

Effective cardinals in the nonstandard universe*

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December 2005

Abstract

We study the structure of effective cardinals in the nonstandard set universe of Hrbaček set theory **HST**. Some results resemble those known in descriptive set theory in the domain of Borel reducibility of equivalence relations.

Introduction

Nonstandard analysis as a domain in mathematics¹ emerged in the beginning of 1960s when A. Robinson [26] demonstrated that nonstandard models (that is, proper elementary extensions) of the real continuum lead to a mathematically rigorous system including infinitesimals and infinitely large numbers. In the course of 1960s, the model theoretic tools used by Robinson were shown to be applicable to variety of mathematical structures, and that such an applicability was based on a few general properties of nonstandard extensions, in particular, elementarity and saturation. For instance any \aleph_1 -saturated elementary extension ${}^*\mathbb{N}$ of the integers \mathbb{N} contains an infinitely large number. Several nonstandard axiomatic systems were proposed, beginning with the mid-1970s, based on those general principles. Unlike the model-theoretic approach, such theories as Nelson's internal set theory [23], two theories of [8, 9], bounded set theory [13], axiomatically described nonstandard extensions of the whole standard set universe of **ZFC** rather than extensions of any particular structure.

In the mid-1990s we formulated *Hrbaček set theory* **HST** [14], based on earlier theories in [8, 9]. This theory accumulated achievements of different

*This project was partially supported by DFG grant 436 RUS 17/68/05.

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¹ See [5, 29] on the precursorial history of *infinitesimal analysis*.

nonstandard set theories and inhibited their faults. The set universe of **HST** is axiomatized as a von Neumann superstructure \mathbb{H} over a fully saturated elementary extension \mathbb{I} (\mathbb{I} = internal sets) of the class \mathbb{WF} of all well-founded sets, see more on this in Section 1. Our monograph [17] presents in detail the structure of the **HST** universe and metamathematical properties of **HST** and some other popular nonstandard set theories.

This paper is devoted to the structure of cardinalities in the nonstandard set universe of **HST**. Note that **HST** does not include the axioms of **Power Set**, **Choice**, and **Regularity**. In fact these axioms contradict **HST**. This is why methods of study of the structure of cardinalities known from **ZFC** are not always applicable in **HST**. Nevertheless there are two rather regular families of cardinalities in **HST**: \mathbb{WF} -cardinals and \mathbb{I} -cardinals. Either family behaves in **ZFC**-like manner simply because both \mathbb{WF} and \mathbb{I} satisfy **ZFC**. The intersection of the two families consists of finite cardinals. But little is known beyond this. Some independence results have been obtained. For instance, the hypothesis that all infinite sets in \mathbb{I} are equinumerous in the whole universe \mathbb{H} , and the hypothesis that \mathbb{I} -cardinals are preserved in \mathbb{H} (except for hyperfinite cardinalities $m < n$ such that $\frac{m}{n}$ is not infinitesimal, [19]) are consistent with **HST**, see [15] or [17], Chapter 7.

Yet an alternative approach seems to be much more promising in the context of **HST**. Instead of abstract “cantorial” cardinalities, we consider here those induced by *effective* embeddings, *i.e.* those definable in some way or given by a certain construction. In this we follow earlier works in nonstandard analysis. For instance studies on collapse of hyperfinite cardinalities by Borel and countably determined maps were carried out in 1980s, see [12, 19, 27]. Further studies revealed a complicated structure of “Borel” and “countably determined” cardinalities of hyperfinite sets [16].

However **HST** admits a much more general concept of effective cardinality than those based on Borel or countably determined maps. This concept involves the class $\mathbb{L}[\mathbb{I}]$ of all sets *constructible over* \mathbb{I} , and the class Δ_2^{ss} of all sets $x \in \mathbb{L}[\mathbb{I}]$, $x \subseteq \mathbb{I}$ (see details below), which includes and greatly exceeds Borel and countably determined sets.

The first part of the paper is devoted to effective cardinalities of internal sets and, generally, sets that consist of internal elements. We prove that effective cardinalities of internal sets are just their \mathbb{I} -cardinals in the \mathbb{I} -infinite domain, and resemble multiplicative galaxies in the hyperfinite domain. Effective cardinalities of Σ_1^{ss} sets (\mathbb{WF} -size unions of internal sets) are still linearly ordered and admit characterization in terms of cuts (initial segments) in the class ${}^*\text{Card}$ of all \mathbb{I} -cardinals. Some results for cardinalities in more complicated classes Π_1^{ss} and Δ_2^{ss} will be presented, too.

The second part of the paper considers effective cardinalities in their generality. Fortunately there is a reduction down to \mathbb{I} : any set in $\mathbb{L}[\mathbb{I}]$ admits an effective bijection onto the quotient structure of the form X/E , where E is a Δ_2^{ss} relation on a Δ_2^{ss} set X (by necessity $X \subseteq \mathbb{I}$).

And this brings us to an analogy with modern descriptive set theory, where cardinality problems for Borel quotient structures in Polish spaces became the focal point since early 1990s — especially in the form of *Borel reducibility* of quotients and the corresponding equivalence relations, see *e.g.* [6, 7, 18]. We pursue essentially the same idea, with Δ_2^{ss} reduction maps in the same role as Borel reductions in descriptive set theory.

Inspired by this analogy, we prove several results related to dichotomy of “large”–“small” sets, a nonstandard form of the Ramsey theorem, a theorem saying that quotients with rather small (for instance countable) classes are “smooth” in a sense similar to the smoothness for quotients in descriptive set theory, and finally consider effective reducibility within the family of *monadic* equivalence relations. Those readers with an experience in descriptive set theory may be interested to recognize similarities and differences with the set-up they are accustomed to.

1 Structure of the nonstandard universe

The language of *Hrbáček set theory* **HST** contains two basic predicates, the membership \in and the standardness st , hence it is called *the st- \in -language*. The axioms of **HST** describe a set universe \mathbb{H} where the following classes are defined,

$$\begin{aligned} \mathbb{S} &= \{x : \text{st } x\} && - \text{ standard sets;} \\ \mathbb{I} &= \{y : \exists^{\text{st}} x (y \in x)\} && - \text{ internal set;} \\ \text{WF} &&& - \text{ well-founded }^2 \text{ sets;} \end{aligned}$$

so that $\mathbb{S} \subseteq \mathbb{I}$, \mathbb{I} is an elementary extension of \mathbb{S} in the \in -language, \mathbb{S} (and \mathbb{I} as well) satisfies **ZFC** in the \in -language, the class \mathbb{I} is transitive, and the universe \mathbb{H} is a von Neumann superstructure over \mathbb{I} . The universe \mathbb{H} satisfies all **ZFC** axioms except for Regularity (weakened to Regularity over \mathbb{I}), Choice (weakened to Standard Size Choice) and Power Set axioms. The axioms of Separation and Replacement are accepted in the $\text{st-}\in$ -language.

Metamathematically, **HST** is equiconsistent with **ZFC**, and **HST** is a conservative extension of **ZFC** in the sense that any \in -formula Φ is a theorem of **ZFC** iff Φ^{st} (the relativization of Φ to \mathbb{S}) is a theorem of **HST**. See [17] on

¹ \exists^{st} and \forall^{st} are shorthands for “there is a standard”, “for all standard”.

² A set x is well-founded iff its transitive closure has no infinite \in -decreasing chains.

axioms, metamathematics, basic set theoretic structures, and the structure of hyperreals in the **HST** universe.

Convention 1.1 We argue in **HST** below unless otherwise stated. \square

Asterisks. An \in -isomorphism $x \mapsto {}^*x$ of \mathbb{WF} onto \mathbb{S} is defined in **HST** so that ${}^*x \cap \mathbb{S} = \{{}^*y : y \in x\}$ for all $x \in \mathbb{WF}$. The map $*$ is an elementary embedding of \mathbb{WF} in \mathbb{I} in the \in -language. The classes \mathbb{S} and \mathbb{WF} are \in -isomorphic and satisfy **ZFC**. Each of them can be unformally identified with the conventional set theoretic universe. The class \mathbb{WF} is somewhat more convenient in this role as it is transitive and contains all its subsets, hence some important set theoretic operations are absolute for \mathbb{WF} in **HST**.

Integers and reals. The sets $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ (integers, rationals, reals) belong to \mathbb{WF} and are equal to resp. $(\mathbb{N})^{\mathbb{WF}}$ (i.e. \mathbb{N} defined in \mathbb{WF}), $(\mathbb{Q})^{\mathbb{WF}}$, $(\mathbb{R})^{\mathbb{WF}}$. In addition ${}^*n = n$ for all $n \in \mathbb{N}$, therefore $\mathbb{N} \subseteq {}^*\mathbb{N}$, moreover \mathbb{N} is an initial segment in ${}^*\mathbb{N}$. The set ${}^*\mathbb{N}$ coincides with the set $(\mathbb{N})^{\mathbb{I}}$ of all \mathbb{I} -natural numbers, similarly ${}^*\mathbb{Q}$ and ${}^*\mathbb{R}$ are equal to, resp., $(\mathbb{Q})^{\mathbb{I}}$ and $(\mathbb{R})^{\mathbb{I}}$. Elements of ${}^*\mathbb{N}, {}^*\mathbb{Q}, {}^*\mathbb{R}$ are often called resp. *hyperintegers*, *hyperrationals*, *hyperreals*.

A hyperreal $x \in {}^*\mathbb{R}$ is *infinitesimal*, $x \simeq 0$ in symbols, if $|x| < {}^*r$ in ${}^*\mathbb{R}$ for all $r \in \mathbb{R}$, $r > 0$, and *infinitely large*, if $x^{-1} \simeq 0$, i.e. $|x| > {}^*r$ for all $r \in \mathbb{R}$. A hyperreal x is *limited*, if it is not infinitely large. In this case there exists a unique $r \in \mathbb{R}$ such that $x \simeq {}^*r$ (that is, $x - {}^*r \simeq 0$). Such a real r is denoted by ${}^\circ x$ (the *shadow*, or standard part, of $x \in {}^*\mathbb{R}$).

Ordinals and cardinals. The operation $*$ extends to proper classes $X \subseteq \mathbb{WF}$ by ${}^*X = \bigcup_{x \in \mathbb{WF}, x \subseteq X} {}^*x$, and this does not yield contradiction provided $X \in \mathbb{WF}$. Then ${}^*\mathbb{WF} = \mathbb{I}$. In **HST**, the classes **Card** and **Ord** (all cardinals, resp., ordinals) satisfy $\mathbf{Card} \subseteq \mathbf{Ord} \subseteq \mathbb{WF}$ and $\mathbf{Ord} = (\mathbf{Ord})^{\mathbb{WF}}$ (that is, ordinals = \mathbb{WF} -ordinals), $\mathbf{Card} = (\mathbf{Card})^{\mathbb{WF}}$. Thus classes ${}^*\mathbf{Card} \subseteq {}^*\mathbf{Ord} \subseteq \mathbb{I}$ are defined (all \mathbb{I} -cardinals, resp., \mathbb{I} -ordinals). Note that ${}^*\mathbb{N} \subseteq {}^*\mathbf{Card}$.

Sets of standard size. Sets equinumerous with sets in \mathbb{WF} are called *sets of standard size*. Note that $\mathbf{card} X \in \mathbf{Card}$ is defined then for any set X of standard size. In **HST**, sets of standard size is the same as well-orderable sets, 1.3.1 in [17]. The axiom of **Saturation** claims that every \cap -closed set $X \subseteq \mathbb{I} \setminus \{\emptyset\}$ of standard size has a non-empty intersection $\bigcap X$. The axiom of **Standard Size Choice** claims the existence of a choice function f for any set X of standard size (i.e. $f(x) \in x$ for all $x \in X$, $x \neq \emptyset$). An easy consequence is the axiom of **Power Set** for sets X of standard size: $\mathcal{P}(X)$ is a set of standard size for any such X . Finite sets are sets of standard size. On the other hand any infinite set $X \in \mathbb{I}$, for instance any set of the form $\{0, 1, 2, \dots, h\}$, where $h \in {}^*\mathbb{N} \setminus \mathbb{N}$, is not a set of standard size.

2 Classes Δ_2^{ss} and $\mathbb{L}[\mathbb{I}]$: effective sets

Which sets should be viewed as effective in **HST**? Following the examples of recursive, Borel, constructible sets, we have to choose an initial class of sets and a set of operations applying to the initial sets. The sets obtained this way are considered as effective. In nonstandard set theoretic systems, internal sets are usually considered as the initial sets, because of their special role in the construction of nonstandard universes. (In particular \mathbb{I} is the von Neumann basis of the **HST** universe of sets.) As for the operations, let us take unions and intersections of families of standard size.

We immediately obtain the classes Σ_1^{ss} , Π_1^{ss} of all sets of the form resp. $\bigcup_{a \in A} X_a$, $\bigcap_{a \in A} X_a$, where $A \in \mathbb{WF}$ and all sets X_a belong to \mathbb{I} , or, that is the same, of the form resp. $\bigcup \mathcal{X}$, $\bigcap \mathcal{X}$, where $\mathcal{X} \subseteq \mathbb{I}$ is a set of standard size. (The index ss indicates that unions and intersections of sets of standard size are taken.) We further define the class Δ_2^{ss} of all sets that can be represented both in the form $\bigcup_{a \in A} \bigcap_{b \in B} X_{ab}$, where $A, B \in \mathbb{WF}$ and all X_{ab} belong to \mathbb{I} , and in the dual form (possibly with different sets A, B, X_{ab}). Note that taking, say, three operations of union and intersection no new sets appear according to the following result (1.4.2, 1.4.3 in [17]).

Proposition 2.1 *If $\mathcal{X} \subseteq \Delta_2^{\text{ss}}$ is a set of standard size then the sets $\bigcup \mathcal{X}$ and $\bigcap \mathcal{X}$ belong to Δ_2^{ss} . In addition, any set $X \subseteq \mathbb{I}$ defined in \mathbb{I} by a $\text{st-}\in$ -formula with sets in \mathbb{I} as parameters belongs to Δ_2^{ss} . \square*

Thus Δ_2^{ss} is a rather large class of sets.³ Yet it consists only of those sets X satisfying $X \subseteq \mathbb{I}$. The class $\mathbb{L}[\mathbb{I}]$ of all sets *constructible over* \mathbb{I} extends Δ_2^{ss} on further levels of the von Neumann hierarchy over \mathbb{I} .

Definition 2.2 $\mathbb{L}[\mathbb{I}]$ consists of all sets x which admit a transfinite construction determined by a well-founded tree T with sets in \mathbb{I} attached to all endpoints of T . The tree T itself and the map which attaches internal sets to the endpoints of T belong to Δ_2^{ss} . In every node t of T that is not an endpoint, the set of all sets, attached to immediate successors of t in T is defined. The final set x is obtained in the root of T . \square

Thus sets in $\mathbb{L}[\mathbb{I}]$ are obtained via effectively coded (in Δ_2^{ss}) transfinite iterations of the operation of assembling of a set from its elements. This

³ There are meaningful subclasses within Δ_2^{ss} , namely *countably determined* sets, i.e. those of the form $X = \bigcup_{b \in B} \bigcap_{n \in \mathbb{N}} X_n$, where $B \subseteq \mathcal{P}(\mathbb{N})$ and all sets X_n are internal (there are different but equivalent formulations), and *Borel* sets that belong to the closure of \mathbb{I} under countable operations of \bigcup and \bigcap . These classes are considered within the model theoretic nonstandard analysis under the assumption of \aleph_1 -Saturation, [19].

enables us to view sets in $\mathbb{L}[\mathbb{I}]$ as effectively definable. Conversely, any effective (uniformly) set belongs to $\mathbb{L}[\mathbb{I}]$. Indeed it follows from theorem 2.3(ii) below that effective constructions have to be absolute for $\mathbb{L}[\mathbb{I}]$, hence the results of such constructions are necessarily sets in $\mathbb{L}[\mathbb{I}]$.

Identifying the unformal notion of effectivity in **HST** with $\mathbb{L}[\mathbb{I}]$, we put

$$\left. \begin{array}{l} x \leq^{\text{eff}} y, \quad \text{iff} \quad \text{there is an injection } f \in \mathbb{L}[\mathbb{I}] \text{ of } x \text{ into } y \\ x \equiv^{\text{eff}} y, \quad \text{iff} \quad \text{there is a bijection } f \in \mathbb{L}[\mathbb{I}] \text{ of } x \text{ onto } y \end{array} \right\}. \quad (1)$$

and $x <^{\text{eff}} y$ iff $x \leq^{\text{eff}} y$ but $y \not\leq^{\text{eff}} x$. The ordinary Cantor – Bernstein argument proves $x \leq^{\text{eff}} y \wedge y \leq^{\text{eff}} x \iff x \equiv^{\text{eff}} y$ for any sets $x, y \in \mathbb{L}[\mathbb{I}]$. Define $|x|^{\text{eff}}$, the *effective cardinality* of $x \in \mathbb{L}[\mathbb{I}]$, to be the \equiv^{eff} -equivalence class $\{y \in \mathbb{L}[\mathbb{I}] : x \equiv^{\text{eff}} y\}$. The inequalities $|x|^{\text{eff}} \leq |y|^{\text{eff}}$ and $|x|^{\text{eff}} < |y|^{\text{eff}}$ will be understood as synonymous to resp. $x \leq^{\text{eff}} y$ and $x <^{\text{eff}} y$.

Theorem 2.3 (i) *If $x \subseteq \mathbb{I}$ then $x \in \Delta_2^{\text{ss}} \iff x \in \mathbb{L}[\mathbb{I}]$.*

(ii) *$\mathbb{L}[\mathbb{I}]$ is a transitive class satisfying **HST**⁴ and $\text{WF} \cup \Delta_2^{\text{ss}} \subseteq \mathbb{L}[\mathbb{I}]$.*

(iii) *For any set $A \in \mathbb{L}[\mathbb{I}]$ there is a set $X \in \mathbb{I}$ and an equivalence relation E on X , $\text{E} \in \Delta_2^{\text{ss}}$, such that $A \equiv^{\text{eff}} X/\text{E}$.*

Proof On (i), (ii) see 5.5.4 in [17] where the class Δ_2^{ss} is denoted by \mathbb{E} .

(iii) According to 5.5.4(8) in [17], there exist a set $X \in \mathbb{I}$ and a map $h \in \mathbb{L}[\mathbb{I}]$, $h : X \xrightarrow{\text{onto}} A$. Define, for $x, y \in X$, $x \text{ E } y$ iff $h(x) = h(y)$, and consider the map $a \in A \mapsto f(a) = \{x \in X : h(x) = a\}$.

Theorem 2.3 allows to suitably replace $\mathbb{L}[\mathbb{I}]$ by Δ_2^{ss} in the context of $|\cdot|^{\text{eff}}$. For instance we conclude from 2.3(i) that (1) is equivalent to the following in the domain of subsets of \mathbb{I} :

$$\left. \begin{array}{l} x \leq^{\text{eff}} y, \quad \text{iff} \quad \text{there is a } \Delta_2^{\text{ss}} \text{ injection } f : x \rightarrow y \\ x \equiv^{\text{eff}} y, \quad \text{iff} \quad \text{there is a } \Delta_2^{\text{ss}} \text{ bijection } f : x \xrightarrow{\text{onto}} y \end{array} \right\} \text{for } x, y \subseteq \mathbb{I}. \quad (2)$$

We begin the study of the structure of effective cardinalities $|\cdot|^{\text{eff}}$ with rather simple classes, internal sets and sets of standard size.

⁴ In fact the least class with these properties. A suitable version of Gödel's definition of relative constructibility leads to exactly the same class $\mathbb{L}[\mathbb{I}]$ in **HST**. See 5.5.6 in [17].

3 Effective cardinalities of internal sets

Generally elements of ${}^*\mathbf{Card}$, that is, $\mathbb{1}$ -cardinals, behave like **ZFC** cardinals since $\mathbb{1}$ is a **ZFC** universe (in the \in -language). Let $|x|^{\mathbf{int}} \in {}^*\mathbf{Card}$ denote the $\mathbb{1}$ -cardinality of a set $x \in \mathbb{1}$. Obviously $|x|^{\mathbf{int}} = |y|^{\mathbf{int}}$ implies $|x|^{\mathbf{eff}} = |y|^{\mathbf{eff}}$ since $\mathbb{1} \subseteq \mathbf{\Delta}_2^{\mathbf{ss}}$. This implication is partially reversible according to Corollary 3.2 below. To figure out the effect of non-internal maps in the domain of internal sets, let us give some definitions. Define, for any x ,

$$\left. \begin{aligned} \|x\|_* &= \{ |y|^{\mathbf{int}} : x \supseteq y \in \mathbb{1} \} && \text{the interior spectrum of } x \\ \|x\|^* &= \{ |y|^{\mathbf{int}} : x \subseteq y \in \mathbb{1} \} && \text{the exterior spectrum of } x \end{aligned} \right\}. \quad (3)$$

Then $\|x\|_*$ is a *cut* (initial segment) in ${}^*\mathbf{Card}$ while $\|x\|^*$ is a proper class and a final segment in ${}^*\mathbf{Card}$. Further, for any $\kappa \in {}^*\mathbb{N}$ define the cuts

$$\kappa\mathbb{N} = \{ \lambda \in {}^*\mathbb{N} : \exists n \in \mathbb{N} (\lambda < n\kappa) \}, \quad \kappa/\mathbb{N} = \{ \lambda \in {}^*\mathbb{N} : \forall n \in \mathbb{N} (\lambda < \kappa/n) \}$$

in ${}^*\mathbb{N}$, and the *multiplicative galaxy* $\mathbf{gal} \kappa = \kappa\mathbb{N} \setminus \kappa/\mathbb{N}$ of κ . Then $\lambda \in \mathbf{gal} \kappa$ iff neither of the fractions $\frac{\kappa}{\lambda}, \frac{\lambda}{\kappa}$ is infinitesimal. To preserve the unity of notation put $\kappa\mathbb{N} = \kappa$, $\mathbf{gal} \kappa = \{ \kappa \}$ for any $\kappa \in {}^*\mathbf{Card} \setminus {}^*\mathbb{N}$.

Define, for $K, L \subseteq {}^*\mathbf{Card}$, $K \leq L$ iff $\forall \kappa \in K \exists \lambda \in L (\kappa \leq \lambda)$. Accordingly, $K < L$ iff $K \leq L$ but $L \not\leq K$. In particular, in two cases when one of the sets K, L is a singleton, we obtain

$$\kappa \leq L \text{ iff } \exists \lambda \in L (\kappa \leq \lambda), \quad \text{and} \quad K < \lambda \text{ iff } \forall \kappa \in K (\kappa < \lambda). \quad (4)$$

Note that galaxies are pairwise disjoint intervals in ${}^*\mathbf{Card}$ (singletons outside of ${}^*\mathbb{N}$), thus for any two galaxies Γ_1, Γ_2 , $\Gamma_1 < \Gamma_2$ means that $\kappa_1 < \kappa_2$ for any (equivalently, for all) $\kappa_1 \in \Gamma_1, \kappa_2 \in \Gamma_2$.

See 1.4.9 and 9.6.12 in [17], or [19], on the next theorem. In the case of $\mathbb{1}$ -infinite sets the factors \mathbb{N} and h in 3.1 vanish by obvious reasons.

Theorem 3.1 (i) *Suppose that $X, Y \in \mathbb{1}$ and $f : X \rightarrow Y$ is a $\mathbf{\Delta}_2^{\mathbf{ss}}$ map. Then $|X|^{\mathbf{int}} h \in \| \mathbf{ran} f \|_*$ for any $h \in {}^*\mathbb{N} \setminus \mathbb{N}$. In addition, (a) if $\mathbf{ran} f = Y$ then $|Y|^{\mathbf{int}} \leq |X|^{\mathbf{int}} \mathbb{N}$, and (b) if f is an injection then $|X|^{\mathbf{int}} \leq |Y|^{\mathbf{int}} \mathbb{N}$.*

(ii) *Suppose that $X \in \mathbb{1}$ is infinite. Then $|X|^{\mathbf{eff}} = |X \times \mathbb{N}|^{\mathbf{eff}}$, in particular, $|Y|^{\mathbf{eff}} \leq |X|^{\mathbf{eff}}$ for any internal Y with $|Y|^{\mathbf{int}} \leq |X|^{\mathbf{int}} \mathbb{N}$. \square*

Corollary 3.2 *If $x, y \in \mathbb{1}$ then $|x|^{\mathbf{eff}} \leq |y|^{\mathbf{eff}}$ is equivalent to $|x|^{\mathbf{int}} \leq |y|^{\mathbf{int}} \mathbb{N}$ provided $|y|^{\mathbf{int}} \in {}^*\mathbb{N} \setminus \mathbb{N}$ and to just $|x|^{\mathbf{int}} \leq |y|^{\mathbf{int}}$ otherwise. \square*

Thus $|x|^{\mathbf{eff}} = |y|^{\mathbf{eff}}$ is equivalent to $\mathbf{gal} |x|^{\mathbf{int}} = \mathbf{gal} |y|^{\mathbf{int}}$ in the domain ${}^*\mathbb{N} \setminus \mathbb{N}$, and equivalent to just $|x|^{\mathbf{int}} = |y|^{\mathbf{int}}$ outside of the domain ${}^*\mathbb{N} \setminus \mathbb{N}$. In the $\mathbb{1}$ -infinite domain ${}^*\mathbf{Card} \setminus {}^*\mathbb{N}$, the two characterizations coincide.

4 Effective cardinalities of sets of standard size

By definition sets of standard size, or *s. s. sets*, are those equinumerous (that is, admit a bijection onto) with sets in \mathbb{WF} . For any s. s. set X define $\mathbf{card} X = \mathbf{card} W \in \mathbf{Card}$, where W is a set in \mathbb{WF} equinumerous with X .

Lemma 4.1 (i) *Any s. s. set $X \subseteq \mathbb{I}$ is Σ_1^{ss} and Δ_2^{ss} .*

(ii) *Any s. s. set W is equinumerous with an s. s. set $X \subseteq \mathbb{I}$.*

(iii) *If $X \subseteq \mathbb{I}$ is a s. s. set then ${}^*\mathbf{Card} \setminus \mathbb{N} \subseteq \|X\|^*$ and $\|X\|^* \subseteq \mathbb{N}$.*

(iv) *If $X, Y \subseteq \mathbb{I}$ are s. s. sets then $\mathbf{card} X = \mathbf{card} Y$ iff $|X|^{\text{eff}} = |Y|^{\text{eff}}$, thus $|X|^{\text{eff}}$ can be identified with $\mathbf{card} X$.*

Proof (ii) We may assume that $W \in \mathbb{WF}$. Then the map $w \mapsto {}^*w$ is a bijection of W onto $X = \{{}^*w : w \in W\}$ and X is a set of standard size, too.

(iii) To prove ${}^*\mathbf{Card} \setminus \mathbb{N} \subseteq \|X\|^*$ fix $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ and apply **Saturation** to the family of all sets $C_u = \{c \in \mathbb{I} : u \subseteq c \wedge |c|^{\text{int}} = h\}$, where $u \subseteq X$ is finite.

(iv) Any bijection f between two sets $X, Y \subseteq \mathbb{I}$ of standard size is itself a set of standard size, then apply (i).

It follows that s. s. sets are adequately represented among Σ_1^{ss} sets in the context of \mathbf{card} , and on the other hand $|\cdot|^{\text{eff}}$ and \mathbf{card} coincide on s. s. sets. The next theorem shows that effective cardinalities of Δ_2^{ss} sets begin with sets of standard size, where they coincide with well-founded cardinals, followed by the domain of Δ_2^{ss} sets not of standard size. It will be demonstrated below that the structure of effective cardinalities in the latter is connected with ${}^*\mathbf{Card}$ in certain way.

Theorem 4.2 (i) *Infinite internal sets are not s. s. sets.*

(ii) *Any Δ_2^{ss} set X not of standard size contains an infinite internal subset, that is formally $\mathbb{N} \subsetneq \|X\|_*$.*

(iii) *If X a s. s. set and Y is a Δ_2^{ss} but not s. s. set then $|X|^{\text{eff}} < |Y|^{\text{eff}}$.*

Proof (i) A simple corollary of Lemma 4.1(iii).

(ii) By definition Δ_2^{ss} sets are s. s. unions of Π_1^{ss} sets. Yet it is another rather simple corollary of **Saturation** that any infinite Π_1^{ss} set contains an infinite internal subset, see 1.4.11 in [17].

(iii) By (ii) some number $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ belongs to $\|Y\|_*$. On the other hand $h \in \|X\|^*$ by Lemma 4.1(iii). This implies $|X|^{\text{eff}} \leq |Y|^{\text{eff}}$. The inequality $|Y|^{\text{eff}} \not\leq |X|^{\text{eff}}$ follows from (i).

5 Exteriors and interiors

It turns out that internal approximations $\|X\|_*$, $\|X\|^*$ are very instrumental in the study of effective cardinalities of Σ_1^{ss} and partly Π_1^{ss} sets X . Now a few words on cuts (initial segments) in $^*\text{Card}$.

Definition 5.1 A cut $U \subseteq ^*\text{Card}$ is *standard size (s. s.) cofinal* resp. *coinitial*, iff there exist a cardinal $\vartheta \in \text{Card}$, infinite or equal to $1 = \{0\}$, and an increasing, resp. decreasing sequence $\{\nu_\xi\}_{\xi < \vartheta}$, of $\nu_\xi \in ^*\text{Card}$ such that $U = \bigcup_{\xi < \vartheta} \{\kappa \in ^*\text{Card} : \kappa < \nu_\xi\}$, resp., $U = \bigcap_{\xi < \vartheta} \{\kappa \in ^*\text{Card} : \kappa < \nu_\xi\}$. \square

Note that s. s. cofinal cuts are Σ_1^{ss} while s. s. coinitial cuts are Π_1^{ss} . Internal cuts, i. e. those of the form $U = \{\kappa \in ^*\text{Card} : \kappa < \nu\}$, $\nu \in ^*\text{Card}$, belong to either of the two “standard size” categories, for take $\vartheta = 1$ and $\nu_0 = \nu$. See 1.4b in [17] on the next result:

Proposition 5.2 *Any Δ_2^{ss} cut in $^*\text{Card}$ is s. s. cofinal or s. s. coinitial. If a cut is both s. s. cofinal and s. s. coinitial then it is internal.* \square

Coming back to $\|X\|_*$ and $\|X\|^*$, note that for any X the intersection $\|X\|_* \cap \|X\|^*$ contains at most one element. If $\kappa \in \|X\|_* \cap \|X\|^*$ then there exist internal sets Y, Z with $Y \subseteq X \subseteq Z$ and $|Y|^{\text{int}} = |Z|^{\text{int}} = \kappa$. In this case, if $\kappa \in ^*\mathbb{N}$ then X itself is internal with $|X|^{\text{int}} = \kappa$, while if κ is \mathbb{l} -infinite then only $|X|^{\text{eff}} = |\kappa|^{\text{eff}}$ holds provided X is Δ_2^{ss} .

Lemma 5.3 (i) *If X is a set in Σ_1^{ss} , resp., Π_1^{ss} then $\|X\|_*$ is a standard size cofinal, resp., standard size coinitial cut in $^*\text{Card}$.*

(ii) *In both cases, $\|X\|_* \cup \|X\|^* = ^*\text{Card}$.*

(iii) *In both cases, if either $\|X\|_*$ contains a largest element κ , or $\|X\|^*$ contains a least element κ , then $\kappa \in \|X\|_* \cap \|X\|^*$.*

Proof (i) Consider a set $\mathcal{X} \subseteq \mathbb{l}$ of standard size. Let $X = \bigcup \mathcal{X}$. Then by Saturation any internal set $Y \subseteq X$ is covered by a set of the form $\bigcup \mathcal{X}'$ where $\mathcal{X}' \subseteq \mathcal{X}$ is finite. On the other hand, by 1.3.3 in [17] the set $\mathcal{P}_{\text{fin}}(\mathcal{X}) = \{\mathcal{X}' \subseteq \mathcal{X} : \mathcal{X}' \text{ is finite}\}$ is still a set of standard size.

Prove (ii) for Σ_1^{ss} . Let $X = \bigcup \mathcal{X}$ be as above. Show that any \mathbb{l} -cardinal $\kappa \notin \|X\|_*$ belongs to $\|X\|^*$. Take any set $Z \in \mathbb{l}$ such that $X \subseteq Z$. If $\mathcal{X}' \subseteq \mathcal{X}$ is finite then by definition $\bigcup \mathcal{X}'$ is covered by an internal set of \mathbb{l} -cardinality κ , hence the set $P_{\mathcal{X}'} = \{C \in \mathbb{l} : \bigcup \mathcal{X}' \subseteq C \subseteq Z \wedge |C|^{\text{int}} \leq \kappa\} \in \mathbb{l}$ is non-empty. Apply Saturation to the family of all these sets $P_{\mathcal{X}'}$.

(iii) Apply Saturation.

Example 5.4 The following example⁵ of a Δ_2^{ss} set X such that $\|X\|_* \cup \|X\|^* \subsetneq {}^*\text{Card}$ employs a nontrivial ultrafilter $U \in \mathbb{W}\mathbb{F}$ over \mathbb{N} . Let $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $D = \{1, 2, \dots, h\}$. The set $P = \mathcal{P}^{\downarrow}(D) = \mathcal{P}(D) \cap \downarrow$ of all internal sets $x \subseteq D$ belongs to \downarrow and satisfies $|P|^{\text{int}} = 2^h$. Then

$$U' = \{x \in P : x \cap \mathbb{N} \in U\} = \bigcup_{b \in U} \bigcap_{n \in b} \{x \in P : n \in x\}$$

is an ultrafilter in P and a Δ_2^{ss} set.⁶ We claim that $\|U'\|_* = 2^h/\mathbb{N}$.

Let $Z' \subseteq P$ be an internal set. By Saturation (see e.g. 9.2.15 in [17] or 1.6 in [19]), $Z = \{x \cap \mathbb{N} : x \in Z'\}$ is a closed subset of U . It follows that the Lebesgue measure of Z in $\mathcal{P}(\mathbb{N})$ (identified with $2^{\mathbb{N}}$) is 0. Then easily the Loeb measure of Z in $\mathcal{P}^{\downarrow}(D)$ is 0, so that $|Z'|^{\text{int}} \in 2^h/\mathbb{N}$. Thus $\|U'\|_* \subseteq 2^h/\mathbb{N}$. To prove the converse note that for any $u \in U$ the set $X = \{x \in P : x \cap \mathbb{N} = u\}$ is a Π_1^{ss} subset of U' that surely satisfies $\|X\|_* = 2^h/\mathbb{N}$.

It follows from $\|U'\|_* = 2^h/\mathbb{N}$ that $\|U'\|^* = 2^h$ — by the symmetry of the sets U' and $P \setminus U' = \{D \setminus x : x \in P\}$ within P . \square

One can easily transform the set U' as in 5.4 to a Δ_2^{ss} set $X \subseteq {}^*\mathbb{N}$ such that $\|X\|_* = 2^h/\mathbb{N}$ and $\|X\|^* = 2^h\mathbb{N}$. The gap ${}^*\text{Card} \setminus (\|X\|^* \cup \|X\|_*)$ consists, in this case, of the whole galaxy $\text{gal } 2^h = 2^h\mathbb{N} \setminus 2^h/\mathbb{N}$ in ${}^*\mathbb{N}$. The next theorem shows that this is a maximal possible gap!

Theorem 5.5 *If X is Δ_2^{ss} and $\kappa \in {}^*\text{Card}$, $\kappa \notin \|X\|^* \cup \|X\|_*$, then $\kappa \in {}^*\mathbb{N}$ and the difference ${}^*\text{Card} \setminus (\|X\|^* \cup \|X\|_*)$ is a subset of $\text{gal } \kappa$.*

Thus if X is Δ_2^{ss} and ${}^\mathbb{N} \subseteq \|X\|_*$ then $\|X\|_* \cup \|X\|^* = {}^*\text{Card}$.*

Proof By definition $X = \bigcup_{a \in A} X_a$ where $A \in \mathbb{W}\mathbb{F}$ and every X_a is a Π_1^{ss} set. Take any \downarrow -cardinal $\kappa \in \|X\|^* \setminus \|X\|_*$. Obviously $\bigcup_{a \in A} \|X_a\|_* \subseteq \|X\|_*$, thus $\kappa \in \bigcap_{a \in A} \|X_a\|^*$ by Lemma 5.3. It suffices to prove that any $\lambda \in {}^*\text{Card}$ belongs to $\|X\|^*$ in either of the two cases: 1) $\lambda = \kappa \notin {}^*\mathbb{N}$, 2) $\lambda \in {}^*\mathbb{N} \setminus \kappa\mathbb{N}$. Note that $n\kappa \leq \lambda$ holds for all $n \in \mathbb{N}$ in both cases.

Using Standard Size Choice, choose, for any $a \in A$, a set $Y_a \in \downarrow$ such that $X_a \subseteq Y_a$ and $|Y_a|^{\text{int}} = \kappa$. Thus X is covered by the union $\bigcup_{a \in A} Y_a$. For any finite $A' \subseteq A$, the finite union $Y_{A'} = \bigcup_{a \in A'} Y_a$ is an internal set satisfying $|Y_{A'}|^{\text{int}} \leq \lambda$ by the above. The same application of Saturation as in the proof of Lemma 5.3 yields an internal set Y still with $|Y|^{\text{int}} \leq \lambda$, satisfying $\bigcup_{a \in A} Y_a \subseteq Y$, and hence $X \subseteq Y$ and $\lambda \in {}^*\text{Card}$.

⁵ Essentially given in [24], see also [19], p. 1172, but with a more complicated proof based on a rather nontrivial combinatorial theorem in [4].

⁶ The set U' is even countably determined.

The following corollary belongs to the “small–large dichotomy” type. (B) witnesses that a given Δ_2^{ss} set is rather large w. r. t. a given cut U (has rather large internal subsets), while (A1) and (A2) witness that X is rather small (can be covered by rather small internal sets). The proof is easy: if $U \not\subseteq \llbracket X \rrbracket_*$ then (B) holds by definition, otherwise apply Theorem 5.5 and get (A1) or (A2) (or Lemma 5.3(ii) – in the case of Σ_1^{ss} and Π_1^{ss} sets).

Corollary 5.6 *If $X \subseteq \mathbb{I}$ is a Δ_2^{ss} set and $U \subseteq {}^*\text{Card}$ is a Δ_2^{ss} cut then at least one of the following conditions holds, and moreover (A2) can be excluded for Σ_1^{ss} and Π_1^{ss} sets X :*

- (A1) *for any $\kappa \notin U$ there is an internal set $Y \supseteq X$ such that $|Y|^{\text{int}} = \kappa$;*
- (A2) *there exists $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $h/\mathbb{N} \subseteq U \subseteq h\mathbb{N}$, and for any $\kappa \in {}^*\text{Card} \setminus h\mathbb{N}$ there exists an internal set $Y \supseteq X$ such that $|Y|^{\text{int}} = \kappa$;*
- (B) *there exists an internal set $Y \subseteq X$ such that $|Y|^{\text{int}} \notin U$. □*

6 Effective cardinalities of Σ_1^{ss} sets

One may expect that the bigger $\llbracket X \rrbracket_*$ (or the smaller $\llbracket X \rrbracket^*$) is the bigger $|X|^{\text{eff}}$ should be. According to the next theorem, such a connection holds for Σ_1^{ss} sets X except those satisfying $\llbracket X \rrbracket_* \subseteq \mathbb{N}$.

Following the notation in Section 3, we define, for any $K \subseteq {}^*\text{Card}$, a cut $K\mathbb{N} = \{\lambda : \exists \kappa \in K \exists n \in \mathbb{N} (\lambda \leq n\kappa)\}$ in ${}^*\text{Card}$.

Theorem 6.1 *If X, Y are Σ_1^{ss} sets and $\mathbb{N} \not\subseteq \llbracket Y \rrbracket_*$ then $|X|^{\text{eff}} \leq |Y|^{\text{eff}}$ is equivalent to $\llbracket X \rrbracket_* \subseteq \llbracket Y \rrbracket_*\mathbb{N}$, and also to $\llbracket X \rrbracket_* \subseteq \llbracket Y \rrbracket_*$ if ${}^*\mathbb{N} \subseteq \llbracket Y \rrbracket_*$.*

The case $\llbracket Y \rrbracket_* \subseteq \mathbb{N}$ will be considered below.

Proof Suppose that $X = \bigcup \mathcal{X}$ and $Y = \bigcup \mathcal{Y}$, where $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{I}$ are sets of standard size. There is a set $D \in \mathbb{I}$ such that $X \cup Y \subseteq D$. Assume w. l. o. g. that \mathcal{X}, \mathcal{Y} are \cap -closed families. By **Saturation**, the sets of \mathbb{I} -cardinals $\{|X'|^{\text{int}} : X' \in \mathcal{X}\}, \{|Y'|^{\text{int}} : Y' \in \mathcal{Y}\}$ are cofinal in resp. $\llbracket X \rrbracket_*, \llbracket Y \rrbracket_*$.

Direction \implies . Suppose otherwise. Then there is an internal set $X' \subseteq X$ such that $|Y'|^{\text{int}}m < |X'|^{\text{int}}/n$ for any internal $Y' \subseteq Y$ and $k, n \in \mathbb{N}$. As $\llbracket Y \rrbracket_*$ is a s. s. cofinal cut in ${}^*\text{Card}$ by Lemma 5.3(i), there exists, by **Saturation**, $\kappa \in {}^*\text{Card}$ such that $|Y'|^{\text{int}}m < \kappa < |X'|^{\text{int}}/n$ for any internal $Y' \subseteq Y$ and $k, n \in \mathbb{N}$. Thus $\llbracket Y \rrbracket_*\mathbb{N} < \kappa$, hence $\kappa \in \llbracket Y \rrbracket^*$ by Lemma 5.3(ii). In other words, there is an internal set Z such that $Y \subseteq Z$ and $|Z|^{\text{int}} = \kappa$. On the other hand, $\kappa\mathbb{N} < |X'|^{\text{int}}$ while by $|X|^{\text{eff}} \leq |Y|^{\text{eff}}$ there exists a Δ_2^{ss} injection $X' \rightarrow Z$, a contradiction to Theorem 3.1(i).

Direction \Leftarrow , in a stronger assumption that simply $\|X\|_* \subseteq \|Y\|_*$.

Case 1: $\|Y\|_*$ contains a maximal element $\kappa = |Y_0|^{\text{int}}$, where $Y_0 \in \mathcal{Y}$, hence $Y_0 \subseteq Y$. Then for any $X' \in \mathcal{X}$ the set

$$H_{X'} = \{h \in \mathbb{I} : h : D \rightarrow D \wedge h \upharpoonright X' \text{ is an injection} \wedge h'' X' \subseteq Y_0\}$$

is non-empty. In addition, $H_{X'' \cup X'} = H_{X''} \cap H_{X'}$. **Saturation** yields an element $h \in \bigcap_{X' \in \mathcal{X}} H_{X'}$. Clearly $h \upharpoonright X$ is an injection of X into Y_0 , and hence $|X|^{\text{eff}} \leq |Y|^{\text{eff}}$, as required.

Case 2: $\|Y\|_*$ does not contain a maximal element, and for every $\alpha \in \|Y\|_* \cap {}^*\mathbb{N}$ there exists $\gamma \in \|Y\|_* \cap {}^*\mathbb{N}$ such that $\alpha \mathbb{N} < \gamma$ – meaning that $\gamma > \alpha n$ for any $n \in \mathbb{N}$. By **Standard Size Choice** there is a map $f : \mathcal{X} \rightarrow \mathcal{Y}$ such that $|X'|^{\text{int}} < |f(X')|^{\text{int}}$ for all $X' \in \mathcal{X}$, and even $|X'|^{\text{int}} \mathbb{N} < |f(X')|^{\text{int}}$ provided $|f(X')|^{\text{int}}$ (then also $|X'|^{\text{int}}$) belongs to ${}^*\mathbb{N}$. Then

$$H_{X'} = \{h \in \mathbb{I} : h : D \rightarrow D \wedge h \upharpoonright X' \text{ is an injection} \wedge h'' X' \subseteq f(X')\}$$

is non-empty for any $X' \in \mathcal{X}$. Then argue as in Case 1.

Case 3: the negation of cases 1, 2. Then there is a number $c \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $c \in \|Y\|_*$ but $2c \notin \|Y\|_*$. Then $|[0, c]|^{\text{eff}} \leq |Y|^{\text{eff}}$ while $|X|^{\text{eff}} \leq |[0, 2c]|^{\text{eff}}$ (see case 1). However $|[0, 2c]|^{\text{eff}} = |[0, c]|^{\text{eff}}$ by Corollary 3.2.

Direction \Leftarrow , *general case.* If $\|X\|_* \subseteq \|Y\|_* \mathbb{N}$, but $\|X\|_* \subseteq \|Y\|_*$ does **not** hold then there exist numbers $c \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $n \in \mathbb{N}$ such that $\|X\|_* \subseteq [0, nc)$ and $[0, c) \subseteq \|Y\|_* \subseteq [0, 2c)$. We have $|X|^{\text{eff}} \leq |[0, nc]|^{\text{eff}}$ by the above, and $|[0, c]|^{\text{eff}} \leq |Y|^{\text{eff}}$. It remains to apply Corollary 3.2.

It remains to consider the case $\|Y\|_* \subseteq \mathbb{N}$ avoided in the theorem. It leads to sets of standard size!

Lemma 6.2 *For a set $X \subseteq \mathbb{I}$ to be of standard size each of the conditions $\|X\|_* \subseteq \mathbb{N}$, ${}^*\mathbb{N} \setminus \mathbb{N} \subseteq \|X\|^*$ is necessary and, if X is Δ_2^{ss} , also sufficient.*

Proof By Theorem 4.2(i) $\|X\|_* \subseteq \mathbb{N}$. On the other hand ${}^*\mathbb{N} \setminus \mathbb{N} \subseteq \|X\|^*$ by Lemma 4.1(iii). The sufficiency follows from Theorem 4.2(ii).

Thus Theorem 6.1 *fails* in the case $\|X\|_* = \mathbb{N}$: take any pair of infinite sets $X, Y \subseteq \mathbb{I}$ of standard size with $\text{card } X \neq \text{card } Y$ and apply 6.2 to show that $\|X\|^* = \|Y\|^* = \mathbb{N}$, and Lemma 4.1 to show that $|X|^{\text{eff}} \neq |Y|^{\text{eff}}$. Nevertheless we easily obtain the following corollary.

Corollary 6.3 *If X, Y are Σ_1^{ss} sets then their effective cardinalities are comparable in the sense that at least one of the following inequalities holds: $|X|^{\text{eff}} \leq |Y|^{\text{eff}}$ or $|Y|^{\text{eff}} \leq |X|^{\text{eff}}$. \square*

7 Effective cardinalities of $\mathbf{\Pi}_1^{\text{ss}}$ sets

The proof of \implies in Theorem 6.1 does not work for $\mathbf{\Pi}_1^{\text{ss}}$ sets since $\|Y\|_*$ is now s.s. coinital and the Saturation argument does not work. On the other hand there is a suitable counterexample.

Example 7.1 Fix $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let S be the set of all internal maps $s : \{0, 1, 2, \dots, h-1, h\} \rightarrow \{0, 1\} = 2$. Define $a_s, b_s \in 2^{\mathbb{N}}$ (hence $\in \mathbb{W}\mathbb{F}$) so that $a_s(k) = s(k)$ and $b_s(k) = s(h-k)$ for all $k \in \mathbb{N}$. For $a, b \in 2^{\mathbb{N}}$ put $S_{ab} = \{s : a_s = a \wedge b_s = b\}$ and $S_a = \{s : a_s = a\}$. Then S is internal, $|S|^{\text{int}} = 2^{h+1}$, while each S_a is a $\mathbf{\Pi}_1^{\text{ss}}$ set with $\|S_a\|_* = 2^h/\mathbb{N}$. Obviously $(2^h/\mathbb{N})\mathbb{N} = (2^h/\mathbb{N})$. To see that S and S_a lead to a counterexample to \implies of Theorem 6.1, it suffices to prove that $|S|^{\text{eff}} = |S_a|^{\text{eff}}$ for some a .

Since either of S, S_a is a union of $2^{\mathbb{N}}$ -many sets of the form S_{ab} , it remains to show that $|S_{ab}|^{\text{eff}} = |S_{a'b'}|^{\text{eff}}$ for all a, b, a', b' . By Saturation there is $\sigma \in S$ such that $a(n) = a'(n) \oplus \sigma(n)$ and $b(n) = b'(n) \oplus \sigma(h-n)$ for all $n \in \mathbb{N}$, where \oplus is addition modulo 2. Finally the internal map $s \mapsto s \oplus \sigma$ (in the termwise sense) easily maps S_{ab} onto $S_{a'b'}$ in 1-1 way. \square

In fact 7.1 is the only possible counterexample for $\mathbf{\Pi}_1^{\text{ss}}$ sets in the following sense: if X, Y are $\mathbf{\Pi}_1^{\text{ss}}$ sets, $\mathbb{N} \not\subseteq \|Y\|_*$, and $|X|^{\text{eff}} \leq |Y|^{\text{eff}}$ then either $\|X\|_* \subseteq \|Y\|_*\mathbb{N}$ or there is a number $\kappa \in \|Y\|_*$, $\kappa \in {}^*\mathbb{N} \setminus \mathbb{N}$, such that $\|X\|_* = \kappa/\mathbb{N}$ while $\|Y\|_* \subseteq \kappa\mathbb{N}$. We skip the proof.

Our further goal is to present what looks like a near-counterexample, (ii) of Theorem 7.2, to \Leftarrow of Theorem 6.1 in the field of $\mathbf{\Pi}_1^{\text{ss}}$ sets.

If X is a $\mathbf{\Pi}_1^{\text{ss}}$ set then $\|X\|^*$ is standard size coinital by Lemma 5.3. If $\|X\|^*$ contains a least element κ then κ is simultaneously the largest element in $\|X\|_*$ still by Lemma 5.3, and then easily $|X|^{\text{eff}} = |\kappa|^{\text{eff}}$. It follows that if in this case Y is another $\mathbf{\Pi}_1^{\text{ss}}$ set with $\|Y\|^* = \|X\|^*$ then $|X|^{\text{eff}} = |Y|^{\text{eff}}$. But if $\|X\|^*$ does not contain a least element then there is an infinite coinital sequence with standard size many terms. This case is considered by the next theorem. It follows from (ii) that there are sets of the largest effective cardinality among all $\mathbf{\Pi}_1^{\text{ss}}$ sets X with the same $\|X\|^*$, while (iii) presents a rather nontrivial partial counterexample to Theorem 6.1 for $\mathbf{\Pi}_1^{\text{ss}}$ sets. We deal with $\mathbb{1}$ -infinite cardinals here, but similar results can be obtained in the hyperfinite domain — we leave it to the reader.

Theorem 7.2 (i) *If $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{1}$ are sets of standard size, $X = \bigcap \mathcal{X}$, $Y = \bigcap \mathcal{Y}$, $\vartheta = \text{card } \mathcal{X} \in \mathbb{W}\mathbb{F}$ is an infinite regular cardinal, $\|X\|^* = \|Y\|^*$, and the coinitality of $\|X\|^*$ is exactly ϑ , then $|Y|^{\text{eff}} \leq |X|^{\text{eff}}$.*

(ii) There exist Π_1^{ss} sets X, Y as in (ii) such that $|X|^{\text{eff}} \leq |Y|^{\text{eff}}$ fails via Δ_2^{ss} injections g of the form $g = \bigcup_{w \in W} \bigcap_{\xi < \vartheta} g_{w\xi}$, where all $g_{w\xi}$ are internal and W is a set of standard size.

Proof (i) Assume w.l.o.g. that there exist sets $X_0 \in \mathcal{X}$, $Y_0 \in \mathcal{Y}$ such that $X \subseteq X_0$ and $Y \subseteq Y_0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, and the families \mathcal{X}, \mathcal{Y} are \cap -closed. We claim that there exists a function $\psi : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying

$$\forall A \in \mathcal{P}_{\text{fin}}(\mathcal{X}) \exists f \in F \forall X \in A (\psi(X) \subseteq \text{dom } f \wedge f''\psi(X) \subseteq X), \quad (5)$$

where $F \in \mathbb{I}$ is the set of all 1-1 functions $f \in \mathbb{I}$ with $\text{dom } f \subseteq Y_0$ and $\text{ran } f \subseteq X_0$. To define ψ fix an enumeration $\mathcal{X} = \{X_\alpha : \alpha < \vartheta\}$. Suppose that $\alpha' < \vartheta$, and the values $\psi(X_\alpha) \in \mathcal{Y}$, $\alpha < \alpha'$, have been defined. In our assumptions, there is a set $Y \in \mathcal{Y}$ such that $|Y|^{\text{int}} < |\bigcap_{\alpha \in A} X_\alpha|^{\text{int}}$ for every finite $A \subseteq [0, \alpha']$. To complete the inductive step put $\psi(X_{\alpha'}) = Y$.

To prove (5) consider a finite set $A = \{\alpha_1 < \dots < \alpha_n\} \subseteq \vartheta$. By the construction $|\psi(X_{\alpha_k})|^{\text{int}} < |\bigcap_{1 \leq i \leq k} X_{\alpha_i}|^{\text{int}}$ for all $k = 1, \dots, n$. Arguing in \mathbb{I} , we easily find a map $f \in F$ such that $f''(\psi(X_{\alpha_k})) \subseteq \bigcap_{1 \leq i \leq k} X_{\alpha_i}$ for every $k = 1, \dots, n$, hence (5) holds.

Yet by Saturation (5) is equivalent to the following:

$$\exists f \in F \forall X \in \mathcal{X} (\psi(X) \subseteq \text{dom } f \wedge f''(\psi(X)) \subseteq X). \quad (6)$$

Thus $f''Y \subseteq X$, for such an f , and hence $|Y|^{\text{eff}} \leq |X|^{\text{eff}}$ holds even by means of an internal map f .

(ii) Fix an infinite cardinal ϑ in WF . It easily follows from Saturation that there exists a strictly decreasing sequence $\vec{\nu} = \{\nu_\xi\}_{\xi < \vartheta}$ of \mathbb{I} -cardinals $\nu_\xi \in {}^*\text{Card} \setminus {}^*\mathbb{N}$. A $\vec{\nu}$ -large set will be any $X \in \mathbb{I}$ such that $\exists \xi < \vartheta (|X|^{\text{int}} \geq \nu_\xi)$.

Let $\tau = \vartheta^+$ (the next cardinal in WF). The counterexample was based on a sequence $\{Y_\gamma\}_{\gamma < \tau}$ of internal sets Y_γ such that

- (a) for any pair of disjoint finite sets $u, v \subseteq \tau$, $u \neq \emptyset$, the set $Y_{uv} = \bigcap_{\alpha \in u} Y_\alpha \setminus \bigcup_{\beta \in v} Y_\beta$ is $\vec{\nu}$ -large;
- (b) $|Y_{uv}|^{\text{int}} = |Y_{u\emptyset}|^{\text{int}}$ for any disjoint finite $u, v \subseteq \tau$;
- (c) if $\xi < \vartheta$, $A \subseteq \tau$, and $|Y_{\{\alpha, \beta\}, \emptyset}|^{\text{int}} \geq \nu_\xi$ (that is, $|Y_\alpha \cap Y_\beta|^{\text{int}} \geq \nu_\xi$) for all $\alpha, \beta \in A$ then $\text{card } A \leq \vartheta$.

We define Y_γ by induction. To begin with put $Y_0 = [0, \nu_0]$ (an initial segment in ${}^*\text{Ord}$). Now suppose that $\gamma < \tau$ and a set $Y_\delta \in \mathbb{I}$ has been defined for every $\delta < \gamma$ so that (a) and (b) hold below γ . Re-enumerate $\{Y_\delta : \delta < \gamma\} = \{Z_\alpha : \alpha < \lambda\}$, where $\lambda = \min\{\gamma, \vartheta\}$, without repetitions.

For any pair of disjoint finite sets $u, v \subseteq \lambda$, $u \neq \emptyset$, define the internal set $Z_{uv} = \bigcap_{\alpha \in u} Z_\alpha \setminus \bigcup_{\beta \in v} Z_\beta$. In our assumptions, the $\mathbb{1}$ -cardinals $\kappa_{uv} = |Z_{uv}|^{\text{int}}$ satisfy $\kappa_{uv} = \kappa_u$, where $\kappa_u = \kappa_{u\emptyset}$, and $\exists \xi < \vartheta$ ($\kappa_u \geq \nu_\xi$). For any finite $u \subseteq \lambda$ let $\xi(u)$ be the least ordinal $\xi < \vartheta$ such that $\nu_\xi < \kappa_u$ and $\xi > \sup u$. We assert that there is an internal set Z satisfying

- (d) $|Z \cap Z_{uv}|^{\text{int}} = \nu_{\xi(u)}$ and $|Z_{uv} \setminus Z|^{\text{int}} = \kappa_u$ for any pair of disjoint finite sets $u, v \subseteq \lambda$, $u \neq \emptyset$, and
- (e) $|Z \setminus \bigcup_{\beta \in v} Z_\beta|^{\text{int}} = |Z|^{\text{int}} \geq \nu_0$ for each finite set $v \subseteq \lambda$.

Indeed as ϑ is a set of standard size it suffices to prove that for any finite $d \subseteq \vartheta$ there is a set $Z \in \mathbb{1}$ satisfying (d), (e) for all $u, v \subseteq d$.

Note that the sets of the form Z_{uv} , where $u \cup v = d$ and $u \cap v = \emptyset$, are mutually disjoint, and by definition satisfy $\nu_{\xi(u)} \leq \kappa_u = |Z_{uv}|^{\text{int}}$. This allows us to define an internal Z satisfying (d) for all pairs u, v with $u \cup v = d$, $u \cap v = \emptyset$, $u \neq \emptyset$, and, adding a sufficient portion out of $\bigcup_{\beta \in d} Z_\beta$, also $|Z \setminus \bigcup_{\beta \in d} Z_\beta|^{\text{int}} = |Z|^{\text{int}} \geq \nu_0$. It remains to show (d) for all disjoint sets $u, v \subseteq d$ not necessarily with $u \cup v = d$.

We show this by backward induction on the cardinality of $u \cup v$. Suppose that $u \cup v \subsetneq d$. Take any $\alpha \in d \setminus (u \cup v)$. Let $u' = u \cup \{\alpha\}$ and $v' = v \cup \{\alpha\}$. Then by the inductive hypothesis $|Z \cap Z_{u'v}|^{\text{int}} = \nu_{\xi(u')}$ and $|Z \cap Z_{uv'}|^{\text{int}} = \nu_{\xi(u)}$. Since $Z_{uv} = Z_{u'v} \cup Z_{uv'}$ and easily $\xi(u) \leq \xi(u')$ whenever $u \subseteq u'$, we conclude that $|Z \cap Z_{uv}|^{\text{int}} = \nu_{\xi(u)}$ as required. Similarly, $|Z_{u'v} \setminus Z|^{\text{int}} = \kappa_{u'}$ and $|Z_{uv'} \setminus Z|^{\text{int}} = \kappa_u$, therefore $|Z_{uv} \setminus Z|^{\text{int}} = \kappa_{u'} + \kappa_u = \kappa_u$ as required.

Take as Y_γ any set $Z \in \mathbb{1}$ satisfying (d), (e). We have to demonstrate that (a), (b) remain true for the sequence $\{Y_\delta\}_{\delta \leq \gamma}$, or, that is equivalent, for the sequence $\{Z_\alpha\}_{\alpha \leq \lambda}$, where $Z_\lambda = Y_\gamma = Z$.

Take any pair of disjoint sets $u, v \subseteq \lambda \cup \{\lambda\}$. If $\lambda \notin u \cup v$ then the set $Z_{uv} = \bigcap_{\alpha \in u} Z_\alpha \setminus \bigcup_{\beta \in v} Z_\beta$ is the same as above so there is nothing to prove. Suppose that $\lambda \in u$; put $u' = u \setminus \{\lambda\}$. Then $Z_{uv} = Z \cap Z_{u'v}$, and hence Z_{uv} is $\vec{\nu}$ -large by (d) (applied for the pair u', v). Separately if $u = \{\lambda\}$ then $u' = \emptyset$, hence $Z_{u'v}$ is not defined, but obviously $Z_{uv} = Z \setminus \bigcup_{\beta \in v} Z_\beta$, therefore Z_{uv} is $\vec{\nu}$ -large by (e). Suppose that $\lambda \in v$; put $v' = v \setminus \{\lambda\}$. Then $Z_{uv} = Z_{uv'} \setminus Z$, and hence Z_{uv} is $\vec{\nu}$ -large still by (d). This proves (a); the derivation of (b) from (d), (e) is similar.

This ends the recursive construction of the sets Y_γ .

Show that such a sequence $\{Y_\gamma\}_{\gamma < \tau}$ also satisfies (c). We prove not only that $\text{card } A \leq \vartheta$ for any set A as in (c), but even more the order type of A in τ is $\leq \vartheta$. Suppose that $\gamma \in A$. Let us come back to the reenumerated system $\{Y_\delta : \delta < \gamma\} = \{Z_\alpha : \alpha < \lambda\}$, where $\lambda = \min\{\gamma, \vartheta\}$, and to the construction of $Y_\gamma = Z_\lambda = Z$ satisfying (d). It follows from (d) that, for any $\alpha < \lambda$,

$|Y_\gamma \cap Z_\alpha|^{\text{int}} = \nu_{\xi(\{\alpha\})} < \nu_\alpha$. In other words, for any $\xi < \vartheta$ the inequality $|Y_\gamma \cap Z_\alpha|^{\text{int}} \geq \nu_\xi$ can be true only for $\alpha < \xi$. Thus there exist ($< \vartheta$)-many sets Y_δ , $\delta < \gamma$, satisfying $|Y_\gamma \cap Y_\delta|^{\text{int}} \geq \nu_\xi$, as required.

Coming back to the proof of (ii) of Theorem 7.2, we fix a sequence $\{Y_\gamma\}_{\gamma < \tau}$ satisfying (a), (b), (c), and put $\mathcal{Y} = \{Y_\gamma : \gamma < \tau\}$. Then $Y = \bigcap_{\gamma < \tau} Y_\gamma$ is a $\mathbf{\Pi}_1^{\text{ss}}$ set. Note that every $\mathbb{1}$ -cardinal ν_ξ , $\xi < \vartheta$, belongs to $\|Y\|^*$ by (c), and on the other hand it follows by **Saturation** that every internal superset H of Y contains a subset of the form $\bigcap_{\alpha \in u} Y_\alpha = Y_{u\emptyset}$, where $u \subseteq \tau$ is finite, and hence $|H|^{\text{int}} \geq \nu_\xi$ for some $\xi < \vartheta$ by (a). It follows that the sequence $\{\nu_\xi\}_{\xi < \vartheta}$ is cointial in $\|Y\|^*$. It follows from Lemma 5.3 that $\|Y\|_*$ coincides with the set $\Omega = \{\kappa \in {}^*\text{Card} : \forall \xi < \vartheta (\kappa < \nu_\xi)\}$.

The other side of the counterexample will be the $\mathbf{\Pi}_1^{\text{ss}}$ set $X = \bigcap_{\xi < \vartheta} X_\xi$, where $X_\xi = \{\kappa \in {}^*\text{Ord} : \kappa < \nu_\xi\} \in \mathbb{1}$. Easily $|X_\xi|^{\text{int}} = \nu_\xi$, therefore the sequence $\{\nu_\xi\}_{\xi < \vartheta}$ is cointial in $\|X\|^*$, too. We conclude that $\|X\|^* = \|Y\|^*$, hence $\|X\|_* = \Omega = \|Y\|_*$ by Lemma 5.3.

To accomplish (ii), suppose towards the contrary that there is an injection $g \in \mathbf{\Delta}_2^{\text{ss}}$, $g : X \rightarrow Y$ of the form $g = \bigcup_{w \in W} \bigcap_{\xi < \vartheta} g_{w\xi}$, where all $g_{w\xi}$ are internal and W a set of standard size. Then each $g_w = \bigcap_{\xi < \vartheta} g_{w\xi}$ is still an injection into Y , whose domain $D_w = \text{dom } g_w \subseteq X$ is still a $\mathbf{\Pi}_1^{\text{ss}}$ set, moreover, an intersection of ($\leq \vartheta$)-many internal sets. (The combination of quantifiers $\exists \forall^{\text{st}} \xi < \vartheta$ converts to $\forall^{\text{st}} p \in \mathcal{P}_{\text{fin}}(\vartheta) \exists$ by **Saturation**.)

We claim that $\|D_w\|_* = \Omega$ for at least one $w \in W$.

(Indeed otherwise choose any $\kappa_w \in \Omega \setminus \|D_w\|_*$ for every $w \in W$; here **Standard Size Choice** is applied. Recall that $\|X\|^*$ is a standard size cointial final segment in ${}^*\text{Card}$, therefore the complement Ω of is not standard size cofinal by **Saturation**. It follows that there is an $\mathbb{1}$ -cardinal $\kappa \in \Omega$ bigger than each κ_w . Then $\kappa \in \|D_w\|^*$ for any $w \in W$, thus any D_w is covered by an internal set of $\mathbb{1}$ -cardinality κ . Still by **Saturation**, the union $\bigcup_{w \in W} D_w$ can be covered by an internal set C , $|C|^{\text{int}} = \kappa$. Then $X \subseteq C$, contradiction.)

This result allows us to replace X by D_w , or, in different words, reduce the task to the case when g , a given injection $X \rightarrow Y$, is equal to $\bigcap_{\xi < \vartheta} g_\xi$, each g_ξ being an internal set. An easy application of **Saturation** shows that there is a finite set $u \subseteq \vartheta$ such that $h = \bigcap_{\xi \in u} g_\xi$ is an injective function. On the other hand h is an internal function extending g , thus $X = \text{dom } g \subseteq h$ and $h''X \subseteq Y$. Let $D = \text{dom } h$ (an internal superset of X).

Recall that $Y = \bigcap_{\gamma < \tau} Y_\gamma$ where all Y_γ are internal. Thus, for any γ , $h''X \subseteq Y_\gamma$, and hence, as $X = \bigcap_{\xi < \vartheta} X_\xi$ and the family of all sets X_ξ is \cap -closed, **Saturation** yields an ordinal $\xi(\gamma) < \vartheta$ such that $X_{\xi(\gamma)} \subseteq D$ and $h''X_{\xi(\gamma)} \subseteq Y_\gamma$. As $\tau = \vartheta^+$, there is at least one $\xi < \vartheta$ such that $G = \{\gamma < \tau : h''X_\xi \subseteq Y_\gamma\}$ is unbounded in τ . Thus all sets Y_γ , $\gamma \in G$ include as a subset

one and the same internal set $R = h'' X_\xi$. Note that $|R|^{\text{int}} = |X_\xi|^{\text{int}}$ because h is an injection. But X_ξ is a $\vec{\nu}$ -large set, a contradiction with (c).

Question 7.3 Is Corollary 6.3 still true for sets in Δ_2^{ss} or in Π_1^{ss} ? □

Theorem 10.2 below shows that a wider category of Δ_2^{ss} *quotients* has plenty of incomparable sets. Note that the existence of countably determined sets incomparable in the sense of countably determined injections, is also an open problem. A counterexample defined in [2] in the **AST** frameworks makes use of the hypothesis that there exist only \aleph_1 -many internal sets, and hence is irreproducible in **HST**.

On the other hand all Borel sets (in the sense of Footnote 3) are Borel-comparable. This result was first obtained by **AST**-followers, see e.g. [12], and then reproved in [27]. See more on this in [17], 9.6 and 9.7.

8 Effective sets in the form of quotients

Sets of the form X/E , where X is Δ_2^{ss} while E is a Δ_2^{ss} equivalence relation on X will be called Δ_2^{ss} *quotients*. These Δ_2^{ss} quotients include the class Δ_2^{ss} itself, for take E to be just the equality on a given Δ_2^{ss} set X , so that the map sending any $x \in X$ to $\{x\}$ is a bijection of X onto X/E .

On the other hand, it follows from Theorem 2.3(iii) that every set in $\mathbb{L}[\mathbb{l}]$, that is, every effective set in the sense explained in Section 2, admits an effective bijection onto a Δ_2^{ss} quotient. Thus Δ_2^{ss} quotients exhaust, in the context of effective cardinalities, all effective ($= \mathbb{L}[\mathbb{l}]$) sets in general.

One may ask whether Δ_2^{ss} quotients produce more effective cardinalities than just Δ_2^{ss} sets. Call *smooth* any Δ_2^{ss} quotient that admits a Δ_2^{ss} bijection onto a Δ_2^{ss} set. We show in Section 9 that every Δ_2^{ss} quotient X/E , such that all E -classes $[x]_E = \{y \in X : x E y\}$, $x \in X$, are sets of standard size, is smooth. A family of non-smooth Δ_2^{ss} quotients, those defined by means of *monadic partitions* of ${}^*\mathbb{N}$, will be studied in Sections 10, 11. We prove there that there exist incomparable effective cardinalities of monadic Δ_2^{ss} quotients, still an open problem for Δ_2^{ss} sets themselves. We also prove a “small–large” type theorem for Δ_2^{ss} quotients in Section 12, similar to 5.6 but not so sharp, with an interesting Ramsey-like corollary.

Note that Δ_2^{ss} quotients consist of subsets of \mathbb{l} which are not necessarily internal sets themselves. Accordingly injections of Δ_2^{ss} quotients are maps whose **dom** and **ran** not necessarily consist of internal sets. Still there is a way to pull the consideration down to the basic level.

Definition 8.1 Let E, F be equivalence relations on sets X, Y . A set $R \subseteq X \times Y$ is a (E, F) -invariant pre-injection of X into Y iff 1) $\text{dom } R = X$ ⁷ and 2) the equivalence $x E x' \iff y F y'$ holds for all $\langle x, y \rangle \in R$ and $\langle x', y' \rangle \in R$.

Such a set R is a *reduction of X/E to Y/F* (or just of E to F) if in addition 3) R is a (graph of a) function $X \rightarrow Y$.

Write $E \leq_{\text{eff}} F$ iff there is a (E, F) -invariant pre-injection $P \subseteq X \times Y$, $P \in \Delta_2^{\text{ss}}$, of X into Y . Write $E \leq_{\text{eff}}^+ F$, in words: *E is effectively reducible to F* , iff there is a reduction $\rho \in \Delta_2^{\text{ss}}$, $\rho : X \rightarrow Y$ of E to F .

An equivalence relation E on a set X and the quotient X/E are Δ_2^{ss} -smooth iff there is a Δ_2^{ss} set Y such that $E \leq_{\text{eff}} D_Y$, where D_Y is the equality on Y considered as an equivalence relation.⁸ \square

This definition resembles some central concepts in modern descriptive set theory, like Borel reducibility and “Borel cardinals” (see, for instance, [6, 7, 18]), where Borel maps are used in approximately the same role as Δ_2^{ss} maps in this paper.

Proposition 8.2 (i) Suppose that E, F are Δ_2^{ss} equivalence relations on Δ_2^{ss} sets X, Y . Then $|X/E|^{\text{eff}} \leq |Y/F|^{\text{eff}}$ iff $E \leq_{\text{eff}} F$.

(ii) An Δ_2^{ss} equivalence relation E on a Δ_2^{ss} set X is Δ_2^{ss} -smooth iff there exists a Δ_2^{ss} set Y such that $|X/E|^{\text{eff}} = |Y|^{\text{eff}}$.

Proof (i) Suppose that $f \in \mathbb{L}[\mathbb{I}]$ is an injection $X/E \rightarrow Y/F$. Then $P = \{\langle x, y \rangle \in X \times Y : f([x]_E) = [y]_F\}$ is a set in $\mathbb{L}[\mathbb{I}]$, hence a Δ_2^{ss} set by 2.3(i), and obviously an invariant pre-injection. The converse is equally simple: if P is an invariant pre-injection then to define an injection $f : X/E \rightarrow Y/F$ put $f([x]_E) = [y]_F$ for any $\langle x, y \rangle \in P$.

(ii) Suppose that $E \leq_{\text{eff}} D_Z$, where Z is a Δ_2^{ss} set. Let this be witnessed by an invariant pre-injection $R \subseteq X \times Z$ of class Δ_2^{ss} . Clearly $R = \rho$ is then a reduction (a map $X \rightarrow Z$ such that $x E x' \iff \rho(x) = \rho(x')$). The set $Y = \text{ran } \rho \subseteq Z$ is as required.

9 Equivalence relations with standard size classes

In modern descriptive set theory, an equivalence relation E is *countable* iff all equivalence classes $[x]_E = \{y : x E y\}$, $x \in \text{dom } E$, are at most countable. See [10] on properties and some open problems related to countable equivalence relations. But in the nonstandard setting the structure of equivalence

⁷ This condition can be weakened to $[x]_E \cap \text{dom } R \neq \emptyset$ for any $x \in X$ without any harm.

⁸ Note that in this case any invariant pre-injection is a partial map that can be immediately extended to a reduction, and hence in fact $E \leq_{\text{eff}}^+ D_Y$ holds.

relations in a much wider class turns out to be considerably simpler: all of them admit effective transversals.

Recall that a *transversal* of an equivalence relation is any set having exactly one element in common in every equivalence class.

Theorem 9.1 *Any Δ_2^{ss} equivalence relation \mathbf{E} , on an internal set H and with s. s. classes, has a Δ_2^{ss} transversal and hence is Δ_2^{ss} -smooth.*

Opposed to this, the Vitali equivalence on the reals is obviously countable but not smooth (via Borel maps), neither it admits a Borel transversal.

Proof First of all, a Δ_2^{ss} transversal implies Δ_2^{ss} -smoothness: let $\rho(x)$ denote the only element of the transversal equivalent to x and apply 2.1 to show that ρ is still Δ_2^{ss} . Let us prove the existence of a Δ_2^{ss} transversal.

By definition $\mathbf{E} = \bigcup_{a \in A} \bigcap_{b \in B} E_{ab}$, where $E_{ab} \subseteq H \times H$ are internal sets while $A, B \in \mathbb{WF}$. Put $P \text{''} x = \{y : \langle x, y \rangle \in P\}$ for $P \subseteq H \times H$ and $x \in H$.

Lemma 9.2 *There exists a standard size family \mathcal{F} of internal maps $F : H \rightarrow H$ such that $[x]_{\mathbf{E}} \subseteq \{F(x) : F \in \mathcal{F}\}$ for all $x \in H$.*

Proof It suffices to prove the lemma for each “constituent” $E_a = \bigcap_{b \in B} E_{ab}$ of \mathbf{E} . According to 1.3.6 in [17], the intersection $\bigcap \mathcal{X}$ of a s. s. family \mathcal{X} of internal sets either is not a s. s. set or it is finite and there is a finite $\mathcal{X}' \subseteq \mathcal{X}$ such that $\bigcap \mathcal{X}' = \bigcap \mathcal{X}$. It follows that every set $E_a[x]$ is finite and moreover there is a finite set $\beta_{ax} \subseteq B$ such that $E_a \text{''} x = \bigcap_{b \in \beta_{ax}} E_{ab} \text{''} x$. Put, for any $n \in \mathbb{N}$ and any finite $\beta \subseteq B$,

$$E_{a\beta} = \bigcap_{b \in \beta} E_{ab} \quad \text{and} \quad P_{a\beta n} = \{\langle x, y \rangle \in E_{a\beta} : \text{card } E_{a\beta} \text{''} x \leq n\}.$$

All sets $P_{a\beta n}$ are internal. We define $F_{a\beta ni}(x) = “i\text{-th element of } P_{a\beta n} \text{ in the sense of a fixed internal linear ordering of } P_{a\beta n}”$ in the case when $1 \leq i \leq n$ and $P_{a\beta n}$ contains at least i elements, and $F_{a\beta ni}(x) = y_0$ otherwise, where y_0 is a once and for all fixed element of H . It remains to define \mathcal{F} to be the family of all functions $F_{a\beta ni}$.

Let \mathcal{F} be as in the lemma. The sets

$$D_F = \text{dom}(\mathbf{E} \cap F) = \{x \in H : x \mathbf{E} F(x)\} \quad (F \in \mathcal{F})$$

belong to Δ_2^{ss} by Proposition 2.1. Let us fix an internal wellordering \prec of the set H . Suppose that $F \in \mathcal{F}$. For any $x \in H$ we carry out the following construction called the *F-construction for x*. Define an internal \prec -decreasing sequence $\{x_{(a)}\}_{a \leq a(x)}$ of length $a(x) + 1 \in {}^*\mathbb{N}$. Its terms $x_{(a)}$ are defined

by induction on a . Put $x_{(0)} = x$. Assume that $x_{(a)}$ has been defined. If $z = F(x_{(a)}) \prec x_{(a)}$ then put $x_{(a+1)} = z$, otherwise put $a(x) = a$ and stop the construction. Eventually the construction ends since $x_{(a+1)} \prec x_{(a)}$ for all a . Put $\nu_F(x) = 0$ if $a(x)$ is even and $\nu_F(x) = 1$ otherwise.

Define $\psi(x)(F) = \nu_F(x)$ for any $x \in H$, $F \in \mathcal{F}$; thus $\psi : H \rightarrow 2^{\mathcal{F}}$.

Lemma 9.3 *If $r \in 2^{\mathcal{F}}$ then $\Psi_r = \{x \in H : \psi(x) = r\}$ belongs to Δ_2^{ss} .*

Proof Note that $x \in \Psi_r$ iff $\nu_F(x) = r(F)$ for all $F \in \mathcal{F}$. On the other hand, all sets $X_F = \{x \in H : \nu_F(x) = 0\}$ ($F \in \mathcal{F}$) are internal because the F -construction is internal. It remains to apply Proposition 2.1.

According to the next lemma, any two different but E-equivalent elements $x \in H$ have different “profiles” $\psi(x)$.

Lemma 9.4 *If $x \neq y \in H$ and $x \text{ E } y$ then $\psi(x) \neq \psi(y)$.*

Proof Suppose that $y \prec x$. There exists a function $F \in \mathcal{F}$ such that $y = F(x)$. Then $y = x_{(1)}$ in the sense of F -construction for x . It follows that the F -construction for y has exactly one step less than the F -construction for x . Thus $\nu_F(x) \neq \nu_F(y)$ and $\psi(x) \neq \psi(y)$.

We continue the proof of Theorem 9.1. Note that $2^{\mathcal{F}}$ and $\mathcal{P}(2^{\mathcal{F}})$ are sets of standard size together with \mathcal{F} (1.3.3 in [17]). Thus by the axiom of Standard Size Choice there is a map $A \mapsto r_A$ such that $r_A \in A$ for any non-empty $A \subseteq 2^{\mathcal{F}}$. Its graph $C = \{\langle A, r \rangle : A \subseteq 2^{\mathcal{F}} \wedge r = r_A\}$ is a s.s. set together with $\mathcal{P}(2^{\mathcal{F}})$. For any $x \in H$ put $A(x) = \{\psi(y) : y \in [x]_{\text{E}}\}$, a non-empty subset of $2^{\mathcal{F}}$. Now $X = \{x \in H : \psi(x) = r_{A(x)}\}$ is a transversal for E by Lemma 9.4.

Prove that X is a Δ_2^{ss} set. By definition $X = \bigcup_{\langle A, r \rangle \in C} Y_A \cap \Psi_r$, where $Y_A = \{x \in H : A(x) = A\}$. However $\Psi_r \in \Delta_2^{\text{ss}}$ by Lemma 9.3. It remains to check that $Y_A \in \Delta_2^{\text{ss}}$ for each $A \subseteq 2^{\mathcal{F}}$. Note that $A(x) = \{\psi(F(x)) : F \in \mathcal{F} \wedge x \in D_F\}$, and hence $A(x) = A$ is equivalent to

$$\begin{aligned} \forall r \in A \exists F \in \mathcal{F} (x \in D_F \wedge F(x) \in \Psi_r) \wedge \\ \wedge \forall F \in \mathcal{F} \exists r \in A (x \in D_F \implies F(x) \in \Psi_r). \end{aligned}$$

Yet the sets Ψ_r and D_F are Δ_2^{ss} (see above), while the domains A and \mathcal{F} are sets of standard size. Now apply Proposition 2.1.

□ (Thm 9.1)

10 Monadic partitions

A cut $U \subseteq {}^*\mathbb{N}$ is *additive* if $a \in U \implies 2a \in U$. Any such cut U induces an equivalence relation $x M_U y$ iff $|x - y| \in U$ on ${}^*\mathbb{N}$. (The additivity implies that M_U is transitive.) Its equivalence classes $[x]_U = \{y : x M_U y\} = \{y : |x - y| \in U\}$, are called *U -monads* and relations of the form M_U , accordingly, *monadic* equivalence relations or *monadic partitions*.

Monads of various kinds are considered in nonstandard analysis. As for those induced by additive cuts in ${}^*\mathbb{N}$, see [11, 20].

The following is an elementary corollary of Proposition 5.2:

Proposition 10.1 *If $\emptyset \neq U \subsetneq {}^*\mathbb{N}$ is an additive Δ_2^{ss} cut then U is non-internal and either standard size cofinal or standard size coinital. \square*

Any additive Δ_2^{ss} cut $U \subseteq {}^*\mathbb{N}$ defines a Δ_2^{ss} quotient ${}^*\mathbb{N}/U = {}^*\mathbb{N}/M_U$, the set of all U -monads. According to the next theorem, effective cardinalities of those quotients are determined by two factors. The first of them is

$$\text{wid } U = \bigcap_{u \in U, u' \in {}^*\mathbb{N} \setminus U} [0, \frac{u'}{u}] = \bigcap_{u \in U} \bigcup_{u' \in U, u' > u} [0, \frac{u'}{u}],$$

the *width* of U .⁹ The second one is the cofinality/coinitiality. The *cofinality* $\text{cof } U$ of a standard size (s. s.) cofinal non-internal cut, is the least cardinal $\vartheta \in \mathbf{Card}$ such that U has an increasing cofinal sequence of type ϑ . The *coinitiality* $\text{coi } U$ of a standard size coinital cut is defined similarly, with a reference to coinital sequences in ${}^*\mathbf{Card} \setminus U$. Note that $\text{cof } U$ and $\text{coi } U$ are infinite regular cardinals.

Additive cuts of lowest possible width are obviously those of the form $U = c\mathbb{N}$, $c \in {}^*\mathbb{N}$ and $U = c/\mathbb{N}$, $c \in {}^*\mathbb{N} \setminus \mathbb{N}$, which we call *slow*; they satisfy $\text{wid } U = \mathbb{N}$. Other additive cuts will be called *fast*.

Theorem 10.2 *Suppose that U, V are additive Δ_2^{ss} cuts in ${}^*\mathbb{N}$ other than \emptyset and ${}^*\mathbb{N}$. Then (i) $|{}^*\mathbb{N}|^{\text{eff}} \leq |{}^*\mathbb{N}/U|^{\text{eff}}$. In addition,*

- (ii) *${}^*\mathbb{N}/U$ is Δ_2^{ss} -smooth iff ${}^*\mathbb{N}/U$ has a Δ_2^{ss} transversal iff U is slow;*
- (iii) *if U is slow then $|{}^*\mathbb{N}/U|^{\text{eff}} \leq |{}^*\mathbb{N}/V|^{\text{eff}}$;*
- (iv) *if both U, V are s. s. cofinal cuts and U is fast then $|{}^*\mathbb{N}/U|^{\text{eff}} \leq |{}^*\mathbb{N}/V|^{\text{eff}}$ iff: $\text{cof } U = \text{cof } V$ and $\text{wid } U \subseteq \text{wid } V$;*
- (v) *if both U, V are s. s. coinital cuts and U is fast then $|{}^*\mathbb{N}/U|^{\text{eff}} \leq |{}^*\mathbb{N}/V|^{\text{eff}}$ iff: $\text{coi } U = \text{coi } V$ and $\text{wid } U \subseteq \text{wid } V$;*

⁹ Also called *the thickness* of U in some papers on AST.

(vi) if U, V are fast cuts, U is s. s. cofinal and V is s. s. coinital then $|\mathbb{N}/U|^{\text{eff}}$ and $|\mathbb{N}/V|^{\text{eff}}$ are incomparable.

Thus either of the two classes of monadic partitions (s. s. cofinal and s. s. coinital) is linearly \leq_{eff} -(pre)ordered in each subclass of the same cofinality (coinitality), slow partitions of both classes form the \leq_{eff} -least type, and there is no other \leq_{eff} -connection between the two classes and their same-cofinality/coinitality subclasses.

See [16] for earlier results of countably determined and Borel reducibility of monadic partitions for *countably* cofinal/coinital cuts.

11 The proof of the reducibility theorem

We begin the proof of Theorem 10.2 with the following observation.

Remark 11.1 Call a set $X \subseteq \mathbb{N}$ *scattered* iff there is a number $c \in \mathbb{N} \setminus \mathbb{N}$ such that $\frac{|X \cap I|^{\text{int}}}{c}$ is infinitesimal for any interval I in \mathbb{N} of length c . It is quite clear that \mathbb{N} is not a finite union of scattered sets, and hence, by **Saturation**, \mathbb{N} is not a standard size union of internal scattered sets. \square

Proof of Theorem 10.2. (i) Choose a number $h \in \mathbb{N} \setminus U$. The map $x \mapsto [xh]_U$ is an injection of \mathbb{N} into \mathbb{N}/U .

(ii) If \mathbb{N}/U admits a Δ_2^{ss} transversal then it is Δ_2^{ss} -smooth. (Let, for $x \in \mathbb{N}$, $\rho(x)$ be the only element of the transversal equivalent to x .) Suppose that \mathbb{N}/U is smooth, i. e. $M_U \leq_{\text{eff}} D_R$ for a suitable Δ_2^{ss} set Z . This is witnessed by a Δ_2^{ss} reduction $\rho : \mathbb{N} \rightarrow Z$. By Theorem 3.1(i) the set $\text{ran } \rho$ can be covered by an internal set Y with $|Y|^{\text{int}} \leq |\mathbb{N}|^{\text{int}}$. Thus $|\mathbb{N}/U|^{\text{eff}} \leq |\mathbb{N}|^{\text{eff}}$. Then $|\mathbb{N}/U|^{\text{eff}} \leq |\mathbb{N}/V|^{\text{eff}}$ for any other additive Δ_2^{ss} cut V by (i), thus U must be slow by (vi). Finally, if U is slow then \mathbb{N}/U has a Δ_2^{ss} transversal by Theorem 1.4.7 in [17].¹⁰

(iii) If U is slow then \mathbb{N}/U is Δ_2^{ss} -smooth, and in fact $|\mathbb{N}/U|^{\text{eff}} \leq |\mathbb{N}|^{\text{eff}}$, see the proof of (ii). It remains to apply (i).

(iv) Thus let U, V be additive s. s. cofinal cuts. Choose increasing sequences $\{u_\xi\}_{\xi < \vartheta}$ and $\{v_\eta\}_{\eta < \tau}$ cofinal in resp. U and V ; $\vartheta = \text{cof } U$ and $\tau = \text{cof } V$ being infinite regular cardinals in \mathbb{WF} . As U is supposed to be fast, we can assume that $\frac{u_{\xi+1}}{u_\xi}$ is infinitely large for all ξ .

¹⁰ Theorem 9.1 yields a Δ_2^{ss} transversal for \mathbb{N}/\mathbb{N} , and hence for any $\mathbb{N}/(h\mathbb{N})$ by multiplication. Transversals defined this way are countably determined but not Borel. Yet partitions of the form $\mathbb{N}/(h/\mathbb{N})$ have no countably determined transversals by 9.7.14 in [17]. These theorems were obtained in [17] on the base of earlier results in [11].

Part 1: assuming $|\ast\mathbb{N}/U|^{\text{eff}} \leq |\ast\mathbb{N}/V|^{\text{eff}}$, we prove that $\text{wid } U \subseteq \text{wid } V$. Let, by 8.2, $R \subseteq \ast\mathbb{N} \times \ast\mathbb{N}$ be a (U, V) -invariant pre-injection, thus $\text{dom } R = \ast\mathbb{N}$, and $|x - x'| \in U \iff |y - y'| \in V$ for all pairs $\langle x, y \rangle$ and $\langle x', y' \rangle$ in R .

Since R is Δ_2^{ss} , we have, by definition, $R = \bigcup_{a \in A} \bigcap_{b \in B} R_{ab}$, where $A, B \in \mathbb{WF}$ and the sets $R_{ab} \subseteq \ast\mathbb{N} \times \ast\mathbb{N}$ are internal.

Let us fix $a \in A$.

Then $R_a = \bigcap_{b \in B} R_{ab} \subseteq R$, hence for any $\eta < \tau$ we have

$$\forall b (x R_{ab} y \wedge x' R_{ab} y') \wedge |y - y'| < v_\eta \implies \exists \xi < \vartheta (|x - x'| < u_\xi)$$

for all $x, x', y, y' \in \ast\mathbb{N}$. We obtain, by **Saturation**,

$$\begin{aligned} \forall \eta < \tau \quad \exists \text{ finite } F \subseteq B \quad \exists \xi < \vartheta \quad \forall x, x', y, y' \in \ast\mathbb{N} : \\ x R_{aF} y \wedge x' R_{aF} y' \wedge |y - y'| < v_\eta \implies |x - x'| < u_\xi, \end{aligned} \quad (7)$$

where $R_{aF} = \bigcap_{b \in F} R_{ab}$. A similar (symmetric) argument yields:

$$\begin{aligned} \forall \xi < \vartheta \quad \exists \text{ finite } F' \subseteq B \quad \exists \eta < \tau \quad \forall x, x', y, y' \in \ast\mathbb{N} : \\ x R_{aF'} y \wedge x' R_{aF'} y' \wedge |x - x'| < u_\xi \implies |y - y'| < v_\eta. \end{aligned} \quad (8)$$

Suppose, towards the contrary, that $\text{wid } U \not\subseteq \text{wid } V$. Then there exists $\eta < \tau$ such that the sequence $\{\frac{v_{\eta'}}{v_\eta}\}_{\eta < \eta' < \tau}$ is not cofinal in $\text{wid } U$.

Keeping $a \in A$ still fixed, we let F and ξ satisfy (7) for this η . By the choice of η , there exists an ordinal $\xi' > \xi$ such that $\frac{u_{\xi'}}{u_\xi} > \frac{v_{\eta'}}{v_\eta}$ for any $\eta' > \eta$, hence in fact $\frac{u_{\xi'}}{u_\xi} > \ell \cdot \frac{v_{\eta'}}{v_\eta}$ for any $\eta' > \eta$ and any $\ell \in \mathbb{N}$. We now let F' and η' satisfy (8) (as F and η) for the ξ' considered. We may assume that $F \subseteq F'$ and $\eta' \geq \eta$ — otherwise take, resp., the union and the maximum of the two. Then we have, for all $\langle x, y \rangle, \langle x', y' \rangle$ in the set $R(a) = R_{aF'}$:

$$\left. \begin{aligned} |y - y'| < v_\eta &\implies |x - x'| < u_\xi \\ |x - x'| < u_{\xi'} &\implies |y - y'| < v_{\eta'} \end{aligned} \right\}; \quad \xi, \xi', \eta, \eta' \text{ depend on } a. \quad (9)$$

Put $D(a) = \text{dom } R(a)$, an internal subset of $\ast\mathbb{N}$ together with $R(a)$.

Note that any interval of length $v_{\eta'}$ in $\ast\mathbb{N}$ consists of approximately $s = \frac{v_{\eta'}}{v_\eta}$ subintervals of length v_η . Accordingly any interval of length $v_{\xi'}$ consists of approximately $t = \frac{u_{\xi'}}{u_\xi}$ subintervals of length u_ξ , while $\frac{s}{t}$ is infinitesimal by the above. It follows by (9) that $\frac{|I \cap D(a)|^{\text{int}}}{|I|^{\text{int}}}$ is infinitesimal for any interval I in $\ast\mathbb{N}$ of length $u_{\xi'}$, hence $D(a)$ is scattered in the sense of 11.1.

On the other hand $\ast\mathbb{N} = \text{dom } R = \bigcup_{a \in A} D_a = \bigcup_{a \in A} D(a)$, where $D_a = \text{dom } R_a$, simply because $R_a \subseteq R(a)$, which is a contradiction with 11.1.

Part 2: in the same assumptions and notation as in Part 1, we prove that $\text{cof } U = \text{cof } V$. This means to prove $\vartheta = \tau$. Suppose $\vartheta \neq \tau$. Let say $\vartheta < \tau$. (The other case is similar.) Then, for a fixed $a \in A$, there is an ordinal $\eta < \tau$, one and the same for all $\xi < \vartheta$, such that (8) takes the form:

$$\begin{aligned} \forall \xi < \vartheta \quad \exists \text{ finite } F' \subseteq B \quad \forall x, x', y, y' \in {}^*\mathbb{N} : \\ x R_{aF'} y \wedge x' R_{aF'} y' \wedge |x - x'| < u_\xi \implies |y - y'| < v_\eta. \end{aligned} \quad (10)$$

Take an ordinal $\xi < \vartheta$ for this η by (7), and then apply (10) for $\xi + 1$. We obtain a finite set $F \subseteq B$ such that, for all $x, x' \in D(a) = \text{dom } R_{aF}$:

$$|x - x'| < u_{\xi+1} \implies |x - x'| < u_\xi. \quad (11)$$

However, as U is fast, the cofinal sequence $\{u_\xi\}$ can be chosen so that $\frac{u_\xi}{u_{\xi+1}}$ is infinitesimal for all ξ . Then the set $D(a)$ is scattered by (11), and so on towards the contradiction as in Part 1.

Part 3. Suppose that $\text{cof } U = \text{cof } V = \vartheta$ (an infinite regular cardinal in Card) and $\text{wid } U \subseteq \text{wid } V$. To prove $|{}^*\mathbb{N}/U|^{\text{eff}} \leq |{}^*\mathbb{N}/V|^{\text{eff}}$ it suffices, by 8.2, to define a reduction of ${}^*\mathbb{N}/U$ to ${}^*\mathbb{N}/V$. Let $\{u_\xi\}_{\xi < \vartheta}$, $\{v_\xi\}_{\xi < \vartheta}$ be increasing cofinal sequences in the cuts resp. U, V . Due to additivity of the cuts, we may w.l.o.g. assume that all terms u_ξ, v_ξ are powers of 2.

We first define subsequences of the cofinal sequences satisfying a certain term-to-term inequality. Note that $\text{wid } U \subseteq \text{wid } V$ basically means

$$\forall v \in V \quad \exists u \in U \quad \forall u' \in U, u' > u \quad \exists v' \in V, v' > v \left(\frac{u'}{u} \leq \frac{v'}{v} \right).$$

This allows us to define an unbounded subsequence of $\{u_\xi\}_{\xi < \vartheta}$ such that, after the reenumeration, the following holds (ξ, η, ζ are ordinals $< \vartheta$):

$$\forall \zeta \quad \forall \xi > \zeta \quad \exists \eta > \zeta \left(\frac{u_\xi}{u_\zeta} \leq \frac{v_\eta}{v_\zeta}, \quad \text{that is,} \quad \frac{v_\zeta}{u_\zeta} \leq \frac{v_\eta}{u_\xi} \right),$$

and then to once again define an unbounded subsection of, now, $\{v_\eta\}_{\eta < \vartheta}$ to satisfy, after the reenumeration, the following:

$$\forall \xi < \eta < \vartheta \left(\frac{v_\xi}{u_\xi} \leq \frac{v_\eta}{u_\eta}, \quad \text{that is,} \quad \frac{u_\eta}{u_\xi} \leq \frac{v_\eta}{v_\xi} \right). \quad (12)$$

Finally, we may assume that $u_0 = 1$. (Replace each u_ξ by $u'_\xi = \frac{u_\xi}{u_0}$. As all u_ξ are powers of 2, these fractions belong to ${}^*\mathbb{N}$. The sequence $\{u'_\xi\}$ is then cofinal in the cut $U' = U/u_0 = \{u : uu_0 \in U\}$. The inequality $|{}^*\mathbb{N}/U|^{\text{eff}} \leq |{}^*\mathbb{N}/U'|^{\text{eff}}$ is witnessed by the map $[x]_U \mapsto [\text{entire part of } \frac{x}{u_0}]_{U'}$.)

Note that the map f sending each u_ξ to v_ξ satisfies the following: $\text{dom } f = \{u_\xi : \xi < \vartheta\}$ is a s.s. set, $\text{dom } f$ and $\text{ran } f$ consist of powers of 2, and $\frac{f(u)}{u} \leq$

$\frac{f(u')}{u'}$ for all $u < u'$ in $\text{dom } f$ by (12). By Saturation there is an internal function F with $D = \text{dom } F$ a hyperfinite subset of ${}^*\mathbb{N} \setminus \{0\}$, such that $\text{dom } f \subseteq \text{dom } F$, $F(u_\xi) = v_\xi$ for all ξ , and still $D = \text{dom } F$ and $Z = \text{ran } F$ consist of powers of 2 and $\frac{f(d)}{d} \leq \frac{f(d')}{d'}$ for all $d < d'$ in D .

Let $h = |D|^{\text{int}} = |Z|^{\text{int}}$ and $D = \{d_1, d_2, \dots, d_h\}$, $Z = \{z_1, z_2, \dots, z_h\}$, in the increasing order of ${}^*\mathbb{N}$ in \mathbb{I} . Then $z_\nu = F(d_\nu)$ for all $\nu = 1, \dots, h$. As all d_ν, z_ν are powers of 2, the fractions $j_\nu = \frac{d_{\nu+1}}{d_\nu}$ and $k_\nu = \frac{z_{\nu+1}}{z_\nu}$ belong to ${}^*\mathbb{N}$ and $j_\nu \leq k_\nu$ by the above. Note also that $d_1 = u_0 = 1$.

Any number $x \in {}^*\mathbb{N}$ admits, in \mathbb{I} , a unique representation in the form $x = \sum_{\nu=1}^h \alpha_\nu d_\nu$, where $\alpha_\nu \in {}^*\mathbb{N}$ and $0 \leq \alpha_\nu < j_\nu$ for all $\nu = 1, \dots, h-1$ (but α_h is not restricted, of course). The first idea that comes to mind is to try $\sigma(x) = \sum_{\nu=1}^h \alpha_\nu z_\nu$ as a reduction of ${}^*\mathbb{N}/U$ to ${}^*\mathbb{N}/V$. However this does not work. Indeed let $x = \sum_{\nu=1}^h d_\nu$ and $x' = \sum_{\nu=1}^{h-1} (j_\nu - 1)d_\nu$, so that $x - x' = 1$ but $|\sigma(x) - \sigma(x')|$ can be very big in the case when, say, $k_\nu > j_\nu$ for all ν . However there is a useful modification.

Suppose that $x = \sum_{\nu=1}^h \alpha_\nu d_\nu \in {}^*\mathbb{N}$, and $0 \leq \alpha_\nu < j_\nu$ for $\nu = 1, \dots, h-1$, as above. Say that x is *type-1* if there exist indices $1 \leq \nu' < \nu'' \leq h-1$ such that $d_{\nu'} \in U$, $d_{\nu''} \notin U$, and $\alpha_\nu = j_\nu - 1$ for all ν such that $\nu' \leq \nu \leq \nu''$. Then take the largest ν'' and the least ν' such that the pair ν', ν'' has this property, and put $\bar{\alpha}_\nu = \alpha_\nu$ for all $\nu < \nu'$ and $\nu > \nu''$, $\bar{\alpha}_\nu = 0$ for $\nu' \leq \nu \leq \nu''$, and $\bar{\alpha}_{\nu''+1} = \alpha_{\nu''+1} + 1$, and define $\bar{x} = \sum_{\nu=1}^h \bar{\alpha}_\nu d_\nu$. Otherwise (x is *type-2*) put $\bar{x} = x$. Easily $\bar{x} - x = \alpha_{\nu'} \in U$ in the type-1 case.

Prove that the map $\rho(x) = \sigma(\bar{x})$ is a reduction of ${}^\mathbb{N}/U$ to ${}^*\mathbb{N}/V$, that is, $|x - x'| \in U \iff |\sigma(\bar{x}) - \sigma(\bar{y})| \in V$ holds for all $x, x' \in {}^*\mathbb{N}$.*

Assume that $x = \sum_{\nu=1}^h \alpha_\nu d_\nu$ and $y = \sum_{\nu=1}^h \gamma_\nu d_\nu$, where $\alpha_\nu, \gamma_\nu < j_\nu$, and $|x - y| \in U$, hence $|x - y| < u_\xi = d_\nu$ for some $\xi < \vartheta$, $\nu < h$. Let $x < y$. Assume w.l.o.g. that x, y are of type-2. (Otherwise change x, y to \bar{x}, \bar{y} .) There exist infinitely (but \mathbb{I} -finitely) many indices $\nu' > \nu$ such that $\alpha_{\nu'} \neq j_{\nu'} - 1$. In this case $\alpha_{\nu'} = \gamma_{\nu'}$ for all $\nu' \geq \nu$ by the assumption $|x - y| < d_\nu$. Thus $|\sigma(x) - \sigma(y)| \in V$ (since $j_\nu \leq k_\nu$ for all ν), as required.

Now suppose that $x < y$ are as above, in particular, of type-2, but $|x - y| \notin U$, hence $|x - y| > u_\xi$ for all $\xi < \vartheta$. Then $D' = \{d_\nu \in D : \alpha_\nu \neq \gamma_\nu\}$ is an internal set, hence it has the largest element, say $d_{\nu''} = \max D'$. Note that $d_{\nu''} \notin U$. (Use the assumption $|x - y| > u$ for all $u \in U$.) We have $\alpha_{\nu''} < \gamma_{\nu''}$ (as $x < y$). Then the only opportunity for $|\sigma(x) - \sigma(y)|$ to belong to V is obviously the existence of an index $\nu' < \nu''$ such that $z_{\nu'} \in V$ and $\gamma_\nu = 0$, $\alpha_\nu = j_\nu - 1 = k_\nu - 1$ for all ν between ν' and ν'' . But this contradicts the assumption that x is of type-2. Thus $|\sigma(x) - \sigma(y)| \notin V$, as required.

(v) The proof of this item follows the same line as the proof of (iv), but with appropriate changes, of course. It will appear elsewhere.

(vi) Suppose that U, V are resp. s. s. cofinal, s. s. coinital additive fast cuts. Prove that $|\mathbb{N}/U|^{\text{eff}} \not\leq |\mathbb{N}/V|^{\text{eff}}$; the proof of $|\mathbb{N}/V|^{\text{eff}} \not\leq |\mathbb{N}/U|^{\text{eff}}$ is similar. Choose an increasing sequence $\{u_\xi\}_{\xi < \vartheta}$ and a decreasing sequence $\{v_\eta\}_{\eta < \tau}$ resp. cofinal in U and coinital in $\mathbb{N} \setminus V$; $\vartheta = \text{cof } U$ and $\tau = \text{coi } V$ being infinite regular cardinals in \mathbb{WF} .

Suppose on the contrary that $R \subseteq \mathbb{N} \times \mathbb{N}$ is an invariant pre-injection of \mathbb{N}/U to \mathbb{N}/V , that is, $|x - x'| \in U \iff |y - y'| \in V$ for any pairs $\langle x, y \rangle$ and $\langle x', y' \rangle$ in R , and $\text{dom } R = \mathbb{N}$. Then $R = \bigcup_{a \in A} \bigcap_{b \in B} R_{ab}$, where $A, B \in \mathbb{WF}$ and R_{ab} are internal sets. Arguing as above in the proof of (iv) (parts 1,2), we obtain by Saturation for any fixed $a \in A$:

$$\begin{aligned} \exists \text{ finite } F \subseteq B \quad \exists \xi < \vartheta \quad \exists \eta < \tau \quad \forall x, x', y, y' \in \mathbb{N} : \\ x R_{aF} y \wedge x' R_{aF} y' \wedge |y - y'| < v_\eta \implies |x - x'| < u_\xi, \end{aligned} \quad (13)$$

where $R_{aF} = \bigcap_{b \in F} R_{ab}$, and, in the opposite direction,

$$\begin{aligned} \forall \xi < \vartheta \quad \forall \eta < \tau \quad \exists \text{ finite } F' \subseteq B \quad \forall x, x', y, y' \in \mathbb{N} : \\ x R_{aF'} y \wedge x' R_{aF'} y' \wedge |x - x'| < u_\xi \implies |y - y'| < v_\eta. \end{aligned} \quad (14)$$

Let $a \in A$. Take ξ, η, F as in (13). Take then F' as in (14) for $\xi + 1$ and η . We may assume that $F \subseteq F'$. Then for all pairs $\langle x, y \rangle, \langle x', y' \rangle$ in the set $D(a) = \text{dom } R(a)$, where $R(a) = R_{aF'}$, we have $|x - x'| < u_{\xi+1} \implies |x - x'| < v_\xi$. Assuming w. l. o. g. that $\frac{u_{\xi+1}}{u_\xi}$ is infinitely large for all ξ , we conclude that each $D(a)$ is an internal scattered set in the sense of 11.1, and so on towards contradiction as above.

□ (Thm 10.2)

12 On small and large effective sets

Here we prove a “small–large” type theorem related to Δ_2^{ss} quotients. The notions of smallness and largeness will be connected with a cut $U \subseteq \mathbb{Card}$, as in Corollary 5.6. By necessity there also will be a gap between the largeness and smallness, but we don’t know whether its size can be reduced.

Recall that a cut (initial segment) $U \subseteq \mathbb{Card}$ is called *exponential* iff $\kappa \in U \implies 2^\kappa \in U$, or, equivalently, $U = 2^U$ holds, where $2^U = \{\vartheta \in \mathbb{Card} : \exists \kappa \in U (\vartheta \leq 2^\kappa)\}$. (2^κ is understood as the cardinal exponentiation in \mathbb{I} .)

We write $\lambda \geq 2^U$ to mean $\lambda \geq 2^\kappa$ for all $\kappa \in U$.

Theorem 12.1 *Suppose that \mathbb{E} is a Δ_2^{ss} equivalence relation on an internal set H and $U \subseteq \mathbb{Card}$ is a Δ_2^{ss} cut such that $\mathbb{N} \subseteq U$. Then at least one of the following conditions holds:*

- (A) for any $\lambda \in {}^*\mathbf{Card}$ with $\lambda \geq 2^U$ and any $m \in {}^*\mathbb{N} \setminus \mathbb{N}$ there is an internal map ρ defined on H such that $|\mathbf{ran} \rho|^{\mathbf{int}} \leq \lambda^m$ ($=\lambda$ whenever $\lambda \notin {}^*\mathbb{N}$) and $\rho(x) = \rho(y) \implies x \mathbf{E} y$ for all $x, y \in H$;
- (B) there exists an internal set $Y \subseteq H$ of pairwise \mathbf{E} -inequivalent elements such that $|Y|^{\mathbf{int}} \notin U$.

If U is an exponential non-internal cut then (A) and (B) are incompatible even in the case when $\Delta_2^{\mathbf{ss}}$ maps ρ are allowed in (A).

In terms of effective cardinals (B) means $\kappa \leq |H/\mathbf{E}|^{\mathbf{eff}}$ (and even by means of an internal reduction) for some $\kappa = |Y|^{\mathbf{int}} \in {}^*\mathbb{N} \setminus U$, that is a restriction of the cardinality of the quotient H/\mathbf{E} from below. Accordingly (A) means that for all $\lambda \geq 2^U$ and $m \in {}^*\mathbb{N} \setminus \mathbb{N}$ and any internal Z with $|Z|^{\mathbf{int}} = \lambda^m$ there is an equivalence relation \mathbf{F} on Z (in terms of (A), $\rho(x) \mathbf{F} \rho(y)$ iff $x \mathbf{E} y$) such that $|H/\mathbf{E}|^{\mathbf{eff}} \leq |Z/\mathbf{F}|^{\mathbf{eff}}$ (still by means of an internal reduction), a restriction of the cardinality of H/\mathbf{E} from above.

Some theorems of this form are known from descriptive set theory, for instance Silver's theorem on $\mathbf{\Pi}_1^1$ equivalence relations in [28], in which “small” means at most countably many equivalence classes while “large” means that there exists a pairwise \mathbf{E} -inequivalent perfect set.

Note that the implication $\rho(x) = \rho(y) \implies x \mathbf{E} y$ in (A) cannot be replaced by the equivalence $\rho(x) = \rho(y) \iff x \mathbf{E} y$: indeed the latter would imply the $\Delta_2^{\mathbf{ss}}$ smoothness of \mathbf{E} , which, generally speaking, is not the case even for equivalence relations of the form \mathbf{M}_U by Theorem 10.2.

Proof (Theorem 12.1). *Case 1:* U is standard size cofinal, including internal cuts. In this case we prove an even stronger result, namely the disjunction $(A') \vee (B)$, where

- (A') there exist a set $D \in \mathbb{W}\mathbb{F}$, and for each $d \in D$ an internal set R_d and an internal map $f_d: H \rightarrow R_d$ such that $|R_d|^{\mathbf{int}} \in 2^U$ and $f(x) = f(y) \implies x \mathbf{E} y$ for all $x, y \in H$, where $f(x) = \{f_d(x)\}_{d \in D}$.

We first show that (A') implies (A). Suppose that $\lambda \geq 2^U$, $m \in {}^*\mathbb{N} \setminus \mathbb{N}$. Recall that the map $d \mapsto {}^*d$ is an injection $D \rightarrow {}^*D$. Its image $D' = \{{}^*d : d \in D\} \subseteq {}^*D$ is a set of standard size together with D . By 4.1(iii), D' can be covered by an internal set $S \subseteq {}^*D$ such that $|S|^{\mathbf{int}} \leq m$. The Extension principle (1.3.13 in [17]) yields an internal function F defined on $S \times H$ so that $F({}^*d, x) = f_d(x)$ for all $d \in D$, $x \in H$. By the same reasons there is an internal map r defined on S so that $r({}^*d) = R_d$ for all $d \in D$. We can assume that for any $s \in S$, $r(s)$ is an internal set with $|r(s)|^{\mathbf{int}} < \lambda$, and $F(s, x) \in r(s)$ for all $x \in H$. (Otherwise redefine r and F by $r(s) = \{0\}$

and $F(s, x) = 0$ for all “bad” s — but none of $s = *d$, $d \in D$, is “bad” in the assumptions of (A’). Put $\rho(x)(s) = F(s, x)$ for $x \in H$, $s \in S$.

We begin the proof of (A’) \vee (B). By definition $\mathbf{E} = \bigcup_{a \in A} \bigcap_{b \in B} E_b^a$, where all sets $E_b^a \subseteq H \times H$ are internal while $A, B \in \mathbb{W}\mathbb{F}$. We may w.l.o.g. assume that every set E_b^a is symmetric (similarly to \mathbf{E} itself), that is, $E_b^a = (E_b^a)^{-1}$, where $E^{-1} = \{\langle y, x \rangle : x E y\}$: indeed

$$\mathbf{E} = \mathbf{E} \cap \mathbf{E}^{-1} = \bigcup_{a \in A} \bigcap_{b, b' \in B} E_b^a \cup (E_{b'}^a)^{-1} = \bigcup_{a \in A} \bigcap_{b, b' \in B} C_{bb'}^a,$$

where the sets $C_{bb'}^a = (E_b^a \cup (E_{b'}^a)^{-1}) \cap (E_{b'}^a \cup (E_b^a)^{-1})$ are symmetric. (We write $x E y$ for $\langle x, y \rangle \in E$ whenever E is a binary relation.)

It follows from the transitivity of \mathbf{E} that for any $x, y \in H$

$$\exists a \in A \exists z \in H \forall b \in B (x E_b^a z \wedge y E_b^a z) \implies x \mathbf{E} y.$$

The axiom of **Saturation** transforms this to

$$\exists a \in A \forall B' \in \mathcal{P}_{\mathbf{fin}}(B) \exists z \in H (x E_{B'}^a z \wedge y E_{B'}^a z) \implies x \mathbf{E} y,$$

where $E_{B'}^a = \bigcap_{b \in B'} E_b^a$. As the two leftmost quantifiers are restricted to the sets A and $\mathcal{P}_{\mathbf{fin}}(B)$ in $\mathbb{W}\mathbb{F}$, the last formula is equivalent to

$$\forall \varphi \in \Phi \exists a \in A \exists z \in H (x E_{\varphi(a)}^a z \wedge y E_{\varphi(a)}^a z) \implies x \mathbf{E} y, \quad (15)$$

where $\Phi \in \mathbb{W}\mathbb{F}$ is the set of all functions $\varphi : A \rightarrow \mathcal{P}_{\mathbf{fin}}(B)$.

As U is standard size cofinal, there is an increasing sequence $\{\nu_\xi\}_{\xi < \vartheta}$ of elements $\nu_\xi \in U$, cofinal in U , with ϑ being an infinite cardinal in \mathbf{Card} , or simply U is internal, $\vartheta = 1 = \{0\}$, and ν_0 is the least element in $*\mathbf{Card} \setminus U$.

Suppose that (B) of the theorem fails, i.e. there is no pairwise \mathbf{E} -inequivalent sets Y with $|Y|^{\mathbf{int}} \notin U$. More formally,

$$\forall Y \in P (\forall \xi < \vartheta (|Y|^{\mathbf{int}} \geq \nu_\xi) \implies \exists x \neq y \in Y \exists a \in A \forall b \in B (x E_b^a y)),$$

where $P = \mathcal{P}^1(H) = \{Y \subseteq H : Y \text{ is internal}\}$. **Saturation** converts the expression to the right of \implies to

$$\exists a \in A \forall B' \in \mathcal{P}_{\mathbf{fin}}(B) \exists x \neq y \in Y (x E_{B'}^a y),$$

and then to $\forall \varphi \in \Phi \exists a \in A \exists x \neq y \in Y (x E_{\varphi(a)}^a y)$. We conclude that for any function $\varphi \in \Phi$

$$\forall Y \in P (\forall \xi < \vartheta (|Y|^{\mathbf{int}} \geq \nu_\xi) \implies \exists a \in A \exists x \neq y \in Y (x E_{\varphi(a)}^a y)).$$

Saturation yields an ordinal $\xi(\varphi) < \vartheta$ and a finite set $A_\varphi \subseteq A$ such that

$$\forall Y \in P (|Y|^{\text{int}} \geq \nu_{\xi(\varphi)} \implies \exists a \in A_\varphi \exists x \neq y \in Y (x E_{\varphi(a)}^a y)). \quad (16)$$

Let Y_φ be any maximal (internal) subset of H such that $\neg x E_{\varphi(a)}^a y$ for all $a \in A_\varphi$ and $x \neq y \in Y_\varphi$. Then (16) implies $|Y_\varphi|^{\text{int}} < \nu_{\xi(\varphi)}$, while the properties of maximality of Y_φ and symmetricity of E_b^a imply

$$\forall x \in H \exists y \in Y_\varphi \exists a \in A_\varphi (x E_{\varphi(a)}^a y). \quad (17)$$

Put $\zeta_x(\varphi, a) = \{y \in Y_\varphi : x E_{\varphi(a)}^a y\}$ for $x \in H$, $\varphi \in \Phi$, $a \in A_\varphi$. Thus ζ_x belongs to the set Z of all functions ζ defined on the set $D = \{\langle \varphi, a \rangle : \varphi \in \Phi \wedge a \in A_\varphi\} \in \mathbb{WF}$ and satisfying $\zeta_x(\varphi, a) \in R_\varphi = \mathcal{P}^1(Y_\varphi)$. The sets R_φ are internal and satisfy $|R_\varphi|^{\text{int}} \in 2^U$ (because $|Y_\varphi|^{\text{int}} \in U$).

We claim that $\zeta_x = \zeta_y$ implies $x E y$. It suffices, by (15), to prove that for every $\varphi \in \Phi$ there exist $a \in A$, $z \in H$ such that $x E_{\varphi(a)}^a z$ and $y E_{\varphi(a)}^a z$. Note that $\zeta_x(\varphi, a) = \zeta_y(\varphi, a) \neq \emptyset$ for some $a \in A_\varphi$ by (17). Take any $z \in \zeta_x(\varphi, a)$. Then $z \in Y_\varphi$, thus $\langle x, z \rangle$ and $\langle y, z \rangle$ belong to $E_{\varphi(a)}^a$, as required.

To accomplish the proof of (A') in the assumption $\neg(B)$, we put $f_d(x) = \zeta_x(\varphi, a)$ and $R_d = R_\varphi$ for all $x \in H$ and $d = \langle \varphi, a \rangle \in D$.

Case 2: U is standard size coinital, but non-internal. Suppose that (B) fails, and consider any $m \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $\lambda \geq 2^U$. Then (B) fails also for the internal, hence, s. s. cofinal, cut $U' = \{\kappa \in {}^*\mathbf{Card} : 2^\kappa \leq \lambda\}$: indeed, $U \subseteq U'$ by the choice of λ . Therefore (A) holds for U' . Thus there is an internal map ρ , $\text{dom } \rho = H$, such that $|\text{ran } \rho|^{\text{int}} \leq \lambda^m$ and $\rho(x) = \rho(y) \implies x E y$.

Incompatibility. Assume that $Y \subseteq H$ witnesses (B), in particular, $\kappa = |Y|^{\text{int}} \notin U = 2^U$. Then $U' = \{\lambda \in {}^*\mathbf{Card} : 2^\lambda < \kappa\}$ is an internal cut with $U \subseteq U'$. Thus $U \subsetneq U'$ since U is non-internal. Therefore there is $\vartheta \notin U$ such that $2^\vartheta < \kappa$. Applying this trick once again, we find $\vartheta \notin U$ with $2^{2^\vartheta} < \kappa$. Suppose on the contrary that ρ witnesses (A) for $\lambda = 2^\vartheta$ and some $m \in {}^*\mathbb{N} \setminus \mathbb{N}$, $m < \vartheta$. Then $\rho \upharpoonright Y$ is an internal injection of Y into an internal set $Z = \rho''Y$ satisfying $|Z|^{\text{int}} \leq 2^{\vartheta m}$. But this contradicts Theorem 3.1, since by definition $2^{\vartheta m} \cdot n < 2^{\vartheta \cdot \vartheta} < 2^{2^\vartheta} < \kappa = |Y|^{\text{int}}$ for any $n \in \mathbb{N}$.

□ (Thm 12.1)

The case $U = \mathbb{N}$ deserves special attention. Since \mathbb{N} is a s. s. cofinal cut, a stronger dichotomy holds: (A') \vee (B). Clearly (B) claims the existence of an infinite internal set of pairwise E-inequivalent elements in this case. On the other hand, the sets R_d in (A') are finite, hence $P = \prod_{d \in D} R_d$ is a set of standard size, and so is any quotient of the form P/F , where F is an equivalence relation on P . Thus (A') implies that H/E itself is a set of

standard size. Such a dichotomy (i.e. standard size of H/E or an infinite internal pairwise inequivalent set) is contained in Theorem 1.4.11 in [17]. Similar dichotomies appeared in [16] for countably determined equivalence relations. P. Zlatoš informed us that a close result for $U = \mathbb{N}$ was earlier obtained by Vencovská (unpublished) in the frameworks of **AST**.

13 Nonstandard version of the finite Ramsey theorem

The following corollary of Theorem 12.1 is a Ramsey-like result. Recall that $[A]^n = \{X \subseteq A : \text{card } X = n\}$. By a *partition* of $[A]^n$ we understand any equivalence relation E on $[A]^n$, and a *homogeneous set* for E is any $H \subseteq A$ such that the sets $X \in [H]^n$ are pairwise E -equivalent.

The finite Ramsey theorem claims (in **ZFC**) that

(*) *for any natural numbers ℓ, n, s there is $k \in \mathbb{N}$ such that $k \rightarrow (\ell)_s^n$.*

Here $k \rightarrow (\ell)_s^n$ means that for any partition of $[k]^n$ into s -many parts there is an ℓ -element homogeneous set $H \subseteq k$. We refer to [25], and also to 3.3.7 in [1], § 6 in [21], or [3] for a modern proof, details and related results.

Let $K(\ell, s, n)$ denote the least k satisfying $k \rightarrow (\ell)_s^n$. It is known that $K(\ell, s, n)$ is rapidly increasing as a function of ℓ for any fixed n, s , see [3]. But of course K is a recursive function.

It is an easy nonstandard corollary of (*) that $\kappa \xrightarrow{\text{int}} (\ell)_s^n$ for all $n, s, \ell \in \mathbb{N}$ and $\kappa \in {}^*\mathbb{N} \setminus \mathbb{N}$ where **int** over the arrow means that the partition and the homogeneous set are assumed to be internal. A nicer nonstandard version, also well-known, is $\kappa \xrightarrow{\text{int}} (\infty)_s^n$ for any $n, s \in \mathbb{N}$ and $\kappa \in {}^*\mathbb{N} \setminus \mathbb{N}$, that is, any internal partition $[\kappa]^n$ into s parts admits an infinite internal homogeneous set. By the way, its quantifier structure is simpler than that of (*):

$$\forall \kappa, \ell, n, s \forall \text{partition} \exists A \forall u, v \in [A]^n.$$

The following theorem contains a much more general claim. In **HST**, define a function K in \mathbb{WF} as above. Then *K is a standard function ${}^*\mathbb{N}^3 \rightarrow {}^*\mathbb{N}$ having in the internal universe \mathbb{I} the same properties as K in \mathbb{WF} .

Theorem 13.1 *Suppose that $U \subsetneq {}^*\mathbb{N}$ is a Δ_2^{ss} cut with $\mathbb{N} \subseteq U$, closed under *K and exponential, $n \in \mathbb{N}$, $\kappa \in {}^*\mathbb{N} \setminus U$, and E is a Δ_2^{ss} equivalence relation on $[\kappa]^n$. If there is no internal pairwise E -inequivalent sets $Y \subseteq [\kappa]^n$ satisfying $|Y|^{\text{int}} \notin U$, then the partition E admits an internal homogeneous set $A \subseteq \kappa$ such that $|A|^{\text{int}} \notin U$.*

A similar result was obtained in [22] in the case $U = \mathbb{N}$ for countably determined equivalence relations. See Theorem 2.8 in [19] for a somewhat weaker result in the case when t in the proof of 13.1 is predefined.

Proof Define, in \mathbb{WF} , $f(s) = K(s, s, n)$ for each $s \in \mathbb{N}$. Then $f : \mathbb{N} \rightarrow \mathbb{N}$ and $s \leq f(s)$, $\forall s$. The map *f has the same properties with respect to ${}^*\mathbb{N}$. As U is *K -closed and exponential, there exist $s, \vartheta \in {}^*\mathbb{N} \setminus U$ and $m \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that ${}^*f(s) = {}^*K(s, s, n) \leq \kappa$ and $2^{\vartheta m} \leq s$.

In our assumptions, (B) of Theorem 12.1 fails, hence (A) holds, that is, there exists an internal map ρ defined on $[\kappa]^n$ such that $|\mathbf{ran} \rho|^{\text{int}} \leq 2^{\vartheta m} \leq s$ and $\rho(u) = \rho(v) \implies u \mathbf{E} v$ for all $u, v \in [\kappa]^n$. On the other hand, we have $\kappa \rightarrow (s)_s^n$ by the choice of s , therefore the partition of $[\kappa]^n$ induced by ρ has an internal homogeneous set A such that $|A|^{\text{int}} = s \notin U$. Thus $\rho(u) = \rho(v)$, and hence $u \mathbf{E} v$, for all $u, v \in [A]^n$.

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