol{\pi})$-generic type, $\boldsymbol{\pi}$ being a multiforcing. We are going to focus on *non-principal* reals, *i.e.*, those different from the principal generic reals $x_{\xi k}[G]$ (Remark 7.4). We'll work towards the goal of making every non-principal real to be non-generic with respect to each of the factor forcing notions $\boldsymbol{\pi}(\xi, k)$.

13. Real names

Our next goal is to introduce a suitable notation related to names of reals in the context of forcing notions of the form $\mathbf{MT}(\boldsymbol{\pi})$.

Definition 13.1. A *real name* is any set $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times \omega)$ such that the sets $K^{\mathbf{c}}_{ni} = \{\boldsymbol{p} \in \mathbf{MT} : \langle \boldsymbol{p}, n, i \rangle \in \mathbf{c}\}$ satisfy the following:

[14] Here $\boldsymbol{r}' \restriction Y$ is the plain restriction of the function $\boldsymbol{r}' : |\boldsymbol{r}'| \to \mathbf{PT}$ to the set $|\boldsymbol{r}'| \cap Y$.

(∗) if $n, k, \ell < \omega$, $k \neq \ell$, and $\boldsymbol{p} \in K^{\mathbf{c}}_{nk}$, $\boldsymbol{q} \in K^{\mathbf{c}}_{n\ell}$, then $\boldsymbol{p}, \boldsymbol{q}$ are SAD (somewhere almost disjoint, see Definition 7.1).

A real name $\mathbf{c}$ is *small* if each set $K^{\mathbf{c}}_{ni}$ is at most countable — then the sets $\mathtt{dom}\,\mathbf{c} = \bigcup_{n,i} K^{\mathbf{c}}_{ni} \subseteq \mathbf{MT}$ and $|\mathbf{c}| = \bigcup_{n,i} \bigcup_{\boldsymbol{p} \in K^{\mathbf{c}}_{ni}} |\boldsymbol{p}| \subseteq \omega_1 \times \omega$, and $\mathbf{c}$ itself, are countable, too. □

Definition 13.2. Let $\mathbf{c}$ be a real name and $G \subseteq \mathbf{MT}$ a pairwise compatible set. Define the *evaluation* $\mathbf{c}[G] \in \omega^\omega$ so that $\mathbf{c}[G](n) = i$ iff:

– either $\exists\, \boldsymbol{p} \in G\, \exists\, \boldsymbol{q} \in K^{\mathbf{c}}_{ni}\, (\boldsymbol{p} \leqslant \boldsymbol{q})$ (recall that $\boldsymbol{p} \leqslant \boldsymbol{q}$ means $\boldsymbol{p}$ is stronger),
– or just $i = 0$ and $\neg\, \exists\, \boldsymbol{p} \in G\, \exists\, \boldsymbol{q} \in \bigcup_i K^{\mathbf{c}}_{ni}\, (\boldsymbol{p} \leqslant \boldsymbol{q})$ (default case). □

Definition 13.3. Let $\boldsymbol{\pi}$ be a multiforcing. A real name $\mathbf{c}$ is said to be a $\boldsymbol{\pi}$*-real prename* if, in addition to (∗) above, the following condition holds:

(†) each set $K^{\mathbf{c}}_n = \bigcup_i K^{\mathbf{c}}_{ni}$ is *pre-dense for* $\mathbf{MT}(\boldsymbol{\pi})$, in the sense that the set $K^{\mathbf{c}}_n{\uparrow}\boldsymbol{\pi} = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \exists\, \boldsymbol{q} \in K^{\mathbf{c}}_n\, (\boldsymbol{p} \leqslant \boldsymbol{q})\}$ is dense (then obviously open dense) in $\mathbf{MT}(\boldsymbol{\pi})$.

Generally speaking, we do not assume that $K^{\mathbf{c}}_n \subseteq \mathbf{MT}(\boldsymbol{\pi})$. However if, in addition to (∗), (†) above, $K^{\mathbf{c}}_n \subseteq \mathbf{MT}(\boldsymbol{\pi})$ holds for all n, then say that $\mathbf{c}$ is a $\boldsymbol{\pi}$*-real name*. Then each set $K^{\mathbf{c}}_n = \bigcup_i K^{\mathbf{c}}_{ni}$ is a pre-dense subset of $\mathbf{MT}(\boldsymbol{\pi})$. □

Remark 13.4. Let $\boldsymbol{\pi}$ be a multiforcing, $\mathbf{c}$ be a $\boldsymbol{\pi}$-real prename, and a set $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ be $\mathbf{MT}(\boldsymbol{\pi})$-generic over the collection of all sets $K^{\mathbf{c}}_n{\uparrow}\boldsymbol{\pi}$ as in (†). (All of $K^{\mathbf{c}}_n{\uparrow}\boldsymbol{\pi}$ are dense by the choice of $\mathbf{c}$.) Then the "or" case in Definition 13.2 never happens, because we have $G \cap (K^{\mathbf{c}}_n{\uparrow}\boldsymbol{\pi}) \neq \varnothing$ by the choice of G. □

Remark 13.5. If $\boldsymbol{\pi}$ is a *regular* multiforcing then the notions of being SAD and being incompatible in $\mathbf{MT}(\boldsymbol{\pi})$ are equivalent by Lemma 5.5(i), so that a $\boldsymbol{\pi}$-real name is the same as a $\mathbf{MT}(\boldsymbol{\pi})$-name for an element of ω^ω in the general theory of forcing. □

Example 13.6. If $\xi < \omega_1$, $k < \omega$, then $\dot{\boldsymbol{x}}_{\xi k}$ is a real name such that if $i = 0, 1$ then the set $K^{\dot{\boldsymbol{x}}_{\xi k}}_{ni}$ consists of a lone multitree $\boldsymbol{r} = \boldsymbol{r}^{\xi k}_{ni}$ with $|\boldsymbol{r}| = \{\langle \xi, k\rangle\}$ and $T^{\boldsymbol{r}}_{\xi k} = \{t \in 2^{<\omega} : \mathtt{lh}(t) \leq n \vee t(n) = i\}$, and if $i \geq 2$ then $K^{\dot{\boldsymbol{x}}_{\xi k}}_{ni} = \varnothing$. □

Remark 13.7. If $\boldsymbol{\pi} \in \mathbf{MT}$ and $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ then $\dot{\boldsymbol{x}}_{\xi k}$ is a $\boldsymbol{\pi}$-real prename of the real $x_{\xi k} = x_{\xi k}[G] \in 2^\omega$, the (ξ, k)th term of a $\mathbf{MT}(\boldsymbol{\pi})$-generic sequence $\langle x_{\xi k}[G]\rangle_{\langle \xi, k\rangle \in |\boldsymbol{\pi}|}$. That is, if $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is generic then the real $x_{\xi k}[G]$ defined by 7.4 coincides with the real $\dot{\boldsymbol{x}}_{\xi k}[G]$ defined by 13.2. □

14. Direct forcing

The following definition of the **direct forcing** relation is not explicitly associated with any concrete forcing notion, but in fact the direct forcing relation (in all three instances) is compatible with any forcing notion of the form $\mathbf{MT}(\boldsymbol{\pi})$.

Let $\mathbf{c}$ be a real name. Let us say that a multitree $\boldsymbol{p}$:

- **directly forces** $\mathbf{c}(n) = i$, where $n, i < \omega$, iff there is a multitree $\boldsymbol{q} \in K^{\mathbf{c}}_{ni}$ such that $\boldsymbol{p} \leqslant \boldsymbol{q}$ (meaning: $\boldsymbol{p}$ is stronger);
- **directly forces** $s \subset \mathbf{c}$, where $s \in \omega^{<\omega}$, iff for all $n < \mathtt{lh}(s)$, $\boldsymbol{p}$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;

- **directly forces** $\mathbf{c} \notin [T]$, where $T \in \mathbf{PT}$, iff there is a string $s \in \omega^{<\omega} \smallsetminus T$ such that $\boldsymbol{p}$ directly forces $s \subset \mathbf{c}$.

Lemma 14.1. *If* $\boldsymbol{\pi}$ *is a multiforcing and* $\mathbf{c}$ *is a* $\boldsymbol{\pi}$*-real prename,* $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$*,* $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$*,* $T \in \mathbf{PT}$*,* $n < \omega$*, then*

(i) *there is a number* $i < \omega$ *and a multitree* $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi}), \boldsymbol{q} \leqslant \boldsymbol{p}$*, which directly forces* $\mathbf{c}(n) = i$*;*
(ii) *there is a multitree* $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi}), \boldsymbol{q} \leqslant \boldsymbol{p}$*, which directly forces* $\mathbf{c} \notin [T(\to 0)]$ *or directly forces* $\mathbf{c} \notin [T(\to 1)]$.

Note that if $T \in \boldsymbol{\pi}(\xi, k)$ then the trees $T(\to i)$, $i = 0, 1$ belong to $\boldsymbol{\pi}(\xi, k)$.

Proof. (i) Use the density of sets $K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}$ by Definition 13.3(†) above.

(ii) Let $r = \mathtt{stem}(T)$, $n = \mathtt{lh}(r)$. By (i), there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi}), \boldsymbol{p}' \leqslant \boldsymbol{p}$, and, for any $m \le n$, — a number $i_m = 0, 1$, such that $\boldsymbol{q}$ directly forces $\mathbf{c}(m) = i_m$, $\forall\, m < n$. Define $s \in 2^{n+1}$ by $s(m) = i_m$ for each $m \le n$. Then $\boldsymbol{q}$ directly forces $s \subset \mathbf{c}$. Yet s cannot belong to both $T(\to 0)$ and $T(\to 1)$. □

15. Sealing real names

The next definition extends Definition 10.2 to real names.

Definition 15.1. Assume that $\boldsymbol{\pi}, \boldsymbol{\varphi}$ are multiforcings, $\mathbf{c}$ is a real name, and $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$. Say that $\boldsymbol{\varphi}$ *seals* $\mathbf{c}$ *over* $\boldsymbol{\pi}$, symbolically $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varphi}$, if $\boldsymbol{\varphi}$ seals, over $\boldsymbol{\pi}$, each set $K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}$ defined in Definition 13.3(†), in the sense of Definition 10.2. □

Corollary 15.2 *(of Theorem 12.1). In the assumptions of Theorem 11.3, if* $\mathbf{c} \in \mathfrak{M}^+$ *and* $\mathbf{c}$ *is a* $\boldsymbol{\pi}$*-real prename then* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varphi}$.

Proof. Note that each set $K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}$ belongs to $\mathfrak{M}^+$ (as so do $\mathbf{c}$ and $\boldsymbol{\pi}$) and is dense in $\mathbf{MT}(\boldsymbol{\pi})$, so it remains to apply Theorem 12.1. □

Lemma 15.3. *Let* $\boldsymbol{\pi}, \boldsymbol{\varphi}, \boldsymbol{\sigma}$ *be multiforcings and* $\mathbf{c}$ *be a real name. Then*

(i) *if* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varphi}$ *then* $\mathbf{c}$ *is a* $\boldsymbol{\pi}$*-real prename and a* $(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi})$*-real prename;*
(ii) *if* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varphi} \sqsubset \boldsymbol{\sigma}$ *then* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\sigma}$*;*
(iii) *if* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{MF}$*,* $0 < \mu < \lambda$*,* $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$*, and* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\pi}_\mu$*, then* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varphi} = \bigcup^{\mathsf{cw}}_{\mu\le\alpha<\lambda} \boldsymbol{\pi}_\alpha$.

Proof. (i) By definition, we have $\boldsymbol{\pi} \sqsubset_{K_n^{\mathbf{c}}\uparrow\boldsymbol{\pi}} \boldsymbol{\varphi}$ for each n, therefore $K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$ (then obviously open dense) and pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi})$ by Lemma 10.3(i). It follows that $K_n^{\mathbf{c}}\!\uparrow\!(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi})$ is dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi})$.

To check (ii), (iii) apply (iv), (v) of Lemma 10.3. □

16. Non-principal names and avoiding refinements

Let $\boldsymbol{\pi}$ be a multiforcing. Then $\mathbf{MT}(\boldsymbol{\pi})$ adds a collection of reals $x_{\xi k}$, $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$, where each *principal real* $x_{\xi k} = x_{\xi k}[G]$ is $\boldsymbol{\pi}(\xi, k)$-generic over the ground set universe. Obviously many more reals are added, and given a $\boldsymbol{\pi}$-real prename $\mathbf{c}$, one can elaborate different requirements for a condition $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$ to force that $\mathbf{c}$ is a name of a real of the form $x_{\xi k}$ or to force the opposite. But we are mostly interested in simple conditions related to the “opposite” part. The next definition provides such a condition.

Definition 16.1. Let $\boldsymbol{\pi}$ be a multiforcing, $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$. A real name $\mathbf{c}$ is *non-principal over* $\boldsymbol{\pi}$ *at* ξ, k if the following set is open dense in $\mathbf{MT}(\boldsymbol{\pi})$:

$$\boldsymbol{D}^{\boldsymbol{\pi}}_{\xi k}(\mathbf{c}) = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T^{\boldsymbol{p}}_{\xi k}]\}. \quad \square$$

We'll show below (Theorem 18.2(i)) that the non-principality implies $\mathbf{c}$ being **not** a name of the real $x_{\xi k}[\underline{G}]$. And further, the avoidance condition in the next definition will be shown to imply $\mathbf{c}$ being a name of a non-generic real.

Definition 16.2. Let $\boldsymbol{\pi}, \boldsymbol{\varphi}$ be multiforcings, $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$, $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$. Say that $\boldsymbol{\varphi}$ *avoids a real name* $\mathbf{c}$ *over* $\boldsymbol{\pi}$ *at* ξ, k, in symbol $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\varphi}$, if for each $Q \in \boldsymbol{\varphi}(\xi, k)$, $\boldsymbol{\varphi}$ seals, over $\boldsymbol{\pi}$, the set

$$\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) = \{\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi}) : \boldsymbol{r} \text{ directly forces } \mathbf{c} \notin [Q]\},$$

in the sense of Definition 10.2 — formally $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q,\boldsymbol{\pi})} \boldsymbol{\varphi}$. $\square$

Lemma 16.3. *Assume that $\boldsymbol{\pi}, \boldsymbol{\varphi}, \boldsymbol{\sigma}$ are multiforcings, $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$, and $\mathbf{c}$ is a $\boldsymbol{\pi}$-real prename. Then:*

(i) *if $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\varphi}$ and $Q \in \boldsymbol{\varphi}(\xi, k)$ then the set $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$ and pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi})$;*
(ii) *if $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\varphi} \sqsubset \boldsymbol{\sigma}$ then $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\sigma}$;*
(iii) *if $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha < \lambda}$ is a $\sqsubset$-increasing sequence in $\mathbf{MF}$, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$, and $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\pi}_\mu$, then $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\varphi} = \bigcup^{\mathsf{cw}}_{\mu \le \alpha < \lambda} \boldsymbol{\pi}_\alpha$.*

Proof. (i) Apply Lemma 10.3(i).

(ii) Let $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$ and $S \in \boldsymbol{\sigma}(\xi, k)$. Then, as $\boldsymbol{\varphi} \sqsubset \boldsymbol{\sigma}$, there is a finite set $\{Q_1, \dots, Q_m\} \subseteq \boldsymbol{\varphi}(\xi, k)$ such that $S \subseteq Q_1 \cup \dots \cup Q_m$. We have $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q_i,\boldsymbol{\pi})} \boldsymbol{\varphi}$ for all i since $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\varphi}$, therefore $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q_i,\boldsymbol{\pi})} \boldsymbol{\sigma}$, $\forall i$, by Lemma 10.3(iv). Note that $\bigcap_i \boldsymbol{D}(\mathbf{c}, Q_i, \boldsymbol{\pi}) \subseteq \boldsymbol{D}(\mathbf{c}, S, \boldsymbol{\pi})$ since $S \subseteq \bigcup_i Q_i$. We conclude that $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},S,\boldsymbol{\pi})} \boldsymbol{\sigma}$ by Lemma 10.3(ii), (iii). Therefore $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\sigma}$. $\square$

17. Generic refinement avoids non-principal names

The following theorem says that generic refinements as in Section 11 avoid nonprincipal names. It resembles Theorem 12.1 to some extent, yet the latter is not directly applicable here as both the multitree Q and the set $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ depend on $\boldsymbol{\varphi}$, and hence the sets $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ do not necessarily belong to $\mathfrak{M}^+$. However the proof will be based on rather similar arguments.

Theorem 17.1. *In the assumptions of Theorem 11.3, if $\langle \eta, K \rangle \in |\boldsymbol{\pi}| \subseteq \mathfrak{M}$ and $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-real prename non-principal over $\boldsymbol{\pi}$ at η, K then $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\eta K} \boldsymbol{\varphi}$.*

Proof. Assume that $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\Phi]$ is obtained from an $\mathfrak{M}^+$-generic sequence Φ of multisystems in $\mathbf{MS}(\boldsymbol{\pi})$, as in Definition 11.1. We stick to the notation of 11.1.

Let $Q \in \boldsymbol{\varphi}(\eta, K)$; we have to prove that $\boldsymbol{\varphi}$ seals the set $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ over $\boldsymbol{\pi}$. By construction $Q = \boldsymbol{Q}^{\Phi}_{\eta K, \widetilde{m}}(s_0) \subseteq \boldsymbol{Q}^{\Phi}_{\eta K, \widetilde{m}}$ for some $\widetilde{m} < \omega$; it can be assumed that $Q = \boldsymbol{Q}^{\Phi}_{\eta K, \widetilde{m}}$. Following the proof of Theorem 12.1, we suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, define $X = |\boldsymbol{u}|$, $Y = |\boldsymbol{\pi}| \smallsetminus X$, and assume that $T^{\boldsymbol{u}}_{\xi k} = \boldsymbol{Q}^{\Phi}_{\xi k, m_{\xi k}}$, where $m_{\xi k} < \omega$, for each $\langle \xi, k \rangle \in X$.

Consider the set Δ of all multisystems $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$ such that there is a number $H > 0$ and a multitree $\boldsymbol{q} \in \mathbf{MS}(\boldsymbol{\pi})$ satisfying conditions

(1) $|\boldsymbol{q}| \cap X = \varnothing$ and $\boldsymbol{q} \leqslant \boldsymbol{p}$;
(2) if $\langle \xi, k\rangle \in X$ then $\langle \xi, k, m_{\xi k}\rangle \in |\boldsymbol{\varphi}|$;
(3) if $\langle \xi, k, m\rangle \in |\boldsymbol{\varphi}|$ then $h^{\boldsymbol{\varphi}}(\xi, k, m) = H$;

(but not (4) though) as in the proof of Theorem 12.1, along with two more requirements

(5) $\langle \eta, K, \widetilde{m}\rangle \in |\boldsymbol{\varphi}|$ — hence still $h^{\boldsymbol{\varphi}}(\eta, K, \widetilde{m}) = H$ by (3);
(6) if $s \in 2^H$ and $\tau \in (2^H)^X$ then $\mathbf{s}(\boldsymbol{\varphi}, \tau) \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\boldsymbol{\varphi}}_{\eta K,\widetilde{m}}(s)]$.

Lemma 17.2. Δ *is dense in* $\mathbf{MS}(\boldsymbol{\pi})$.

Proof. Suppose that $\boldsymbol{\psi} \in \mathbf{MS}(\boldsymbol{\pi})$; we can assume that $\boldsymbol{\psi}$ already satisfies

(a) if $\langle \xi, k\rangle \in X$ then $\langle \xi, k, m_{\xi k}\rangle \in |\boldsymbol{\psi}|$;
(b) there is a number $g < \omega$ such that $h^{\boldsymbol{\psi}}(\xi, k, m) = g$ for all $\langle \xi, k, m\rangle \in |\boldsymbol{\psi}|$;

as in Lemma 12.2, and in addition $\langle \eta, K, \widetilde{m}\rangle \in |\boldsymbol{\psi}|$.

Let $H = g + 1$. Define a multisystem $\boldsymbol{\chi} \in \mathbf{MS}(\boldsymbol{\pi})$ so that $|\boldsymbol{\chi}| = |\boldsymbol{\psi}|$, and $h^{\boldsymbol{\chi}}(\xi, k, m) = H$, $T^{\boldsymbol{\chi}}_{\xi k,m}(s^\frown i) = T^{\boldsymbol{\psi}}_{\xi k,m}(s)(\to i)$ for all $\langle \xi, k, m\rangle \in |\boldsymbol{\psi}|$ and $s^\frown i \in 2^H$; then $\boldsymbol{\psi} \preccurlyeq \boldsymbol{\chi}$. We claim that there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying (1), and a multisystem $\boldsymbol{\varphi} \in \mathbf{MS}(\boldsymbol{\pi})$ satisfying (6) and such that still $|\boldsymbol{\varphi}| = |\boldsymbol{\psi}|$ and $h^{\boldsymbol{\varphi}}(\xi, k, m) = H$ for all $\langle \xi, k, m\rangle \in |\boldsymbol{\psi}|$, and in addition

(c) if $\langle \xi, k\rangle \in X$ and $s \in 2^H$ then $T^{\boldsymbol{\varphi}}_{\xi k,m_{\xi k}}(s) \subseteq T^{\boldsymbol{\chi}}_{\xi k,m_{\xi k}}(s)$, and we also have $T^{\boldsymbol{\varphi}}_{\eta K,\widetilde{m}}(s) \subseteq T^{\boldsymbol{\chi}}_{\eta K,\widetilde{m}}(s)$;
(d) $T^{\boldsymbol{\varphi}}_{\xi k,m}(s) = T^{\boldsymbol{\chi}}_{\xi k,m}(s)$ for all applicable ξ, k, m, s not covered by (c).

To achieve (6) in one step for one particular $\tau \in (2^H)^X$, consider the multitree $\boldsymbol{r} = \mathbf{s}(\boldsymbol{\chi}, \tau) \cup \boldsymbol{p}$. By Lemma 14.1 and the density assumption of the theorem, there is a multitree $\boldsymbol{r}' \in \mathbf{MT}(\boldsymbol{\varphi})$, $\boldsymbol{r}' \leqslant \boldsymbol{r}$, which directly forces $\mathbf{c} \notin [T^{\boldsymbol{r}'}_{\eta K}]$, and there are multitrees $U_s \in \mathbf{MT}(\boldsymbol{\varphi})$, $s \in 2^H$, such that $U_s \subseteq T^{\boldsymbol{\chi}}_{\eta K,\widetilde{m}}(s)$ and $\boldsymbol{r}'$ directly forces $\mathbf{c} \notin [U_s]$, $\forall s$. Let $\boldsymbol{\chi}'$ be obtained from $\boldsymbol{\chi}$ by the following reassignment.

(I) We set $T^{\boldsymbol{\chi}'}_{\xi k,m_{\xi k}}(\tau(\xi, k)) = T^{\boldsymbol{r}'}_{\xi k}$ for all $\langle \xi, k\rangle \in X$.
(II) If $s \in 2^H$, and either $\langle \eta, K\rangle \notin X$, or $\widetilde{m} \neq m_{\eta K}$, or $s \neq \tau(\eta, K)$ then we set $T^{\boldsymbol{\chi}'}_{\eta K,\widetilde{m}}(s) = U_s$. (Note that if $\langle \eta, K\rangle \in X$ and $\widetilde{m} = m_{\eta K}$ then the tree $T^{\boldsymbol{\chi}'}_{\eta K,\widetilde{m}}(\tau(\eta, K)) = T^{\boldsymbol{r}'}_{\eta K}$ is already defined by (I).)

Let $\boldsymbol{p}' = \boldsymbol{r}' \restriction Y$, so that $\boldsymbol{r}' = \mathbf{s}(\boldsymbol{\chi}', \tau) \cup \boldsymbol{p}'$. By construction the tree $\boldsymbol{p}'$ satisfies (6), for the system τ chosen, in the case $\langle \eta, K\rangle \in X$, $\widetilde{m} = m_{\eta K}$, $s = \tau(\eta, K)$ by (I) and in all other cases by (II).

Now consider another $\tau' \in (2^H)^X$ and the multitree $\boldsymbol{r}' = \mathbf{s}(\boldsymbol{\chi}', \tau') \cup \boldsymbol{p}'$. There is a multitree $\boldsymbol{r}'' \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{r}'' \leqslant \boldsymbol{r}'$, which directly forces $\mathbf{c} \notin [T^{\boldsymbol{r}'}_{\eta K}]$ and $\mathbf{c} \notin [U'_s]$ for each $s \in 2^H$, where $U'_s \in \mathbf{MT}(\boldsymbol{\varphi})$ and $U'_s \subseteq T^{\boldsymbol{\chi}'}_{\eta K,\widetilde{m}}(s)$. Let $\boldsymbol{\chi}''$ be obtained from $\boldsymbol{\chi}'$ by the same reassignment (for τ' instead of τ).

And so on. The final multisystem and multitree of this construction will be $\boldsymbol{\varphi}$ and $\boldsymbol{q}$ satisfying (1), (2), (3), (5), (6). □ (Lemma)

Come back to the theorem. Note that $\Delta \in \mathfrak{M}^+$, similarly to the proof of Theorem 12.1. Therefore, by the lemma, there is an index j such that the system $\boldsymbol{\varphi}(j)$ belongs to Δ. Let this be witnessed by a number $H > 0$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, such that conditions (1), (2), (3), (5), (6) are satisfied for $\boldsymbol{\varphi} = \boldsymbol{\varphi}(j)$.

It remains to prove that $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. Let V consist of all multitrees $\boldsymbol{v} = \mathbf{s}(\boldsymbol{\varphi}(j), \tau)$, where $\tau \in (2^H)^X$; $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v} \in V} [\boldsymbol{v}]$ by construction.

Further, if $s \in 2^H$ and $\boldsymbol{v} \in V$ then $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\boldsymbol{\varphi}(j)}_{\eta K,\widetilde{m}}(s)]$ by (6), that is, directly forces $\mathbf{c} \notin [\boldsymbol{T}^{\Phi}_{\eta K,\widetilde{m}}(s)]$ in the notation of Definition 11.1. Therefore $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [\boldsymbol{Q}^{\Phi}_{\eta K,\widetilde{m}}(s)]$ since $\boldsymbol{Q}^{\Phi}_{\eta K,\widetilde{m}}(s) \subseteq \boldsymbol{T}^{\Phi}_{\eta K,\widetilde{m}}(s)$ by Lemma 11.3(b). However $Q = \boldsymbol{Q}^{\Phi}_{\eta K,\widetilde{m}} = \bigcup_{s \in 2^H} \boldsymbol{Q}^{\Phi}_{\eta K,\widetilde{m}}(s)$ by Lemma 11.3(a). It follows that $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [Q]$, that is, $\boldsymbol{v} \in \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})^{|\boldsymbol{u}|}_{\boldsymbol{q}}$.

We conclude that V is a (finite) subset of $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. And this accomplishes the proof of $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. $\square$

18. Consequences for reals in generic extensions

We first prove a result saying that all reals in $\mathbf{MT}(\boldsymbol{\pi})$-generic extensions are adequately represented by real names. Then Theorem 18.2 will show effects of the property of being a non-principal name.

Proposition 18.1. *Suppose that $\boldsymbol{\pi}$ is a regular multiforcing, $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is generic over the ground set universe $\mathbf{V}$, and $x \in \mathbf{V}[G] \cap \omega^\omega$. Then*

(i) *there is a $\boldsymbol{\pi}$-real name $\mathbf{c} \in \mathbf{V}$ such that $x = \mathbf{c}[G]$;*
(ii) *if $\mathbf{MT}(\boldsymbol{\pi})$ is a CCC forcing in $\mathbf{V}$ then there is a small $\boldsymbol{\pi}$-real name $\mathbf{d} \in \mathbf{V}$ with $x = \mathbf{d}[G]$.*

Proof. (i) is an instance of a general forcing theorem (see Remark 13.5 on the effect of regularity). To prove (ii), pick a real name $\mathbf{c}$ by (i), extend each set $K^{\mathbf{c}}_n = \bigcup_i K^{\mathbf{c}}_{ni}$ to an open dense set O_n by closing strongwards, choose maximal antichains $A_n \subseteq O_n$ in those sets — which have to be countable by CCC, and then let $A_{ni} = A_n \cap K^{\mathbf{c}}_{ni}$ and $\mathbf{d} = \{\langle \boldsymbol{p}, n, i\rangle : \boldsymbol{p} \in A_{ni}\}$. $\square$

Theorem 18.2. *Let $\boldsymbol{\pi}$ be a regular multiforcing. Then*

(i) *if a set $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is generic over the ground set universe $\mathbf{V}$, $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, and $x \in \mathbf{V}[G] \cap \omega^\omega$, then $x \neq x_{\xi k}[G]$ if and only if there is a $\boldsymbol{\pi}$-real name $\mathbf{c}$, non-principal over $\boldsymbol{\pi}$ at ξ, k and such that $x = \mathbf{c}[G]$.*
(ii) *if $\mathbf{c}$ is a $\boldsymbol{\pi}$-real prename, $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, φ is a multiforcing, $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \varphi$, and a set $G \subseteq \mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \varphi)$ is generic over $\mathbf{V}$ then $\mathbf{c}[G] \notin \bigcup_{Q \in \varphi(\xi,k)} [Q]$.*

Proof. (i) Suppose that $x \neq x_{\xi k}[G]$. By a known forcing theorem, there is a $\boldsymbol{\pi}$-real name $\mathbf{c}$ such that $x = \mathbf{c}[G]$ and $\mathbf{MT}(\boldsymbol{\pi})$ forces that $\mathbf{c} \neq x_{\xi k}[\underline{G}]$. It remains to show that $\mathbf{c}$ is a non-principal name over $\boldsymbol{\pi}$ at ξ, k. We have to prove that the set

$$\boldsymbol{D}^{\boldsymbol{\pi}}_{\xi k}(\mathbf{c}) = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T^{\boldsymbol{p}}_{\xi k}]\}$$

is open dense in $\mathbf{MT}(\boldsymbol{\pi})$. The openness is clear, let us prove the density. Consider an arbitrary $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$. Then $\boldsymbol{q}$ $\mathbf{MT}(\boldsymbol{\pi})$-forces $\mathbf{c} \neq x_{\xi k}[\underline{G}]$ by the choice of $\mathbf{c}$, hence we can assume that, for some n, it is $\mathbf{MT}(\boldsymbol{\pi})$-forced by $\boldsymbol{q}$ that $\mathbf{c}(n) \neq x_{\xi k}[\underline{G}](n)$. Then by Lemma 14.1(i) there is a multitree $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{p} \leqslant \boldsymbol{q}$, and a string $s \in \omega^{n+1}$, such that $\boldsymbol{p}$ $\mathbf{MT}(\boldsymbol{\pi})$-forces $s \subseteq \mathbf{c}$. Now it suffices to show that $s \notin T^{\boldsymbol{p}}_{\xi k}$. Suppose otherwise: $s \in T^{\boldsymbol{p}}_{\xi k}$. Then the tree $T = T^{\boldsymbol{p}}_{\xi k} \restriction_s$ still belongs to $\mathbf{MT}(\boldsymbol{\pi})$. Therefore the multitree $\boldsymbol{r}$ defined by $T^{\boldsymbol{r}}_{\xi k} = T$ and $T^{\boldsymbol{r}}_{\xi' k'} = T^{\boldsymbol{p}}_{\xi' k'}$ for each pair $\langle \xi', k'\rangle \neq \langle \xi, k\rangle$, belongs to $\mathbf{MT}(\boldsymbol{\pi})$ and satisfies $\boldsymbol{r} \leqslant \boldsymbol{p} \leqslant \boldsymbol{q}$. However $\boldsymbol{r}$ directly forces both $\mathbf{c}(n)$ and $x_{\xi k}[\underline{G}](n)$ to be equal to one and the same value $\ell = s(n)$, which contradicts to the choice of n.

To prove the converse let $\mathbf{c} \in \mathbf{V}$ be a real name non-principal over $\boldsymbol{\pi}$ at ξ, k, and $x = \mathbf{c}[G]$. Assume to the contrary that $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ and $x = x_{\xi k}[G]$. There is a multitree $\boldsymbol{q} \in G$ which $\mathbf{MT}(\boldsymbol{\pi})$-forces $\mathbf{c} = x_{\xi k}[\underline{G}]$. As $\mathbf{c}$ is non-principal, there is a stronger multitree $\boldsymbol{p} \in G \cap \boldsymbol{D}^{\boldsymbol{\pi}}_{\xi k}(\mathbf{c})$, $\boldsymbol{p} \leqslant \boldsymbol{q}$. Thus $\boldsymbol{p}$ directly forces $\mathbf{c} \notin [T^{\boldsymbol{p}}_{\xi k}]$, and hence $\mathbf{MT}(\boldsymbol{\pi})$-forces the same statement. Yet $\boldsymbol{p}$ $\mathbf{MT}(\boldsymbol{\pi})$-forces $\dot{\boldsymbol{x}}_{\xi k} \in [T^{\boldsymbol{p}}_{\xi k}]$, of course, and this is a contradiction.

(ii) Suppose towards the contrary that $Q \in \text{♀}(\xi, k)$ and $\mathbf{c}[G] \in [Q]$. By definition, ♀ seals, over $\boldsymbol{\pi}$, the set

$$\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) = \{\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi}) : \boldsymbol{r} \text{ directly forces } \mathbf{c} \notin [Q]\}\,.$$

Therefore in particular $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \text{♀})$ by Lemma 10.3. We conclude that $G \cap \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) \neq \varnothing$. In other words, there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ which directly forces $\mathbf{c} \notin [Q]$. It easily follows that $\mathbf{c}[G] \notin [Q]$, which is a contradiction. □

19. Combining refinement types

Here we summarize the properties of generic refinements considered above. The next definition combines the refinement types $\sqsubset_D$, $\sqsubset_{\boldsymbol{D}}$, $\sqsubset^{\mathbf{c}}_{\xi k}$.

Definition 19.1. Suppose that $\boldsymbol{\pi} \sqsubset \text{♀}$ are multiforcings and $\mathfrak{M} \in \mathrm{HC}$ is any set. Let $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \text{♀}$ mean that the four following requirements hold:

(1) if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$, $D \in \mathfrak{M}$, $D \subseteq \boldsymbol{\pi}(\xi, k)$, D is pre-dense in $\boldsymbol{\pi}(\xi, k)$, then $\boldsymbol{\pi}(\xi, k) \sqsubset_D \text{♀}(\xi, k)$;
(2) if $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$, then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \text{♀}$;
(3) if $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-real prename then $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \text{♀}$;
(4) if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ and $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-real prename, *non-principal over* $\boldsymbol{\pi}$ *at* ξ, k, then $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \text{♀}$. □

Corollary 19.2 *(of Lemmas 9.4, 10.3, 15.3, 16.3). Let $\boldsymbol{\pi}, \text{♀}, \boldsymbol{\sigma}$ be multiforcings and $\mathfrak{M}$ be a countable set. Then:*

(i) *if $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \text{♀}$ and $\mathfrak{M}' \subseteq \mathfrak{M}$ then $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}'} \boldsymbol{\sigma}$;*
(ii) *if $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \text{♀} \sqsubset \boldsymbol{\sigma}$ then $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \boldsymbol{\sigma}$;*
(iii) *if $\langle \boldsymbol{\pi}_\alpha\rangle_{\alpha<\lambda}$ is a $\sqsubset$-increasing sequence in $\mathbf{MF}$, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$, and $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \boldsymbol{\pi}_\mu$, then $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \text{♀} = \bigcup^{\mathsf{cw}}_{\mu\le\alpha<\lambda} \boldsymbol{\pi}_\alpha$.* □

Corollary 19.3. *If $\boldsymbol{\pi}$ is a small multiforcing, $\mathfrak{M} \in \mathrm{HC}$, and ♀ is an $\mathfrak{M}$-generic refinement of $\boldsymbol{\pi}$* (exists by Proposition 11.2!)*, then $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \text{♀}$.*

Proof. We have $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \text{♀}$ by a combination of 11.3(ii), 12.1, 15.2, and 17.1. □

IV. The forcing notion

In this chapter we define the forcing notion to prove the main theorem. It will have the form $\mathbf{MT}(\sqcap\!\!\sqcap)$, for a certain multiforcing $\sqcap\!\!\sqcap$ with $|\sqcap\!\!\sqcap| = \omega_1 \times \omega$. The multiforcing $\sqcap\!\!\sqcap$ will be equal to the componentwise union of terms of a certain *increasing sequence* $\vec{\sqcap\!\!\sqcap}$ of small multiforcings. And quite a complicated construction of this sequence in $\mathbf{L}$, the constructible universe, will employ some ideas related to diamond-style constructions, as well as to some sort of *definable genericity*.

20. Increasing sequences of small multiforcings

Recall that $\mathbf{MF}$ is the set of all multiforcings (Section 7). Let $\mathbf{sMF} \subseteq \mathbf{MF}$ be the set of all *small special* multiforcings; s accounts for both *small* and *special*. Thus a multiforcing $\boldsymbol{\pi} \in \mathbf{MF}$ belongs to $\mathbf{sMF}$ if $|\boldsymbol{\pi}|$ is (at most) countable and if $\langle \xi, k\rangle \in |\boldsymbol{\pi}|$ then $\boldsymbol{\pi}(\xi, k)$ is a small special (Definition 5.4) forcing in $\mathbf{PTF}$.

Definition 20.1. Let $\overrightarrow{\mathbf{sMF}}$, resp., $\overrightarrow{\mathbf{sMF}}_{\omega_1}$ be the set of all $\sqsubset$-increasing sequences $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa}$ of multiforcings $\boldsymbol{\pi}_\alpha \in \mathbf{sMF}$, of length $\kappa = \mathtt{dom}(\vec{\boldsymbol{\pi}}) < \omega_1$, resp., $\kappa = \omega_1$, which are *domain-continuous*, in the sense that if $\lambda < \kappa$ is a limit ordinal then $|\boldsymbol{\pi}_\lambda| = \bigcup_{\alpha<\lambda} |\boldsymbol{\pi}_\alpha|$. Sequences in $\overrightarrow{\mathbf{sMF}} \cup \overrightarrow{\mathbf{sMF}}_{\omega_1}$ are called *multisequences*. We order $\overrightarrow{\mathbf{sMF}} \cup \overrightarrow{\mathbf{sMF}}_{\omega_1}$ by the usual relations $\subseteq$ and $\subset$ of extension of sequences.

- Thus $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varphi}}$ iff $\kappa = \mathtt{dom}(\vec{\boldsymbol{\pi}}) < \lambda = \mathtt{dom}(\vec{\boldsymbol{\varphi}})$ and $\boldsymbol{\pi}_\alpha = \boldsymbol{\varphi}_\alpha$ for all $\alpha < \kappa$.
- In this case, if $\mathfrak{M}$ is any set, and $\boldsymbol{\varphi}_\kappa$ (the first term of $\vec{\boldsymbol{\varphi}}$ absent in $\vec{\boldsymbol{\pi}}$) satisfies $\boldsymbol{\pi} \sqsubset_{\mathfrak{M}} \boldsymbol{\varphi}_\kappa$, where $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$, then we write $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$.

If $\vec{\boldsymbol{\pi}}$ is a multisequence in $\overrightarrow{\mathbf{sMF}} \cup \overrightarrow{\mathbf{sMF}}_{\omega_1}$ then let $\mathbf{MT}(\vec{\boldsymbol{\pi}}) = \mathbf{MT}(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathsf{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$ (componentwise union), and $\kappa = \mathtt{dom}\, \vec{\boldsymbol{\pi}}$. Accordingly a $\vec{\boldsymbol{\pi}}$*-real (pre)name* will mean a $\boldsymbol{\pi}$-real (pre)name. □

Corollary 20.2. *Suppose that $\kappa < \lambda < \omega_1$, $\mathfrak{M}$ is a countable set, and $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa}$ is a multisequence in $\overrightarrow{\mathbf{sMF}}$. Then:*

(i) *the componentwise union $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathsf{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$ is a regular multiforcing;*
(ii) *there is a multisequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{sMF}}$ satisfying $\mathtt{dom}(\vec{\boldsymbol{\varphi}}) = \lambda$ and $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$;*
(iii) *if moreover $\langle s_\alpha \rangle_{\alpha<\lambda}$ is an $\subset$-increasing sequence of countable sets $s_\alpha \subseteq \omega_1 \times \omega$, $s_\alpha = |\boldsymbol{\pi}_\alpha|$ for all $\alpha < \kappa$, and $s_\gamma = \bigcup_{\alpha<\gamma} s_\alpha$ for all limit $\gamma < \lambda$, then there is a multisequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{sMF}}$ satisfying $\mathtt{dom}(\vec{\boldsymbol{\varphi}}) = \lambda$, $|\boldsymbol{\varphi}_\alpha| = s_\alpha$ for all $\alpha < \lambda$, and $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$;*
(iv) *if $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\rho}}, \vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{sMF}}$ and $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\rho}} \subseteq \vec{\boldsymbol{\varphi}}$ then $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$;*
(v) *if $\vec{\boldsymbol{\varphi}} = \langle \boldsymbol{\varphi}_\alpha \rangle_{\alpha<\lambda} \in \overrightarrow{\mathbf{sMF}}$, $\kappa < \lambda$, and $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$ then $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha \sqsubset_{\mathfrak{M}} \boldsymbol{\varphi}_\beta$ whenever $\kappa \le \beta < \lambda$, and also $\boldsymbol{\pi} \sqsubset_{\mathfrak{M}} \boldsymbol{\varphi}' = \bigcup^{\mathsf{cw}}_{\kappa\le\beta<\lambda} \boldsymbol{\varphi}_\beta$, therefore*

  (a) *$\mathbf{MT}(\boldsymbol{\varphi}')$ is open dense in $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$,*
  (b) *if $\langle \xi, k \rangle \in |\boldsymbol{\pi}|$, $D \in \mathfrak{M}$, $D \subseteq \boldsymbol{\pi}(\xi, k)$, D is pre-dense in $\boldsymbol{\pi}(\xi, k)$, then D remains pre-dense in $\boldsymbol{\pi}(\xi, k) \cup \boldsymbol{\varphi}(\xi, k)$,*
  (c) *if $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\vec{\boldsymbol{\pi}})$, $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}})$, then $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varphi}') = \mathbf{MT}(\vec{\boldsymbol{\varphi}})$.*

Proof. (i) Make use of Lemma 9.2(iv).

(ii) We define terms $\boldsymbol{\varphi}_\alpha$ of the multisequence $\boldsymbol{\varphi}$ required by induction.

Naturally put $\boldsymbol{\varphi}_\alpha = \boldsymbol{\pi}_\alpha$ for each $\alpha < \kappa$.

Now suppose that $\kappa \le \gamma < \lambda$, multiforcings $\boldsymbol{\varphi}_\alpha$, $\alpha < \gamma$, have been defined, and $\vec{\boldsymbol{\rho}} = \langle \boldsymbol{\varphi}_\alpha \rangle_{\alpha<\gamma}$ is a multisequence in $\overrightarrow{\mathbf{sMF}}$. To define $\boldsymbol{\varphi}_\gamma$, consider the componentwise union $\boldsymbol{\rho} = \bigcup^{\mathsf{cw}} \vec{\boldsymbol{\rho}} = \bigcup^{\mathsf{cw}}_{\alpha<\gamma} \boldsymbol{\varphi}_\alpha$. (Note that $\boldsymbol{\rho}$ is not equal to $\boldsymbol{\varphi}_\delta$ in case $\gamma = \delta + 1$ is a successor ordinal.) We can assume that $\mathfrak{M}$ contains $\vec{\boldsymbol{\rho}}$ and satisfies $\gamma \subseteq \mathfrak{M}$ (otherwise take a suitably bigger set). By Proposition 11.2, there is an $\mathfrak{M}$-generic refinement $\boldsymbol{\varphi}$ of $\boldsymbol{\rho}$. By Theorem 11.3, $\boldsymbol{\varphi}$ is small special multiforcing, $\boldsymbol{\rho} \sqsubset \boldsymbol{\varphi}$, and $\boldsymbol{\rho}_\alpha \sqsubset \boldsymbol{\varphi}$ for all $\alpha < \gamma$. In addition $\boldsymbol{\rho} \sqsubset_{\mathfrak{M}} \boldsymbol{\varphi}$ by Corollary 19.3. We let $\boldsymbol{\rho}_\gamma = \boldsymbol{\varphi}$. The extended multisequence $\vec{\boldsymbol{\rho}}_+ = \langle \boldsymbol{\rho}_\alpha \rangle_{\alpha<\gamma+1}$ belongs to $\overrightarrow{\mathbf{sMF}}$ and satisfies $\vec{\boldsymbol{\rho}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\rho}}_+$. It remains to define $\boldsymbol{\varphi}_\gamma := \boldsymbol{\rho}_\gamma = \boldsymbol{\varphi}$.

(iii) The proof is similar, with the extra care of $|\boldsymbol{\varphi}_\alpha| = s_\alpha$.

(iv) The relation $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$ involves only the first term of $\vec{\boldsymbol{\varphi}}$ absent in $\vec{\boldsymbol{\pi}}$.

To prove the main claim of (v) make use of Corollary 19.2.

To prove (v)(a) apply Corollary 10.1.

(v)(b) As $\boldsymbol{\pi} \sqsubset_{\mathfrak{M}} \boldsymbol{\varphi}'$ and $D \in \mathfrak{M}$, we have $\boldsymbol{\pi}(\xi, k) \sqsubset_D \boldsymbol{\varphi}(\xi, k)$. Therefore D is pre-dense in $\boldsymbol{\varphi}(\xi, k)$ by Lemma 9.4(ii).

(v)(c) Similarly $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}'$, $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$ by Lemma 10.3(i). □

Our plan regarding the forcing notion for Theorem 1.1 will be to define a certain multisequence $\vec{\Pi}$ in $\overrightarrow{\mathbf{sMF}}_{\omega_1}$ and the ensuing multiforcing $\Pi = \bigcup^{\mathsf{cw}} \vec{\Pi}$ with remarkable properties related to definability and its own genericity of some sort. But we need first to introduce an important notion involved in the construction.

21. Layer restrictions of multiforcings and deciding sets

The construction of the mentioned multiforcing Π will be maintained in such a way that different *layers* $\langle \Pi(k,\xi)\rangle_{\xi<\omega_1}$, $k<\omega$, appear rather independent of each other, albeit the principal inductive parameter will be ξ rather than k. To reflect this feature, we introduce here a suitable notation related to layer restrictions. If $m<\omega$ then, using a special "layer restriction" symbol $\upharpoonright\!\upharpoonright$ to provide a transparent distinction from the ordinary restriction $\upharpoonright$, we define sets of multitrees:

$$\begin{aligned}
\mathbf{MT}\upharpoonright\!\upharpoonright_{<m} &= \text{all multitrees } \boldsymbol{p} \in \mathbf{MT} \text{ such that } |\boldsymbol{p}| \subseteq \omega_1 \times m,\\
\mathbf{MT}\upharpoonright\!\upharpoonright_{\ge m} &= \text{all multitrees } \boldsymbol{p} \in \mathbf{MT} \text{ with } |\boldsymbol{p}| \subseteq \omega_1 \times (\omega \smallsetminus m),\\
\mathbf{MT}\upharpoonright\!\upharpoonright_{m} &= \text{all multitrees } \boldsymbol{p} \in \mathbf{MT} \text{ such that } |\boldsymbol{p}| \subseteq \omega_1 \times \{m\},
\end{aligned}$$

and, given a multiforcing $\boldsymbol{\pi}$, define $\mathbf{MT}(\boldsymbol{\pi})\upharpoonright\!\upharpoonright_{<m}$, $\mathbf{MT}(\boldsymbol{\pi})\upharpoonright\!\upharpoonright_{\ge m}$, $\mathbf{MT}(\boldsymbol{\pi})\upharpoonright\!\upharpoonright_{m}$ similarly. Accordingly if $\boldsymbol{p} \in \mathbf{MT}$ then define the *layer restriction* $\boldsymbol{p}\upharpoonright\!\upharpoonright_{<m} \in \mathbf{MT}\upharpoonright\!\upharpoonright_{<m}$ so that $|\boldsymbol{p}\upharpoonright\!\upharpoonright_{<m}| = \{\langle\xi,k\rangle \in |\boldsymbol{p}| : k<m\}$ and $\boldsymbol{p}\upharpoonright\!\upharpoonright_{<m}(\xi,k) = \boldsymbol{\pi}(\xi,k)$ whenever $\langle\xi,k\rangle \in |\boldsymbol{p}\upharpoonright\!\upharpoonright_{<m}|$. Define $\boldsymbol{p}\upharpoonright\!\upharpoonright_{\ge m} \in \mathbf{MT}\upharpoonright\!\upharpoonright_{\ge m}$, $\boldsymbol{p}\upharpoonright\!\upharpoonright_{m} \in \mathbf{MT}\upharpoonright\!\upharpoonright_{m}$ similarly.

The same definitions are maintained with multiforcings:

$$\begin{aligned}
\mathbf{sMF}\upharpoonright\!\upharpoonright_{<m} &= \text{all multiforcings } \boldsymbol{\pi} \in \mathbf{sMF} \text{ such that } |\boldsymbol{\pi}| \subseteq \omega_1 \times m,\\
\mathbf{sMF}\upharpoonright\!\upharpoonright_{\ge m} &= \text{all multiforcings } \boldsymbol{\pi} \in \mathbf{sMF} \text{ with } |\boldsymbol{\pi}| \subseteq \omega_1 \times (\omega \smallsetminus m),\\
\mathbf{sMF}\upharpoonright\!\upharpoonright_{m} &= \text{all multiforcings } \boldsymbol{\pi} \in \mathbf{sMF} \text{ such that } |\boldsymbol{\pi}| \subseteq \omega_1 \times \{m\},
\end{aligned}$$

and $\mathbf{MF}\upharpoonright\!\upharpoonright_{<m}$, $\mathbf{MF}\upharpoonright\!\upharpoonright_{\ge m}$, $\mathbf{MF}\upharpoonright\!\upharpoonright_{m}$ are defined similarly.

Accordingly if $\boldsymbol{\pi} \in \mathbf{MF}$ (in particular if $\boldsymbol{\pi} \in \mathbf{sMF}$) and $m<\omega$ then define the *layer restriction* $\boldsymbol{\pi}\upharpoonright\!\upharpoonright_{<m} \in \mathbf{MF}\upharpoonright\!\upharpoonright_{<m}$ (resp., $\in \mathbf{sMF}\upharpoonright\!\upharpoonright_{<m}$), so that $|\boldsymbol{\pi}\upharpoonright\!\upharpoonright_{<m}| = \{\langle\xi,k\rangle \in |\boldsymbol{\pi}| : k<m\}$ and $\boldsymbol{\pi}\upharpoonright\!\upharpoonright_{<m}(\xi,k) = \boldsymbol{\pi}(\xi,k)$ whenever $\langle\xi,k\rangle \in |\boldsymbol{\pi}\upharpoonright\!\upharpoonright_{<m}|$. Define $\boldsymbol{\pi}\upharpoonright\!\upharpoonright_{\ge m} \in \mathbf{MF}\upharpoonright\!\upharpoonright_{\ge m}$, $\boldsymbol{\pi}\upharpoonright\!\upharpoonright_{m} \in \mathbf{MF}\upharpoonright\!\upharpoonright_{m}$ similarly.

A similar notation applies to multisequences (Definition 20.1). If $m<\omega$ then we let $\overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{<m}$, $\overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{\ge m}$, $\overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{m}$ be the set of all multisequences in $\overrightarrow{\mathbf{sMF}}$ whose all terms belong to resp. $\mathbf{sMF}\upharpoonright\!\upharpoonright_{<m}$, $\mathbf{sMF}\upharpoonright\!\upharpoonright_{\ge m}$, $\mathbf{sMF}\upharpoonright\!\upharpoonright_{m}$. Define similarly $\overrightarrow{\mathbf{sMF}}_{\omega_1}\upharpoonright\!\upharpoonright_{<m}$, $\overrightarrow{\mathbf{sMF}}_{\omega_1}\upharpoonright\!\upharpoonright_{\ge m}$, $\overrightarrow{\mathbf{sMF}}_{\omega_1}\upharpoonright\!\upharpoonright_{m}$ (multisequences of length ω_1).

And further, if $\vec{\boldsymbol{\pi}} = \langle\boldsymbol{\pi}_\alpha\rangle_{\alpha<\kappa} \in \overrightarrow{\mathbf{sMF}}$ and $m<\omega$ then define $\vec{\boldsymbol{\pi}}\upharpoonright\!\upharpoonright_{<m} = \langle\boldsymbol{\pi}_\alpha\upharpoonright\!\upharpoonright_{<m}\rangle_{\alpha<\kappa} \in \overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{<m}$, and define $\vec{\boldsymbol{\pi}}\upharpoonright\!\upharpoonright_{\ge m} \in \overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{\ge m}$, $\vec{\boldsymbol{\pi}}\upharpoonright\!\upharpoonright_{m} \in \overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{m}$ similarly. The same for $\vec{\boldsymbol{\pi}} = \langle\boldsymbol{\pi}_\alpha\rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{sMF}}_{\omega_1}$

Definition 21.1. Assume that $m<\omega$. A multisequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$ *m-decides* a set W if either $\vec{\boldsymbol{\pi}}\upharpoonright\!\upharpoonright_{\ge m}$ belongs to W (*positive* decision) or there is no multisequence $\vec{\boldsymbol{\varphi}} \in W \cap \overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{\ge m}$ extending $\vec{\boldsymbol{\pi}}\upharpoonright\!\upharpoonright_{\ge m}$ (*negative* decision). In particular, $\vec{\boldsymbol{\pi}}$ *0-decides* W if either $\vec{\boldsymbol{\pi}} \in W$ or there is no $\vec{\boldsymbol{\varphi}} \in W \cap \overrightarrow{\mathbf{sMF}}$ extending $\vec{\boldsymbol{\pi}}$. □

Lemma 21.2. *If $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$, $\mathfrak{M}$ is countable, W is any set, and $m<\omega$, then there is a multisequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{sMF}}$ such that $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$ and $\vec{\boldsymbol{\varphi}}$ m-decides W.*

Proof. By Corollary 20.2, there is a multisequence $\vec{\boldsymbol{\rho}} \in \overrightarrow{\mathbf{sMF}}$ such that $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\rho}}$. Then either $\vec{\boldsymbol{\rho}}$ outright m-decides W negatively, or there is a sequence $\vec{\boldsymbol{\sigma}} \in W \cap \overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{\ge m}$ satisfying $\vec{\boldsymbol{\rho}}\upharpoonright\!\upharpoonright_{\ge m} \subseteq \vec{\boldsymbol{\sigma}}$.

On the other hand, using Corollary 20.2(iii), we get a multisequence $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{sMF}}\upharpoonright\!\upharpoonright_{<m}$ of the same length as $\vec{\boldsymbol{\sigma}}$, such that $\vec{\boldsymbol{\rho}}\upharpoonright\!\upharpoonright_{<m} \subseteq \vec{\boldsymbol{\tau}}$. Therefore there exists a multisequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{sMF}}$ of that same length, satisfying $\vec{\boldsymbol{\varphi}}\upharpoonright\!\upharpoonright_{\ge m} = \vec{\boldsymbol{\sigma}}$ and $\vec{\boldsymbol{\varphi}}\upharpoonright\!\upharpoonright_{<m} = \vec{\boldsymbol{\tau}}$ — then obviously $\vec{\boldsymbol{\rho}} \subseteq \vec{\boldsymbol{\varphi}}$ and by definition $\vec{\boldsymbol{\varphi}}$ decides W positively. Finally we have $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\rho}} \subseteq \vec{\boldsymbol{\varphi}}$, and hence $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$ by Corollary 20.2(iv). □

22. Auxiliary diamond sequences

Recall that HC is the set of all *hereditarily countable* sets (those with finite or countable transitive closures).

Definition 22.1. We use standard notation $\varSigma_n^{\rm HC}$, $\varPi_n^{\rm HC}$, $\varDelta_n^{\rm HC}$ (slanted $\varSigma, \varPi, \varDelta$) for classes of *lightface* definability in HC (no parameters allowed), and $\mathbf{\Sigma}_n^{\rm HC}$, $\mathbf{\Pi}_n^{\rm HC}$, $\mathbf{\Delta}_n^{\rm HC}$ (foldface upright $\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{\Delta}$) for classes of *boldface* definability in HC (parameters in HC allowed). □

The next theorem employs the technique of diamond sequences in $\mathbf{L}$.

Theorem 22.2 *(in $\mathbf{L}$). There exist $\varDelta_1^{\rm HC}$ sequences $\langle \vec{\boldsymbol\pi}[\mu] \rangle_{\mu<\omega_1}$, $\langle D[\mu] \rangle_{\mu<\omega_1}$, $\langle z[\mu] \rangle_{\mu<\omega_1}$, such that, for every μ, $D[\mu]$ and $z[\mu]$ are sets in* HC*, $\vec{\boldsymbol\pi}[\mu] \in \overrightarrow{\mathbf{sMF}}$, $\mathtt{dom}(\vec{\boldsymbol\pi}[\mu]) = \mu$, and in addition if $\vec{\sqcap} = \langle \sqcap_\nu \rangle_{\nu<\omega_1} \in \overrightarrow{\mathbf{sMF}}_{\omega_1}$, $z \in$ HC, and $D \subseteq \mathbf{MT}(\vec{\sqcap})$, then the set M of all ordinals $\mu < \omega_1$ such that*

(a) *$z[\mu] = z$;*
(b) *$\vec{\boldsymbol\pi}[\mu]$ is equal to the restricted sub-multisequence $\vec{\sqcap}\restriction\mu = \langle \sqcap_\nu \rangle_{\nu<\mu}$;*
(c) *$D[\mu] = D \cap \mathbf{MT}(\vec{\sqcap}\restriction\mu)$;*

is stationary in ω_1.

Proof. *Arguing in $\mathbf{L}$, the constructible universe*, we let $\leqslant_{\mathbf{L}}$ be the canonical wellordering of $\mathbf{L}$. It is known that $\leqslant_{\mathbf{L}}$ orders HC similarly to ω_1, and that $\leqslant_{\mathbf{L}}$ is $\varDelta_1^{\rm HC}$ and has the *goodness* property: the set of all $\leqslant_{\mathbf{L}}$-initial segments $I_x(\leqslant_{\mathbf{L}}) = \{y : y \leqslant_{\mathbf{L}} x\}$, where $x \in$ HC, is still $\varDelta_1^{\rm HC}$.

The diamond principle $\diamondsuit_{\omega_1}$ is true in $\mathbf{L}$ by [16, Theorem 13.21], hence there is a $\varDelta_1^{\rm HC}$ sequence of sets $S_\alpha \subseteq \alpha$, $\alpha < \omega_1$, such that

(A) if $X \subseteq$ HC then the set $\{\alpha < \omega_1 : S_\alpha = X \cap \alpha\}$ is stationary in ω_1.

The $\varDelta_1^{\rm HC}$-definability property is achieved by taking the $\leqslant_{\mathbf{L}}$-least possible S_α at each step α. We get the following two results (B), (C) as easy corollaries.

First, let $A_\mu = \{c_\alpha : \alpha \in S_\mu\}$, where c_α is the α-th element of HC in the sense of the ordering $\leqslant_{\mathbf{L}}$. Then $\langle A_\mu \rangle_{\mu<\omega_1}$ is still a $\varDelta_1^{\rm HC}$ sequence, and

(B) if $X_\alpha \in$ HC for all $\alpha < \omega_1$ then the set $\{\mu : A_\mu = \{X_\alpha : \alpha < \mu\}\}$ is stationary in ω_1.

Second, for any α, if $A_\alpha = \langle a_\gamma \rangle_{\gamma<\alpha}$, where each a_γ itself is equal to an ω-sequence $\langle a_\gamma^n \rangle_{n<\omega}$, then let $B_\alpha^n = \langle a_\gamma^n \rangle_{\gamma<\alpha}$ for all n. Otherwise let $B_\alpha^n = \varnothing$, $\forall n$. Then $\langle B_\alpha \rangle_{\alpha<\omega_1}^{n<\omega}$ is still a $\varDelta_1^{\rm HC}$ system of sets in HC, such that

(C) if $X_\alpha^n \in$ HC for all $\alpha < \omega_1$, $n < \omega$, then, for every $\mu < \omega_1$, the set $\{\mu : \forall n\ (B_\mu^n = \{X_\alpha^n : \alpha < \mu\}\}$ is stationary in ω_1.

Now things become somewhat more complex.

Let $\mu < \omega_1$. We define $z[\mu] = \bigcup B_\mu^0$. If $B_\mu^1 \in \overrightarrow{\mathbf{sMF}}$ and $\mathtt{dom}(B_\mu^1) = \mu$ then let $\vec{\boldsymbol\pi}[\mu] = B_\mu^1$; otherwise let $\vec{\boldsymbol\pi}[\mu]$ be equal to the $\leqslant_{\mathbf{L}}$-least multisequence in $\overrightarrow{\mathbf{sMF}}$ of length μ. (Those exist by Corollary 20.2(ii).) Finally we let $D[\mu] = \bigcup B_{\mu+1}^2$.

Let's show that the sequences of sets $\vec{\boldsymbol\pi}[\mu]$, $D[\mu]$, $z[\mu]$ prove the theorem. Suppose that $\vec{\sqcap} = \langle \sqcap_\nu \rangle_{\nu<\omega_1} \in \overrightarrow{\mathbf{sMF}}_{\omega_1}$, $z \in$ HC, and $D \subseteq \mathbf{MT}(\vec{\sqcap})$. Let $X_\alpha^0 = z$, $X_\alpha^1 = \langle \alpha, \sqcap_\alpha \rangle$, $X_\alpha^2 = D \cap \mathbf{MT}(\vec{\sqcap}\restriction\alpha)$ for all α. The set

$$M = \{\mu < \omega_1 : B^n_\mu = \{X^n_\alpha : \alpha < \mu\} \text{ for } n = 0, 1, 2\}$$

is stationary by (C). Assume that $\mu \in M$. Then $B^0_\mu = \{X^0_\alpha : \alpha < \mu\} = \{z\}$, therefore $z[\mu] = z$. Further $B^1_\mu = \{X^1_\alpha : \alpha < \mu\} = \{\langle\alpha, ⫪_\alpha\rangle : \alpha < \mu\} = \vec{⫪}\restriction\mu \in \overrightarrow{\mathbf{sMF}}$, therefore $\vec{\boldsymbol{\pi}}[\mu] = \vec{⫪}\restriction\mu$. Finally we have $D[\mu] = \bigcup B^2_{\mu+1} = \bigcup_{\alpha\le\mu} X^2_\alpha = D \cap \mathbf{MT}(\vec{⫪}\restriction\mu)$, as required. □

23. Key sequence theorem

Now we prove a theorem which introduces the key multisequence $\vec{⫪}$.

Theorem 23.1 *($\mathbf{V} = \mathbf{L}$). There exists a multisequence $\vec{⫪} = \langle ⫪_\alpha\rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{sMF}}_{\omega_1}$ satisfying the following requirements:*

- (i) *if $m < \omega$ then the multisequence $\vec{⫪}\upharpoonleft\!\!\upharpoonleft_m$ belongs to the class $\Delta^{\rm HC}_{m+2}$;*
- (ii) *if $m' < \omega$ and $W \subseteq \overrightarrow{\mathbf{sMF}}$ is a boldface $\mathbf{\Sigma}^{\rm HC}_{m'+1}$ set then there is an ordinal $\gamma < \omega_1$ such that the multisequence $\vec{⫪}\restriction\gamma$ m'-decides W;*
- (iii) *if a set $D \subseteq \mathbf{MT}(\vec{⫪})$ is dense in $\mathbf{MT}(\vec{⫪})$, then the set Z of all ordinals $\gamma < \omega_1$ such that $\vec{⫪}\restriction\gamma \subset_{\{D\cap\mathbf{MT}(\vec{⫪}\restriction\gamma)\}} \vec{⫪}$, is stationary in ω_1.*

Proof. If $m < \omega$ then let $\mathbf{un}_m(p, x)$ be a canonical universal Σ_{m+1} formula, so that the family of all boldface $\mathbf{\Sigma}^{\rm HC}_{m+1}$ sets $X \subseteq \mathrm{HC}$ (those definable in HC by Σ_{m+1} formulas with parameters in HC) is equal to the family of all sets of the form $\Upsilon_m(p) = \{x \in \mathrm{HC} : \mathrm{HC} \models \mathbf{un}_m(p, x)\}$, $p \in \mathrm{HC}$.

- (I) Fix $\Delta^{\rm HC}_1$ sequences $\langle\vec{\boldsymbol{\pi}}[\mu]\rangle_{\mu<\omega_1}$, $\langle D[\mu]\rangle_{\mu<\omega_1}$, and $\langle z[\mu]\rangle_{\mu<\omega_1}$ satisfying Theorem 22.2; the terms $D[\mu], z[\mu], \vec{\boldsymbol{\pi}}[\mu]$ of the sequences belong to HC, and in addition $\vec{\boldsymbol{\pi}}[\mu] \in \overrightarrow{\mathbf{sMF}}$, $\mathtt{dom}(\vec{\boldsymbol{\pi}}[\mu]) = \mu$.
- (II) Let $\mu < \omega_1$. If $z[\mu]$ is a pair of the form $z[\mu] = \langle m, p\rangle$ then let $m[\mu] = m$ and $p[\mu] = p$, otherwise let $m[\mu] = p[\mu] = 0$.
- (III) If $m < \omega$ then let, by Lemma 21.2, $\vec{\boldsymbol{\pi}}[\mu, m] \in \overrightarrow{\mathbf{sMF}}$ be the $\leqslant_{\mathbf{L}}$-least multisequence in $\overrightarrow{\mathbf{sMF}}$ which satisfies $\vec{\boldsymbol{\pi}}[\mu] \subset_{\{D[\mu]\}} \vec{\boldsymbol{\pi}}[\mu, m]$ and m-decides the set $\Upsilon_m(p[\mu])$. Let $[\mu, m]^+ = \mathtt{dom}(\vec{\boldsymbol{\pi}}[\mu, m])$; then $\mu < [\mu, m]^+ < \omega_1$.

Proposition 23.2 *(in $\mathbf{L}$). The sequences $\langle m[\mu]\rangle_{\mu<\omega_1}$ and $\langle p[\mu]\rangle_{\mu<\omega_1}$ belong to the lightface definability class $\Delta^{\rm HC}_1$. If $m < \omega$ then the sequences $\langle\vec{\boldsymbol{\pi}}[\mu, m]\rangle_{\mu<\omega_1}$ and $\langle[\mu, m]^+\rangle_{\mu<\omega_1}$ belong to the lightface class $\Delta^{\rm HC}_{m+2}$.*

Proof. Routine. Note that $\vec{\boldsymbol{\pi}}[\mu, m]$ and $[\mu, m]^+$ depend on m through the formulas $\mathbf{un}_m(\cdot, \cdot)$, whose complexity strictly increases with $m \to \infty$. □

Now define a multisequence $\vec{⫪} = \langle ⫪_\alpha\rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{sMF}}_{\omega_1}$ and a family of strictly increasing, continuous maps $\mu_m : \omega_1 \to \omega_1$, $m < \omega$, as follows:

1°. Let $\mu_m(0) = 0$ and $\mu_m(\lambda) = \mathtt{sup}_{\gamma<\lambda}\mu_m(\gamma)$ for all m and all limit $\lambda < \omega_1$.

2°. Suppose that $m < \omega$, $\gamma < \omega_1$, $\mu = \mu_m(\gamma)$, and the twofold-restricted sequence $(\vec{⫪}\restriction\mu)\upharpoonleft\!\!\upharpoonleft_m = (\vec{⫪}\upharpoonleft\!\!\upharpoonleft_m)\restriction\mu$ is already defined. If the following holds:

(∗) $m \ge m' = m[\mu]$ and $(\vec{⫪}\restriction\mu)\upharpoonleft\!\!\upharpoonleft_m$ coincides with $\vec{\boldsymbol{\pi}}[\mu]\upharpoonleft\!\!\upharpoonleft_m$,

then let $\mu_m(\gamma + 1) = [\mu, m']^+$ and $(\vec{⫪}\restriction[\mu, m']^+)\upharpoonleft\!\!\upharpoonleft_m = \vec{\boldsymbol{\pi}}[\mu, m']\upharpoonleft\!\!\upharpoonleft_m$.

3°. In the assumptions of 2°, if 2°(∗) fails, then let $\vec{\boldsymbol{\rho}}$ be the $\leqslant_{\mathbf{L}}$-least multisequence in $\overrightarrow{\mathbf{sMF}}$ with $(\vec{\Pi}\restriction\mu){\upharpoonleft\!\upharpoonright}_m \subset \vec{\boldsymbol{\rho}}$ (we refer to Corollary 20.2), and define $\mu_m(\gamma+1) = \mathtt{dom}(\vec{\boldsymbol{\rho}})$ and $(\vec{\Pi}\restriction\mu_m(\gamma+1)){\upharpoonleft\!\upharpoonright}_m = \vec{\boldsymbol{\rho}}{\upharpoonleft\!\upharpoonright}_m$.

To conclude, given $\gamma < \omega_1$ and m, if an ordinal $\mu = \mu_m(\gamma)$, and a multisequence $(\vec{\Pi}\restriction\mu){\upharpoonleft\!\upharpoonright}_m = (\vec{\Pi}{\upharpoonleft\!\upharpoonright}_m)\restriction\mu$ are defined, then items 2°, 3°define a bigger ordinal $\mu_m(\gamma+1) > \mu = \mu_m(\gamma)$ and a longer multisequence $(\vec{\Pi}\restriction\mu_m(\gamma+1)){\upharpoonleft\!\upharpoonright}_m$ satisfying $(\vec{\Pi}\restriction\mu){\upharpoonleft\!\upharpoonright}_m \subset (\vec{\Pi}\restriction\mu_m(\gamma+1)){\upharpoonleft\!\upharpoonright}_m$. Thus overall items 1°, 2°, 3°of the definition contain straightforward instructions as how to uniquely define the layers $\vec{\Pi}{\upharpoonleft\!\upharpoonright}_m$ and maps μ_m for different $m < \omega$, independently from each other.

From now on, fix a multisequence $\vec{\Pi} = \langle \Pi_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{sMF}}_{\omega_1}$ of multiforcings $\Pi_\alpha \in \mathbf{sMF}$ and increasing continuous maps $\mu_m : \omega_1 \to \omega_1$ defined by 1°, 2°, 3°. As the maps μ_m are continuous, the following holds:

Proposition 23.3 *(in* **L***).* $\mathbb{C} = \{\gamma < \omega_1 : \forall m\, (\gamma = \mu_m(\gamma))\}$ *is a club in* ω_1. □

To show that $\vec{\Pi}$ proves Theorem 23.1, we check items (i), (ii), (iii).

(i) Let $m < \omega$. Then the multisequence $\vec{\Pi}{\upharpoonleft\!\upharpoonright}_m$ and the map μ_m belong to the class $\Delta^{\mathrm{HC}}_{m+2}$ by Proposition 23.2; a routine proof is omitted.

(ii) Suppose that $m' < \omega$ and $W \subseteq \overrightarrow{\mathbf{sMF}}$ is a $\boldsymbol{\Sigma}^{\mathrm{HC}}_{m'+1}$ set. Pick $p \in \mathrm{HC}$ such that $W = \Upsilon_{m'}(p)$. Let $z = \langle m', p\rangle$. As $\mathbb{C}$ is a club, it follows from the choice of terms $\vec{\boldsymbol{\pi}}\lceil\mu\rceil$, $D\lceil\mu\rceil$, and $z\lceil\mu\rceil$, by (I) and Theorem 22.2, that there is an ordinal $\gamma \in \mathbb{C}$ such that $\vec{\boldsymbol{\pi}}\lceil\gamma\rceil = \vec{\Pi}\restriction\gamma$ and $z\lceil\gamma\rceil = z$ — hence, $m\lceil\gamma\rceil = m'$ and $p\lceil\gamma\rceil = p$.

Let $\mu = \gamma$; then also $\mu = \mu_m(\gamma)$, $\forall m$ — since $\gamma \in \mathbb{C}$, and $\vec{\Pi}\restriction\mu = \vec{\boldsymbol{\pi}}\lceil\mu\rceil$.

Then it follows from the choice of $\vec{\Pi}$ that item 2°of the construction applies for the ordinal γ chosen and all $m \geq m'$. It follows that the multisequence $\vec{\boldsymbol{\rho}} = \vec{\boldsymbol{\pi}}\lceil\mu, m'\rceil$ and the ordinal $\nu = \mu_m(\gamma+1) = \lceil\mu, m'\rceil^+$ satisfy $\nu = \mathtt{dom}(\vec{\boldsymbol{\rho}})$ and $(\vec{\Pi}\restriction\nu){\upharpoonleft\!\upharpoonright}_m = \vec{\boldsymbol{\rho}}{\upharpoonleft\!\upharpoonright}_m$ for all $m \geq m'$. In other words, $(\vec{\Pi}\restriction\nu){\upharpoonleft\!\upharpoonright}_{\geq m'} = \vec{\boldsymbol{\rho}}{\upharpoonleft\!\upharpoonright}_{\geq m'}$.

However by definition $\vec{\boldsymbol{\rho}}$ m'-decides the set $W = \Upsilon_{m'}(p)$, and the definition of this property depends only on $\vec{\boldsymbol{\rho}}{\upharpoonleft\!\upharpoonright}_{\geq m'}$.

(iii) Assume that a set $D \subseteq \mathbf{MT}(\vec{\Pi})$ is dense in $\mathbf{MT}(\vec{\Pi})$, and $C \subseteq \mathbb{C}$ is a club in ω_1. Following the proof of (ii), we find an ordinal $\gamma \in C$ such that $\vec{\boldsymbol{\pi}}\lceil\gamma\rceil = \vec{\Pi}\restriction\gamma$, $m\lceil\gamma\rceil = 0$, and $D\lceil\gamma\rceil = D \cap \mathbf{MT}(\vec{\Pi}\restriction\gamma)$. Note that $\gamma = \mu_m(\gamma)$, $\forall m$. We have $\vec{\boldsymbol{\pi}}\lceil\gamma\rceil \subset_{\{D\lceil\gamma\rceil\}} \vec{\boldsymbol{\pi}}\lceil\gamma, 0\rceil$ by (III) (with $\mu = \gamma$), that is,

$$\vec{\boldsymbol{\pi}}\lceil\gamma\rceil \subset_{\{D\cap\mathbf{MT}(\vec{\Pi}\restriction\gamma)\}} \vec{\boldsymbol{\pi}}\lceil\gamma, 0\rceil. \tag{†}$$

Yet it follows from the choice of γ that condition 2°(∗) holds (for $\mu = \gamma$) for all $m \geq 0$. Then, by definition 2°, the ordinal $\mu^+ = \lceil\gamma, m\rceil^+$ satisfies $\mu^+ = \mu_m(\gamma+1)$ and $(\vec{\Pi}\restriction\mu^+){\upharpoonleft\!\upharpoonright}_m = (\vec{\boldsymbol{\pi}}\lceil\gamma, 0\rceil){\upharpoonleft\!\upharpoonright}_m$ for all m, that is, just $\vec{\Pi}\restriction\mu^+ = \vec{\boldsymbol{\pi}}\lceil\gamma, 0\rceil$. We conclude that $\vec{\Pi}\restriction\gamma \subset_{\{D\cap\mathbf{MT}(\vec{\Pi}\restriction\gamma)\}} \vec{\Pi}\restriction\mu^+$ by (†), therefore we have $\vec{\Pi}\restriction\gamma \subset_{\{D\cap\mathbf{MT}(\vec{\Pi}\restriction\gamma)\}} \vec{\Pi}$, as required. □(Theorem 23.1)

Definition 23.4 *(in* **L***).* From now on we fix a multisequence $\vec{\Pi} = \langle \Pi_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{sMF}}_{\omega_1}$ satisfying requirements of Theorem 23.1, that is,

(i) if $m < \omega$ then the multisequence $\vec{\Pi}{\upharpoonleft\!\upharpoonright}_m$ belongs to the lightface $\Delta^{\mathrm{HC}}_{m+2}$;

(ii) if $m' < \omega$ and $W \subseteq \overrightarrow{\mathbf{sMF}}$ is a boldface $\boldsymbol{\Sigma}^{\mathrm{HC}}_{m'+1}$ set then there is an ordinal $\gamma < \omega_1$ such that the multisequence $\vec{\Pi}\restriction\gamma$ m'-decides W;

(iii) if a set $D \subseteq \mathbf{MT}(\vec{\Pi})$ is dense in $\mathbf{MT}(\vec{\Pi})$, then the set Z of all ordinals $\gamma < \omega_1$ such that $\vec{\Pi}\restriction\gamma \subset_{\{D\cap\mathbf{MT}(\vec{\Pi}\restriction\gamma)\}} \vec{\Pi}$, is stationary in ω_1.

We call this fixed $\vec{\Pi}$ *the key multisequence.* □

As usual, a set $U \subseteq \mathbf{sMF}{\Vert\upharpoonright}_{\geq m}$ is *dense in* $\mathbf{sMF}{\Vert\upharpoonright}_{\geq m}$ if for each $\vec{\boldsymbol{\pi}} \in \mathbf{sMF}{\Vert\upharpoonright}_{\geq m}$ there is a multisequence $\vec{\boldsymbol{\varphi}} \in U$ satisfying $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\varphi}}$.

Lemma 23.5. *If $m < \omega$ and $W \subseteq \mathbf{sMF}{\Vert\upharpoonright}_{\geq m}$ is a $\mathbf{\Sigma}^{\mathrm{HC}}_{m+1}$ set dense in $\mathbf{sMF}{\Vert\upharpoonright}_{\geq m}$ then there is an ordinal $\gamma < \omega_1$ such that $(\vec{\Pi}\upharpoonright\gamma){\Vert\upharpoonright}_{\geq m} \in W$. In particular, if $W \subseteq \overrightarrow{\mathbf{sMF}}$ is a $\mathbf{\Sigma}^{\mathrm{HC}}_1$ set dense in $\overrightarrow{\mathbf{sMF}}$ then there is $\gamma < \omega_1$ such that $\vec{\Pi}\upharpoonright\gamma \in W$.*

Proof. Apply 23.4(ii). The negative decision is impossible by the density. $\square$

24. Key forcing notion

We continue to argue in $\mathbf{L}$, and we'll make use of the key multisequence $\vec{\Pi} = \langle \Pi_\alpha \rangle_{\alpha<\omega_1}$ introduced by Definition 23.4. Π Π

Definition 24.1 *(in $\mathbf{L}$).* Define the multiforcings

$$\begin{array}{llll} \Pi & = \bigcup^{\mathrm{cw}} \vec{\Pi} & = \bigcup^{\mathrm{cw}}_{\alpha<\omega_1} \Pi_\alpha & \in \mathbf{MF}, \\ \Pi_{<\gamma} & = \bigcup^{\mathrm{cw}} (\vec{\Pi}\upharpoonright\gamma) & = \bigcup^{\mathrm{cw}}_{\alpha<\gamma} \Pi_\alpha & \in \mathbf{sMF}, \text{ for each } \gamma < \omega_1 \\ \Pi_{\geq\gamma} & = \bigcup^{\mathrm{cw}} (\vec{\Pi}\upharpoonright(\omega_1 \smallsetminus \gamma)) & = \bigcup^{\mathrm{cw}}_{\gamma\leq\alpha<\omega_1} \Pi_\alpha & \in \mathbf{MF}, \quad \text{for each } \gamma < \omega_1. \end{array}$$

We further define $\mathbb{P} = \mathbf{MT}(\Pi) = \mathbf{MT}(\vec{\Pi})$, and, for all $\gamma < \omega_1$,

$$\mathbb{P}_{<\gamma} = \mathbf{MT}(\Pi_{<\gamma}) = \mathbf{MT}(\vec{\Pi}\upharpoonright\gamma), \quad \mathbb{P}_{\geq\gamma} = \mathbf{MT}(\Pi_{\geq\gamma}) = \mathbf{MT}(\vec{\Pi}\upharpoonright(\omega_1 \smallsetminus \gamma)). \quad \square$$

The set $\mathbb{P} = \mathbf{MT}(\Pi)$ will be our principal forcing notion, *the key forcing*.

Lemma 24.2 *(in $\mathbf{L}$). Π is a regular multiforcing. In addition, $|\Pi| = \omega_1 \times \omega$, thus if $\xi < \omega_1$ and $k < \omega$ then there is an ordinal $\alpha < \omega_1$ such that $\langle \xi, k \rangle \in |\Pi_\alpha|$. Therefore $\mathbb{P} = \prod_{\xi<\omega_1,\, k<\omega} \Pi(\xi, k)$ (with finite support).*

Proof. To prove the additional claim, note that the set W of all multisequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$ satisfying $\langle \xi, k \rangle \in |\bigcup^{\mathrm{cw}} \vec{\boldsymbol{\pi}}|$ is $\mathbf{\Sigma}^{\mathrm{HC}}_1$ (with ξ as a parameter of definition). In addition W is dense in $\overrightarrow{\mathbf{sMF}}$. (First extend $\vec{\boldsymbol{\pi}}$ by Corollary 20.2 so that is has a non-limit length and the last term, then make use of Corollary 11.4.) Therefore by Lemma 23.5 there is an ordinal $\gamma < \omega_1$ such that $\vec{\Pi}\upharpoonright\gamma \in W$, as required. $\square$

If $\xi < \omega_1$ and $k < \omega$ then, following the lemma, let $\alpha(\xi, k) < \omega_1$ be the least ordinal α satisfying $\langle \xi, k \rangle \in |\Pi_\alpha|$. Thus a forcing $\Pi_\alpha(\xi, k) \in \mathbf{PTF}$ is defined whenever α satisfies $\alpha(\xi, k) \leq \alpha < \omega_1$, and $\langle \Pi_\alpha(\xi, k) \rangle_{\alpha(\xi,k)\leq\alpha<\omega_1}$ is a $\sqsubset$-increasing sequence of countable special forcings in $\mathbf{PTF}$.

Note that $\Pi(\xi, k) = \bigcup_{\alpha(\xi,k)\leq\alpha<\omega_1} \Pi_\alpha(\xi, k)$ by construction.

Corollary 24.3 *(in $\mathbf{L}$). If $k < \omega$ then the sequence of ordinals $\langle \alpha(\xi, k) \rangle_{\xi<\omega_1}$ and the sequence of multiforcings $\langle \Pi_\alpha(\xi, k) \rangle_{\xi<\omega_1,\, \alpha(\xi,k)\leq\alpha<\omega_1}$ are $\Delta^{\mathrm{HC}}_{k+2}$.*

Proof. By construction the following double equivalence holds:

$$\begin{aligned} \alpha < \alpha(\xi, k) \iff & \exists\, \boldsymbol{\pi} (\boldsymbol{\pi} = \Pi_\alpha{\Vert\upharpoonright}_k \wedge \langle \xi, k \rangle \in \mathtt{dom}\, \boldsymbol{\pi}) \iff \\ \iff & \forall\, \boldsymbol{\pi} (\boldsymbol{\pi} = \Pi_\alpha{\Vert\upharpoonright}_k \implies \langle \xi, k \rangle \in \mathtt{dom}\, \boldsymbol{\pi}) . \end{aligned}$$

However $\boldsymbol{\pi} = \Pi_\alpha{\Vert\upharpoonright}_k$ is a $\Delta^{\mathrm{HC}}_{k+2}$ relation by Theorem 23.1(i). It follows that so is the sequence $\langle \alpha(\xi, k) \rangle_{\xi<\omega_1}$. The second claim easily follows by the same Definition 23.4(i). $\square$

Corollary 24.4 *(in* $\mathbf{L}$*, of Lemma 9.2(v)). If* $\xi < \omega_1$*,* $k < \omega$*, and* $\alpha(\xi,k) \le \alpha < \omega_1$ *then the set* $\sqcap_\alpha(\xi,k)$ *is pre-dense in* $\sqcap(\xi,k)$ *and in* $\sqcap$*.* □

In spite of Lemma 24.2, the sets $|\sqcap_{<\gamma}|$ can be quite arbitrary (countable) subsets of $\omega_1 \times \omega$. However we easily get the next corollary:

Corollary 24.5 *(in* $\mathbf{L}$*, of Lemma 24.2). The set* $\mathbb{C}' = \{\gamma < \omega_1 : |\sqcap_{<\gamma}| = \gamma \times \omega\}$ *is a club in* ω_1*.* □

Lemma 24.6 *(in* $\mathbf{L}$*).* $\mathbb{P}$ *is CCC.*

Proof. Let $A \subseteq \mathbb{P}$ be a maximal antichain in $\mathbb{P}$. The set

$$C = \{\gamma < \omega_1 : A \cap \mathbb{P}_{<\gamma} \text{ is a maximal antichain in } \mathbb{P}_{<\gamma}\}$$

is a club in ω_1. Let $D = \{\boldsymbol{p} \in \mathbb{P} : \exists\, \boldsymbol{q} \in A\,(p \leqslant q)\}$; this is an open dense set. By Definition 23.4(iii), there is an ordinal $\gamma \in C$ such that $\vec{\sqcap}\restriction\gamma \subset_{\{D\cap\mathbb{P}_{<\gamma}\}} \vec{\sqcap}$. Recall that $\gamma \in C$, hence $A \cap \mathbb{P}_{<\gamma}$ is a maximal antichain in $\mathbb{P}_{<\gamma}$, thus $D \cap \mathbb{P}_{<\gamma}$ is open dense in $\mathbb{P}_{<\gamma}$. Therefore the set $D \cap \mathbb{P}_{<\gamma}$ is pre-dense in the forcing $\mathbf{MT}(\vec{\sqcap}) = \mathbb{P}$ by Corollary 20.2(v)(c). We claim that $A = A \cap \mathbb{P}_{<\gamma}$, so A is countable.

Indeed suppose that $\boldsymbol{r} \in A \smallsetminus \mathbb{P}_{<\gamma}$. Then $\boldsymbol{r}$ is compatible with some $\boldsymbol{q} \in D \cap \mathbb{P}_{<\gamma}$; let $\boldsymbol{p} \in D \cap \mathbb{P}_{<\gamma}$, $\boldsymbol{p} \leqslant \boldsymbol{q}$, $\boldsymbol{p} \leqslant \boldsymbol{r}$. As $\boldsymbol{q} \in D$, there is some $\boldsymbol{r}' \in A$ with $\boldsymbol{q} \leqslant \boldsymbol{r}'$. Then $\boldsymbol{r} = \boldsymbol{r}'$ as A is an antichain; thus $\boldsymbol{q} \leqslant \boldsymbol{r} \in A \smallsetminus \mathbb{P}_{<\gamma}$. However $\boldsymbol{q} \in \mathbb{P}_{<\gamma}$ and $A \cap \mathbb{P}_{<\gamma}$ is a maximal antichain in $\mathbb{P}_{<\gamma}$, thus $\boldsymbol{q}$, and hence $\boldsymbol{r}$, is compatible with some $\boldsymbol{r}'' \in A \cap \mathbb{P}_{<\gamma}$. Which is a contradiction. □

Corollary 24.7 *(in* $\mathbf{L}$*). If a set* $D \subseteq \mathbb{P}$ *is pre-dense in* $\mathbb{P}$ *then there is an ordinal* $\gamma < \omega_1$ *such that* $D \cap \mathbb{P}_{<\gamma}$ *is already pre-dense in* $\mathbb{P}$*.*

Proof. We can assume that in fact D is dense. Let $A \subseteq D$ be a maximal antichain in D; then A is a maximal antichain in $\mathbb{P}$ because of the density of D. Then $A \subseteq \mathbb{P}_{<\gamma}$ for some $\gamma < \omega_1$ by Lemma 24.6. But A is pre-dense in $\mathbb{P}$. □

V. Auxiliary forcing relation

Recall that $\mathbb{P} = \mathbf{MT}(\sqcap)$, the key forcing notion, is a product forcing notion defined (in $\mathbf{L}$) in Section 24. Its components $\sqcap(\xi,k)$ have different complexity in HC, depending on k by Corollary 24.3, hence it's difficult to hope that the forcing notion $\mathbb{P}$ (or $\sqcap$) as a whole is definable in HC. Somewhat surprisingly, the $\mathbb{P}$-forcing relation turns out to be definable in HC when restricted to analytic formulas of a certain level of complexity within the usual hierarchy. This will be established on the base of an auxiliary forcing relation.

25. Preliminaries

We argue in L.

Consider the 2nd order arithmetic language, with variables $k, l, m, n, \dots$ of type 0 over ω and variables $a, b, x, y, \dots$ of type 1 over ω^ω, whose atomic formulas are those of the form $x(k) = n$. Let $\mathscr{L}$ be the extension of this language, which allows to substitute free variables of type 0 with natural numbers (as usual) and free variables of type 1 with **small real names** (Definition 13.1) $\mathbf{c} \in \mathbf{L}$. By *$\mathscr{L}$-formulas* we understand formulas of this extended language.

We define natural classes $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$ $(n \ge 1)$ of $\mathscr{L}$-formulas. Let $\mathscr{L}(\Sigma\Pi)^1_1$ be the closure of $\mathscr{L}\Sigma^1_1 \cup \mathscr{L}\Pi^1_1$ under $\neg, \wedge, \vee$ and quantifiers over ω. If φ is a formula in $\mathscr{L}\Sigma^1_n$ (resp., $\mathscr{L}\Pi^1_n$), then let φ^- be the result of canonical transformation of the negation $\neg\,\varphi$ to the $\mathscr{L}\Pi^1_n$ (resp., $\mathscr{L}\Sigma^1_n$) form.

Definition 25.1. If φ is a $\mathscr{L}$-formula and $G \subseteq \mathbf{MT}$ is a pairwise compatible set of multitrees then let $\varphi[G]$ be the result of substitution of $\mathbf{c}[G]$ for any real name $\mathbf{c}$ in φ. (Recall Definition 13.2.) Thus $\varphi[G]$ is an ordinary 2nd order arithmetic formula, with natural numbers and elements of ω^ω as parameters. □

The definition of the auxiliary forcing depends on a two more definitions.

Definition 25.2. If $m < \omega$ then $\overrightarrow{\mathbf{sMF}}[\vec{\Pi}\upharpoonright\upharpoonright_{<m}]$ consists of all multisequences $\vec{\pi} \in \overrightarrow{\mathbf{sMF}}$ such that $\vec{\pi}\upharpoonright\upharpoonright_{<m} \subset \vec{\Pi}\upharpoonright\upharpoonright_{<m}$, that is, $\vec{\pi}\upharpoonright\upharpoonright_{<m} = (\vec{\Pi}\upharpoonright\upharpoonright_{<m})\upharpoonright\delta$, where $\delta = \mathtt{dom}(\vec{\pi})$ — multisequences which agree with the key multisequence $\vec{\Pi}$ on layers below m. Obviously $\overrightarrow{\mathbf{sMF}}[\vec{\Pi}\upharpoonright\upharpoonright_{<m+1}] \subseteq \overrightarrow{\mathbf{sMF}}[\vec{\Pi}\upharpoonright\upharpoonright_{<m}] \subseteq \overrightarrow{\mathbf{sMF}}[\vec{\Pi}\upharpoonright\upharpoonright_{<0}] = \overrightarrow{\mathbf{sMF}}$. □

If $\gamma < \omega_1$ then the subsequence $\vec{\Pi}\upharpoonright\gamma$ of the key multisequence $\vec{\Pi}$ belongs to $\bigcap_{m<\omega} \overrightarrow{\mathbf{sMF}}[\vec{\Pi}\upharpoonright\upharpoonright_{<m}]$, of course. To prove the next lemma use 23.4(i).

Lemma 25.3. *$\overrightarrow{\mathbf{sMF}}[\vec{\Pi}\upharpoonright\upharpoonright_{<m}]$ is a subset of* HC *of definability class $\Delta^{\mathrm{HC}}_{m+1}$.* □

The other definition deals with models of a subtheory of **ZFC**.

Definition 25.4. Let $\mathbf{ZFL}^-$ be the theory containing all axioms of $\mathbf{ZFC}^-$ (minus the Power Set axiom) plus the axiom of constructibility $\mathbf{V} = \mathbf{L}$. Any transitive model (TM) of $\mathbf{ZFL}^-$ has the form $\mathbf{L}_\alpha$, where $\alpha \in \mathbf{Ord}$. Therefore it is true in $\mathbf{L}$ that for any set x there is a least TM $\mathfrak{L}(x) \models \mathbf{ZFL}^-$ containing x. □

If $x \in$ HC (HC= all hereditarily countable sets) then $\mathfrak{L}(x)$ is a *countable* transitive model (CTM) of $\mathbf{ZFL}^-$.

26. Auxiliary forcing relation

We continue to argue in L.

Here we define a relation $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \varphi$ between multitrees $\boldsymbol{p}$, multisequences $\vec{\pi}$, and $\mathscr{L}$-formulas φ, which will suitably approximate the true $\mathbb{P}$-forcing relation. The definition goes on by induction on the complexity of φ.

1°. Let φ be a closed $\mathscr{L}(\Sigma\Pi)^1_1$ formula, $\vec{\pi} \in \overrightarrow{\mathbf{sMF}}$, and $\boldsymbol{p} \in \mathbf{MT}$. (Not necessarily $\boldsymbol{p} \in \mathbf{MT}(\vec{\pi})$.) We define:

- (a) $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \varphi$ iff there is a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$, an ordinal $\vartheta < \mathtt{dom}\,\vec{\pi}$, and a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\pi}\upharpoonright\vartheta)$, such that $\boldsymbol{p} \leqslant \boldsymbol{p}_0$ (meaning: $\boldsymbol{p}$ is stronger), the model $\mathfrak{M}$ contains $\vec{\pi}\upharpoonright\vartheta$ (then contains $\mathbf{MT}(\vec{\pi}\upharpoonright\vartheta)$ as well) and contains φ (that is, all real names in φ), $\vec{\pi}\upharpoonright\vartheta \subset_{\mathfrak{M}} \vec{\pi}$, and $\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\pi}\upharpoonright\vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$ in the usual sense — that is, if $G \subseteq \mathbf{MT}(\vec{\pi}\upharpoonright\vartheta)$ is a $\mathbf{MT}(\vec{\pi}\upharpoonright\vartheta)$-generic filter over $\mathfrak{M}$ then the formula $\varphi[G]$ (defined by 25.1) is true in $\mathfrak{M}[G]$ [15];
- (b) $\boldsymbol{p}\ \mathtt{wforc}_{\vec{\pi}}\ \varphi$ (weak forcing) iff there is no multisequence $\vec{\tau} \in \overrightarrow{\mathbf{sMF}}$ and $\boldsymbol{p}' \in \mathbf{MT}(\vec{\tau})$ such that $\vec{\pi} \subseteq \vec{\tau}$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\tau}}\ \neg\,\varphi$.

2°. If $\varphi(x)$ is a $\mathscr{L}\Pi^1_n$ formula, $n \geq 1$, then we define $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \exists\, x\, \varphi(x)$ iff there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\pi}}\ \varphi(\mathbf{c})$.

[15] Note that not only we require $\varphi[\underline{G}]$ to be $\mathbf{MT}(\vec{\pi}\upharpoonright\vartheta)$-forced, but also that this status is suitably sealed by $\vec{\pi}\upharpoonright\vartheta \subset_{\mathfrak{M}} \vec{\pi}$, so that further extensions of $\vec{\pi}$ will not bring up a contradiction.

3°. If φ is a closed $\mathscr{L}\Pi^1_n$ formula, $n \ge 2$, then we define $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$ iff $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$, and there is no multisequence $\vec{\boldsymbol\tau} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$ and multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\boldsymbol\tau})$ such that $\vec{\boldsymbol\pi} \subseteq \vec{\boldsymbol\tau}$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\boldsymbol\tau}}\ \varphi^-$.[16]

Remark 26.1. It easily holds by induction that if $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$ then

(1) $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{sMF}}$,
(2) φ is a closed formula in one of the classes $\mathscr{L}(\Sigma\Pi)^1_1$, $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$, $n \ge 2$,
(3) if $n \ge 2$ and $\varphi \in \mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$ then $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$. □

Remark 26.2. We claim that the condition

$$\text{“}\boldsymbol{p}_0\ \ \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)\text{-forces } \varphi[\underline{G}] \text{ over } \mathfrak{M}\text{”}$$

in 1°a does not depend on the choice of a CTM $\mathfrak{M}$. (Note that independence of ϑ is not asserted here!) Indeed consider another CTM $\mathfrak{M}' \models \mathbf{ZFL}^-$ still containing $\vec{\boldsymbol\pi}\restriction\vartheta$ and φ. Suppose towards the contrary that a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$ $\mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$ but does not $\mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$-force $\varphi[\underline{G}]$ over $\mathfrak{M}'$.

Then there is a stronger multitree $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$ which $\mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$-forces $\neg\,\varphi[\underline{G}]$ over $\mathfrak{M}'$. Now consider any filter $G \subseteq \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$, $\mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$-generic both over $\mathfrak{M}$ and over $\mathfrak{M}'$ and containing $\boldsymbol{p}$, then containing $\boldsymbol{p}_0$ as well. Then one and the same formula $\varphi[G]$ is true in $\mathfrak{M}[G]$ but false in $\mathfrak{M}'[G]$. However it is known by the Mostowski absoluteness theorem [16, Theorem 25.4] that all transitive models agree on Σ^1_1 formulas. This is a contradiction. □

Corollary 26.3. *Let φ is a closed $\mathscr{L}(\Sigma\Pi)^1_1$ formula, $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{sMF}}$, and $\boldsymbol{p} \in \mathbf{MT}$. Assume that an ordinal $\vartheta < \mathtt{dom}\,\vec{\boldsymbol\pi}$, a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol\pi}\restriction\vartheta)$, and a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$ witness $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$ in the sense of 1°a. Then ϑ, $\boldsymbol{p}_0$, and the model $\mathfrak{N} = \mathfrak{L}(\varphi, \vec{\boldsymbol\pi}\restriction\vartheta)$[17] witness $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$ as well.*

Proof. First of all $\mathfrak{N} \subseteq \mathfrak{M}$, and hence we have $\vec{\boldsymbol\pi}\restriction\vartheta \subset_{\mathfrak{N}} \vec{\boldsymbol\pi}$ by Corollary 19.2(i). It remains to make use of Remark 26.2. □

Lemma 26.4. *Assume that multisequences $\vec{\boldsymbol\pi} \subseteq \vec{\boldsymbol\varrho}$ belong to $\overrightarrow{\mathbf{sMF}}$, $\boldsymbol{q}, \boldsymbol{p} \in \mathbf{MT}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, φ is an $\mathscr{L}$-formula as in 26.1, and if $n \ge 2$ and $\varphi \in \mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$ then $\vec{\boldsymbol\pi}, \vec{\boldsymbol\varrho} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$. Then $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$ implies $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol\varrho}}\ \varphi$, and if φ belongs to $\mathscr{L}(\Sigma\Pi)^1_1$ then $\boldsymbol{p}\ \mathtt{wforc}_{\vec{\boldsymbol\pi}}\ \varphi$ implies $\boldsymbol{q}\ \mathtt{wforc}_{\vec{\boldsymbol\varrho}}\ \varphi$ as well.*

Proof. If φ is a $\mathscr{L}(\Sigma\Pi)^1_1$ formula, $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$, and this is witnessed by $\mathfrak{M}$, ϑ, $\boldsymbol{p}_0$ as in 1°a, then the exactly same $\mathfrak{M}$, ϑ, $\boldsymbol{p}_0$ witness $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol\varrho}}\ \varphi$.

The induction step $\exists$, as in 2°, is elementary.

Now the induction step $\forall$, as in 3°. Let φ be a closed $\mathscr{L}\Pi^1_n$-formula, $n \ge 2$, and $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$. Assume to the contrary that $\boldsymbol{q}\ \mathtt{forc}_{\vec{\boldsymbol\varrho}}\ \varphi$ fails. Then by 3° there exist: a multisequence $\vec{\boldsymbol\varrho}' \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$ and multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol\varrho}')$ such that $\vec{\boldsymbol\varrho} \subseteq \vec{\boldsymbol\varrho}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\ \mathtt{forc}_{\vec{\boldsymbol\varrho}'}\ \varphi^-$. But then $\vec{\boldsymbol\pi} \subseteq \vec{\boldsymbol\varrho}'$ and $\boldsymbol{q}' \leqslant \boldsymbol{p}$, hence $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$ fails by 3°, which is a contradiction.

The additional result for $\mathtt{wforc}$ and $\mathscr{L}(\Sigma\Pi)^1_1$ formulas is entirely similar to the induction step $\forall$ as just above. □

[16] If $\vec{\boldsymbol\pi}$ does not belong to $\overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$ in 3°, then $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol\pi}}\ \varphi$ holds for any $\mathscr{L}\Pi^1_n$ formula φ by default as a multisequence not in $\overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$ is definitely not extendable to a multisequence in $\overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$. This motivates the condition $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$ in 3°.

[17] Thus, by 25.4, $\mathfrak{N} \models \mathbf{ZFL}^-$ is the least CTM containing $\vec{\boldsymbol\pi}\restriction\vartheta$ and (all parameters in) φ.

Definition 26.5. If K is one of the classes $\mathscr{L}(\Sigma\Pi)^1_1$, $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$ $(n \ge 2)$, then let $\mathbf{FORC}[K]$ consist of all triples $\langle \vec{\boldsymbol{\pi}}, \boldsymbol{p}, \varphi \rangle$ such that (1), (2), (3) of 26.1 hold, and in addition $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ. Then $\mathbf{FORC}[K]$ is a subset of HC. □

Lemma 26.6 *(definability, in* $\mathbf{L}$*).* $\mathbf{FORC}[\mathscr{L}(\Sigma\Pi)^1_1]$ *belongs to* Δ^{HC}_1*. If* $n \ge 2$ *then* $\mathbf{FORC}[\mathscr{L}\Sigma^1_n]$ *belongs to* $\Sigma^{\mathrm{HC}}_{n-1}$ *and* $\mathbf{FORC}[\mathscr{L}\Pi^1_n]$ *belongs to* Π^{HC}_{n-1}*.*

Proof. The following is a semi-formal structure of the definition of $\langle \vec{\boldsymbol{\pi}}, \boldsymbol{p}, \varphi \rangle \in \mathbf{FORC}[\mathscr{L}(\Sigma\Pi)^1_1]$ by 1°a, modified via Corollary 26.3[18]:

$$\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}} \wedge \boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\rho}}) \wedge \varphi \text{ is a formula in } \mathscr{L}(\Sigma\Pi)^1_1 \wedge \tag{1}$$

$$\wedge\ \exists \vartheta < \mathtt{dom}\, \vec{\boldsymbol{\pi}}\ \exists \boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta) \tag{2}$$

$$\big(\vartheta,\ \boldsymbol{p}_0, \text{ and the model } \mathfrak{M} = \mathfrak{L}(\varphi, \vec{\boldsymbol{\pi}} \restriction \vartheta) \text{ witness } \boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi \text{ as in } 1°\text{a}\big). \tag{3}$$

Relations in line (1) are definable in HC by bounded formulas, hence Δ^{HC}_1. The quantifiers in line (2) are bounded. The quantifier $\exists \mathfrak{M}$ in 1°a is eliminated by an explicit reference to $\mathfrak{M} = \mathfrak{L}(\varphi, \vec{\boldsymbol{\pi}} \restriction \vartheta)$ via Corollary 26.3, where $\langle \varphi, \vec{\boldsymbol{\pi}} \restriction \vartheta \rangle \mapsto \mathfrak{M} = \mathfrak{L}(\varphi, \vec{\boldsymbol{\pi}} \restriction \vartheta)$ is clearly a Δ^{HC}_1 map (in $\mathbf{L}$). Finally line (3) is Δ^{HC}_1 as well because forcing over a CTM $\mathfrak{M}$ is definable in $\mathfrak{M}$, hence is Δ^{HC}_1. This wraps up the Δ^{HC}_1 estimation for $\mathscr{L}(\Sigma\Pi)^1_1$.

The inductive step by 2° is quite elementary.

Now the step by 3°. Assume that $n \ge 2$, and it is already established that $\mathbf{FORC}[\mathscr{L}\Sigma^1_n] \in \Sigma^{\mathrm{HC}}_{n-1}$. Then $\langle \vec{\boldsymbol{\pi}}, \boldsymbol{p}, \varphi \rangle \in \mathbf{FORC}[\mathscr{L}\Pi^1_n]$ iff $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\mathbb{\Pi}} \upharpoonleft\upharpoonleft_{<n-2}]$, $\boldsymbol{p} \in \mathbf{MT}$, φ is a closed $\mathscr{L}\Pi^1_n$ formula in $\mathfrak{M}(\vec{\boldsymbol{\pi}})$, and, by 3°, there exist no triple $\langle \vec{\boldsymbol{\tau}}, \boldsymbol{p}', \psi \rangle \in \mathbf{FORC}[\mathscr{L}\Sigma^1_n]$ such that $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{sMF}}[\vec{\mathbb{\Pi}} \upharpoonleft\upharpoonleft_{<n-2}]$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\tau}}$, $\boldsymbol{p}' \in \mathbf{MT}(\vec{\boldsymbol{\tau}})$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and ψ is φ^-. Evaluating the key term $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{sMF}}[\vec{\mathbb{\Pi}} \upharpoonleft\upharpoonleft_{<n-2}]$ by Lemma 25.3 as $\Delta^{\mathrm{HC}}_{n-1}$, we get the required estimation Π^{HC}_{n-1} of $\mathbf{FORC}[\mathscr{L}\Pi^1_n]$. □

27. Forcing simple formulas

We still argue in L. The following results are mainly related to the relation $\mathtt{forc}$ with respect to formulas in the class $\mathscr{L}(\Sigma\Pi)^1_1$.

Lemma 27.1 *(in* $\mathbf{L}$*). Assume that* $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$*,* $\vec{\boldsymbol{\varsigma}} \in \overrightarrow{\mathbf{sMF}} \cup \overrightarrow{\mathbf{sMF}}_{\omega_1}$*,* $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\varsigma}}$*,* $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$*,* φ *is a formula in* $\mathscr{L}(\Sigma\Pi)^1_1$*,* $\mathfrak{N} \models \mathbf{ZFL}^-$ *is a TM containing* $\vec{\boldsymbol{\varsigma}}$ *and* φ*, and* $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ*. Then* $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\varsigma}})$*-forces* $\varphi[\underline{G}]$ *over* $\mathfrak{N}$*.*

Proof. By definition there is an ordinal $\vartheta < \mathtt{dom}\, \vec{\boldsymbol{\pi}}$, a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$, and a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$ containing φ and $\vec{\boldsymbol{\pi}} \restriction \vartheta$, such that $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol{\pi}}$, $\boldsymbol{p} \leqslant \boldsymbol{p}_0$, and $\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$. By Corollary 26.3, we can w.l.o.g. assume that $\mathfrak{M} = \mathfrak{L}(\varphi, \vec{\boldsymbol{\pi}} \restriction \vartheta)$ (the smallest CTM of $\mathbf{ZFL}^-$ containing φ and $\vec{\boldsymbol{\pi}} \restriction \vartheta$, Definition 25.4). Then $\mathfrak{M} \subseteq \mathfrak{N}$.

Now suppose that $G \subseteq \mathbf{MT}(\vec{\boldsymbol{\varsigma}})$ is a set $\mathbf{MT}(\vec{\boldsymbol{\varsigma}})$-generic over $\mathfrak{N}$ and $\boldsymbol{p} \in G$ — then $\boldsymbol{p}_0 \in G$, too. We have to prove that $\varphi[G]$ is true in $\mathfrak{N}[G]$.

We claim that the set $G' = G \cap \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$ is $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-generic over $\mathfrak{M}$. Indeed, let a set $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$, be open dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$. Then, as $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol{\varsigma}}$ by Corollary 20.2(v)(c), $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol{\varsigma}})$ by 20.2(c), and hence $G \cap \boldsymbol{D} \ne \varnothing$ by the choice of G. It follows that $G' \cap \boldsymbol{D} \ne \varnothing$.

We claim that $\mathbf{c}[G] = \mathbf{c}[G']$ for any name $\mathbf{c} \in \mathfrak{M}$, in particular, for any name in φ. Indeed, as $G' \subseteq G$, the otherwise occurs by Definition 13.2 only if for some n, i and $\boldsymbol{q}' \in K^{\mathbf{c}}_{ni}$ there is $\boldsymbol{q} \in G$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{q}'$,

[18] See footnote 17.

but there is no such $\boldsymbol{q}$ in G'. Let $\boldsymbol{D}$ consist of all multitrees $\boldsymbol{r} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$ either satisfying $\boldsymbol{r} \leqslant \boldsymbol{q}'$ or somewhere AD with $\boldsymbol{q}'$. Then $\boldsymbol{D} \in \mathfrak{M}$ and $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$. Therefore $\boldsymbol{D} \cap G' \neq \varnothing$ by the above, so let $\boldsymbol{r} \in \boldsymbol{D} \cap G'$. If $\boldsymbol{r} \leqslant \boldsymbol{q}'$ then we get a contradiction with the choice of $\boldsymbol{q}'$. If $\boldsymbol{r}$ is somewhere AD with $\boldsymbol{q}'$ then we get a contradiction with the choice of $\boldsymbol{q}$ as both $\boldsymbol{q}, \boldsymbol{r}$ belong to the generic filter G.

It follows that $\varphi[G]$ coincides with $\varphi[G']$. Note also that $\boldsymbol{p}_0 \in G'$. We conclude that $\varphi[G']$ is true in $\mathfrak{M}[G']$ as $\boldsymbol{p}_0$ forces $\varphi[\underline{G}]$ over $\mathfrak{M}$. The same formula $\varphi[G]$ is true in $\mathfrak{N}[G]$ by the Mostowski absoluteness. □

Lemma 27.2. *Let $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$, $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, φ be a formula in $\mathscr{L}(\Sigma\Pi)^1_1$. Then*

(i) *$\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ and $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ $\neg\varphi$ cannot hold together;*
(ii) *if $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ then $\boldsymbol{p}$ $\mathtt{wforc}_{\vec{\boldsymbol{\pi}}}$ φ;*
(iii) *if $\boldsymbol{p}$ $\mathtt{wforc}_{\vec{\boldsymbol{\pi}}}$ φ then there exists a multisequence $\vec{\boldsymbol{\varrho}} \in \overrightarrow{\mathbf{sMF}}$ such that $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}})} \vec{\boldsymbol{\varrho}}$ and $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\varrho}}}$ φ* (see Definition 25.4 on models $\mathfrak{L}(x)$);
(iv) *$\boldsymbol{p}$ $\mathtt{wforc}_{\vec{\boldsymbol{\pi}}}$ φ and $\boldsymbol{p}$ $\mathtt{wforc}_{\vec{\boldsymbol{\pi}}}$ $\neg\varphi$ cannot hold together.*

Proof. (i) Otherwise $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}})$-forces both $\varphi[\underline{G}]$ and $\neg\varphi[\underline{G}]$ over a large enough CTM $\mathfrak{M}$, by Lemma 27.1, which cannot happen.

(ii) Assume that $\boldsymbol{p}$ $\mathtt{wforc}_{\vec{\boldsymbol{\pi}}}$ φ fails, hence there is a multisequence $\vec{\boldsymbol{\varrho}} \in \overrightarrow{\mathbf{sMF}}$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\varrho}})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ $\neg\varphi$. But Lemma 26.4 implies $\boldsymbol{q}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ, which contradicts to (i).

(iii) Let $\mathfrak{M} = \mathfrak{L}(\vec{\boldsymbol{\pi}})$. By Corollary 20.2(ii), there is a multisequence $\vec{\boldsymbol{\varrho}} \in \overrightarrow{\mathbf{sMF}}$ satisfying $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varrho}}$. We claim that $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}})$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$ in the usual sense — then by definition $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\varrho}}}$ φ (via $\mathfrak{M}$, $\vartheta = \mathtt{dom}\,\vec{\boldsymbol{\pi}}$, and $\boldsymbol{p}_0 = \boldsymbol{p}$), and we are done. To prove the claim suppose otherwise. Then there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}})$-forces $\neg\varphi[\underline{G}]$ over $\mathfrak{M}$, thus $\boldsymbol{q}$ $\mathtt{forc}_{\vec{\boldsymbol{\varrho}}}$ $\neg\varphi$. But this contradicts to $\boldsymbol{p}$ $\mathtt{wforc}_{\vec{\boldsymbol{\pi}}}$ φ.

(iv) There is a multisequence $\vec{\boldsymbol{\varrho}} \in \overrightarrow{\mathbf{sMF}}$ by (iii), such that $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varrho}}$ and $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\varrho}}}$ φ. Note that still $\boldsymbol{p}$ $\mathtt{wforc}_{\vec{\boldsymbol{\varrho}}}$ $\neg\varphi$ by Lemma 26.4. Extend $\vec{\boldsymbol{\varrho}}$ once again, getting a contradiction with (i). □

Corollary 27.3. *Let $n \geq 2$, $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$, $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, φ be a formula in $\mathscr{L}\Sigma^1_n$. Then $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ and $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ^- cannot hold together.*

Proof. If $n = 1$ then apply Lemma 27.2(i). If $n \geq 2$ then the result immediately follows by definition (3° in Section 26). □

The following is similar to the case $\vec{\boldsymbol{\rho}} = \vec{\boldsymbol{\pi}}$ in 27.1, but with $\mathtt{wforc}$.

Corollary 27.4 *(in $\mathbf{L}$). Assume that $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$, $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, φ is a formula in $\mathscr{L}(\Sigma\Pi)^1_1$, $\mathfrak{N} \models \mathbf{ZFL}^-$ is a TM containing $\vec{\boldsymbol{\pi}}$ and φ, and $\boldsymbol{p}$ $\mathtt{wforc}_{\vec{\boldsymbol{\pi}}}$ φ. Then $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}})$-forces $\varphi[\underline{G}]$ over $\mathfrak{N}$ in the usual sense.*

Proof. Otherwise there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which $\mathbf{MT}(\vec{\boldsymbol{\pi}})$-forces $\neg\varphi[\underline{G}]$ over $\mathfrak{N}$. On the other hand, by Lemma 27.2(iii), there is a multisequence $\vec{\boldsymbol{\varrho}} \in \overrightarrow{\mathbf{sMF}}$ such that $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}})} \vec{\boldsymbol{\varrho}}$ and $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\varrho}}}$ φ, hence, $\boldsymbol{q}$ $\mathtt{forc}_{\vec{\boldsymbol{\varrho}}}$ φ by Lemma 26.4. However we have $\boldsymbol{q}$ $\mathtt{forc}_{\vec{\boldsymbol{\varrho}}}$ $\neg\varphi$ by definition (1°a in Section 26 with $\vartheta = \mathtt{dom}\,\vec{\boldsymbol{\pi}}$), which contradicts to Lemma 27.2(i). □

28. Forcing with subsequences of the key multisequence

The following Theorem 28.3 will show that the auxiliary relation $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$, considered with countable initial segments $\vec{\boldsymbol{\pi}} = \vec{\sqcap}\restriction\alpha$ of the key sequence $\vec{\sqcap}$, essentially coincides with the true forcing relation of the key forcing notion $\mathbb{P} = \mathbf{MT}(\vec{\sqcap})$.

We argue in L.

Recall that the key multisequence $\vec{\Pi} = \langle \Pi_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{sMF}}_{\omega_1}$, satisfying (i), (ii), (iii) of Theorem 23.1, was fixed by 23.4, and $\mathbb{P} = \mathbf{MT}(\vec{\Pi})$ is our forcing notion. If $\gamma < \omega_1$ then the subsequence $\vec{\Pi}\restriction\gamma$ belongs to $\overrightarrow{\mathbf{sMF}}[\vec{\Pi}\restriction\restriction_{<m}]$, $\forall m$.

Definition 28.1. We write $\boldsymbol{p}$ $\mathtt{forc}_\alpha$ φ instead of $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\Pi}\restriction\alpha}$ φ, for the sake of brevity. Let $\boldsymbol{p}$ $\mathtt{forc}$ φ mean: $\boldsymbol{p}$ $\mathtt{forc}_\alpha$ φ for some $\alpha < \omega_1$. □

Lemma 28.2 *(in* $\mathbf{L}$*). Assume that* $\boldsymbol{p} \in \mathbb{P}$, $\alpha < \omega_1$, *and* $\boldsymbol{p}$ $\mathtt{forc}_\alpha$ φ. *Then:*

(i) *if* $\alpha \le \beta < \omega_1$, $\boldsymbol{q} \in \mathbb{P}_{<\beta} = \mathbf{MT}(\vec{\Pi}\restriction\beta)$, *and* $\boldsymbol{q} \leqslant \boldsymbol{p}$, *then* $\boldsymbol{q}$ $\mathtt{forc}_\beta$ φ;
(ii) *if* $\boldsymbol{q} \in \mathbb{P}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, *then* $\boldsymbol{q}$ $\mathtt{forc}_\beta$ φ *for some* β; $\alpha \le \beta < \omega_1$;
(iii) *if* $\boldsymbol{q} \in \mathbb{P}$ *and* $\boldsymbol{q}$ $\mathtt{forc}$ φ^- *then* $\boldsymbol{p}, \boldsymbol{q}$ *are somewhere almost disjoint;*
(iv) *therefore, 1st, if* $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, *and* $\boldsymbol{p}$ $\mathtt{forc}$ φ *then* $\boldsymbol{q}$ $\mathtt{forc}$ φ, *and 2nd,* $\boldsymbol{p}$ $\mathtt{forc}$ φ, $\boldsymbol{p}$ $\mathtt{forc}$ φ^- *cannot hold together.*

Proof. To prove (i) apply Lemma 26.4. To prove (ii) let β satisfy $\alpha < \beta < \omega_1$ and $\boldsymbol{q} \in \mathbf{MT}(\vec{\Pi}\restriction\beta)$, and apply (i). Finally to prove (iii) note that $\boldsymbol{p}, \boldsymbol{q}$ have to be incompatible in $\mathbb{P}$, as otherwise (i) leads to contradiction, but the incompatibility in $\mathbb{P}$ implies being somewhere AD by Corollary 7.2. □

Theorem 28.3. *If* φ *is a closed formula as in 26.1(2), and* $\boldsymbol{p} \in \mathbb{P}$, *then* $\boldsymbol{p}$ $\mathbb{P}$*-forces* $\varphi[\underline{G}]$ *over* $\mathbf{L}$ *in the usual sense, if and only if* $\boldsymbol{p}$ $\mathtt{forc}$ φ.

Proof. Let $\Vdash$ denote the usual $\mathbb{P}$-forcing relation over $\mathbf{L}$.

Part 1: φ is a formula in $\mathscr{L}(\Sigma\Pi)^1_1$. If $\boldsymbol{p}$ $\mathtt{forc}$ φ then $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\Pi}\restriction\gamma}$ φ for some $\gamma < \omega_1$, and then $\boldsymbol{p} \Vdash \varphi[\underline{G}]$ by Lemma 27.1 with $\vec{\varphi} = \vec{\Pi}$ and $\mathfrak{N} = \mathbf{L}$.

Suppose now that $\boldsymbol{p} \Vdash \varphi[\underline{G}]$. There is an ordinal $\gamma_0 < \omega_1$ such that $\boldsymbol{p} \in \mathbb{P}_{\gamma_0} = \mathbf{MT}(\vec{\Pi}\restriction\gamma_0)$ and φ belongs to $\mathfrak{L}(\vec{\Pi}\restriction\gamma_0)$. The set U of all multisequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$ such that $\gamma_0 < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$ and there is an ordinal ϑ, $\gamma_0 < \vartheta < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$, such that $\vec{\boldsymbol{\pi}}\restriction\vartheta \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)} \vec{\boldsymbol{\pi}}$, is dense in $\vec{\boldsymbol{\pi}}$ by Corollary 20.2(ii), and is $\boldsymbol{\Delta}^{\mathrm{HC}}_1$. Therefore by Corollary 23.5 there is an ordinal $\gamma < \omega_1$ such that $\vec{\boldsymbol{\pi}} = \vec{\Pi}\restriction\gamma \in U$.

Let this be witnessed by an ordinal ϑ, $\gamma_0 < \vartheta < \gamma = \mathtt{dom}\,\vec{\boldsymbol{\pi}}$ and $\vec{\boldsymbol{\pi}}\restriction\vartheta \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)} \vec{\boldsymbol{\pi}}$. We claim that $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$ in the usual sense — then by definition $\boldsymbol{p}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ, and we are done. To prove the claim, assume otherwise. Then there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\Pi}\restriction\vartheta)$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$-forces $\neg\varphi[\underline{G}]$ over $\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$. Then by definition $\boldsymbol{q}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ $\neg\varphi$ holds, hence $\boldsymbol{q}$ $\mathtt{forc}$ $\neg\varphi$, and then $\boldsymbol{q} \Vdash \neg\varphi[\underline{G}]$ (see above), with a contradiction to $\boldsymbol{p} \Vdash \varphi[\underline{G}]$.

Part 2: the step $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$ $(n \ge 1)$. Consider a $\mathscr{L}\Pi^1_n$ formula $\varphi(x)$. Assume $\boldsymbol{p}$ $\mathtt{forc}$ $\exists\, x\, \varphi(x)$. By definition there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}$ $\mathtt{forc}$ $\varphi(\mathbf{c})$. By inductive hypothesis, $\boldsymbol{p} \Vdash \varphi(c)[\underline{G}]$, that is, $\boldsymbol{p} \Vdash \exists\, x\, \varphi(x)[\underline{G}]$. Conversely, assume that $\boldsymbol{p} \Vdash \exists\, x\, \varphi(x)[\underline{G}]$. As $\mathbb{P}$ is CCC, there is a small real name $\mathbf{c}$ (in $\mathbf{L}$) such that $\boldsymbol{p} \Vdash \varphi(\mathbf{c})[\underline{G}]$. We have $\boldsymbol{p}$ $\mathtt{forc}$ $\varphi(\mathbf{c})$ by the inductive hypothesis, hence $\boldsymbol{p}$ $\mathtt{forc}$ $\exists\, x\, \varphi(x)$.

Part 3: the step $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$ $(n \ge 2)$. Consider a closed $\mathscr{L}\Sigma^1_n$ formula φ. Assume that $\boldsymbol{p}$ $\mathtt{forc}$ φ^-. By Lemma 28.2(iv), there is no multitree $\boldsymbol{q} \in \mathbb{P}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, with $\boldsymbol{q}$ $\mathtt{forc}$ φ. This implies $\boldsymbol{p} \Vdash \varphi^-[G]$ by the inductive hypothesis.

Conversely, suppose that $\boldsymbol{p} \Vdash \varphi^-[G]$. There is an ordinal $\gamma_0 < \omega_1$ such that $\boldsymbol{p} \in \mathbb{P}_{\gamma_0} = \mathbf{MT}(\vec{\Pi}\restriction\gamma_0)$ and φ belongs to $\mathfrak{L}(\vec{\Pi}\restriction\gamma_0)$. Consider the set U of all multisequences of the form $\vec{\boldsymbol{\pi}}\restriction\restriction_{\ge n-2}$, where $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\Pi}\restriction\restriction_{<n-2}]$, $\mathtt{dom}\,\vec{\boldsymbol{\pi}} > \gamma_0$, and there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{p}$ ($\boldsymbol{q}$ is stronger) and $\boldsymbol{q}$ $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ φ. It follows from Lemma 25.3 and Lemma 26.6 that U belongs to $\boldsymbol{\Sigma}^{\mathrm{HC}}_{n-1}$ (with φ and $\boldsymbol{p}_0$ as parameters). Therefore by 23.4(ii) there is an ordinal $\gamma < \omega_1$ such that the subsequence $\vec{\Pi}\restriction\gamma$ $(n-2)$-decides U.

Case 1: $(\vec{\Pi}\restriction\gamma)\restriction\restriction_{\ge n-2} \in U$. Let this be witnessed by a multisequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\Pi}\restriction\restriction_{<n-2}]$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, so that in particular $(\vec{\Pi}\restriction\gamma)\restriction\restriction_{\ge n-2} = \vec{\boldsymbol{\pi}}\restriction\restriction_{\ge n-2}$ and $\mathtt{dom}\,\vec{\boldsymbol{\pi}} = \gamma > \gamma_0$. Then by definition

(Definition 25.2) we also have $\vec{\boldsymbol{\pi}}{\upharpoonright\upharpoonright}_{<n-2} = (\vec{\mathbb{\Pi}}{\upharpoonright}\gamma){\upharpoonright\upharpoonright}_{<n-2}$, so that overall $\vec{\boldsymbol{\pi}} = \vec{\mathbb{\Pi}}{\upharpoonright}\gamma$. Thus $\boldsymbol{q} \in \mathbf{MT}(\vec{\mathbb{\Pi}}{\upharpoonright}\gamma)$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and $\boldsymbol{q}\ \mathtt{forc}_{\vec{\mathbb{\Pi}}{\upharpoonright}\gamma}\ \varphi$, that is, $\boldsymbol{q} \Vdash \varphi[\underline{G}]$ by the inductive hypothesis, contrary to the choice of $\boldsymbol{p}$. Therefore Case 1 cannot happen, and we have:

Case 2: negative decision, no multisequence in U extends $(\vec{\mathbb{\Pi}}{\upharpoonright}\gamma){\upharpoonright\upharpoonright}_{\geq n-2}$. We can assume that $\gamma > \gamma_0$. (Otherwise replace γ by γ_0+1.) We claim that $\boldsymbol{p}\ \mathtt{forc}_\gamma\ \varphi^-$. Indeed otherwise by 3° there is a multisequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\mathbb{\Pi}}{\upharpoonright\upharpoonright}_{<n-2}]$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, such that $\vec{\mathbb{\Pi}}{\upharpoonright}\gamma \subseteq \vec{\boldsymbol{\pi}}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and $\boldsymbol{q}\ \mathtt{forc}_{\vec{\varrho}}\ \varphi$. But then $\vec{\boldsymbol{\pi}}$ and $\boldsymbol{q}$ witness that $\vec{\boldsymbol{\sigma}} = \vec{\boldsymbol{\pi}}{\upharpoonright\upharpoonright}_{\geq n-2}$ belongs to U. On the other hand, $\vec{\boldsymbol{\sigma}}$ obviously extends $\vec{\mathbb{\Pi}}{\upharpoonright}\gamma{\upharpoonright\upharpoonright}_{\geq n-2}$, since $\vec{\mathbb{\Pi}}{\upharpoonright}\gamma \subseteq \vec{\boldsymbol{\pi}}$, contrary to the Case 2 assumption. Thus indeed $\boldsymbol{p}\ \mathtt{forc}_\gamma\ \varphi^-$, as required. □

The next corollary provides a useful strengthening.

Corollary 28.4. *If $n < \omega$, Φ is a Δ_1^{HC} collection of closed $\mathscr{L}\Pi^1_{n+2}$ formulas, $\boldsymbol{p}_0 \in \mathbb{P}$, and $\boldsymbol{p}_0$ $\mathbb{P}$-forces $\varphi[\underline{G}]$ over* $\mathbf{L}$ *for each $\varphi \in \Phi$, then there is an ordinal $\gamma < \omega_1$ such that if $\varphi \in \Phi$ then $\boldsymbol{p}_0\ \mathtt{forc}_{\vec{\mathbb{\Pi}}{\upharpoonright}\gamma}\ \varphi$.* (Same γ for all φ.)

Proof. Let U consist of all multisequences of the form $\vec{\boldsymbol{\pi}}{\upharpoonright\upharpoonright}_{\geq n}$, where $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\mathbb{\Pi}}{\upharpoonright\upharpoonright}_{<n}]$, and there is a formula $\varphi \in \Phi$ and $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ such that $\boldsymbol{p} \leqslant \boldsymbol{p}_0$ and $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi^-$. It follows from Lemmas 25.3 and 26.6 that U is a $\mathbf{\Sigma}^{\mathrm{HC}}_{n+1}$ set, so by 23.4(ii) there is an ordinal $\gamma < \omega_1$ such that $\vec{\mathbb{\Pi}}{\upharpoonright}\gamma$ n-decides U.

Case 1: $(\vec{\mathbb{\Pi}}{\upharpoonright}\gamma){\upharpoonright\upharpoonright}_{\geq n} \in U$, that is, the multisequence $\vec{\boldsymbol{\pi}} = \vec{\mathbb{\Pi}}{\upharpoonright}\gamma$ satisfies the condition that there exist $\varphi \in \Phi$ and a multitree $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ such that $\boldsymbol{p} \leqslant \boldsymbol{p}_0$ and $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi^-$, and hence $\boldsymbol{p}$ $\mathbb{P}$-forces $\varphi^-[\underline{G}]$ over $\mathbf{L}$ by Theorem 28.3, contrary to the choice of $\boldsymbol{p}_0$. Therefore Case 1 leads to a contradiction.

Case 2: no multisequence in U extends $(\vec{\mathbb{\Pi}}{\upharpoonright}\gamma){\upharpoonright\upharpoonright}_{\geq n}$. We can assume that $\gamma > \gamma_0$. (Otherwise replace γ by γ_0+1.) We claim that γ is as required. Indeed otherwise $\boldsymbol{p}_0\ \mathtt{forc}_{\vec{\mathbb{\Pi}}{\upharpoonright}\gamma}\ \varphi$ *fails* for a formula $\varphi \in \Phi$, thus (3° in Section 26), there is a multisequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\mathbb{\Pi}}{\upharpoonright\upharpoonright}_{<n}]$ and $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ such that $\vec{\mathbb{\Pi}}{\upharpoonright}\gamma \subseteq \vec{\boldsymbol{\pi}}$, $\boldsymbol{p} \leqslant \boldsymbol{p}_0$, and $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi^-$. It follows that $\vec{\boldsymbol{\pi}}{\upharpoonright\upharpoonright}_{\geq n} \in U$. In addition, $\vec{\boldsymbol{\pi}}{\upharpoonright\upharpoonright}_{\geq n}$ extends $(\vec{\mathbb{\Pi}}{\upharpoonright}\gamma){\upharpoonright\upharpoonright}_{\geq n}$ by construction. But this contradicts to the Case 2 assumption. □

29. Tail invariance

Various invariance theorems are very typical for all kinds of forcing. We prove two major invariance theorems on the auxiliary forcing. The first one shows its independence of formally unrelated tails of multisequences $\vec{\boldsymbol{\pi}}$ involved, while the other one (Section 30) explores the permutational invariance.

If $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda} \in \overrightarrow{\mathbf{sMF}}$ and $\gamma < \lambda = \mathtt{dom}\,\vec{\boldsymbol{\pi}}$ then let the γ-*tail* $\vec{\boldsymbol{\pi}}{\upharpoonright}_{\geq\gamma}$ be the restriction $\vec{\boldsymbol{\pi}}{\upharpoonright}[\gamma,\lambda)$ to the ordinal semiinterval $[\gamma,\lambda) = \{\alpha : \gamma \leq \alpha < \lambda\}$. Then the multiforcing $\mathbf{MT}(\vec{\boldsymbol{\pi}}{\upharpoonright}_{\geq\gamma}) = \bigcup^{\mathsf{cw}}_{\gamma\leq\alpha<\lambda} \vec{\boldsymbol{\pi}}(\alpha)$ is open dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}})$ by Corollary 20.2(v)(a). Therefore it can be expected that if $\vec{\varrho}$ is another multisequence of the same length $\lambda = \mathtt{dom}\,\vec{\varrho}$, and $\vec{\varrho}{\upharpoonright}_{\geq\gamma} = \vec{\boldsymbol{\pi}}{\upharpoonright}_{\geq\gamma}$, then the relation $\mathtt{forc}_{\vec{\boldsymbol{\pi}}}$ coincides with $\mathtt{forc}_{\vec{\varrho}}$. And indeed this turns out to be the case (almost).

Theorem 29.1. *Assume that $\vec{\boldsymbol{\pi}}, \vec{\varrho}$ are multisequences in $\overrightarrow{\mathbf{sMF}}$, $\gamma < \lambda = \mathtt{dom}\,\vec{\boldsymbol{\pi}} = \mathtt{dom}\,\vec{\varrho}$, $\vec{\varrho}{\upharpoonright}_{\geq\gamma} = \vec{\boldsymbol{\pi}}{\upharpoonright}_{\geq\gamma}$, $\boldsymbol{p} \in \mathbf{MT}$, and φ is an $\mathscr{L}$-formula. Then*

(i) *if $\varphi \in \mathscr{L}(\Sigma\Pi)^1_1$ then $\boldsymbol{p}\ \mathtt{wforc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ iff $\boldsymbol{p}\ \mathtt{wforc}_{\vec{\varrho}}\ \varphi$;*
(ii) *if $n \geq 2$, $\vec{\boldsymbol{\pi}}, \vec{\varrho} \in \overrightarrow{\mathbf{sMF}}[\vec{\mathbb{\Pi}}{\upharpoonright\upharpoonright}_{<n-2}]$, and $\varphi \in \mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$, then $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ iff $\boldsymbol{p}\ \mathtt{forc}_{\vec{\varrho}}\ \varphi$.*

Proof. (i) Suppose to the contrary that $\boldsymbol{p}\ \mathtt{wforc}_{\vec{\boldsymbol{\pi}}}\ \varphi$, but $\boldsymbol{p}\ \mathtt{wforc}_{\vec{\varrho}}\ \varphi$ fails, so there is a multisequence $\vec{\varrho}' \in \overrightarrow{\mathbf{sMF}}$ and $\boldsymbol{p}' \in \mathbf{MT}(\vec{\varrho}')$ such that $\vec{\varrho} \subset \vec{\varrho}'$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\varrho}'}\ \neg\,\varphi$. Let $\lambda' = \mathtt{dom}\,\vec{\varrho}'$. By Corollary 20.2(v)(a), there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\vec{\varrho}'{\upharpoonright}_{\geq\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}'$. Then still $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r}\ \mathtt{forc}_{\vec{\varrho}'}\ \neg\,\varphi$, by Lemma 26.4.

Define a multisequence $\vec{\boldsymbol{\pi}}'$ so that $\mathtt{dom}\,\vec{\boldsymbol{\pi}}' = \lambda' = \mathtt{dom}\,\vec{\mathfrak{P}}'$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$, and $\vec{\boldsymbol{\pi}}'\restriction_{\geq\lambda} = \vec{\mathfrak{P}}'\restriction_{\geq\lambda}$. Then $\boldsymbol{r} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}')$, and $\boldsymbol{r}\ \mathtt{wforc}_{\vec{\boldsymbol{\pi}}'}\ \varphi$ by Lemma 26.4.

Consider any CTM $\mathfrak{N} \models \mathbf{ZFL}^-$ containing φ, $\vec{\boldsymbol{\pi}}'$, $\vec{\mathfrak{P}}'$. Then, by Corollary 27.1, one and the same multitree $\boldsymbol{r}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$-forces $\varphi[\underline{G}]$ but $\mathbf{MT}(\vec{\mathfrak{P}}')$-forces $\neg\,\varphi[\underline{G}]$ over $\mathfrak{N}$. But this contradicts to the fact that the forcing notions $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$, $\mathbf{MT}(\vec{\mathfrak{P}}')$ contain one and the same dense set $\mathbf{MT}(\vec{\boldsymbol{\pi}}'\restriction_{\geq\lambda}) = \mathbf{MT}(\vec{\mathfrak{P}}'\restriction_{\geq\lambda})$.

(ii)**, the** $\mathscr{L}\Pi^1_2$ **case.** Let $\varphi(x)$ be a $\mathscr{L}\Sigma^1_1$ formula, $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \forall\,x\,\varphi(x)$. Suppose to the contrary that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\mathfrak{P}}}\ \forall\,x\,\varphi(x)$ fails, so, by 3° (with $n = 2$) of § 26, there is a multisequence $\vec{\mathfrak{P}}' \in \overrightarrow{\mathbf{sMF}}$ and a multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\mathfrak{P}}')$ such that $\vec{\mathfrak{P}} \subseteq \vec{\mathfrak{P}}'$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\mathfrak{P}}'}\ \exists\,x\,\varphi^-(x)$. By definition there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\mathfrak{P}}'}\ \varphi^-(\mathbf{c})$. There is a multitree $\boldsymbol{r} \in \mathbf{MT}(\vec{\mathfrak{P}}'\restriction_{\geq\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}'$. Then still $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r}\ \mathtt{forc}_{\vec{\mathfrak{P}}'}\ \varphi^-(\mathbf{c})$, hence $\boldsymbol{r}\ \mathtt{wforc}_{\vec{\mathfrak{P}}'}\ \varphi^-(\mathbf{c})$ as well by Lemma 27.2(ii). As above there is a multisequence $\vec{\boldsymbol{\pi}}'$ such that $\mathtt{dom}\,\vec{\boldsymbol{\pi}}' = \lambda' = \mathtt{dom}\,\vec{\mathfrak{P}}'$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$, and $\vec{\boldsymbol{\pi}}'\restriction_{\geq\lambda} = \vec{\mathfrak{P}}'\restriction_{\geq\lambda}$. Then $\boldsymbol{r} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}')$ and $\boldsymbol{r}\ \mathtt{wforc}_{\vec{\boldsymbol{\pi}}'}\ \varphi^-(\mathbf{c})$ by Claim (i) already established. By Lemma 27.2(iii), there is a multisequence $\vec{\boldsymbol{\sigma}}$ such that $\vec{\boldsymbol{\pi}}' \subset \vec{\boldsymbol{\sigma}}$ and $\boldsymbol{r}\ \mathtt{forc}_{\vec{\boldsymbol{\sigma}}}\ \varphi^-(\mathbf{c})$, hence, $\boldsymbol{r}\ \mathtt{forc}_{\vec{\boldsymbol{\sigma}}}\ \exists\,x\,\varphi^-(x)$. But this contradicts to $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \forall\,x\,\varphi(x)$ by 3°of § 26, since $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\pi \subset \pi' \subset \sigma$.

(ii)**, the step** $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$**,** $n \geq 2$**.** Let $\varphi(x)$ be a formula in $\mathscr{L}\Pi^1_n$. Assume that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \exists\,x\,\varphi(x)$. By definition (see 2° in Section 26), there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi(\mathbf{c})$. Then we have $\boldsymbol{p}\ \mathtt{forc}_{\vec{\mathfrak{P}}}\ \varphi(\mathbf{c})$ by the inductive assumption, thus $\boldsymbol{p}\ \mathtt{forc}_{\vec{\mathfrak{P}}}\ \exists\,x\,\psi(x)$.

(ii)**, the step** $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$**,** $n \geq 3$**.** Assume that φ is a $\mathscr{L}\Pi^1_n$ formula, $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$, but to the contrary $\boldsymbol{p}\ \mathtt{forc}_{\vec{\mathfrak{P}}}\ \varphi$ fails. Then by 3° of Section 26, as $\vec{\mathfrak{P}} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\restriction\restriction_{<n-2}]$, there is a multisequence $\vec{\mathfrak{P}}' \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\restriction\restriction_{<n-2}]$ and a multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\mathfrak{P}}')$ such that $\vec{\mathfrak{P}} \subseteq \vec{\mathfrak{P}}'$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\ \mathtt{forc}_{\vec{\mathfrak{P}}'}\ \varphi^-$. By Corollary 20.2(v)(a), there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\vec{\mathfrak{P}}'\restriction_{\geq\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}'$. Then still $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r}\ \mathtt{forc}_{\vec{\mathfrak{P}}'}\ \varphi^-$. As above in the proof of (i), there is a multisequence $\vec{\boldsymbol{\pi}}'$ such that $\mathtt{dom}\,\vec{\boldsymbol{\pi}}' = \lambda' = \mathtt{dom}\,\vec{\mathfrak{P}}'$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$, and $\vec{\boldsymbol{\pi}}'\restriction_{\geq\lambda} = \vec{\mathfrak{P}}'\restriction_{\geq\lambda}$.

We claim that $\vec{\boldsymbol{\pi}}' \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\restriction\restriction_{<n-2}]$. Indeed if $\alpha < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$ then $\vec{\boldsymbol{\pi}}'(\alpha) = \vec{\boldsymbol{\pi}}(\alpha)$ (as $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$), hence $\vec{\boldsymbol{\pi}}'(\alpha)\restriction\restriction_{<m-2} = \vec{\boldsymbol{\pi}}(\alpha)\restriction\restriction_{<m-2} = \sqcap_\alpha\restriction\restriction_{<m-2}$ (as $\vec{\boldsymbol{\pi}}$ belongs to $\overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\restriction\restriction_{<n-2}]$). If $\mathtt{dom}\,\vec{\boldsymbol{\pi}} \leq \alpha < \mathtt{dom}\,\vec{\boldsymbol{\pi}}'$ then $\vec{\boldsymbol{\pi}}'(\alpha) = \vec{\mathfrak{P}}'(\alpha)$ (as $\vec{\boldsymbol{\pi}}'\restriction_{\geq\lambda} = \vec{\mathfrak{P}}'\restriction_{\geq\lambda}$), hence $\vec{\boldsymbol{\pi}}'(\alpha)\restriction\restriction_{<m-2} = \vec{\mathfrak{P}}'(\alpha)\restriction\restriction_{<m-2} = \sqcap_\alpha\restriction\restriction_{<m-2}$ (as $\vec{\mathfrak{P}}'$ belongs to $\overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\restriction\restriction_{<n-2}]$). Thus $\vec{\boldsymbol{\pi}}'(\alpha)\restriction\restriction_{<m-2} = \sqcap_\alpha\restriction\restriction_{<m-2}$ for all α, meaning that $\vec{\boldsymbol{\pi}}'\restriction\restriction_{<m-2} \subset \vec{\sqcap}\restriction\restriction_{<m-2}$ and $\vec{\boldsymbol{\pi}}' \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\restriction\restriction_{<n-2}]$. To conclude, $\vec{\boldsymbol{\pi}}' \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\restriction\restriction_{<n-2}]$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$, $\boldsymbol{r} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}'\restriction_{\geq\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}$, and also $\boldsymbol{r}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}'}\ \varphi^-$ by the inductive hypothesis. But this contradicts to the assumption $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$. □

30. Permutations

Still **arguing in L**, we let PERM be the set of all bijections $\boldsymbol{h} : \omega_1 \times \omega \xrightarrow{\text{onto}} \omega_1 \times \omega$, such that the *non-identity domain* $\mathbf{NID}(\boldsymbol{h}) = \{\langle\xi,k\rangle : \boldsymbol{h}(\xi,k) \neq \langle\xi,k\rangle\}$ is at most countable. Elements of PERM will be called *permutations*. If $m < \omega$ then let PERM_m consist of those permutations $\boldsymbol{h} \in \mathrm{PERM}$ satisfying $\mathbf{NID}(\boldsymbol{h}) \subseteq \omega_1 \times (\omega \smallsetminus m)$, in other words, $\boldsymbol{h}(\xi,k) = \langle\xi,k\rangle$ for all $\xi < \omega_1$, $k < m$.

Let $\boldsymbol{h} \in \mathrm{PERM}$. We extend the action of $\boldsymbol{h}$ as follows.

- if $\boldsymbol{p}$ is a multitree then $\boldsymbol{hp}$ is a multitree, $|\boldsymbol{hp}| = \boldsymbol{h}``\boldsymbol{p} = \{\boldsymbol{h}(\xi,k) : \langle\xi,k\rangle \in |\boldsymbol{p}|\}$, and $(\boldsymbol{hp})(\boldsymbol{h}(\xi,k)) = \boldsymbol{p}(\xi,k)$ whenever $\langle\xi,k\rangle \in |\boldsymbol{p}|$, in other words, $\boldsymbol{hp}$ coincides with the superposition $\boldsymbol{p} \circ (\boldsymbol{h}^{-1})$;
- if $\boldsymbol{\pi} \in \mathbf{MT}$ is a multiforcing then $\boldsymbol{h} \cdot \boldsymbol{\pi} = \boldsymbol{\pi} \circ (\boldsymbol{h}^{-1})$ is a multiforcing, $|\boldsymbol{h} \cdot \boldsymbol{\pi}| = \boldsymbol{h}``\boldsymbol{\pi}$ and $(\boldsymbol{h} \cdot \boldsymbol{\pi})(\boldsymbol{h}(\xi,k)) = \boldsymbol{\pi}(\xi,k)$ whenever $\langle\xi,k\rangle \in |\boldsymbol{\pi}|$;
- if $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times \omega)$ is a real name, then put $\boldsymbol{h}\mathbf{c} = \{\langle\boldsymbol{hp},n,i\rangle : \langle\boldsymbol{p},n,i\rangle \in \mathbf{c}\}$, thus easily $\boldsymbol{h}\mathbf{c}$ is a real name as well;
- if $\vec{\boldsymbol{\pi}} = \langle\boldsymbol{\pi}_\alpha\rangle_{\alpha<\kappa}$ is a multisequence, then $\boldsymbol{h}\vec{\boldsymbol{\pi}} = \langle\boldsymbol{h} \cdot \boldsymbol{\pi}_\alpha\rangle_{\alpha<\kappa}$, still a multisequence.
- if $\varphi := \varphi(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ is a $\mathscr{L}$-formula (with all real names explicitly indicated), then $\boldsymbol{h}\varphi$ is $\varphi(\boldsymbol{h}\mathbf{c}_1, \ldots, \boldsymbol{h}\mathbf{c}_n)$.

Many notions and relations defined above are clearly PERM-invariant, *e.g.*, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$ iff $\boldsymbol{hp} \in \mathbf{MT}(\boldsymbol{h} \cdot \boldsymbol{\pi})$, $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$ iff $\boldsymbol{h} \cdot \boldsymbol{\pi} \sqsubset \boldsymbol{h} \cdot \boldsymbol{\varphi}$, *et cetera.* The invariance also takes place with respect to the relation $\texttt{forc}$, at least to some extent.

Theorem 30.1. *Assume that* $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$, $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, φ *is an* $\mathscr{L}$*-formula, and* $\boldsymbol{h} \in \mathrm{PERM}$. *Then*

(i) *if* φ *belongs to* $\mathscr{L}(\Sigma\Pi)^1_1$ *and* $\boldsymbol{p}\ \texttt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$, *then* $(\boldsymbol{hp})\ \texttt{wforc}_{\boldsymbol{h}\vec{\boldsymbol{\pi}}}\ (\boldsymbol{h}\varphi)$;
(ii) *if* $n \ge 2$, $\boldsymbol{h} \in \mathrm{PERM}_{n-2}$, *and* φ *belongs to* $\mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$, *then* $\boldsymbol{p}\ \texttt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ *iff* $(\boldsymbol{hp})\ \texttt{forc}_{\boldsymbol{h}\vec{\boldsymbol{\pi}}}\ (\boldsymbol{h}\varphi)$.

Proof. Let $\vec{\boldsymbol{\varphi}} = \boldsymbol{h}\vec{\boldsymbol{\pi}}$, $\boldsymbol{q} = \boldsymbol{hp}$, $\psi := \boldsymbol{h}\varphi$.

(i) Suppose to the contrary that $\boldsymbol{p}\ \texttt{wforc}_{\vec{\boldsymbol{\pi}}}\ \varphi$, but $\boldsymbol{q}\ \texttt{wforc}_{\vec{\boldsymbol{\varphi}}}\ \psi$ fails, so that there is a multisequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{sMF}}$ and $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subset \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\ \texttt{forc}_{\vec{\boldsymbol{\varphi}}'}\ \neg\psi$. The multisequence $\vec{\boldsymbol{\pi}}' = \boldsymbol{h}^{-1}\vec{\boldsymbol{\varphi}}'$ then satisfies $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\pi}}'$, and the multitree $\boldsymbol{p}' = \boldsymbol{h}^{-1}\boldsymbol{q}'$ belongs to $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$ and $\boldsymbol{p}' \leqslant \boldsymbol{p}$, hence we have $\boldsymbol{p}'\ \texttt{wforc}_{\vec{\boldsymbol{\pi}}'}\ \varphi$ by Lemma 26.4.

Now let $\mathfrak{M} \models \mathbf{ZFL}^-$ be an arbitrary CTM containing $\vec{\boldsymbol{\pi}}', \vec{\boldsymbol{\varphi}}', \varphi, \psi, \boldsymbol{h}\restriction|\boldsymbol{h}|$. Then, by Corollary 27.4, $\boldsymbol{p}'$ $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$-forces $\varphi[\underline{G}]$, but $\boldsymbol{q}'$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}}')$-forces $\psi[\underline{G}]$, over $\mathfrak{M}$. However the sets $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$, $\mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ belong to the same model $\mathfrak{M}$, where they are order-isomorphic via the isomorphism induced by $\boldsymbol{h}\restriction|\boldsymbol{h}|$. Therefore, and since $\boldsymbol{q} = \boldsymbol{hp}$ and $\psi = \boldsymbol{h}\varphi$, it cannot happen that both $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$-forces $\varphi[\underline{G}]$ and $\boldsymbol{q}$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}}')$-forces $\neg\psi[\underline{G}]$. But this contradicts to the above.

(ii)**, the** $\mathscr{L}\Pi^1_2$ **case.** Assume that $\varphi(x)$ is a $\mathscr{L}\Sigma^1_1$ formula, $\psi(x) := \boldsymbol{h}\varphi(x)$, $\boldsymbol{p}\ \texttt{forc}_{\vec{\boldsymbol{\pi}}}\ \forall x\,\varphi(x)$, but to the contrary $\boldsymbol{q}\ \texttt{forc}_{\vec{\boldsymbol{\varphi}}}\ \forall x\,\psi(x)$ fails. Thus there is a multisequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{sMF}}$ and a multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subset \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\ \texttt{forc}_{\vec{\boldsymbol{\varphi}}'}\ \exists x\,\psi^-(x)$. By definition there is a small real name $\mathbf{d}$ such that $\boldsymbol{q}'\ \texttt{forc}_{\vec{\boldsymbol{\varphi}}'}\ \psi^-(\mathbf{d})$. The multisequence $\vec{\boldsymbol{\pi}}' = \boldsymbol{h}^{-1}\vec{\boldsymbol{\varphi}}'$ then satisfies $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varphi}}$, the multitree $\boldsymbol{p}' = \boldsymbol{h}^{-1}\boldsymbol{q}'$ belongs to $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$ and $\boldsymbol{p}' \leqslant \boldsymbol{p}$, $\mathbf{c} = \boldsymbol{h}^{-1}\mathbf{d}$ is a small real name, and we have $\boldsymbol{p}'\ \texttt{wforc}_{\vec{\boldsymbol{\pi}}'}\ \varphi^-(\mathbf{c})$ by (i). Then by Lemma 27.2 there is a longer multisequence $\vec{\boldsymbol{\sigma}} \in \overrightarrow{\mathbf{sMF}}$ satisfying $\vec{\boldsymbol{\pi}}' \subset \vec{\boldsymbol{\sigma}}$ and $\boldsymbol{p}'\ \texttt{forc}_{\vec{\boldsymbol{\sigma}}}\ \varphi^-(\mathbf{c})$, that is, we have $\boldsymbol{p}'\ \texttt{forc}_{\vec{\boldsymbol{\sigma}}}\ \exists x\,\varphi^-(x)$. But by definition (3° in Section 26) this contradicts to the assumption $\boldsymbol{p}\ \texttt{forc}_{\vec{\boldsymbol{\pi}}}\ \forall x\,\varphi(x)$.

(ii)**, the step** $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$**,** $n \ge 2$**.** Let $\varphi(x)$ be a formula in $\mathscr{L}\Pi^1_n$, $\psi(x) := \boldsymbol{h}\varphi(x)$, and $\boldsymbol{h} \in \mathrm{PERM}_{n-2}$. Assume that $\boldsymbol{p}\ \texttt{forc}_{\vec{\boldsymbol{\pi}}}\ \exists x\,\varphi(x)$. By definition (see 2° in Section 26), there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\ \texttt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi(\mathbf{c})$. Then we have $\boldsymbol{q}\ \texttt{forc}_{\vec{\boldsymbol{\varphi}}}\ \psi(\mathbf{d})$ by inductive assumption, where $\mathbf{d} = \boldsymbol{h}\mathbf{c}$ is a small real name itself. Thus $\boldsymbol{q}\ \texttt{forc}_{\vec{\boldsymbol{\varphi}}}\ \exists x\,\psi(x)$.

(ii)**, the step** $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$**,** $n \ge 3$**.** Let φ be a formula in $\mathscr{L}\Pi^1_n$, and $\boldsymbol{h} \in \mathrm{PERM}_{n-2}$. Let $\boldsymbol{p}\ \texttt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$, in particular $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$, but, to the contrary, $\boldsymbol{q}\ \texttt{forc}_{\vec{\boldsymbol{\varphi}}}\ \psi$ fails, where $\boldsymbol{q} = \boldsymbol{hp}$, $\vec{\boldsymbol{\varphi}} = \boldsymbol{h}\vec{\boldsymbol{\pi}}$, and ψ is $\boldsymbol{h}\varphi$, as above. Then in our assumptions, $\vec{\boldsymbol{\varphi}}\upharpoonright\upharpoonright_{<n-2} = \vec{\boldsymbol{\pi}}\upharpoonright\upharpoonright_{<n-2}$, hence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$ as well. Therefore by definition (3° in Section 26) there is a multisequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$ and $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\ \texttt{forc}_{\vec{\boldsymbol{\varphi}}'}\ \psi^-$.

Now let $\boldsymbol{p}' = \boldsymbol{h}^{-1}\boldsymbol{q}'$ and $\vec{\boldsymbol{\pi}}' = \boldsymbol{h}^{-1}\vec{\boldsymbol{\varphi}}'$, so that $\boldsymbol{p}' \leqslant \boldsymbol{p}$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$, and, that is most important, $\vec{\boldsymbol{\pi}}'$ belongs to $\overrightarrow{\mathbf{sMF}}[\vec{\sqcap}\upharpoonright\upharpoonright_{<n-2}]$ since so does $\vec{\boldsymbol{\varphi}}'$ and $\boldsymbol{h}^{-1} \in \mathrm{PERM}_{n-2}$. Moreover we have $\boldsymbol{p}'\ \texttt{forc}_{\vec{\boldsymbol{\pi}}'}\ \varphi^-$ by inductive assumption. We conclude that $\boldsymbol{p}\ \texttt{forc}_{\vec{\boldsymbol{\pi}}}\ \varphi$ fails, which is a contradiction. □

VI. The model

In this conclusive section we gather the results obtained above towards the proof of Theorem 1.1. We begin with the analysis of definability of key generic reals in $\mathbb{P}$-generic extensions of $\mathbf{L}$, which will lead to (I) and (II) of Theorem 1.1. Then we proceed to (III) (elementary equivalence) and (IV) (the non-wellorderability).

31. Key generic extension and subextensions

Recall that the key multisequence $\vec{\sqcap} = \langle \sqcap_\alpha \rangle_{\alpha<\omega_1}$ of small multiforcings $\sqcap_\alpha$ is defined in $\mathbf{L}$ by 23.4, the componentwise union $\sqcap = \bigcup^{\mathsf{cw}}_{\alpha<\omega_1} \sqcap_\alpha$ is a multiforcing, $|\sqcap| = \omega_1 \times \omega$ in $\mathbf{L}$, and $\mathbb{P} = \mathbf{MT}(\vec{\sqcap}) = \mathbf{MT}(\sqcap) \in \mathbf{L}$ is our key forcing notion, equal to the finite-support product $\prod_{\xi<\omega_1, k<\omega} \sqcap(\xi, k)$ of perfect-tree forcings $\sqcap(\xi, k)$ in $\mathbf{L}$. See Section 24, where some properties of $\mathbb{P}$ are established, including CCC and definability of the factors $\sqcap(\xi, k)$ in $\mathbf{L}$.

Remark 31.1. From now on, we'll typically argue in $\mathbf{L}$ and in $\mathbb{P}$-generic extensions of $\mathbf{L}$, so by Lemma 24.6 it will always be true that $\omega_1^{\mathbf{L}} = \omega_1$. This allows us to still think that $|\sqcap| = \omega_1 \times \omega$ (rather than $\omega_1^{\mathbf{L}} \times \omega$). □

Definition 31.2. Let a set $G \subseteq \mathbb{P}$ be generic over the constructible set universe $\mathbf{L}$. If $\langle \xi, k\rangle \in \omega_1 \times \omega$ then following Remark 7.4, we

- define $G(\xi, k) = \{T^{\boldsymbol{p}}_{\xi k} : \boldsymbol{p} \in G \wedge \langle \xi, k\rangle \in |\boldsymbol{p}|\} \subseteq \sqcap(\xi, k)$;
- let $x_{\xi k} = x_{\xi k}[G] \in 2^\omega$ be the only real in $\bigcap_{T \in G(\xi,k)} [T]$.

Thus $\mathbb{P}$ adjoins an array $\boldsymbol{X} = \boldsymbol{X}[G] = \langle x_{\xi k}\rangle_{\langle \xi,k\rangle \in \omega_1\times\omega}$ of reals to $\mathbf{L}$, where each $x_{\xi k} = x_{\xi k}[G] \in 2^\omega \cap \mathbf{L}[G]$ is a $\sqcap(\xi, k)$-generic real over $\mathbf{L}$, and $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{X}]$.

Define a subarray $\boldsymbol{X}_m = \boldsymbol{X}_m[G] = \langle x_{\xi k}[G]\rangle_{\xi<\omega_1 \wedge k<m}$ for each m. □

Let $G \subseteq \mathbb{P}$ be a set (filter) $\mathbb{P}$-generic over $\mathbf{L}$. If $m < \omega$ then following the notation in Section 21 we define

$$G{\restriction\restriction}_{<m} = G \cap \mathbf{MT}{\restriction\restriction}_{<m} = \{\boldsymbol{p}{\restriction\restriction}_{<m} : \boldsymbol{p} \in G\}\,,$$

so that the set $G{\restriction\restriction}_{<m}$ is $\mathbb{P}{\restriction\restriction}_{<m}$-generic over $\mathbf{L}$, where accordingly

$$\mathbb{P}{\restriction\restriction}_{<m} = \mathbb{P} \cap \mathbf{MT}{\restriction\restriction}_{<m} = \{\boldsymbol{p}{\restriction\restriction}_{<m} : \boldsymbol{p} \in \mathbb{P}\}\,.$$

Each subextension $\mathbf{L}[G{\restriction\restriction}_{<m}] \subseteq \mathbf{L}[G]$ coincides with $\mathbf{L}[\boldsymbol{X}_m]$. Our goal will be to demonstrate that the model $\mathbf{L}[\boldsymbol{X}] = \mathbf{L}[G]$, proves Theorem 1.1.

32. Definability of generic reals

Recall that the factors $\sqcap(\xi, k)$ of the forcing notion $\sqcap$ are defined by $\sqcap(\xi, k) = \bigcup_{\alpha(\xi,k)\le\alpha<\omega_1} \sqcap_\alpha(\xi, k)$, where $\alpha(\xi, k) < \omega_1$, and the sets $\sqcap_\alpha(\xi, k)$ are countable sets of perfect trees, whose definability in $\mathbf{L}$ is determined by Corollary 24.3. We'll freely use the notation introduced by Definition 31.2.

Theorem 32.1. *Assume that a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$, $\xi < \omega_1$, $k < \omega$, and $x \in \mathbf{L}[G] \cap 2^\omega$. The following are equivalent:*

(1) $x = x_{\xi k}[G]$;
(2) *x is $\sqcap(\xi, k)$-generic over $\mathbf{L}$;*
(3) $x \in \bigcap_{\alpha(\xi,k)\le\alpha<\omega_1} \bigcup_{T\in\sqcap_\alpha(\xi,k)} [T]$.

Proof. (1) $\Longrightarrow$ (2) is a routine (see Remark 7.4). To check (2) $\Longrightarrow$ (3) recall that each set $\sqcap_\alpha(\xi, k)$ is pre-dense in $\sqcap(\xi, k)$ by Lemma 9.2(v). It remains to establish (3) $\Longrightarrow$ (1). Suppose towards the contrary that a real $x \in \mathbf{L}[G] \cap 2^\omega$ satisfies (3) but $x \neq x_{\xi k}[G]$. By Theorem 18.2(i) there is a $\sqcap$-real name $\mathbf{c} = \langle C_{ni}\rangle_{n,i<\omega}$, non-principal over Π at ξ, k and such that $x = \mathbf{c}[G]$. Being non-principal means that the following set is open dense in $\mathbb{P} = \mathbf{MT}(\Pi)$:

$$\boldsymbol{D}^{\Pi}_{\xi k}(\mathbf{c}) = \{\boldsymbol{p} \in \mathbb{P} = \mathbf{MT}(\Pi) : \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T^{\boldsymbol{p}}_{\xi k}]\}\,.$$

And as $\mathbb{P} = \mathbf{MT}(\Pi)$ is a CCC forcing by Lemma 24.6, we can assume that the name $\mathbf{c}$ is small, that is, each set $C_{ni} \subseteq \mathbb{P}$ is countable. Then there is an ordinal $\gamma_0 < \omega_1$ such that $C_{ni} \subseteq \mathbb{P}_{<\gamma_0}$ for all n, i. Then $\mathbf{c}$ is a $\Pi_{<\gamma_0}$-real name. Moreover we can assume by Corollary 24.7 that $\boldsymbol{D}^{\Pi}_{\xi k}(\mathbf{c}) \cap \mathbb{P}_{<\gamma_0}$ is pre-dense in $\mathbb{P}$.

Now consider the set W of all multisequences $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha < \mathtt{dom}(\vec{\boldsymbol{\pi}})} \in \overrightarrow{\mathbf{sMF}}$ such that $\mathtt{dom}(\vec{\boldsymbol{\pi}}) > \gamma_0$ and

- either (I) $\vec{\Pi} \restriction \gamma_0 \not\subset \vec{\boldsymbol{\pi}}$;
- or (II) $\vec{\Pi} \restriction \gamma_0 \subset \vec{\boldsymbol{\pi}}$ and $\mathbf{c}$ is **not** non-principal over $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}} \vec{\boldsymbol{\pi}}$ at ξ, k ;
- or (III) $\vec{\Pi} \restriction \gamma_0 \subset \vec{\boldsymbol{\pi}}$, $\mathtt{dom}(\vec{\boldsymbol{\pi}}) = \delta + 1$ is a successor, and $\bigcup^{\mathsf{cw}}_{\alpha<\delta} \boldsymbol{\pi}_\alpha \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\pi}_\delta$.

We assert that W is *dense* in $\overrightarrow{\mathbf{sMF}}$: any multisequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}$ can be extended to some $\vec{\boldsymbol{\varrho}} \in W$. Indeed first extend $\vec{\boldsymbol{\pi}}$ by Corollary 20.2 so that is has a length $\mathtt{dom}(\vec{\boldsymbol{\pi}}) = \delta > \gamma_0$. If now $\vec{\Pi} \restriction \gamma_0 \not\subset \vec{\boldsymbol{\pi}}$ then immediately $\vec{\boldsymbol{\pi}} \in W$ via (I), so we assume that $\vec{\Pi} \restriction \gamma_0 \subset \vec{\boldsymbol{\pi}}$. We can also assume that $\mathbf{c}$ is non-principal over $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}} \vec{\boldsymbol{\pi}}$ at ξ, k by similar reasons related to (II). The multisequence $\vec{\boldsymbol{\pi}}$ can be extended, by Corollary 20.2, by an extra term $\boldsymbol{\pi}_\delta$, so that the extended multisequence $\vec{\boldsymbol{\pi}}_+$ satisfies $\vec{\boldsymbol{\pi}} \subset_{\{\mathbf{c}\}} \vec{\boldsymbol{\pi}}_+$, that is, $\boldsymbol{\pi} \sqsubset_{\{\mathbf{c}\}} \boldsymbol{\pi}_\delta$. By Definition 19.1 and the nonprincipality of $\mathbf{c}$, we get $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi k} \boldsymbol{\pi}_\delta$. Therefore $\vec{\boldsymbol{\pi}}_+ \in W$ via (III).

Since W is $\boldsymbol{\Sigma}^{\mathrm{HC}}_1$, by Definition 23.4(ii) there is an ordinal $\gamma < \omega_1$ such that the multisequence $\vec{\Pi} \restriction \gamma$ 0-decides W. However the negative decision is impossible by the density (see the proof of Lemma 24.2). We conclude that $\vec{\Pi} \restriction \gamma \in W$; hence, $\gamma > \gamma_0$. Option (I) for $\vec{\boldsymbol{\pi}} = \vec{\Pi} \restriction \gamma$ clearly fails, and (II) fails either because the set $\boldsymbol{D}^{\Pi}_{\xi k}(\mathbf{c}) \cap \mathbb{P}_{<\gamma_0}$ is pre-dense in $\mathbb{P}$ and $\gamma > \gamma_0$. Therefore $\vec{\Pi} \restriction \gamma$ belongs to W via (III), that is, $\gamma = \delta + 1$ and $\Pi_{<\delta} = \bigcup^{\mathsf{cw}}_{\alpha<\delta} \Pi_\alpha \sqsubset^{\mathbf{c}}_{\xi k} \Pi_\delta$. Then $\Pi_{<\delta} \sqsubset^{\mathbf{c}}_{\xi k} \Pi_{\geq\delta} = \bigcup^{\mathsf{cw}}_{\delta \leq \alpha < \omega_1} \Pi_\delta$ by Lemma 16.3(iii).

Now Theorem 18.2(ii) with $\boldsymbol{\pi} = \Pi_{<\delta}$ and $\boldsymbol{\varrho} = \Pi_{\geq\delta}$ (note that $\boldsymbol{\pi} \cup^{\mathsf{cw}} \boldsymbol{\varrho} = \Pi$) implies $x = \mathbf{c}[G] \notin \bigcup_{Q \in \Pi_{\geq\delta}(\xi,k)} [Q]$, which contradicts to the Assumption (3). □

Corollary 32.2. *Assume that $k < \omega$ and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$. Then*

$$\mathbb{W}_k = \{\langle \xi, x_{\xi k}[G] \rangle : \xi < \omega_1\} \subseteq \omega_1 \times 2^\omega$$

is a set of definability class Π^{HC}_{k+2} in $\mathbf{L}[G]$ and in any transitive model $M \models \mathbf{ZFC}$ satisfying $\mathbf{L} \subseteq M \subseteq \mathbf{L}[G]$ and $\{x_{\xi k}[G] : \xi < \omega_1\} \subseteq M$.

Proof. By the theorem, it is true in $\mathbf{L}[G]$ that $\langle \xi, x \rangle \in \mathbb{W}_k$ iff

$$\forall\, \alpha < \omega_1\, \exists T \in \Pi_\alpha(\xi, k) \big(\alpha(\xi, k) \leq \alpha \implies x \in [T]\big),$$

which can be re-written as

$$\forall\, \alpha < \omega_1\, \forall\, \mu < \omega_1\, \forall\, Y\, \exists T \in Y \big(\mu = \alpha(\xi, k) \wedge Y = \Pi_\alpha(\xi, k) \wedge \mu \leq \alpha \implies x \in [T]\big).$$

Here the equality $\mu = \alpha(\xi, k)$ (with a fixed k) is $\Delta^{\mathrm{HC}}_{k+2}$ by Corollary 24.3, and so is the equality $Y = \Pi_\alpha(\xi, k)$ by Corollary 24.3. It follows that the whole relation is Π^{HC}_{k+2}, since the quantifier $\exists T \in Y$ is bounded. □

The next corollary is the first cornerstone in the proof of Theorem 1.1.

Corollary 32.3 *(= (I), (II) of Theorem 1.1). Assume that $m<\omega$ and a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$. Then $\omega^\omega \cap \mathbf{L}[G{\restriction\restriction}_{<m}]$ is a Σ^1_{m+3} set in $\mathbf{L}[G]$, and it holds in $\mathbf{L}[G{\restriction\restriction}_{<m}]$ that there is a Δ^1_{m+3} wellordering of ω^ω of length ω_1.*

Proof. If $\gamma<\omega_1$ then let $\boldsymbol{X}_{\gamma m} = \langle x_{\xi k}[G]\rangle_{\xi<\gamma \wedge k<m}$. The equality $Y=\boldsymbol{X}_{\gamma m}$ is a Π^{HC}_{m+1} relation in $\mathbf{L}[G]$ (with γ, Y as arguments) by Corollary 32.2. Therefore

$$\omega^\omega \cap \mathbf{L}[G{\restriction\restriction}_{<m}] = \{x \in \omega^\omega : \exists\,\gamma<\omega_1 (x \in \mathbf{L}[\boldsymbol{X}_{\gamma m}])\}$$

is a set in $\Sigma^{\mathrm{HC}}_{m+2}$, hence, a Σ^1_{m+3} set in $\mathbf{L}[G]$. If $x \in \omega^\omega \cap \mathbf{L}[G{\restriction\restriction}_{<m}]$ then let $\gamma(x)$ be the least $\gamma<\omega_1$ such that $x \in \mathbf{L}[\boldsymbol{X}_{\gamma m}]$, and $\nu(x)<\omega_1$ be the index of x in the canonical wellordering of ω^ω in $\mathbf{L}[\boldsymbol{X}_{\gamma m}]$. We wellorder $\omega^\omega \cap \mathbf{L}[G{\restriction\restriction}_{<m}]$ according to the lexicographical ordering of the triples $\langle \mathtt{max}\{\gamma(x),\nu(x)\},\gamma(x),\nu(x)\rangle$. □

33. Elementary equivalence

Here we prove the following elementary equivalence theorem for key generic extensions. The result is essentially (III) of Theorem 1.1.

Theorem 33.1. *Assume that $m<\omega$ and a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$. Then $\mathbf{L}[G{\restriction\restriction}_{<m}]$ is an elementary submodel of $\mathbf{L}[G]$ w.r.t. all Σ^1_{m+2} formulas with real parameters in $\mathbf{L}[G{\restriction\restriction}_{<m}]$.*

Proof. Suppose that this is not the case. Then there is a Π^1_{m+1} formula $\varphi(r,x)$ with $r \in \omega^\omega \cap \mathbf{L}[G{\restriction\restriction}_{<m}]$ as the only parameter, and a real $x_0 \in \omega^\omega \cap \mathbf{L}[G]$ such that $\varphi(r,x_0)$ is true in $\mathbf{L}[G]$ but there is no $x \in \omega^\omega \cap \mathbf{L}[G{\restriction\restriction}_{<m}]$ such that $\varphi(r,x)$ is true in $\mathbf{L}[G]$. By a version of Proposition 18.1(ii), we have $r=\mathbf{c}_0[G]$, where $\mathbf{c}_0$ is a small $(\mathbb{P}{\restriction\restriction}_{<m})$-real name. (See Section 31 on notation.) And there is a small $\mathbb{P}$-real name $\mathbf{c}$ such that $x_0=\mathbf{c}[\underline{G}]$.

By Theorem 28.3, there is a multitree $\boldsymbol{p}_0 \in G$ such that

(1) $\boldsymbol{p}_0$ $\mathbb{P}$-forces '$\varphi(\mathbf{c}_0[\underline{G}],\mathbf{c}[\underline{G}]) \wedge \neg\,\exists\,x \in \mathbf{L}[\underline{G}{\restriction\restriction}_{<m}]\,\varphi(\mathbf{c}_0[\underline{G}],x)$' over $\mathbf{L}$;
(2) $\boldsymbol{p}_0$ $\mathtt{forc}$ $\varphi(\mathbf{c}_0,\mathbf{c})$, that is, $\boldsymbol{p}_0$ $\mathtt{forc}_{\vec{\sqcap}\restriction\gamma_0}$ $\varphi(\mathbf{c}_0,\mathbf{c})$, where $\gamma_0<\omega_1$ — and we can assume that $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\sqcap}\restriction\gamma_0)$ as well.

As $\mathbf{c},\mathbf{c}_0$ are small real names, there is an ordinal $\delta<\omega_1$ satisfying

(3) $|\mathbf{c}_0| \subseteq \delta\times m$, $|\mathbf{c}| \subseteq \delta\times\omega$, and $|\boldsymbol{p}_0| \subseteq \delta\times\omega$,

and we can enlarge γ_0, if necessary, using the equality $|\vec{\sqcap}|=\omega_1\times\omega$ of Lemma 24.2, to make sure that

(4) $\delta\times\omega \subseteq |\vec{\sqcap}\restriction\gamma_0|$, that is, if $\eta<\delta$ and $k<\omega$ then $\langle\eta,k\rangle \in |\sqcap_{\alpha'}|$ for some $\alpha'=\alpha'(\eta,k)<\gamma_0$.

We start from here towards a contradiction.

Let U consist of all multisequences of the form $\vec{\boldsymbol{\pi}}{\restriction\restriction}_{\geq m}$, where

(A) $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}{\restriction\restriction}_{<m}]$, $\vec{\sqcap}\restriction\gamma_0 \subset \vec{\boldsymbol{\pi}}$, and hence $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ by (2);

and there is an ordinal $\zeta<\omega_1$ and a transformation $\boldsymbol{h} \in \mathrm{PERM}_{m-1}$ such that

(B) $\boldsymbol{h}=\boldsymbol{h}^{-1}$, $\mathbf{NID}(\boldsymbol{h})=D\cup R$, and $\boldsymbol{h}$ maps D onto R and R onto D, where $D=\delta\times[m,\omega)$, $R=\{\langle\xi,m-1\rangle : \nu_0 \leq \xi < \nu_1\}$, and $\delta<\nu_0<\nu_1<\omega_1$;

(C) $\gamma_0 \le \zeta < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$ and $(\boldsymbol{h}\vec{\boldsymbol{\pi}})\restriction_{\ge\zeta} = \vec{\boldsymbol{\pi}}\restriction_{\ge\zeta}$, or equivalently $\boldsymbol{h}(\vec{\boldsymbol{\pi}}(\alpha)) = \vec{\boldsymbol{\pi}}(\alpha)$ whenever $\zeta \le \alpha < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$.

It follows from Lemma 25.3 that U is a $\mathbf{\Sigma}^{\mathrm{HC}}_{m+1}$ set (with $\vec{\mathbb{\Pi}}\restriction\gamma_0$, δ as parameters). Therefore by 23.4(ii) there is an ordinal $\gamma < \omega_1$ such that $\vec{\mathbb{\Pi}}\restriction\gamma$ m-decides U.

Case 1: $(\vec{\mathbb{\Pi}}\restriction\gamma){\restriction\restriction}_{\ge m} \in U$. Basically this means that there is a transformation $\boldsymbol{h} \in \mathrm{PERM}_{m-1}$ such that (A), (B), (C) hold for $\boldsymbol{h}$ and $\vec{\boldsymbol{\pi}} = \vec{\mathbb{\Pi}}\restriction\gamma$, via ordinals $\delta < \nu_0 < \nu_1$ and $\gamma_0 < \zeta < \gamma$ as in (B), (C).

By Lemma 26.4 and (2), we have $\boldsymbol{p}_0\ \mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\ \varphi(\mathbf{c}_0, \mathbf{c})$. We further get, by Theorem 30.1, $\boldsymbol{h}\boldsymbol{p}_0\ \mathtt{forc}_{\boldsymbol{h}\cdot(\vec{\mathbb{\Pi}}\restriction\gamma)}\ \varphi(\boldsymbol{h}\mathbf{c}_0, \boldsymbol{h}\mathbf{c})$, because φ is a $\mathscr{L}\Pi^1_{n+1}$ formula and $\boldsymbol{h}$ belongs to PERM_{m-1}. However $\boldsymbol{h}\mathbf{c}_0 = \mathbf{c}_0$ since $|\mathbf{c}_0| \cap \mathbf{NID}(\boldsymbol{h}) = \varnothing$ by (B). Thus $\boldsymbol{p}'_0\ \mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\ \varphi(\mathbf{c}_0, \mathbf{c}')$ holds by Theorem 29.1 and (C), where $\mathbf{c}' = \boldsymbol{h}\mathbf{c}$, $\boldsymbol{p}'_0 = \boldsymbol{h}\boldsymbol{p}_0$. Note that the common part $|\boldsymbol{p}_0| \cap |\boldsymbol{p}'_0|$ of the domains of $\boldsymbol{p}_0, \boldsymbol{p}'_0$ does not intersect $\mathbf{NID}(\boldsymbol{h})$ by (B) since $|\boldsymbol{p}_0| \subseteq \delta \times \omega$ by (3). It follows that $\boldsymbol{p}_0, \boldsymbol{p}'_0$ are compatible, basically $\boldsymbol{p} = \boldsymbol{p}_0 \cup \boldsymbol{p}'_0$ is a multitree in $\mathbf{MT}(\vec{\mathbb{\Pi}}\restriction\gamma)$. Thus $\boldsymbol{p} \leqslant \boldsymbol{p}'_0$ and still $\boldsymbol{p}\ \mathtt{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\ \varphi(\mathbf{c}_0, \mathbf{c}')$. It follows by Theorem 28.3 that

(5) $\boldsymbol{p}$ $\mathbb{P}$-forces $\varphi(\mathbf{c}_0[\underline{G}], \mathbf{c}'[\underline{G}])$ over $\mathbf{L}$.

However $|\mathbf{c}'| \subseteq \omega_1 \times m$ by construction because $|\mathbf{c}| \subseteq \delta \times \omega$ by (3), and hence $\mathbf{c}'[\underline{G}] \in \mathbf{L}[\underline{G}{\restriction\restriction}_{<m}]$ is forced. Thus $\boldsymbol{p}$ $\mathbb{P}$-forces $\exists\, x \in \mathbf{L}[\underline{G}{\restriction\restriction}_{<m}]\, \varphi(\mathbf{c}_0[\underline{G}], x)$ over $\mathbf{L}$ by (5), contrary to (1). The contradiction closes Case 1.

Case 2: negative decision, no multisequence in U extends $(\vec{\mathbb{\Pi}}\restriction\gamma){\restriction\restriction}_{\ge m}$. We can assume that $\gamma > \gamma_0$. (Otherwise replace γ by $\gamma_0 + 1$.) Let ν_0 be the lest ordinal, bigger than δ and satisfying $|\vec{\mathbb{\Pi}}\restriction\gamma| \subseteq \nu_0 \times \omega$. Let $\nu_1 = \nu_0 + \omega$. Then countable sets $D = \delta \times [m, \infty)$ and R as in (B) are defined and $D \cap R = \varnothing$, so we can fix a transformation $\boldsymbol{h} \in \mathrm{PERM}_{m-1}$ satisfying (B). Note that $D \subseteq \delta \times \omega \subseteq |\vec{\mathbb{\Pi}}\restriction\gamma|$ by (4) but $R \cap |\vec{\mathbb{\Pi}}\restriction\gamma| = \varnothing$ by the choice of ν_0.

Pick $\lambda < \omega_1$ such that $\lambda > \gamma > \gamma_0$. Then the multisequence $\vec{\text{ϙ}} = \vec{\mathbb{\Pi}}\restriction\lambda$ clearly satisfies (A), (B) and extends $\vec{\mathbb{\Pi}}\restriction\gamma$. Our plan is now to slightly modify $\vec{\text{ϙ}}$ in order to fulfill (C) as well, with $\zeta = \gamma$. Such a minor modification consists in the replacement of the R-part of $\vec{\text{ϙ}}$ above γ by the $\boldsymbol{h}$-copy of its D-part.

To present this in detail, recall that $\vec{\text{ϙ}} = \vec{\mathbb{\Pi}}\restriction\lambda = \langle \mathbb{\Pi}_\alpha \rangle_{\alpha<\lambda}$, where each $\mathbb{\Pi}_\alpha$ is a small multiforcing, whose domain $d_\alpha = |\mathbb{\Pi}_\alpha| \subseteq \omega_1 \times \omega$ is countable. If $\alpha < \gamma$ then put $\boldsymbol{\pi}_\alpha = \mathbb{\Pi}_\alpha$. Suppose that $\gamma \le \alpha < \lambda$. Then $D \subseteq |\mathbb{\Pi}_\alpha|$ by (4). Define a modified multiforcing $\boldsymbol{\pi}_\alpha$ such that

(a) $|\boldsymbol{\pi}_\alpha| = d_\alpha \cup R$ — note that $D \subseteq d_\alpha \subseteq |\boldsymbol{\pi}_\alpha|$ in this case because $D \subseteq |\vec{\mathbb{\Pi}}\restriction\gamma|$ by (4) (as $\gamma_0 \le \gamma$), and hence $D \subseteq d_\alpha = |\mathbb{\Pi}_\alpha|$ (as $\alpha \ge \gamma$),

(b) if $\langle \xi, k \rangle \in d_\alpha \smallsetminus R$ then $\boldsymbol{\pi}_\alpha(\xi, k) = \mathbb{\Pi}_\alpha(\xi, k)$,

(c) if $\langle \xi, k \rangle \in D$, so $\boldsymbol{h}(\xi, k) = \langle \eta, m-1 \rangle \in R$, then $\boldsymbol{\pi}_\alpha(\eta, m-1) = \mathbb{\Pi}_\alpha(\xi, k)$.

We claim that $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ is a multisequence, that is, if $\alpha < \beta < \lambda$ then $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\pi}_\beta$. This amounts to the following: if $\langle \eta, k \rangle \in |\boldsymbol{\pi}_\alpha|$ then $\boldsymbol{\pi}_\alpha(\eta, k) \sqsubset \boldsymbol{\pi}_\beta(\eta, k)$. Note that $\boldsymbol{\pi}_\alpha(\eta, k) = \mathbb{\Pi}_\alpha(\eta, k)$ in case $\langle \eta, k \rangle \notin R$.

Thus it remains to check that $\boldsymbol{\pi}_\alpha(\eta, m-1) \sqsubset \boldsymbol{\pi}_\beta(\eta, m-1)$ whenever $\alpha < \beta < \lambda$, $\langle \eta, m-1 \rangle = \boldsymbol{h}(\xi, k) \in R \cap |\boldsymbol{\pi}_\alpha|$, and $\langle \xi, k \rangle \in D$. If now $\alpha < \gamma$ then $R \cap |\boldsymbol{\pi}_\alpha| = \varnothing$ by the choice of ν_0, so it remains to consider the case when $\gamma \le \alpha$. Then the pairs $\langle \xi, k \rangle$, $\langle \eta, m-1 \rangle$ belong to $|\boldsymbol{\pi}_\alpha|$ by construction, and we have $\boldsymbol{\pi}_\alpha(\eta, m-1) = \mathbb{\Pi}_\alpha(\xi, k)$ and $\boldsymbol{\pi}_\beta(\eta, m-1) = \mathbb{\Pi}_\beta(\xi, k)$. Therefore $\boldsymbol{\pi}_\alpha(\xi, m) \sqsubset \boldsymbol{\pi}_\beta(\xi, m)$ since $\vec{\mathbb{\Pi}}$ is a multisequence, and we are done.

Now we claim that the multisequence $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ satisfies (A), (B), (C). Indeed as the difference between each $\boldsymbol{\pi}_\alpha$ and the corresponding $\mathbb{\Pi}_\alpha$ is fully located in the domain $R = \{\langle \xi, m-1 \rangle : \nu_0 \le \xi < \nu_1\}$, we have $\vec{\boldsymbol{\pi}}{\restriction\restriction}_{<m-1} = \vec{\text{ϙ}}{\restriction\restriction}_{<m-1}$, therefore $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\mathbb{\Pi}}{\restriction\restriction}_{<m}]$. We also note that $\vec{\boldsymbol{\pi}}\restriction\gamma = \vec{\text{ϙ}}\restriction\gamma$ by construction, hence $\vec{\mathbb{\Pi}}\restriction\gamma = \vec{\text{ϙ}}\restriction\gamma \subset \vec{\boldsymbol{\pi}}$. This implies (A).

We also have (B) by construction. We finally claim that (C) is satisfied with $\zeta = \gamma$, that is, if $\gamma \le \alpha < \lambda$ then $\boldsymbol{h} \cdot \boldsymbol{\pi}_\alpha = \boldsymbol{\pi}_\alpha$. Indeed we have $D \cup R \subseteq |\boldsymbol{\pi}_\alpha|$, see (a). Now the invariance of $\boldsymbol{\pi}_\alpha$ under $\boldsymbol{h}$ holds by (b), (c).

It follows that $\vec{\pi}{\restriction\restriction}_{\geq m} \in U$. In addition, $\vec{\pi}{\restriction\restriction}_{\geq m}$ extends $(\vec{\sqcap}{\restriction}\gamma){\restriction\restriction}_{\geq m}$, since $\vec{\sqcap}{\restriction}\gamma \subset \vec{\pi}$. But this contradicts to the Case 2 assumption.

To conclude, either case leads to a contradiction, proving the theorem. □

34. Non-wellorderability

We finally prove that the reals are not wellorderable by a (lightface) analytically definable relation in $\mathbb{P}$-generic extensions, that is, (IV) of Theorem 1.1.

Theorem 34.1. *Assume that $m < \omega$ and a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$. Then it is true in $\mathbf{L}[G]$ that the reals are not wellorderable by an analytically definable relation.*

Proof. Suppose to the contrary that, in $\mathbf{L}[G]$, a Σ^1_{m+2} relation $\ll$ strictly wellorders ω^ω, $m \geq 1$. Let $\psi(x, y)$ be a parameter-free Σ^1_{m+2} formula, which defines $\ll$, so that $x \ll y$ iff $\psi(x, y)$ in $\mathbf{L}[G]$. Note that $\ll$ is essentially a Δ^1_{m+2} relation, since $x \ll y \iff y \not\ll x \wedge x \neq y$.

Of all nonconstructible reals $x_{\xi m}[G]$, $\xi < \omega_1$, there is a $\ll$-least one. We suppose that $x_{0m}[G]$ is such. (If it is some $x_{\xi_0 m}[G]$, $\xi_0 \neq 0$, then the arguments suitably change in obvious way.) That is, $x_{0m}[G] \ll x_{\xi m}[G]$ whenever $\xi > 0$. Accordingly there is a multitree $\boldsymbol{p}_0 \in G \subseteq \mathbb{P}$ that $\mathbb{P}$-forces, over $\mathbf{L}$, that

(i) $\ll$ (that is, the relation defined by ψ) is a wellordering of ω^ω, and
(ii) $\forall \xi > 0\, (x_{0m}[\underline{G}] \ll x_{\xi m}[\underline{G}])$.

Therefore, if $\xi > 0$ then $\boldsymbol{p}_0$ $\mathbb{P}$-forces $(\dot{\boldsymbol{x}}_{\xi m}[\underline{G}] \ll \dot{\boldsymbol{x}}_{0m}[\underline{G}])^-$ over $\mathbf{L}$. (We make use of the real names $\dot{\boldsymbol{x}}_{\xi k}$ introduced by 13.6, 13.7.) By Corollary 28.4, there is an ordinal $\gamma_1 < \omega_1$, such that if $\xi < \omega_1$ then $\boldsymbol{p}_0 \ \mathtt{forc}_{\vec{\sqcap}\restriction\gamma_1} (\dot{\boldsymbol{x}}_{\xi m} \ll \dot{\boldsymbol{x}}_{0m})^-$. We can enlarge γ_1, if necessary, using Lemma 24.2, to make sure that 1) $\langle 0, m\rangle \in |\vec{\sqcap}{\restriction}\gamma_1|$, that is, $\langle 0, m\rangle \in |\sqcap_{\alpha'}|$ for some $\alpha' < \gamma_1$, and 2) $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\sqcap}{\restriction}\gamma_1) = \mathbf{MT}(\sqcap_{<\gamma_1})$; $\sqcap_{<\gamma_1} = \bigcup^{\mathsf{cw}}_{\xi<\gamma_1} \sqcap_\xi$ is a small multiforcing.

Let $\delta < \omega_1$ be the least ordinal satisfying $|\sqcap_{<\gamma_1}| \subseteq \delta \times \omega$; then $|\boldsymbol{p}_0| \subseteq \delta \times \omega$.

The remainder of the proof is somewhat similar to the proof of Theorem 33.1. If $\xi < \omega_1$ then let $\boldsymbol{h}_\xi \in \mathrm{PERM}_m$ be the permutation of $\langle 0, m\rangle$ and $\langle \xi, m\rangle$, so that $\mathbf{NID}(\boldsymbol{h}_\xi) = \{\langle 0, m\rangle, \langle \xi, m\rangle\}$, $\boldsymbol{h}_\xi(0, m) = \langle \xi, m\rangle$, $\boldsymbol{h}_\xi(\xi, m) = \langle 0, m\rangle$, $\boldsymbol{h}_\xi(\eta, n) = \langle \eta, n\rangle$ for any pair $\langle \eta, n\rangle$ different from both $\langle 0, m\rangle$ and $\langle \xi, m\rangle$. Let U consist of all multisequences of the form $\vec{\pi}{\restriction\restriction}_{\geq m}$, where

(A) $\vec{\pi} \in \overrightarrow{\mathbf{sMF}}[\vec{\sqcap}{\restriction\restriction}_{<m}]$, $\vec{\sqcap}{\restriction}\gamma_1 \subset \vec{\pi}$, and hence $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\pi})$ by construction;

and there exist ordinals $\xi, \zeta < \omega_1$ such that

(B) $\delta < \xi < \mathtt{dom}\, \vec{\pi}$;
(C) $\gamma_1 \leq \zeta < \mathtt{dom}\, \vec{\pi}$ and $(\boldsymbol{h}_\xi \vec{\pi}){\restriction}_{\geq\zeta} = \vec{\pi}{\restriction}_{\geq\zeta}$ (a common tail!), or equivalently $\boldsymbol{h}_\xi \cdot (\vec{\pi}(\alpha)) = \vec{\pi}(\alpha)$ whenever $\zeta \leq \alpha < \mathtt{dom}\, \vec{\pi}$.

It follows from Lemma 25.3 that U is a $\mathbf{\Sigma}^{\mathrm{HC}}_{m+1}$ set (with $\vec{\sqcap}{\restriction}\gamma_1$ as a parameter). Therefore by 23.4(ii) there is an ordinal $\gamma < \omega_1$ such that $\vec{\sqcap}{\restriction}\gamma$ m-decides U.

Case 1: $(\vec{\sqcap}{\restriction}\gamma){\restriction\restriction}_{\geq m} \in U$. Basically this means that $\gamma > \gamma_1$ and there are ordinals $\xi < \omega_1$ and $\zeta < \gamma$ such that (A), (B), (C) hold for ξ and the multisequence $\vec{\pi} = \vec{\sqcap}{\restriction}\gamma$. By Lemma 26.4 and the choice of γ_1, $\boldsymbol{p}_0 \ \mathtt{forc}_{\vec{\sqcap}\restriction\gamma} (\dot{\boldsymbol{x}}_{\xi m} \ll \dot{\boldsymbol{x}}_{0m})^-$. This implies $\boldsymbol{p}'_0 \ \mathtt{forc}_{\boldsymbol{h}_\xi(\vec{\sqcap}\restriction\gamma)} (\dot{\boldsymbol{x}}_{0m} \ll \dot{\boldsymbol{x}}_{\xi m})^-$ by Theorem 30.1, where $\boldsymbol{p}'_0 = \boldsymbol{h}_\xi \boldsymbol{p}_0$, because $(\dot{\boldsymbol{x}}_{0m} \ll \dot{\boldsymbol{x}}_{\xi m})^-$ is a $\mathscr{L}\Pi^1_{m+2}$ formula and $\boldsymbol{h}_\xi$ belongs to PERM_m. We conclude that $\boldsymbol{p}'_0 \ \mathtt{forc}_{\vec{\sqcap}\restriction\gamma} (\dot{\boldsymbol{x}}_{0m} \ll \dot{\boldsymbol{x}}_{\xi m})^-$ by (C) and Theorem 29.1.

Note that $\langle \xi, m\rangle \notin |\boldsymbol{p}_0|$ by (B). It follows by the definition of $\boldsymbol{h}_\xi$ that $\boldsymbol{p}_0, \boldsymbol{p}_0'$ are compatible, basically $\boldsymbol{p} = \boldsymbol{p}_0 \cup \boldsymbol{p}_0'$ is a multitree in $\mathbf{MT}(\vec{\Pi}\upharpoonright\gamma)$ and $\boldsymbol{p} \leqslant \boldsymbol{p}_0$, $\boldsymbol{p} \leqslant \boldsymbol{p}_0'$. Thus both $\boldsymbol{p}\ \mathtt{forc}_{\vec{\Pi}\upharpoonright\gamma}\ (\dot{\boldsymbol{x}}_{0m} \ll \dot{\boldsymbol{x}}_{\xi m})^-$ and $\boldsymbol{p}\ \mathtt{forc}_{\vec{\Pi}\upharpoonright\gamma}\ (\dot{\boldsymbol{x}}_{\xi m} \ll \dot{\boldsymbol{x}}_{0m})^-$ hold. Therefore $\boldsymbol{p}$ $\mathbb{P}$-forces both $(x_{0m}[\underline{G}] \ll x_{\xi m}[\underline{G}])^-$ and $(x_{\xi m}[\underline{G}] \ll x_{0m}[\underline{G}])^-$ over $\mathbf{L}$ by Theorem 28.3. This is a contradiction since $\boldsymbol{p}_0$, a weaker condition, forces $\ll$ to be a wellordering. The contradiction closes Case 1.

Case 2: negative decision, no multisequence in U extends $(\vec{\Pi}\upharpoonright\gamma)\upharpoonright\upharpoonright_{\geq m}$. We can assume that $\gamma > \gamma_1$ (otherwise replace γ by $\gamma_1 + 1$). The set $d = |\vec{\Pi}\upharpoonright\gamma| = \bigcup_{\alpha<\gamma} |\Pi_\alpha| \subseteq \omega_1 \times \omega$ is countable, hence there is an ordinal ξ, $\delta < \xi < \omega_1$, such that $\langle \xi, m\rangle \notin d$. Finally pick an ordinal λ, $\gamma < \lambda < \omega_1$. Then $\vec{\qoppa} = \vec{\Pi}\upharpoonright\lambda$ (as $\vec{\boldsymbol{\pi}}$) and ξ clearly satisfy (A) and (B), and $\vec{\qoppa}$ extends $\vec{\Pi}\upharpoonright\gamma$. Let's modify $\vec{\qoppa}$ a little bit, in order to fulfill (C) as well.

As above, $\vec{\qoppa} = \vec{\Pi}\upharpoonright\lambda = \langle \Pi_\alpha\rangle_{\alpha<\lambda}$, each Π_α is a small multiforcing, and its domain $d_\alpha = |\Pi_\alpha| \subseteq \omega_1 \times \omega$ is countable. If $\alpha < \gamma$ then put $\boldsymbol{\pi}_\alpha = \Pi_\alpha$. Suppose that $\gamma \leq \alpha < \lambda$. Then $\alpha \geq \gamma_1$, and hence $\langle 0, m\rangle \in |\Pi_\alpha|$ by the choice of γ_1. Define a modified multiforcing $\boldsymbol{\pi}_\alpha$ such that $|\boldsymbol{\pi}_\alpha| = d_\alpha \cup \{\langle \xi, m\rangle\}$, if $\langle \eta, k\rangle \in d_\alpha \smallsetminus \{\langle \xi, m\rangle\}$ then $\boldsymbol{\pi}_\alpha(\eta, k) = \Pi_\alpha(\eta, k)$, and finally $\boldsymbol{\pi}_\alpha(\xi, m) = \Pi_\alpha(0, m)$.

We claim that $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha\rangle_{\alpha<\lambda}$ is a multisequence, that is, if $\alpha < \beta < \lambda$ then $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\pi}_\beta$. This amounts to the following: if $\langle \eta, k\rangle \in |\boldsymbol{\pi}_\alpha|$ then $\boldsymbol{\pi}_\alpha(\eta, k) \sqsubset \boldsymbol{\pi}_\beta(\eta, k)$. Note that $\boldsymbol{\pi}_\alpha(\eta, k) = \Pi_\alpha(\eta, k)$ whenever $\langle \eta, k\rangle \neq \langle \xi, m\rangle$. Thus it remains to check that $\boldsymbol{\pi}_\alpha(\xi, m) \sqsubset \boldsymbol{\pi}_\beta(\xi, m)$ given $\alpha < \beta < \lambda$ such that $\langle \xi, m\rangle \in |\boldsymbol{\pi}_\alpha|$. If $\alpha < \gamma$ then $|\boldsymbol{\pi}_\alpha| = |\Pi_\alpha| = d_\alpha$ by construction, and hence $\langle \xi, m\rangle \notin d_\alpha$ by the choice of ξ. It remains to consider the case $\gamma \leq \alpha < \lambda$. Then $\langle 0, m\rangle \in d_\alpha$ (see above), hence the pairs $\langle 0, m\rangle$, $\langle \xi, m\rangle$ belong to $|\boldsymbol{\pi}_\alpha|$ by construction, and then obviously belong to $|\boldsymbol{\pi}_\beta|$ as $\alpha < \beta$. Now $\boldsymbol{\pi}_\alpha(\xi, m) = \Pi_\alpha(0, m)$ and $\boldsymbol{\pi}_\beta(\xi, m) = \Pi_\beta(0, m)$, and we have $\boldsymbol{\pi}_\alpha(\xi, m) \sqsubset \boldsymbol{\pi}_\beta(\xi, m)$ since $\vec{\Pi}$ is a multisequence.

Now we claim that the multisequence $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha\rangle_{\alpha<\lambda}$ satisfies (A), (B), (C) with $\zeta = \gamma$. If $\alpha < \lambda$ then the difference between Π_α and $\boldsymbol{\pi}_\alpha$ is located in the one-element domain $\{\langle \xi, m\rangle\}$, therefore $\boldsymbol{\pi}_\alpha\upharpoonright\upharpoonright_{<m} = \Pi_\alpha\upharpoonright\upharpoonright_{<m}$. It follows that $\vec{\boldsymbol{\pi}}\upharpoonright\upharpoonright_{<m} = (\vec{\Pi}\upharpoonright\lambda)\upharpoonright\upharpoonright_{<m}$, hence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{sMF}}[\vec{\Pi}\upharpoonright\upharpoonright_{<m}]$. We further have $\vec{\boldsymbol{\pi}}\upharpoonright\gamma = \vec{\Pi}\upharpoonright\gamma$ by construction. Thus $\vec{\Pi}\upharpoonright\gamma \subset \vec{\boldsymbol{\pi}}$, hence $\vec{\Pi}\upharpoonright\gamma_1 \subset \vec{\boldsymbol{\pi}}$, and we have (A).

We also have (B) and (C) (with $\zeta = \gamma$) by construction.

Thus $\vec{\boldsymbol{\pi}}\upharpoonright\upharpoonright_{\geq m} \in U$. In addition, $\vec{\boldsymbol{\pi}}\upharpoonright\upharpoonright_{\geq m}$ extends $(\vec{\Pi}\upharpoonright\gamma)\upharpoonright\upharpoonright_{\geq m}$, since even more $\vec{\Pi}\upharpoonright\gamma \subset \vec{\boldsymbol{\pi}}$ by construction. But this contradicts to the Case 2 assumption.

To conclude, either case leads to a contradiction, proving the theorem. $\square$

35. Proof of the main theorem

Proof *(Theorem 1.1).* We consider a $\mathbb{P}$-generic extension $\mathbf{L}[G]$ of $\mathbf{L}$ and present it in the form $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{X}]$ as in Section 31, where $\boldsymbol{X} = \langle x_{\xi k}\rangle_{\langle \xi,k\rangle\in\omega_1\times\omega}$, and each $x_{\xi k} = x_{\xi k}[G]$ is a real in $2^\omega \cap \mathbf{L}[G]$. We also consider the subextensions $\mathbf{L}[G\upharpoonright\upharpoonright_{<m}] = \mathbf{L}[\langle x_{\xi k}\rangle_{\xi<\omega_1 \wedge k<m}]$ of $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{X}]$. Then (I) and (II) of Theorem 1.1 hold by Corollary 32.3, (III) holds by Theorem 33.1, and finally (IV) holds by Theorem 34.1. $\square$

36. Problems

It would be interesting to prove that the model $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{X}]$ admits no *projective* (= analytically definable *with real parameters*) wellorderings of the reals. The argument in Section 34 does not go through in this case.

Another problem emerges from the fact that the partial wellorderings defined in the proof of Corollary 32.3 can be glued in $\mathbf{L}[G]$ to form an OD (ordinal-definable) wellordering of the whole set $\omega^\omega \cap \mathbf{L}[G]$. Is there a generic extension of $\mathbf{L}$, in which the full basis theorem (as in Theorem 1.1) holds, but there is no OD wellordering of the reals?

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