VARIA
Ideals and Equivalence Relations

Vladimir Kanovei

Feb 2005, Caltech

Annotation

A selection of basic results on Borel reducibility of ideals and ERs, especially those with comparably short proofs. This is an unfinished text as yet. Some proofs have missing parts and loose ends. ¹

¹ kanovei@mccme.ru and vkanovei@math.uni-wuppertal.de are my contact addresses.
## Contents

1 Reducibility ............................................. 4
   1.a Borel reducibility ..................................... 4
   1.b “Algebraic” Borel reducibility ......................... 5
   1.c Borel, continuous, and Baire measurable reductions .... 5
   1.d Reducibility via maps between the underlying sets ...... 6
   1.e Isomorphism ...................................... 7
   1.f Remarks .................................... 8

2 Introduction to ideals .................................. 8
   2.a Notation .......................................... 8
   2.b P-ideals and submeasures ......................... 9
   2.c Polishable ideals ................................ 10
   2.d Some $F_\sigma$ ideals .......................... 10
   2.e Erdös – Ulam and density ideals ............... 11
   2.f Some transfinite sequences of Borel ideals .......... 12
   2.g “Other” ideals ................................ 13

3 Introduction to equivalence relations ................. 13
   3.a Basic equivalence relations ........................ 13
   3.b Borel reducibility of basic equivalence relations .. 15
   3.c Operations on equivalence relations ............. 18
   3.d Orbit equivalence relations of group actions ..... 18
   3.e Forcings associated with pairs of equivalence relations ...... 21

4 “Elementary” stuff ..................................... 21
   4.a $E_3$ and $T_2$: outcasts ........................ 21
   4.b Discretization and generation by ideals .......... 22
   4.c Summables irreducible to density-0 ............. 24
   4.d The family $\ell^p$ ................................ 26
   4.e $\ell^\infty$: maximal $K_\sigma$ .................. 28

5 Smooth ERs and the first dichotomy ............... 30
   5.a Smooth and below ................................ 30
   5.b Assembling smooth equivalence relations .......... 31
   5.c The 1st dichotomy theorem ...................... 32

6 Hyperfinite and countable ERs ...................... 34
   6.a $\text{Fin}$ is the least! ......................... 34
   6.b Countable equivalence relations .................. 35
   6.c Hyperfinite equivalence relations ........... 36
   6.d Non-hyperfinite countable equivalence relations .. 39
   6.e Assembling hyperfinite equivalence relations .... 39
## CONTENTS

7 The 2nd dichotomy \( \text{40} \)
   7.a The Gandy – Harrington closure \( \text{40} \)
   7.b Restricted product forcing \( \text{42} \)
   7.c Splitting system \( \text{43} \)
   7.d Construction of a splitting system \( \text{44} \)
   7.e A forcing notion associated with \( \mathbf{E}_0 \) \( \text{45} \)

8 Ideal \( \mathcal{I}_1 \) and P-ideals \( \text{47} \)
   8.a Ideals below \( \mathcal{I}_1 \) \( \text{47} \)
   8.b \( \mathcal{I}_1 \) and P-ideals \( \text{49} \)
   8.c The hard part \( \text{51} \)

9 Equivalence relation \( \mathbf{E}_1 \) \( \text{52} \)
   9.a \( \mathbf{E}_1 \) and hypersmoothness \( \text{52} \)
   9.b The 3rd dichotomy \( \text{53} \)
   9.c Case 2 \( \text{54} \)
   9.d The construction \( \text{56} \)
   9.e Above \( \mathbf{E}_1 \) \( \text{58} \)

10 Actions of the infinite symmetric group \( \text{60} \)
   10.a Infinite symmetric group \( \mathcal{S}_\infty \) \( \text{61} \)
   10.b Borel invariant sets \( \text{62} \)
   10.c ERs classifiable by countable structures \( \text{63} \)
   10.d Reduction to countable graphs \( \text{64} \)
   10.e Borel countably classified ERs: reduction to \( \mathbf{T}_\xi \) \( \text{65} \)

11 Turbulent group actions \( \text{67} \)
   11.a Local orbits and turbulence \( \text{67} \)
   11.b Ergodicity \( \text{68} \)
   11.c “Generical” reduction of countably classified ERs to \( \mathbf{T}_\xi \) \( \text{70} \)
   11.d Ergodicity of turbulent actions w.r.t. \( \mathbf{T}_\xi \) \( \text{71} \)
   11.e Inductive step of countable power \( \text{71} \)
   11.f Inductive step of the Fubini product \( \text{73} \)
   11.g Other inductive steps \( \text{74} \)
   11.h An application to the shift actions of ideals \( \text{75} \)

12 Ideal \( \mathcal{I}_3 \) and the equivalence relation \( \mathbf{E}_3 \) \( \text{77} \)
   12.a Ideals below \( \mathcal{I}_3 \) \( \text{77} \)
   12.b Assembling equivalence relations \( \text{77} \)
   12.c The 6th dichotomy \( \text{79} \)
   12.d Case 1 \( \text{80} \)
   12.e Case 2 \( \text{81} \)

13 Summable ideals \( \text{82} \)
   13.a A useful lemma \( \text{82} \)
   13.b Under the summable ideal \( \text{83} \)
   13.c Case 1 \( \text{84} \)
   13.d Case 2 \( \text{86} \)
   13.e Construction \( \text{87} \)
1 REDUCIBILITY

14 c₀-equalities
  14.a Some examples and simple results ............................................ 88
  14.b Classification ................................................................. 90
  14.c LV-equalities ................................................................. 93

15 \(T_2 \) is not reducible to ... 96
  15.a Pinned ERs do not reduce \(T_2 \) .................................................. 96
  15.b Fubini product of pinned ERs is pinned ................................... 97
  15.c Complete left-invariant actions produce pinned ERs .................... 98
  15.d All \(F_\sigma \) ideals are pinned .................................................. 99
  15.e Another family of pinned ideals ............................................. 100

16 Universal analytic ERs and reduction to ideals 101

A Technical introduction
  A.a Notation .................................................................................. 101
  A.b Descriptive set theory ............................................................ 102
  A.c Trivia of “effective” descriptive set theory ............................... 102
  A.d Polish–like families and the Gandy – Harrington topology .......... 104

Список литературы 105

1 Reducibility

There are several reasonable ways to compare ERs, usually formalized in terms of existence of a reduction, i.e., a map of certain kind which allows to derive one of the ERs from the other one. Borel reducibility \(\leq_B \) is the key one, yet there are several special types of \(\leq_B \), in particular, those induced by a low-level maps, useful in many cases. Generally, the most of research on reducibility of Borel ERs or ideals is concentrated around the following notions of reducibility.

1.a Borel reducibility

If \(E \) and \(F \) are ERs on Polish spaces resp. \(X, Y \), then

* \(E \leq_B F \) (Borel reducibility) means that there is a Borel map \(\vartheta : X \to Y \) (called reduction) such that \(x E y \iff \vartheta(x) F \vartheta(y) \) for all \(x, y \in X \);

* \(E \sim_B F \) iff \(E \leq_B F \) and \(F \leq_B E \) (Borel bi-reducibility);

* \(E <_B F \) iff \(E \leq_B F \) but not \(F \leq_B E \) (strict Borel reducibility);

* \(E \subsetneq_B F \) means that there is a Borel embedding, i.e., a \(1 \to 1 \) reduction;

* \(E \approx_B F \) iff \(E \subseteq_B F \) and \(F \subseteq_B E \) (a rare form, [18, § 0]);
1 REDUCIBILITY

* $E \subseteq_n F$ means that there is a Borel invariant embedding, i.e., an embedding $\vartheta$ such that $\text{ran}\vartheta = \{\vartheta(x) : x \in X\}$ is an F-invariant set (meaning that the F-saturation $[\text{ran}\vartheta]_F = \{y' : \exists x (y \in F \vartheta(x))\}$ equals $\text{ran}\vartheta$);

* $E \leq C$, $E \subseteq C$, $E \subseteq^1 C$ mean that there is a continuous resp. reduction, embedding, invariant embedding.

Sometimes they write $X/E \leq_B Y/F$ instead of $E \leq_B F$.

Borel reducibility of ideals: $\mathcal{I} \leq_B \mathcal{J}$ iff $E_\mathcal{I} \leq_B E_\mathcal{J}$. Thus it is required that there is a Borel map $\vartheta : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ such that $x \Delta y \in \mathcal{I}$ iff $\vartheta(x) \Delta \vartheta(y) \in \mathcal{J}$. (Here $\mathcal{I}$ is an ideal on $A$ and $\mathcal{J}$ is an ideal on $B$.)

Versions $\mathcal{I} \leq_C \mathcal{J}$, $\mathcal{I} \subseteq_B \mathcal{J}$, $\mathcal{I} \subseteq_C \mathcal{J}$ have the corresponding meaning.

1.b “Algebraic” Borel reducibility

This is a more special version of Borel reducibility of ideals, characterized by the property that the reduction must respect a chosen algebraic structure. We shall be especially interested in the Boolean algebra structure and a weaker $\Delta$-group structure of sets of the form $\mathcal{P}(A)$. Let $\mathcal{I}$, $\mathcal{J}$ be ideals on resp. $A$, $B$.

Borel BA reducibility: $\mathcal{I} \leq_{BA,BA} \mathcal{J}$ if there is a Borel $\mathcal{J}$-approximate Boolean algebra homomorphism $\vartheta : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ with $x \in \mathcal{I} \iff \vartheta(x) \in \mathcal{J}$.

A version: $\mathcal{I} \leq_{BA,BA}^+ \mathcal{J}$ if there is a set $A \in \mathcal{J}^+$ with $\mathcal{I} \leq_{BA,BA} (\mathcal{J} \upharpoonright A)$.

Here, $\vartheta : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ is an $\mathcal{J}$-approximate Boolean algebra homomorphism if the sets $(\vartheta(x) \cup \vartheta(y)) \Delta \vartheta(x \cup y)$ and $(\vartheta(\complement x) \Delta \vartheta(\complement x) \Delta \vartheta(x \Delta y)$ always belong to $\mathcal{J}$ whenever $x, y \subseteq A$. Let further a $\mathcal{J}$-approximate $\Delta$-homomorphism be any map $\vartheta : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ such that $(\vartheta(x) \Delta \vartheta(y)) \Delta \vartheta(x \Delta y)$ always belongs to $\mathcal{J}$. This leads to a weaker reducibility:

Borel $\Delta$-reducibility: $\mathcal{I} \leq_{BA,\Delta} \mathcal{J}$ iff there is a Borel $\mathcal{J}$-approximate $\Delta$-homomorphism $\vartheta : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ such that $x \in \mathcal{I} \iff \vartheta(x) \in \mathcal{J}$.

1.c Borel, continuous, and Baire measurable reductions

Many properties of Borel reductions hold for a bigger family of Baire measurable (BM, for brevity) maps. Any reducibility definition in §§ 1.a, 1.b admits a weaker BM version, which claims that the reduction postulated to exist is only BM, not necessarily Borel. Such a version will be denoted with a subscript BM instead of B, for instance, $E \leq_{BM} F$ means that there is a BM reduction, i.e., a BM map $\vartheta : X = \text{dom} E \rightarrow Y = \text{dom} F$ such that $x E y \iff \vartheta(x) F \vartheta(y)$ for all $x, y \in X$.

On the other hand, a continuous reducibility can sometimes be derived.
1 REDUCIBILITY

Lemma 1 (Louveau ?). If \( \mathcal{I} \) is a Borel ideal on a countable \( A \), \( E \) an equivalence relation on a Polish \( \mathcal{X} \), and \( E, \mathcal{I} \leq_{B, \text{BM}} E \), then \( E, \mathcal{I} \leq C \) \( E \times E \) (via a continuous reduction), that is, there exist continuous maps \( \vartheta_0, \vartheta_1 : \mathcal{P}(A) \to \mathcal{X} \) such that, for any \( x, y \in \mathcal{P}(N) \), \( x \Delta y \in \mathcal{I} \) iff both \( \vartheta_0(x) E \vartheta_0(y) \) and \( \vartheta_1(x) E \vartheta_1(y) \). 

Proof. We w.l.o.g. suppose that \( A = \mathbb{N} \). Let \( \vartheta : \mathcal{P}(N) \to \mathcal{X} \) witness that \( E, \mathcal{I} \leq_{B, \text{BM}} E \). Then \( \vartheta \) is continuous on a dense \( G_\delta \) set \( D = \bigcap_i D_i \subseteq \mathcal{P}(N) \), all \( D_i \) dense open and \( D_{i+1} \subseteq D_i \). A sequence \( 0 = n_0 < n_1 < n_2 < \ldots \) and, for any \( i \), a set \( u_i \subseteq [n_i, n_{i+1}) \) can be easily defined, by induction on \( i \), so that \( x \cap [n_i, n_{i+1}) = u_i \implies x \in D_i \). \( \square \) Let 

\[
N_1 = \bigcup_i [n_{2i}, n_{2i+1}), \quad N_2 = \bigcup_i [n_{2i+1}, n_{2i+2}), \quad U_1 = \bigcup_i u_{2i}, \quad U_2 = \bigcup_i u_{2i+1}.
\]

Now set \( \vartheta_1(x) = \vartheta((x \cap N_1) \cup U_2) \) and \( \vartheta_2(x) = \vartheta((x \cap N_2) \cup U_1) \) for \( x \subseteq \mathbb{N} \). \( \square \)

The following question should perhaps be answered in the negative in general and be open for some particular cases.

Question 2. Suppose that \( E \leq B \) \( F \) are Borel ERs. Does there always exist a continuous reduction? \( \square \)

1.d Reducibility via maps between the underlying sets

This is an even more special kind of Borel reducibility. Let \( \mathcal{I}, \mathcal{J} \) be ideals on resp. \( A, B \), as above.

Rudin–Keisler order: \( \mathcal{I} \leq_{R,K} \mathcal{J} \) iff there exists a function \( b : \mathbb{N} \to \mathbb{N} \) (a Rudin–Keisler reduction) such that \( x \in \mathcal{J} \iff b^{-1}(x) \in \mathcal{J} \).

Rudin–Blass order: \( \mathcal{I} \leq_{R,B} \mathcal{J} \) iff there is a finite-to-one function \( b : \mathbb{N} \to \mathbb{N} \) (a Rudin–Blass reduction) with the same property.

A version: \( \mathcal{I} \leq_{R,B}^+ \mathcal{J} \) allows \( b \) to be defined on a proper subset of \( \mathbb{N} \), in other words, we have pairwise disjoint finite non-empty sets \( w_k = b^{-1}([k]) \) such that \( x \in \mathcal{J} \iff w_x = \bigcup_{k \in x} w_k \in \mathcal{J} \).

Another version: \( \mathcal{I} \leq_{R,B}^{++} \mathcal{J} \) requires that, in addition, the sets \( w_k = b^{-1}([k]) \) satisfy \( \max w_k < \min w_{k+1} \).

There is a “clone” of the Rudin–Blass order which applies in a much more general situation. Suppose that \( X = \prod_{k \in \mathbb{N}} X_k \) and \( Y = \prod_{k \in \mathbb{N}} Y_k \), \( 0 = n_0 < n_1 < n_2 < \ldots \), and \( H_i : X_i \to \prod_{n_i \leq k < n_{i+1}} Y_k \) for any \( i \). Then, we can define

\[
\Psi(x) = H_0(x_0) \cup H_1(x_1) \cup H_2(x_2) \cup \ldots \in Y
\]

\footnote{Sets like \( u_i \) are called stabilizers, they are of much help in study of Borel ideals.}
1 REDUCIBILITY

for each \( x = \{ x_i \}_{i \in \mathbb{N}} \in X \). Maps \( \Psi \) of this kind were called additive by Farah [7]. More generally, if, in addition, \( 0 = m_0 < m_1 < m_2 < \ldots \), and \( H_i : \prod_{j \leq i < m_{i+1}} X_j \to \prod_{k \leq j < n_{i+1}} Y_k \) for any \( i \), then we can define

\[
\Psi(x) = H_0(x \upharpoonright [m_0, m_1]) \cup H_1(x \upharpoonright [m_1, m_2]) \cup H_2(x \upharpoonright [m_2, m_3]) \cup \ldots \in Y
\]

for each \( x \in X \). Farah calls maps \( \Psi \) of this kind asymptotically additive. All of them are Borel functions \( X \to Y \); provided all sets \( X_j \) and \( Y_k \) are finite.

Suppose now that \( E \) and \( F \) are ERs on resp. \( X = \prod_k X_k \) and \( Y = \prod_k Y_k \).

**Additive reducibility:** \( E \leq_A F \) if there is an additive reduction \( E \) to \( F \). \( E \leq_{AA} F \) if there is an asymptotically additive reduction \( E \) to \( F \).

**Lemma 3** (Farah [7]). Suppose that \( \mathcal{I} \) and \( \mathcal{J} \) are Borel ideals on \( \mathbb{N} \). Then \( \mathcal{I} \leq^{++}_{\mathbb{R}B} \mathcal{J} \) iff \( E_{\mathcal{I}} \leq_A E_{\mathcal{J}} \).

(By definition \( E_{\mathcal{I}} \) and \( E_{\mathcal{J}} \) are ERs on \( \mathcal{P}(\mathbb{N}) \), yet we can consider them as ERs on \( 2^\mathbb{N} = \prod_{k \in \mathbb{N}} \{0, 1\} \), as usual, which yields the intended meaning for \( E_{\mathcal{I}} \leq_A E_{\mathcal{J}} \).)

**Proof.** If \( \mathcal{I} \leq^{++}_{\mathbb{R}B} \mathcal{J} \) via a sequence of finite sets \( w_i \) with \( \max w_i < \min w_{i+1} \) then we put \( n_0 = 0 \) and \( n_i = \min w_i \) for \( k \geq 1 \), so that \( w_i \subseteq [n_i, n_{i+1}) \), and, for any \( i \), put \( H_i(0) = [n_i, n_{i+1}) \setminus \{0\} \) and let \( H_i(1) \) be the characteristic function of \( w_i \) within \( [n_i, n_{i+1}) \). Conversely, if \( E_{\mathcal{I}} \leq_A E_{\mathcal{J}} \) via a sequence \( 0 = n_0 < n_1 < \ldots \) and a family of maps \( H_i : \{0, 1\} \to 2^{[n_i, n_{i+1})} \) then \( \mathcal{I} \leq^{++}_{\mathbb{R}B} \mathcal{J} \) via the sequence of sets \( w_i = \{ k \in [n_i, n_{i+1}) : H_i(0)(k) \neq H_i(1)(k) \} \).

The following definition is taken from [19]. Let \( \mathcal{I}, \mathcal{J} \) be ideals on \( \mathbb{N} \).

**Reducibility via inclusion:** \( \mathcal{I} \leq_1 \mathcal{J} \) if there is a map \( b : \mathbb{N} \to \mathbb{N} \) such that \( x \in \mathcal{I} \implies b^{-1}(x) \in \mathcal{J} \). (Note \( \implies \) instead of \( \iff \!).

In particular if \( \mathcal{I} \subseteq \mathcal{J} \) then \( \mathcal{I} \leq_1 \mathcal{J} \) via \( b(k) = k \). It follows that this order is not fully compatible with \( \leq_{\mathbb{R}} \) because \( \mathcal{I}_{1/\mathbb{N}} \leq_{\mathbb{R}} \mathcal{I} \) while the summand ideal \( \mathcal{I}_{1/n} \) and the density-0 ideal \( \mathbb{Z} \) are known to be \( \leq_{\mathbb{R}} \)-incomparable.

1.e Isomorphism

Let \( \mathcal{I}, \mathcal{J} \) be ideals on resp. \( A, B \). **Isomorphism** \( \mathcal{I} \cong \mathcal{J} \) means that there is a bijection \( \beta : A \rightarrow B \) such that we have \( x \in \mathcal{I} \iff \beta^*x \in \mathcal{J} \) for all \( x \subseteq A \).

Sometimes they use a weaker definition: let \( \mathcal{I} \cong^* \mathcal{J} \) mean that there are sets \( A' \in \mathcal{I}^c \) and \( B' \in \mathcal{J}^c \) such that \( \mathcal{I} \upharpoonright A' \cong \mathcal{J} \upharpoonright B' \). Yet this implies \( \mathcal{I} \cong \mathcal{J} \) in most usual cases, the only notable exception (among nontrivial ideals), is produced by the ideals \( \mathcal{I} = \text{Fin} \) and \( \mathcal{J} = \text{Fin} \oplus \mathcal{P}(\mathbb{N}) \cong \{ x \subseteq \mathbb{N} : x \cap D \in \text{Fin} \} \), where \( D \) is an infinite and coinfinitive set \(^3\); then \( \mathcal{I} \cong^* \mathcal{J} \) but not \( \mathcal{I} \cong \mathcal{J} \).

\(^3\) Kechris [27] called ideals \( \mathcal{J} \) of this kind trivial variations of \( \text{Fin} \).
1. Remarks

The following shows simple relationships between different reducibilities:

\[ \mathcal{I} \leq_{\text{RB}} \mathcal{J} \Rightarrow \mathcal{I} \leq_{\text{RK}} \mathcal{J} \Rightarrow \mathcal{I} \leq_{\text{BE}} \mathcal{J} \Rightarrow \mathcal{I} \leq_{+} \mathcal{J} \Rightarrow \mathcal{I} \leq_{\Delta} \mathcal{J} \Rightarrow \mathcal{I} \leq_{\text{B}} \mathcal{J} \]

For instance if \( b : \mathbb{N} \to \mathbb{N} \) witnesses \( \mathcal{I} \leq_{\text{RK}} \mathcal{J} \) then \( \vartheta_b(X) = b^{-1}(X) \) witnesses \( \mathcal{I} \leq_{\text{BE}} \mathcal{J} \). Note that any \( \vartheta_b \) is an exact Boolean algebra homomorphism \( \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \); moreover, it is known that any BM Boolean algebra homomorphism \( \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \) is \( \vartheta_b \) for an appropriate \( b : \mathbb{N} \to \mathbb{N} \). Approximate homomorphisms are liftings of homomorphisms into quotients of \( \mathcal{P}(\mathbb{N}) \), thus, any \( \mathcal{J} \)-approximate \( \vartheta : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) \) induces the map \( \Theta(X) = \{ \vartheta(X) \Delta Y : Y \in \mathcal{J} \} \), which is a homomorphism \( \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{J} \). Farah [6], and Kanovei and Reeken [24] demonstrated that in some important cases (of “non-pathological” P-ideals and, generally, for all Fatou, or Fubini, ideals) we have \( \mathcal{J} \leq_{\text{RK}} \mathcal{J} \iff \mathcal{J} \leq_{\text{BE}} \mathcal{J} \). On the other hand \( \mathcal{J} \leq_{\text{RK}} \mathcal{J} \iff \mathcal{J} \leq_{\text{BE}} \mathcal{J} \) fails for rather artificial P-ideals.

The right-hand end is the most intriguing: is there a pair of Borel ideals \( \mathcal{I}, \mathcal{J} \) such that \( \mathcal{I} \leq_{\text{B}} \mathcal{J} \) but not \( \mathcal{I} \leq_{\Delta} \mathcal{J} \)? If we actually have the equivalence then the whole theory of Borel reducibility for Borel ideals can be greatly simplified because reduction maps which are \( \Delta \)-homomorphisms are much easier to deal with.

2 Introduction to ideals

As many interesting ERs appear as \( E_{\mathcal{I}} \) for a Borel ideal \( \mathcal{I} \), we take space to discuss a few basic items related to Borel ideals. We begin with several examples and notation, and then continue with some important types of ideals.

- \( \text{Fin} = \{ x \subseteq \mathbb{N} : x \text{ is finite} \} \), the ideal of all finite sets;
- \( \mathcal{I}_1 = \text{Fin} \times 0 = \{ x \subseteq \mathbb{N}^2 : \{ k : (x)_k \neq \emptyset \} \in \text{Fin} \} \);
- \( \mathcal{I}_2 = \{1/n\} = \{ x \subseteq \mathbb{N} : \sum_{n \in x} \frac{1}{n+1} < +\infty \}, \text{ the summable ideal} ;\)
- \( \mathcal{I}_3 = 0 \times \text{Fin} = \{ x \subseteq \mathbb{N}^2 : \forall k ((x)_k \in \text{Fin}) \} ;\)
- \( \mathcal{I}_0 = EU_{\{1\}} = \{ x \subseteq \mathbb{N} : \lim_{n \to +\infty} \#(x \cap [0,n)) = 0 \}, \text{ the density ideal} .\)

2.a Notation

- For any ideal \( \mathcal{I} \) on a set \( A \), we define \( \mathcal{I}^+ = \mathcal{P}(A) \setminus \mathcal{I} \) (\( \mathcal{I} \)-positive sets) and \( \mathcal{I}^c = \{ X : \overline{X} \in \mathcal{I} \} \) (the dual filter). Clearly \( \emptyset \neq \mathcal{I}^c \subseteq \mathcal{I}^+ .\)
- If \( B \subseteq A \), then we put \( \mathcal{I} \mid B = \{ x \cap B : x \in \mathcal{I} \} .\)
2 INTRODUCTION TO IDEALS

- If \( I, J \) are ideals on resp. \( A, B \), then \( I \oplus J \) (the disjoint sum) is the ideal of all sets \( x \subseteq C = (\{0\} \times A) \cup (\{1\} \times B) \) with \( (x)_0 \in I \) and \( (x)_1 \in J \) (where \( (x)_i = \{c : \langle i, c \rangle \in x\} \), as usual).

  If the sets \( A, B \) are disjoint then \( I \oplus J \) can be equivalently defined as the ideal of all sets \( x \subseteq A \cup B \) with \( x \cap A \in I \) and \( x \cap B \in J \).

- The Fubini product \( \prod_{a \in A} J_a / I \) of ideals \( J_a \) on sets \( B_a \), over an ideal \( I \) on a set \( A \) is the ideal on the set \( B = \{\{a, b\} : a \in A \land b \in B_a\} \), which consists of all sets \( y \subseteq B \) such that the set \( \{a : (y)_a \not\in J_a\} \) belongs to \( I \), where \( (y)_a = \{b : \langle a, b \rangle \in y\} \) (the cross-section).

- In particular, the Fubini product \( I \otimes J \) of two ideals \( I, J \) on sets resp. \( A, B \), is equal to \( \prod_{a \in A} J_a / I \), where \( J_a = J, \forall a \). Thus \( I \otimes J \) consists of all sets \( y \subseteq A \times B \) such that \( \{a : (y)_a \not\in J \} \in I \).

2.b P-ideals and submeasures

Many important Borel ideals belong to the class of P-ideals.

**Definition 4.** An ideal \( I \) on \( \mathbb{N} \) is a P-ideal if for any sequence of sets \( x_n \in I \) there is a set \( x \in I \) such that \( x_n \subseteq^* x \) (i.e., \( x_n \cap x \in \text{Fin} \)) for all \( n \).

For instance, the ideals \( \text{Fin}, I_2, I_3, I_0 \) (but not \( I_1 \)) are P-ideals.

This class admits several apparently different but equivalent characterizations, one of which is connected with submeasures.

- A submeasure on a set \( A \) is any map \( \varphi : \mathcal{P}(A) \to [0, +\infty] \), satisfying \( \varphi(\emptyset) = 0 \), \( \varphi(\{a\}) < +\infty \) for all \( a \), and \( \varphi(x) \leq \varphi(x \cup y) \leq \varphi(x) + \varphi(y) \).

- A submeasure \( \varphi \) on \( \mathbb{N} \) is lower semicontinuous, or l.s.c. for brevity, if we have \( \varphi(x) = \sup_n \varphi(x \cap [0, n)) \) for all \( x \in \mathcal{P}(\mathbb{N}) \).

To be a measure, a submeasure \( \varphi \) has to satisfy, in addition, that \( \varphi(x \cup y) = \varphi(x) + \varphi(y) \) whenever \( x, y \) are disjoint. Note that any \( \sigma \)-additive measure is l.s.c., but if \( \varphi \) is l.s.c. then \( \varphi_\infty \) is not necessarily l.s.c. itself.

Suppose that \( \varphi \) is a submeasure on \( \mathbb{N} \). Define the tailsubmeasure \( \varphi_\infty(x) = ||x||_{\varphi} = \inf_n (\varphi(x \cap [n, \infty))) \). The following ideals are considered:

\[
\begin{align*}
\text{Fin}_\varphi &= \{ x \in \mathcal{P}(\mathbb{N}) : \varphi(x) < +\infty \} ; \\
\text{Null}_\varphi &= \{ x \in \mathcal{P}(\mathbb{N}) : \varphi(x) = 0 \} ; \\
\text{Exh}_\varphi &= \{ x \in \mathcal{P}(\mathbb{N}) : \varphi_\infty(x) = 0 \} = \text{Null}_\varphi^\infty .
\end{align*}
\]

**Example 5.** \( \text{Fin} = \text{Exh}_\varphi = \text{Null}_\varphi \), where \( \varphi(x) = 1 \) for any \( x \neq \emptyset \). We also have \( 0 \times \text{Fin} = \text{Exh}_\psi \), where \( \psi(x) = \sum_k 2^{-k} \varphi(\{l : \langle k, l \rangle \in x\}) \) is l.s.c..

It turns out (Solecki, see Theorem 41 below) that analytic P-ideals are the same as ideals of the form \( \text{Exh}_\varphi \), where \( \varphi \) is a l.s.c. submeasure on \( \mathbb{N} \). It follows that any analytic P-ideal is \( \Pi^0_3 \).
2. Polishable ideals

There is one more characterization of Borel $\mathbb{P}$-ideals. Let $T$ be the ordinary Polish product topology on $\mathcal{P}(\mathbb{N})$. Then $\mathcal{P}(\mathbb{N})$ is a Polish group in the sense of $T$ and the symmetric difference as the operation, and any ideal $\mathcal{I}$ on $\mathbb{N}$ is a subgroup of $\mathcal{P}(\mathbb{N})$.

**Definition 6.** An ideal $\mathcal{I}$ on $\mathbb{N}$ is **polishable** if there is a Polish group topology $\tau$ on $\mathcal{I}$ which produces the same Borel subsets of $\mathcal{I}$ as $T \upharpoonright \mathcal{I}$. 

The same Solecki’s theorem (Theorem 41) proves that, for analytic ideals, to be a $\mathbb{P}$-ideal is the same as to be polishable. It follows (see Example 5) that, for instance, $\text{Fin}$ and $\mathcal{I}_3 = 0 \times \text{Fin}$ are polishable, but $\mathcal{I}_1 = \text{Fin} \times 0$ is not. The latter will be shown directly after the next lemma.

**Lemma 7.** Suppose that an ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is polishable. Then there is only one Polish group topology $\tau$ on $\mathcal{I}$. This topology refines $T \upharpoonright \mathcal{I}$ and is metrizable by a $\Delta$-invariant metric. If $Z \in \mathcal{I}$ then $\tau \upharpoonright \mathcal{P}(Z)$ coincides with $T \upharpoonright \mathcal{P}(Z)$. In addition, $\mathcal{I}$ itself is $T$-Borel.

**Proof.** Let $\tau$ witness that $\mathcal{I}$ is polishable. The identity map $f(x) = x : \langle \mathcal{I} ; \tau \rangle \to \langle \mathcal{P}(\mathbb{N}) ; T \rangle$ is a $\Delta$-homomorphism and is Borel-measurable because all $(T \upharpoonright \mathcal{I})$-open sets are $\tau$-Borel, hence, by the Pettis theorem (Kechris [26, ??]), $f$ is continuous. It follows that all $(T \upharpoonright \mathcal{I})$-open subsets of $\mathcal{I}$ are $\tau$-open, and that $\mathcal{I}$ is $T$-Borel in $\mathcal{P}(\mathbb{N})$ because $1 - 1$ continuous images of Borel sets are Borel.

A similar “identity map” argument shows that $\tau$ is unique if exists.

It is known (Kechris [26, ??]) that any Polish group topology admits a left-invariant compatible metric, which, in this case, is right-invariant as well since $\Delta$ is an abelian operation.

Let $Z \in \mathcal{P}(\mathbb{N})$. Then $\mathcal{P}(Z)$ is $T$-closed, hence, $\tau$-closed by the above, subgroup of $\mathcal{I}$, and $\tau \upharpoonright \mathcal{P}(Z)$ is a Polish group topology on $\mathcal{P}(Z)$. Yet $T \upharpoonright \mathcal{P}(Z)$ is another Polish group topology on $\mathcal{P}(Z)$, with the same Borel sets. The same “identity map” argument proves that $T$ and $\tau$ coincide on $\mathcal{P}(Z)$. 

**Example 8.** $\mathcal{I}_1 = \text{Fin} \times 0$ is not polishable. Indeed we have $\text{Fin} \times 0 = \bigcup_n W_n$, where $W_n = \{x : x \subseteq \{0, 1, ..., n\} \times \mathbb{N}\}$. Let, on the contrary, $\tau$ be a Polish group topology on $\mathcal{I}_1$. Then $\tau$ and the ordinary topology $T$ coincide on each set $W_n$ by the lemma, in particular, each $W_n$ remains $\tau$-nowhere dense in $W_{n+1}$, hence, in $\mathcal{I}_1$, a contradiction with the Baire category theorem for $\tau$. 

2.d Some $\mathbf{F}_\sigma$ ideals

Any sequence $\{r_n\}_{n \in \mathbb{N}}$ of positive reals $r_n$ with $\sum r_n = +\infty$ defines the ideal

$$ \mathcal{I}_{\{r_n\}} = \{X \subseteq \mathbb{N} : \sum_{n \in X} r_n < +\infty\} = \{X : \mu_{\{r_n\}}(X) < +\infty\}, $$
where $\mu_{\{r_n\}}(X) = \sum_{n \in X} r_n$. These ideals are called *summable ideals*; all of them are $F_\sigma$. References [33, 35, 6]. Any summable ideal is easily a $P$-ideal: indeed, $I_{\{r_n\}} = \text{Exh}_\varphi$, where $\varphi(X) = \sum_{n \in X} r_n$ is a $\sigma$-additive measure.

Summable ideals are perhaps the easiest to study among all $P$-ideals. Further entries: 1) Farah [6, § 1.12] on summable ideals under $\leq_{BE}$; 2) Hjorth: $\leq_{n}$-structure of ideals $\leq_{n}$-reducible to summable ideals, in [13].

**Lemma 9** (Folklore ?). Suppose that $r_n \geq 0$, $r_n \to 0$, and $\sum n r_n = +\infty$. Then any summable ideal $I$ satisfies $I \leq_{RB} I_{\{r_n\}}$.

**Proof.** Let $I = I_{\{p_n\}}$, where $p_n \geq 0$ (no other requirements !). Under the assumptions of the lemma we can associate a finite set $w_n \subseteq \mathbb{N}$ to any $n$ so that $\max w_n < \min w_{n+1}$ and $|r_n - \sum_{j \in w_n} r_j| < 2^{-n}$. □

Farah [6, § 1.10] defines a non-summable $F_\sigma$ $P$-ideal as follows. Let $I_k = [2^k, 2^{k+1})$ and $\psi_k(s) = k^{-2} \min\{k, \#s\}$ for all $k$ and $s \subseteq I_k$, and then

$$\psi(X) = \sum_{k=0}^{\infty} \psi_k(X \cap I_k) \quad \text{and} \quad I = \text{Fin}_\psi;$$

it turns out that $I$ is an $F_\sigma$ $P$-ideal, but not summable. To show that $I$ distinguishes from any $I_{\{r_n\}}$, Farah notes that there is a set $X$ (which depends on $\{r_n\}$) such that the differences $|\mu_{\{r_n\}}(X \cap I_k) - \psi_k(X \cap I_k)|$, $k = 0, 1, 2, \ldots$, are unbounded.


### 2.e Erdős – Ulam and density ideals

These are other types of Borel $P$-ideals. Any sequence $\{r_n\}_{n \in \mathbb{N}}$ of positive reals $r_n$ with $\sum r_n = +\infty$ defines the ideal

$$EU_{\{r_n\}} = \left\{ x \subseteq \mathbb{N} : \lim_{n \to +\infty} \frac{\sum_{i \in x \cap [0,n)} r_i}{\sum_{i \in [0,n)} r_i} = 0 \right\}.$$ 

These ideals are called *Erdős – Ulam* (or: $EU$) *ideals*. Examples: $\mathcal{Z}_0 = EU_{\{1\}}$ and $\mathcal{Z}_{\log} = EU_{\{1/n\}}$.

This definition can be generalized. Let $\text{supp} \mu = \{ n : \mu(\{n\}) > 0 \}$, for any measure $\mu$ on $\mathbb{N}$. Measures $\mu, \nu$ are *orthogonal* if we have $\text{supp} \mu \cap \text{supp} \nu = \emptyset$. Now suppose that $\tilde{\mu} = \{ \mu_n \}_{n \in \mathbb{N}}$ is a sequence of pairwise orthogonal measures on $\mathbb{N}$, with finite sets $\text{supp} \mu_i$. Define $\varphi_{\tilde{\mu}}(X) = \text{sup} \mu_n(X)$ : this is a l.s.c. submeasure on $\mathbb{N}$. Let finally $\mathcal{Z}_{\tilde{\mu}} = \text{Exh}(\varphi_{\tilde{\mu}}) = \{ X : \| X \|_{\varphi_{\tilde{\mu}}} = 0 \}$. Ideals of this form are called *density ideals* by Farah [6, § 1.13]. This class includes all $EU$ ideals (although this is not immediately transparent), and some other ideals: for instance, $0 \times \text{Fin}$ is a density but non-$EU$ ideal. Generally density ideals are...
more complicated than summables. We obtain an even wider class if the requirement, that the sets \( \text{supp} \mu_n \) are finite, is dropped: this wider family includes all summable ideals, too.

References [21], [6, § 1.13].

Further entries: 1) Farah: structure of density ideals under \( \leq_{BE} \); 2) Farah: \( c_0 \)-equalities, 3) Relation to Banach spaces: Hjorth, SuGao.

Which ideals are both summable and density?

2.6 Some transfinite sequences of Borel ideals

We consider three interesting families of Borel ideals (mainly, non-P-ideals), unit-ed by their relation to countable ordinals. Note that the underlying sets of the ideals below are countable sets different from \( \mathbb{N} \).

**Fréchet ideals.** This family consists of ideals \( \text{Fr}_\xi, \xi < \omega_1 \), obtained by inductive construction using Fubini products. We put \( \text{Fr}_1 = \text{Fin} \) and \( \text{Fr}_{\xi+1} = \text{Fin} \otimes \text{Fr}_\xi \) for all \( \xi \). Limit steps cause a certain problem. The most natural idea would be to define \( \text{Fr}_\lambda = \prod_{\xi<\lambda} \text{Fr}_\xi / \text{Fin}_\lambda \) for any limit \( \lambda, \) where \( \text{Fin}_\lambda \) is the ideal of all finite subsets of \( \lambda \), or perhaps \( \text{Fr}_\lambda = \prod_{\xi<\lambda} \text{Fr}_\xi / \text{Bou}_\lambda \), where \( \text{Bou}_\lambda \) is the ideal of all bounted subsets of \( \lambda \), or even \( \text{Fr}_\lambda = \prod_{\xi<\lambda} \text{Fr}_\xi / 0 \), where 0 is the ideal containing only the empty set, yet this appears not to be fully satisfactory in [19], where they define \( \text{Fr}_\lambda = \prod_{\alpha \in \lambda} \text{Fr}_{\xi_{\alpha}} / \text{Fin} \), where \( \{ \xi_n \} \) is a once and for all fixed cofinal increasing sequence of ordinals below \( \lambda \), with understanding that the result is independent of the choice of \( \xi_n \), modulo a certain equivalence.

**Indecomposable ideals.** Let \( \text{otp} X \) be the order type of \( X \subseteq \text{Ord} \). For any ordinals \( \xi, \vartheta < \omega_1 \) define:

\[
\mathcal{J}_\vartheta^\xi = \{ A \subseteq \vartheta : \text{otp} A < \omega^\xi \} \quad \text{(nontrivial only if \( \vartheta \geq \omega^\xi \))}
\]

To see that the sets \( \mathcal{J}_\vartheta^\xi \) are really ideals note that ordinals of the form \( \omega^\xi \) and only those ordinals are *indecomposable*, i.e., are not sums of a pair of smaller ordinals, hence, the set \( \{ A \subseteq \vartheta : \text{otp} A < \gamma \} \) is an ideal iff \( \gamma = \omega^\xi \) for some \( \xi \).

**Weiss ideals.** Let \( |X|_{\text{CB}} \) be the Cantor-Bendixson rank of \( X \subseteq \text{Ord} \), i.e., the least ordinal \( \alpha \) such that \( X^{(\alpha)} = \emptyset \). Here \( X^{(\alpha)} \) is defined by induction on \( \alpha : X^{(0)} = X, X^{(\alpha)} = \bigcap_{\alpha < \lambda} X^{(\alpha)} \) at limit steps \( \lambda \), and finally \( X^{(\alpha+1)} = (X^{(\alpha)})' \), where \( A' \), the Cantor-Bendixson derivative, is the set of all ordinals \( \gamma \in x \) which are limit points of \( X \) in the interval topology. For any ordinals \( \xi, \vartheta < \omega_1 \) define:

\[
\mathcal{W}_\vartheta^\xi = \{ A \subseteq \vartheta : |A|_{\text{CB}} < \omega^\xi \} \quad \text{(nontrivial only if \( \vartheta \geq \omega^{\omega^\xi} \))}
\]

It is less transparent that all \( \mathcal{W}_\vartheta^\xi \) are ideals (Weiss, see Farah [6, § 1.14]) while \( \{ A \subseteq \vartheta : |A|_{\text{CB}} < \gamma \} \) is not an ideal if \( \gamma \) is not of the form \( \omega^\xi \).
2.g “Other” ideals

This title intends to include those interesting ideals which have not yet been subject of comprehensive study. A common method to obtain interesting ideals is to consider a countable set bearing a nontrivial structure, as the underlying set. In principle, there is no difference between different countable set as which of them is taken as the underlying set for the ideals considered. Yet if the set bears a nontrivial structure (i.e., more than just countability) then this gives additional insights as which ideals are meaningful. This is already transparent for the ideals defined in §2.f.

We give two examples.

Ideals on finite sequences. The set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers is countable, yet its own order structure is quite different from that of $\mathbb{N}$. We can exploit this in several ways, for instance, with ideals of sets $X \subseteq \mathbb{N}^{<\omega}$ which intersect every branch in $\mathbb{N}^{<\omega}$ by a set which belongs to a given ideal on $\mathbb{N}$.

3 Introduction to equivalence relations

The structure of Borel and analytic ERs under $\leq_B$ includes key ERs which play distinguished role. The plan of this section is to define some of them and outline their properties, then introduce some classes of ERs.

3.a Basic equivalence relations

Equalities can be considered as the most elementary type of ERs. Let $D(X)$ denote the equality on a set $X$, considered as an equivalence relation on $X$.

A much more diverse family is made of equivalence relations generated by ideals. Recall that for any ideal $\mathcal{I}$ on a set $A$, $E_\mathcal{I}$ is an ER on $\mathcal{P}(A)$, defined so that $X E_\mathcal{I} Y$ iff $X \Delta Y \in \mathcal{I}$. Equivalently, $E_\mathcal{I}$ can be considered as an ER on $2^A$ defined so that $f E_\mathcal{I} g$ iff $f \Delta g \in \mathcal{I}$, where $f \Delta g = \{a \in A : f(a) \neq g(a)\}$. Note that $E_\mathcal{I}$ is Borel provided so is $\mathcal{I}$.

This leads us to the following all-important ERs:

- $E_0 = E_{\text{Fin}}$, thus, $E_0$ is a ER on $\mathcal{P}(\mathbb{N})$ and $x E_0 y$ iff $x \Delta y \in \text{Fin}$.
- $E_1 = E_{\mathcal{I}_1}$, thus, $E_1$ is a ER on $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ and $x E_0 y$ iff $(x)_k = (y)_k$ for all but finite $k$, where, we recall, $(x)_k = \{n : (k,n) \in x\}$ for $x \subseteq \mathbb{N} \times \mathbb{N}$.
- $E_2 = E_{\mathcal{I}_2}$, thus, $E_2$ is a ER on $\mathcal{P}(\mathbb{N})$ and $x E_2 y$ iff $\sum_{k \leq x} \Delta y^k < \infty$.
- $E_3 = E_{\mathcal{I}_3}$, thus, $E_3$ is a ER on $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ and $x E_3 y$ iff $(x)_k E_0 (y)_k$, $\forall k$.

Alternatively, $E_0$ can be viewed as an equivalence relation on $2^B$ defined as $a E_1 b$ iff $a(k) = b(k)$ for all but finite $k$. Similarly, $E_1$ can be viewed as a ER on
\(\mathcal{P}(\mathbb{N})^N\), or even on \((2^\mathbb{N})^N\), defined as \(x \mathcal{E}_1 y \) if \(x(k) = y(k)\) for all but finite \(k\), for all \(x, y \in \mathcal{P}(\mathbb{N})^N\), while \(\mathcal{E}_3\) can be viewed as a ER on \(\mathcal{P}(\mathbb{N})^N\), or on \((2^\mathbb{N})^N\), defined as \(x \mathcal{E}_3 y \) if \(x(k) E_0 y(k)\) for all \(k\).

Relations of the form \(\mathcal{E}_x\) are special case of a wider family of ERs induced by group actions, see §3.d below.

The main structure relation between Borel equivalence relations is \(\leq_\text{B}\), Borel reducibility. Some variations (see §1.a) are involved in special cases.

**Definition 10.** A Borel equivalence relation \(\mathcal{E}\) on a space \(\mathcal{X}\) is:
- countable, if every \(\mathcal{E}\)-class \([x]_\mathcal{E} = \{y \in \mathcal{X} : x \mathcal{E} y\}\), \(x \in \mathcal{X}\), is countable;
- essentially countable, if \(\mathcal{E} \preceq_\text{B} \mathcal{F}\), where \(\mathcal{F}\) is a countable Borel ER;
- finite, if every \(\mathcal{E}\)-class \([x]_\mathcal{E} = \{y \in \mathcal{X} : x \mathcal{E} y\}\), \(x \in \mathcal{X}\), is finite;
- hyperfinite, if \(\mathcal{E} = \bigcup N E_n\) for an increasing sequence of Borel finite ERs \(E_n\);
- smooth, if \(\mathcal{E} \preceq_\text{B} D(2^\mathbb{N})\) — then \(\mathcal{E}\) is obviously Borel;
- hypersmooth, if \(\mathcal{E} = \bigcup N E_n\) for an increasing sequence of smooth ERs \(E_n\).

Countable equivalence relations form a widely studied family.

- \(E_\infty\) is the \(\leq_\text{B}\)-largest, or universal countable Borel ER.

See Theorem 31 on the existence and exact definition of \(E_\infty\).

The next group includes equivalence relations induced by actions of (the additive groups of) some Banach spaces, in particular the following ones well known from textbooks:

\[
\begin{align*}
\ell^p &= \{x \in \mathbb{R}^N : \sum_n |x_n|^p < \infty\} \; (p \geq 1); \quad ||x||_p = \left(\sum_n |x_n|^p\right)^{\frac{1}{p}}; \\
\ell^\infty &= \{x \in \mathbb{R}^N : \sup_n |x_n| < \infty\}; \quad ||x||_\infty = \sup_n |x_n|; \\
c &= \{x \in \mathbb{R}^N : \lim_n x_n < \infty \text{ exists}\}; \quad ||x|| = \sup_n |x_n|; \\
c_0 &= \{x \in \mathbb{R}^N : \lim_n x_n = 0\}; \quad ||x|| = \sup_n |x_n|.
\end{align*}
\]

Note that \(\ell^p, c, c_0\) are separable while \(\ell^\infty\) is non-separable. The domain of each of the four spaces consists of infinite sequences \(x = \{x_n\}_{n \in \mathbb{N}}\) of reals, and is a subgroup of the group \(\mathbb{R}^N\) (with the componentwise addition). The latter can be naturally equipped with the Polish product topology, so that \(\ell^p, \ell^\infty, c, c_0\) are Borel subgroups of \(\mathbb{R}^N\). (But not topological subgroups since the distances are different. The metric definitions as in \(\ell^p\) or \(\ell^\infty\) do not work for \(\mathbb{R}^N\).)

Each of the four mentioned Banach spaces defines an orbit equivalence — a Borel equivalence relation on \(\mathbb{R}^N\) also denoted by, resp., \(\ell^p, \ell^\infty, c, c_0\). For instance, \(x \ell^p y\) if and only if \(\sum_k |x_k - y_k|^p < +\infty\) (for all \(x, y \in \mathbb{R}^N\)). It is known (see Section 4) that \(\ell^1 \sim_\text{B} E_2\) and \(\ell^p \preceq_\text{B} \ell^q\) whenever \(1 \leq p < q\), in particular, \(\ell^1 \sim_\text{B} E_2 \preceq_\text{B} \ell^q\) for any \(q > 1\). On the other hand, \(c_0 \sim_\text{B} Z_0\), where \(Z_0\) is the “density 0” equivalence relation.
$3 \ INTRODUCTION \ TO \ EQUIVALENCE \ RELATIONS$

- $Z_0 = E \varepsilon_0$, thus, for $x$, $y$ in $\mathcal{P}(\mathbb{N})$, $x Z_0 y$ iff $\lim_{n \to \infty} \frac{\#(x \Delta y)}{n} = 0$.

Another important ER is
- $T_2$, often called "the equality of countable sets of reals".

There is no reasonable way to turn $\mathcal{P}_{ctbl}(\mathbb{N}^\mathbb{N})$, the set of all at most countable subsets of $\mathbb{N}^\mathbb{N}$, into a Polish space, in order to directly define the equality of countable sets of reals in terms of $D(\cdot)$. However, nonempty members of $\mathcal{P}_{ctbl}(\mathbb{N}^\mathbb{N})$ can be identified with equivalence classes in $(\mathbb{N}^\mathbb{N})^\mathbb{N}/T_2$, where $g T_2 h$ iff ran $g = \text{ran } h$ : for $g, h \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$. (See below in Section 10 on equivalence relations $T_\alpha$ for all $\alpha < \omega_1$.)

In addition to the families of equivalence relations introduced by Definition 10, some more complicated families will be considered below, including ERs induced by Polish group actions, turbulent ERs, ERs classifiable by countable structures, pinned ERs, and some more.

3.b \ Borel reducibility of basic equivalence relations

The diagram on page 16 begins, at the low end, with cardinals $1 \leq n < \omega$, $\aleph_0$, $\mathfrak{c}$, which denote the ERs of equality on resp. finite, countable, uncountable Polish spaces. As all uncountable Polish spaces are Borel isomorphic, the equivalence relations $D(X)$, $X$ a Polish space, are characterized, modulo $\leq _B$, or even modulo Borel isomorphism between the domains, by the cardinality of the domain, which can be any finite $1 \leq n < \omega$, or $\aleph_0$, or $\mathfrak{c} = 2^{\aleph_0}$.

The $E_0$ splitting is the key element of the diagram on page 16. That $D(2^\mathbb{N}) \leq _B E_0$ can be proved by a rather simple embedding while the strictness can be derived from an old result of Sierpiński [39]: any linear ordering of all $E_0$-classes yields a Lebesgue non-measurable set of the same descriptive complexity as the ordering. That every ER $\leq _B E_0$ is $\sim _B$ to some $n \geq 1$, $D(\mathbb{N})$, $D(2^\mathbb{N})$, or $E_0$ itself, is witnessed by the following two classical results:

1st dichotomy (Thm 29 below). Any Borel, even any $\Pi^1_1$ ER $E$ either has at most countably many equivalence classes, formally, $E \leq _B \aleph_0 = D(\mathbb{N})$, or satisfies $\mathfrak{c} = D(2^\mathbb{N}) \leq _B E$.

2nd dichotomy (Thm 35). Any Borel ER $E$ satisfies either $E \leq _B \mathfrak{c}$ or $E_0 \leq _B E$.

The linearity breaks above $E_0$ : each one of the four equivalence relations $E_1$, $E_2$, $E_3$, $E_\infty$ of the next level is strictly $\leq _B$-bigger than $E_0$, and they are pairwise $\leq _B$-incomparable with each other, see § 2.8.

One naturally asks what is going on in the intervals between $E_0$ and these four equivalence relations. The following results provide some answers.

3rd dichotomy (Thm 46). Any ER $E \leq _B E_1$ satisfies $E \leq _B E_0$ or $E \sim _B E_1$. 
3 INTRODUCTION TO EQUVALENCE RELATIONS

Pict. 1: Reducibility between some basic ERs
Connecting lines here indicate Borel reducibility of lower ERs to upper ones.

4th dichotomy (Thm 67). Any ER $E \leq_B E_2$ either is essentially countable or satisfies $E \sim_n E_2$.

See Definition 10 regarding essentially countable ERs in the 4th dichotomy. The “either” case there remains mysterious: any countable Borel ERs $E \leq_n E_2$ known so far are $\leq_B E_0$. It is a problem whether the “either” case can be improved to $\leq_B E_0$. This is marked by the framebox ʔ on the diagram.

The fifth dichotomy theorem is a bit more special, it will be addressed below.

6th dichotomy (Thm 64). Any ER $E \leq_B E_3$ satisfies $E \leq_B E_0$ or $E \sim_B E_3$.

Adams–Kechris theorem (not to be proved here). There is continuum many pairwise $\leq_B$-incomparable countable Borel ERs.

The framebox $\Box_0$-eqs denotes $0_0$-equalities, a family of Borel ERs introduced by Farah [7], all of them are $\leq_n$-between $E_3$ and $0_0 \sim_n Z_0$, and there is continuum-many $\leq_B$-incomparable among them.
3 INTRODUCTION TO EQUIVALENCE RELATIONS

The non-P domain denotes the family of all ERs $E_\mathcal{S}$, where $\mathcal{S}$ is a Borel ideal which is not a P-ideal. By Solecki [42, 43], for a Borel ideal $\mathcal{S}$ to be not a P-ideal it is necessary and sufficient that $\mathcal{S}_1 \leq_B \mathcal{S}$, or, equivalently, $E_1 \leq_B E_\mathcal{S}$.

**Question 1.** It there any reasonable “basis” of Borel ERs above $E_0$? 

It was once considered [16] as a plausible hypothesis that any Borel ER which is not $\leq_B E_\infty$, i.e., not an essentially countable ER, satisfies $E_i \leq_B E$ for at least one $i = 1, 2, 3$. This turns out to be not the case: Farah [4, 5] and Velickovic [46] found an independent family of uncountable Borel ERs, based on Tsirelson ideals, $\leq_B$-incomparable with $E_1$, $E_2$, $E_3$, see below.

It is the most interesting question whether the diagram on page 16 is complete in the sense that there is no $\leq_B$-connections between the equivalence relations mentioned in the diagram except for those explicitly indicated by lines. Basically, one may want to prove the following non-reducibility claims:

1. $E_1 \not\leq_B E_2$, $T_2$, $c_0$;
2. $\ell^\infty \not\leq_B E_1$, $E_2$, $T_2$, $c_0$;
3. $E_2 \not\leq_B E_1$, $T_2$, $c_0$;
4. $E_\infty \not\leq_B E_1$, $E_2$, $c_0$;
5. $E_3 \not\leq_B \ell^\infty$;
6. $T_2 \not\leq_B \ell^\infty$, $c_0$;
7. $c_0 \not\leq_B \ell^\infty$, $T_2$.

Beginning with (1), we note that $E_1$ is not Borel reducible to any equivalence relation induced by a Polish action (of a Polish group) by Theorem 48 below. On the other hand, $E_2$, $T_2$, $c_0$ obviously belong to this category of ERs.

(2) follows from (1) and (3) and can be omitted.

In (3), $E_2 \not\leq_B E_1$ can be proved by an argument rather similar to the proof of Theorem 22. Alternatively, it will follow from Theorem 40 that any Borel ideal $\mathcal{S}$ with $E_\mathcal{S} \leq_B E_1$ is isomorphic, via a bijection between the underlying sets, to $\mathcal{S}_1$ or to a trivial variation of $\text{Fin}$, but $\mathcal{S}_2$ does not belong to this category. The result $E_2 \not\leq_B c_0$ in (3) is Theorem 22(ii).

The results $E_2 \not\leq_B T_2$ and $c_0 \not\leq_B T_2$ in (3) and (7) are proved below in Section 11 (Corollary 60); this will involve the turbulence theory.

The result of (5) is Lemma 15. It implies $c_0 \not\leq_B \ell^\infty$ in (7).

(6) will be established in Section 15.

This leaves us with (4). We don’t know how to prove $E_\infty \not\leq_B E_1$ easily and directly. The indirect way is to use Theorem 46 below, according to which $E_\infty \leq_B E_1$ would imply either $E_\infty \sim E_1$ — impossible, see above, or $E_\infty \leq E_0$. The latter conclusion is also a contradiction since $E_0 \not< B E_\infty$ is known in the theory of countable Borel equivalence relations (see [2, p. 210]).

**Question 2.** Is $E_\infty$ Borel reducible to $c_0$? to $\ell^1$ or any other $\ell^p$?
3. Operations on equivalence relations

The following operations over ERs are in part parallel to the operations on ideals in § 2.a.

(o1) countable union (if it results in a ER) and countable intersection of ERs on one and the same space;

(o2) countable disjoint union \( E = \bigvee_k E_k \) of ERs \( E_k \) on Polish spaces \( S_k \), that is, an ER on \( S = \bigcup_k (\{k\} \times S_k) \) (with the topology generated by sets of the form \( \{k\} \times U \), where \( U \subseteq S_k \) is open) defined as follows: \( \langle k, x \rangle \in E \langle l, y \rangle \iff k = l \) and \( x \in E_k y \). (If \( S_k \) are pairwise disjoint and open in \( S' = \bigcup_k S_k \) then we can equivalently define \( E = \bigvee_k E_k \) on \( S' \) so that \( x \in E y \iff x, y \) belong to the same \( S_k \) and \( x \in E_k y \));

(o3) product \( E = \prod_k E_k \) of ERs \( E_k \) on spaces \( S_k \), that is, the ER on the product space \( \prod_k S_k \) defined by: \( x \in E y \iff x_k \in E_k y_k \) for all \( k \).

(o4) the \textit{Fubini product} (ultraproduct) \( \prod_{k \in \mathbb{N}} E_k / \mathcal{I} \) of ERs \( E_k \) on spaces \( S_k \), modulo an ideal \( \mathcal{I} \) on \( \mathbb{N} \), that is, the ER on the product space \( \prod_{k \in \mathbb{N}} S_k \) defined as follows: \( x \in E y \iff \{x_k \in E_k y_k : k \in \mathbb{N}\} \in \mathcal{I} \);

(o5) countable power \( E \infty \) of a ER \( E \) on a space \( S \) is an ER on \( S^n \) defined as follows: \( x \in E y \iff \{[x_k] \in E_{k} : k \in \mathbb{N}\} = \{[y_k] : k \in \mathbb{N}\} \), so that for any \( k \) there is \( l \) with \( x_k \in E y_l \) and for any \( l \) there is \( k \) with \( x_k \in E y_l \).

These operations allow us to obtain a lot of interesting ERs starting just with very primitive ones. For instance, we can define the sequence of ERs \( T_\xi \), \( \xi < \omega_1 \), of H. Friedman [9] as follows \(^4\). Let \( T_0 = D(\mathbb{N}) \), the equality relation on \( \mathbb{N} \). We put \( T_{\xi+1} = T_\xi \infty \). If \( \lambda < \omega_1 \) is a limit ordinal, then put \( T_\lambda = \bigvee_{\xi<\lambda} T_\xi \).

In particular \textbf{dom} \( T_0 = \mathbb{N} \) and \( x \in T_1 y \iff \text{ran } x = \text{ran } y \), for \( x, y \in \mathbb{N} \). Thus the map \( \vartheta(x) = \text{ran } x \) witnesses that \( T_1 \leq B D(\mathcal{P}(\mathbb{N})) \). To show the converse, define, for any infinite \( u \subseteq \mathbb{N} \), \( \beta(u) \) be the increasing bijection \( \mathbb{N} \xrightarrow{\text{onto}} u \), while if \( u = \{k_0, \ldots, k_n\} \) is finite, put \( \beta(u)(i) = k_i \) for \( i < n \) and \( \beta(u)(i) = k_n \) for \( i \geq n \). Then \( \beta \) witnesses \( D(\mathcal{P}(\mathbb{N})) \leq B T_1 \), thus, \( T_1 \sim B D(\mathcal{P}(\mathbb{N})) \). It easily follows that \( T_2 \sim B D(\mathcal{P}(\mathbb{N}))^\infty \), in fact, \( T_2 \sim B D(X)^\infty \) for any uncountable Polish space \( X \) as any such \( X \) is Borel isomorphic to \( \mathcal{P}(\mathbb{N}) \) (or to \( 2^\mathbb{N} \), which is essentially the same). With \( \mathbb{X} = \mathbb{N}^\mathbb{N} \) we obtain the definition of \( T_2 \) in § 3.a.

3.d Orbit equivalence relations of group actions

An \textit{action} of a group \( G \) on a space \( \mathbb{X} \) is any map \( a : G \times \mathbb{X} \rightarrow \mathbb{X} \), usually written as \( a(g, x) = g \cdot x \), such that 1) \( e \cdot x = x \), and 2) \( g \cdot (h \cdot x) = (gh) \cdot x \). Then, for any \( g \in G \), the map \( x \mapsto g \cdot x \) is a bijection \( \mathbb{X} \) onto \( \mathbb{X} \) with \( x \mapsto g^{-1} \cdot x \) as the

\(^4\) Hjorth [15] uses \( F_\xi \) instead of \( T_\xi \).
3 INTRODUCTION TO EQUIVALENCE RELATIONS

inverse map. A $\mathcal{G}$-space is a pair $\langle X; a \rangle$, where $a$ is an action of $\mathcal{G}$ on $X$; in this case $X$ itself is also called a $\mathcal{G}$-space, and the orbit $ER$, or $ER$ induced by the action, $E^X_\mathcal{G} = E^X_\mathcal{G}$ is defined on $X$ so that $x E^X_\mathcal{G} y$ iff there is $a \in \mathcal{G}$ with $y = a \cdot x$. $E^X_\mathcal{G}$-classes are the same as $\mathcal{G}$-orbits, i.e.,

$$[x]_\mathcal{G} = [x] E^X_\mathcal{G} = \{y : \exists g \in \mathcal{G} \ (g \cdot x = y)\}.$$ 

A homomorphism (or $\mathcal{G}$-homomorphism) of a $\mathcal{G}$-space $X$ into a $\mathcal{G}$-space $Y$ is any map $F : X \to Y$ compatible with the actions in the sense that $F(g \cdot x) = g \cdot F(x)$ for any $x \in X$ and $g \in \mathcal{G}$. A 1-1 homomorphism is an embedding. An embedding $\to$ is an isomorphism. Note that a homomorphism $\langle X; a \rangle \to \langle Y; b \rangle$ is a reduction of $E^X_\mathcal{G}$ to $E^Y_\mathcal{B}$, but not conversely.

A Polish group is a group whose underlying set is a Polish space and the operations are continuous; a Borel group is a group whose underlying set is a Borel set (in a Polish space) and the operations are Borel maps. A Borel group is Polishable if there is a Polish topology on the underlying set which 1) produces the same Borel sets as the original topology and 2) makes the group Polish.

- If both $X$ and $\mathcal{G}$ are Polish and the action continuous, then $\langle X; a \rangle$ (and also $X$) is called a Polish $\mathcal{G}$-space. If both $X$ and $\mathcal{G}$ are Borel and the action is a Borel map, then $\langle X; a \rangle$ (and also $X$) is called a Borel $\mathcal{G}$-space.

Example 11. (i) Any ideal $\mathcal{I} \subseteq P(\mathbb{N})$ is a group with $\Delta$ as the operation. We cannot expect this group to be Polish in the product topology inherited from $P(\mathbb{N})$ (indeed, $\mathcal{I}$ would have to be $G_\delta$). However if $\mathcal{I}$ is a P-ideal then it is Polishable (see §2.c), in other words, $\langle \mathcal{I}; \Delta \rangle$ is a Polish group in an appropriate Polish topology compatible with the Borel structure of $\mathcal{I}$. Given such a topology, the $\Delta$-action of a (P-ideal) $\mathcal{I}$ on $P(\mathbb{N})$ is Polish, too.

(ii) Consider $\mathcal{G} = P_{1\Delta}(\mathbb{N})$ a countable subgroup of $\langle P(\mathbb{N}); \Delta \rangle$. Define an action of $\mathcal{G}$ on $2^\mathbb{N}$ as follows: $(w \cdot x)(n) = x(n)$ whenever $n \notin w$ and $(w \cdot x)(n) = 1 - x(n)$ otherwise. The orbit equivalence relation $E^X_\mathcal{G}$ of this action is obviously $E_0$. Note that this action is free: $x = w \cdot x$ implies $w = \emptyset$ (the neutral element of $\mathcal{G}$) for any $x \in 2^\mathbb{N}$.

Now consider any Borel pairwise $E_0$-inequivalent set $T \subseteq 2^\mathbb{N}$. Then $w \cdot T \cap T = \emptyset$ for any $w \neq \emptyset$ by the above. It easily follows that $T$ is meager in $2^\mathbb{N}$. (Otherwise $T$ is co-meager on a basic clopen set $\mathcal{O}_s(2^\mathbb{N}) = \{x \in 2^\mathbb{N} : s \subset x\}$, where $s \in 2^{<\omega}$. Put $w = \{n\}$, where $n = 1$ for $s \in 2^{<\omega}$. Then $w \in \mathcal{G}$ maps $T \cap \mathcal{O}_{s \setminus 1}(2^\mathbb{N})$ onto $T \cap \mathcal{O}_{s \setminus 1}(2^\mathbb{N})$. Thus $w \cdot T \cap 2^\mathbb{N} = \emptyset$ — contradiction.) We conclude that $\mathcal{G} \cdot T = \bigcup_{w \in \mathcal{G}} w \cdot T$ is still a meager subset of $2^\mathbb{N}$ in this case, and hence $T$ cannot be a full (Borel) transversal for $E_0$.

(iii) The canonical (or shift) action of a group $\mathcal{G}$ on a set of the form $X^G$ ($X$ any set) is defined as follows: $g \cdot \{x_f\}_{f \in \mathcal{G}} = \{x_{g^{-1}f}\}_{f \in \mathcal{G}}$ for any element $\{x_f\}_{f \in \mathcal{G}} \in X^G$ and any $g \in \mathcal{G}$. This is easily a Polish action provided $G$ is
countable, $X$ a Polish space, and $X^G$ given the product topology. The equivalence relation on $X^G$ induced by this action is denoted by $E(G, X)$.

The next theorem (rather difficult to be proved here) shows that the type of the group is the essential component in the difference between Polish and Borel actions: roughly, any Borel action of a Polish group $G$ is a Polish action of $G$.

**Theorem 12** ([1, 5.2.1]). Suppose that $G$ is a Polish group and $\langle X; a \rangle$ is a Borel $G$-space. Then $X$ admits a Polish topology which 1) produces the same Borel sets as the original topology, and 2) makes the action to be Polish. ☐

If $\langle X; a \rangle$ is a Borel $G$-space (and $G$ is a Borel group) then $E^X_G$ is easily a $\Sigma^1_1$ ER on $X$. Sometimes $E^X_G$ is even Borel: for instance, when $G$ is a countable group and the action is Borel, or if $G = F \subseteq \mathcal{P}(\mathbb{N})$ is a Borel ideal, considered as a group with $\Delta$ as the operation, which acts on $X = \mathcal{P}(\mathbb{N})$ by $\Delta$, so that $E^F_G = E_F$ is Borel because $x E^F_G y$ iff $x \Delta y \in F$. Several much less trivial cases when $E^X_G$ is Borel are described in [1, Chapter 7], for instance, if all $E^X_G$-classes are Borel sets of bounded rank then $E^X_G$ is Borel [1, 7.1.1]. Yet rather surprisingly equivalence classes generated by Borel actions are always Borel.

**Theorem 13** (see [26, 15.14]). If $G$ is a Polish group and $\langle X; a \rangle$ is a Borel $G$-space then every equivalence class of $E^X_G$ is Borel.

**Proof.** It can be assumed, by Theorem 12, that the action is continuous. Then for any $x \in X$ the stabilizer $G_x = \{g : g \cdot x = x\}$ is a closed subgroup of $G$. \footnote{Kechris [26, 9.17] gives an independent proof. Both $G_x$ and its topological closure, say, $G'$ are subgroups, moreover, $G'$ is a closed subgroup, hence, we can assume that $G' = G$, in other words, that $G_x$ is dense in $G$. and the aim is to prove that $G_x = G$. By a simple argument, $G_x$ is either comeager or meager in $G$. But a comeager subgroup easily coincides with the whole $G$. Hence, assume that $G_x$ is meager (and dense) in $G$ and draw a contradiction. Let $\{N_n\}_{n \in \mathbb{N}}$ be a basis of the topology of $X$, and $A_n = \{g \in G : g \cdot x \in V_n\}$. Easily $A_n h = A_n$ for any $h \in G_x$. It follows, because $G_x$ is dense, that every $A_n$ is either meager or comeager. Now, if $g \in G$ then $\{g\} = \bigcap_{n \in \mathbb{N}} N(g) A_n$, where $N(g) = \{n : g \cdot x \in V_n\}$, thus, at least one of the $A_n$ containing $g$ is meager. It follows that $G$ is meager, contradiction.}

We can consider $G_x$ as continuously acting on $G$ by $g \cdot h = gh$ for all $g, h \in G$. Let $F$ denote the associated orbit ER. Then every $F$-class $[g]F = g G_x$ is a shift of $G_x$, hence, $[g]F$ is closed. On the other hand, the saturation $[G]F$ of any open set $G \subseteq G$ is obviously open. Therefore, by Lemma 27(iv) below, $F$ admits a Borel transversal $S \subseteq G$. Yet $g \mapsto g \cdot x$ is a Borel $1-1$ map of a Borel set $S$ onto $[x]_E$, hence, $[x]_E$ is Borel by Countable-to-1 Projection. ☐

It follows that not all $\Sigma^1_1$ ERs are orbit ERs of Borel actions of Polish groups: indeed, take a non-Borel $\Sigma^1_1$ set $X \subseteq \mathbb{N}^\mathbb{N}$, define $xEy$ if either $x = y$ or $x, y \in X$, this is a $\Sigma^1_1$ ER with a non-Borel class $X$. ☐
3.e Forcings associated with pairs of equivalence relations

The range of applications of this comparably new topic is not yet clear, but at least it offers interesting technicalities.

Definition 14 (Zapletal [47]). Suppose that $E$ is a Borel equivalence relation on a Polish space $X$, and $F \leq_{B} E$ is another Borel equivalence relation.

$\mathcal{I}_{E/F}$ is the collection of all Borel sets $X \subseteq X$ such that $E \upharpoonright X \leq_{B} F$. Clearly $\mathcal{I}_{E/F}$ is an ideal in the algebra of all Borel subsets of $X$. The associated forcing $\mathbb{P}_{E/F}$ consists of all Borel sets $X \subseteq X$ if $X \notin \mathcal{I}_{E/F}$.

For instance, the ideal $\mathcal{I}_{D(2\mathbb{N})/D(\mathbb{N})}$ consists of all countable Borel sets $X \subseteq 2^\mathbb{N}$, therefore $\mathbb{P}_{D(2\mathbb{N})/D(\mathbb{N})}$ contains all uncountable Borel sets $X \subseteq 2^\mathbb{N}$ and is equal to the Sacks forcing. The ideal $\mathcal{I}_{E_5/D(2\mathbb{N})}$ consists of all Borel sets $X \subseteq 2^\mathbb{N}$ such that $E_5 \upharpoonright X$ is non-smooth (since smoothness is equivalent to being $\leq_{B} D(2^\mathbb{N})$). See §7.e on the associated forcing $\mathbb{P}_{E_5/D(2^\mathbb{N})}$.

4 “Elementary” stuff

This Section gathers proofs of some reducibility/irreducibility results related to the diagram on page 16, elementary in the sense that they do not involve any special concepts. Some of them are really simple, some other quite tricky.

4.a $E_3$ and $T_2$: outcasts

These equivalence relations, together with $c_0 \sim_{B} Z_0$, are the only non-$\Sigma^0_2$ equivalences explicitly mentioned on the diagram.

Lemma 15. $E_3$ is Borel irreducible to $\ell^\infty$.

Proof. Suppose towards the contrary that $\vartheta : 2^{\mathbb{N} \times \mathbb{N}} \to \mathbb{R}^\mathbb{N}$ is a Borel reduction of $E_3$ to $\ell^\infty$. 6 Since obviously $\ell^\infty \sim_{B} \ell^\infty \times \ell^\infty$, Lemma 1 reduces the general case to the case of continuous $\vartheta$. Define $0, 1 \in 2^\mathbb{N}$ by $0(n) = 0$, $1(n) = 1$, $\forall n$. Define $0 \in 2^{\mathbb{N} \times \mathbb{N}}$ by $0(k, n) = 0$ for all $k, n$, thus $(0)_k = 0$, $\forall k$. Finally, for any $k$ define $z_k \in 2^\mathbb{N}$ by $z_k(n) = 1$ for $n < k$ and $z_k(n) = 0$ for $n \geq k$.

We claim that there are increasing sequences of natural numbers $\{k_m\}$ and $\{j_m\}$ such that $|\vartheta(x)(j_m) - \vartheta(0)(j_m)| > m$ for any $m$ and any $x \in 2^{\mathbb{N} \times \mathbb{N}}$ satisfying

\[
(x)_k = \begin{cases} z_{k_i} & \text{whenever } i < m \text{ and } k = k_i \\ 0 & \text{for all } k < k_m \text{ not of the form } k_i. \end{cases}
\]

6 Recall that, for $x, y \in 2^{\mathbb{N} \times \mathbb{N}}$, $x \equiv_{E_3} y$ means $(x)_k \equiv_{E_3} (y)_k$, $\forall k$, where $(x)_k \in 2^\mathbb{N}$ is defined by $(x)_k(n) = x(k, n)$ for all $n$ while $a \equiv_{E_3} b$ means that $a \Delta b = \{m : a(m) \neq b(m)\}$ is finite.
To see that this implies contradiction define \( x \in 2^{\mathbb{N} \times \mathbb{N}} \) so that \( (x)_k = z_k, \forall i \) and \( (x)_k = 0 \) whenever \( k \) does not have the form \( k_i \). Then obviously \( x \mathcal{E}_3 0 \), but \( |\vartheta(x)(j_m) - \vartheta(0)(j_m)| > m \) for all \( m \), hence \( \vartheta(x) \ell^\infty \vartheta(0) \) fails, as required.

We put \( k_0 = 0 \). To define \( j_0 \) and \( k_1 \), consider \( x_0 \in 2^{\mathbb{N} \times \mathbb{N}} \) defined by \( (x_0)_0 = 1 \) but \( (x_0)_k = 0 \) for all \( k \geq 1 \). Then \( x_0 \mathcal{E}_3 0 \) fails, and hence \( \vartheta(x_0) \ell^\infty \vartheta(0) \) fails either. Take any \( j_0 \) with \( |\vartheta(x_0)(j_0) - \vartheta(0)(j_0)| > 0 \). As \( \vartheta \) is continuous, there is a number \( k_1 > 0 \) such that \( |\vartheta(x)(j_0) - \vartheta(0)(j_0)| > 0 \) holds for any \( x \in 2^{\mathbb{N} \times \mathbb{N}} \) with \( (x)_0 = z_{k_1} \) and \( (x)_k = 0 \) for all \( 0 < k < k_1 \).

To define \( j_1 \) and \( k_2 \), consider \( x_1 \in 2^{\mathbb{N} \times \mathbb{N}} \) defined so that \( (x_1)_0 = z_{k_1} \), \( (x_1)_k = 0 \) whenever \( 0 < k < k_1 \), and \( (x_1)_{k_1} = 1 \). Once again there is a number \( j_1 \) with \( |\vartheta(x_1)(j_1) - \vartheta(0)(j_1)| > 1 \), and a number \( k_2 > k_1 \) such that \( |\vartheta(x)(j_1) - \vartheta(0)(j_1)| > 1 \) for any \( x \in 2^{\mathbb{N} \times \mathbb{N}} \) with \( (x)_0 = z_{k_1}, (x)_{k_1} = z_{k_1}, \) and \( (x)_k = 0 \) for all \( 0 < k < k_1 \) and \( k_1 < k < k_2 \).

Et cetera.

Lemma 16. \( \mathcal{E}_3 \) is Borel reducible to both \( T_2 \) and \( c_0 \).

Proof. (1) If \( a \in \mathbb{N} \) and \( s \in 2^{<\omega} \) then define \( sx \in 2^\mathbb{N} \) by \( (sx)(k) = x(k) + 2s(k) \) for \( k < 1h s \) and \( (sx)(k) = x(k) \) for \( k \geq 1h s \). If \( m \in \mathbb{N} \) then \( m^\wedge x \in 2^\mathbb{N} \) denotes the concatenation. In these terms, if \( x, y \in 2^{\mathbb{N} \times \mathbb{N}} \) then obviously

\[
x \mathcal{E}_3 y \iff \{ m^\wedge (s(x)_m) : s \in 2^{<\omega}, m \in \mathbb{N} \} = \{ m^\wedge (s(y)_m) : s \in 2^{<\omega}, m \in \mathbb{N} \}.
\]

Now any bijection \( 2^{<\omega} \times \mathbb{N} \overset{\text{onto}}{\rightarrow} \mathbb{N} \) yields a Borel reduction of \( \mathcal{E}_3 \) to \( T_2 \).

(2) To reduce \( \mathcal{E}_3 \) to \( c_0 \) consider a Borel map \( \vartheta : 2^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathbb{R}^\mathbb{N} \) such that \( \vartheta(x)(2^n(2k + 1) - 1) = n^{-1}(x)_n(k) \).

Lemma 17. Any countable Borel ER is Borel reducible to \( T_2 \).

Proof. Let \( E \) be a countable Borel ER on \( 2^\mathbb{N} \). It follows from Countable-to-1 Enumeration that there is a Borel map \( f : 2^\mathbb{N} \times \mathbb{N} \rightarrow 2^\mathbb{N} \) such that \( [a]_E = \{ f(a, n) : n \in \mathbb{N} \} \) for all \( a \in 2^\mathbb{N} \). The map \( \vartheta \) sending any \( a \in 2^\mathbb{N} \) to \( x = \vartheta(a) \in 2^{\mathbb{N} \times \mathbb{N}} \) such that \( (x)_n = f(a, n), \forall n \), is a reduction required.

See further study on \( T_2 \) in Section 15, where it will be shown that \( T_2 \) is not Borel reducible to a big family of equivalence relations that includes \( c_0, \ell^p, \ell^\infty, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_\infty \). On the other hand, the equivalence relations in this list, with the exception of \( \mathcal{E}_3, \mathcal{E}_\infty \), are not Borel reducible to \( T_2 \) — this follows from the turbulence theory presented in Section 11.

4.b Discretization and generation by ideals

Some equivalence relations on the diagram on page 16 are explicitly generated by ideals, like \( \mathcal{E}_i, i = 0, 1, 2, 3 \). Some other ERs are defined differently. It will be shown below (Section 16) that any Borel ER \( E \) is Borel reducible to a ER of
the form $E_x$, $I$ a Borel ideal. On the other hand, $c_0$, $c_1$, $c_\infty$ turn out to be Borel equivalent to some meaningful Borel ideals. Moreover, these equivalence relations admit “discretization” by means of restriction to certain subsets of $\mathbb{R}^N$.

**Definition 18.** We define $\mathcal{X} = \prod_{n \in \mathbb{N}} X_n = \{x \in \mathbb{R}^N : \forall n \ (x(n) \in X_n)\}$, where $X_n = \{0, \frac{1}{2^n}, \ldots, \frac{2^n}{2^n}\}$.

**Lemma 19.** $c_0 \leq_B c_0 \upharpoonright \mathcal{X}$ and $c_p \leq_B c_p \upharpoonright \mathcal{X}$ for any $1 \leq p < \infty$.

On the other hand, $c_\infty \leq_B c_\infty \upharpoonright \mathbb{Z}^N$.

**Proof.** We first show that $c_0 \leq_B c_0 \upharpoonright [0,1]^N$. Let $\pi$ be any bijection of $N \times Z$ onto $N$. For $x \in \mathbb{R}^N$, define $\vartheta(x) \in [0,1]^N$ as follows. Suppose that $k = \pi(n, \eta)$ ($\eta \in \mathbb{Z}$). If $\eta \leq x(n) < \eta + 1$ then let $\vartheta(x)(k) = x(n)$. If $x(n) \geq \eta + 1$ then put $\vartheta(x)(k) = 1$. If $x(n) < \eta$ then put $\vartheta(x)(k) = 0$. Then $\vartheta$ is a Borel reduction of $c_0$ to $c_0 \upharpoonright [0,1]^N$. Now we prove that $c_0 \upharpoonright [0,1]^N \leq_B c_0 \upharpoonright \mathcal{X}$. For $x \in [0,1]^N$ define $\psi(x) \in \mathcal{X}$ so that $\psi(x)(n)$ the largest number of the form $\frac{\eta}{2^n}$, $0 \leq \eta \leq 2^n$ smaller than $x(n)$. Then obviously $x \varphi(x)$ holds for any $x \in [0,1]^N$, and hence $\psi$ is a Borel reduction of $c_0 \upharpoonright [0,1]^N$ to $c_0 \upharpoonright \mathcal{X}$.

Thus $c_0 \leq_B c_0 \upharpoonright \mathcal{X}$, and hence in fact $c_0 \sim_B c_0 \upharpoonright \mathcal{X}$.

The argument for $c_1$ is pretty similar. The result for $c_\infty$ is obvious: given $x \in \mathbb{R}^N$, replace any $x(n)$ by the largest integer value $\leq x(n)$.

The version for $c_p$, $1 < p < \infty$, needs some comments in the first part (reduction to $[0,1]^N$). Note that if $\eta \in \mathbb{Z}$ and $\eta - 1 \leq x(n) < \eta < \zeta \leq y(n) < \zeta + 1$ then the value $(y(n) - x(n))_p$ in the distance $\|y - x\|_p = (\sum_n |y(n) - x(n)|^p)^{\frac{1}{p}}$ is replaced by $(\zeta - \eta) + (\eta - x(n))_p + (y(n) - \zeta)_p$ in $\|\vartheta(y) - \vartheta(x)\|_p$. Thus if this happens infinitely many times then both distances are infinite, while otherwise this case can be neglected. Further, if $\eta - 1 \leq x(n) < \eta \leq y(n) < \eta + 1$ then $(y(n) - x(n))_p$ in $\|y - x\|_p$ is replaced by $(\eta - x(n))_p + (y(n) - \eta)_p$ in $\|\vartheta(y) - \vartheta(x)\|_p$. However $(\eta - x(n))_p + (y(n) - \eta)_p \leq (y(n) - x(n))_p \leq 2p-1(\eta - x(n))_p + (y(n) - \eta)_p$, and hence these parts of the sums in $\|y - x\|_p$ and $\|\vartheta(y) - \vartheta(x)\|_p$ differ from each other by a factor between 1 and $2p-1$. Finally, if $\eta \leq x(n)$, $y(n) < \eta + 1$ for one and the same $\eta \in \mathbb{Z}$ then the term $(y(n) - x(n))_p$ in $\|y - x\|_p$ appears unchanged in $\|\vartheta(y) - \vartheta(x)\|_p$. Thus totally $\|y - x\|_p$ is finite iff so is $\|\vartheta(y) - \vartheta(x)\|_p$.

**Lemma 20** (Oliver [37]). $c_0$ is $\sim_B$ to the ER $Z_0 = E_{x_0}$.

**Proof.** Prove that $c_0 \leq_B Z_0$. It suffices, by Lemma 19, to define a Borel reduction $c_0 \upharpoonright \mathcal{X} \to Z_0$, i.e., a Borel map $\vartheta : \mathcal{X} \to \mathcal{P}(N)$ such that $x \varphi(y) \iff \vartheta(x) \Delta \vartheta(y) \in Z_0$ for all $x$, $y \in \mathcal{X}$. Let $x \in \mathcal{X}$. Then, for any $n$, we have $x(n) = \frac{k(n)}{2^n}$ for some natural $k(n) \leq 2^n$. The value of $k(n)$ determines the intersection $\vartheta(x) \cap [2^n, 2^{n+1})$: for each $j < 2^n$, we define $2^n + j \in \vartheta(x)$ iff $j < k(n)$. Then $x(n) = \frac{#(\vartheta(x) \cap [2^n, 2^{n+1}))}{2^n}$ for any $n$, and moreover $|y(n) - x(n)| = \leq 2^n$.
4 "ELEMENTARY" STUFF

\[
\|([\vartheta(x) \Delta \vartheta(y)] \cap [2^n, 2^{n+1})] \| \leq 2^n
\]

dля всъ x, y ∈ X и n. This easily implies that \( \vartheta \) is as required.

To prove \( Z_0 \leq \_0 \), we have to define a Borel map \( \vartheta : \mathcal{P}(\mathbb{N}) \to \mathbb{R}^\mathbb{N} \) such that \( X \Delta Y \in Z_0 \iff \vartheta(X) \_0 \vartheta(Y) \). Most elementary ideas like \( \vartheta(X)(n) = \frac{\#(X \cap \{0, n\})}{n} \) do not work, the right way is based on the following observation: for any sets \( s, t \subseteq [0, n) \) to satisfy \( \#(s \Delta t) \leq k \) it is necessary and sufficient that \( \#(s \Delta z) - \#(t \Delta z) \leq k \) for any \( z \subseteq [0, n) \). To make use of this fact, let us fix an enumeration (with repetitions) \( \{z_j\}_{j \in \mathbb{N}} \) of all finite subsets of \( \mathbb{N} \) such that

\[
\{z_j : 2^n \leq j < 2^{n+1}\} = \text{ all subsets of } [0, n)
\]

for every \( n \). Define, for any \( X \in \mathcal{P}(\mathbb{N}) \) and \( 2^n \leq j < 2^{n+1} \), \( \vartheta(X)(j) = \frac{\#(X \cap \{z_j\})}{n} \).

Then \( \vartheta : \mathcal{P}(\mathbb{N}) \to [0, 1]^\mathbb{N} \) is a required reduction. \( \square \)

Recall that for any sequence of reals \( r_n \geq 0 \), \( E_{\{r_n\}} \) is an equivalence relation on \( \mathcal{P}(\mathbb{N}) \) generated by the ideal \( \mathcal{I}_{\{r_n\}} = \{x \subseteq \mathbb{N} : \sum_{n \in x} r_n < +\infty\} \).

**Lemma 21** (Attributed to Kechris in [13, 2.4]). If \( r_n \geq 0 \), \( r_n \to 0 \), \( \sum_{n \in x} = +\infty \) then \( E_{\{r_n\}} \sim \mathcal{I}_1 \). In particular, \( E_2 = E_{\{1/n\}} \) satisfies \( E_2 \sim \mathcal{I}_1 \).

**Proof.** To prove \( E_{\{r_n\}} \leq \_0 \mathcal{I}_1 \), define \( \vartheta(x) \in \mathbb{R}^\mathbb{N} \) for any \( x \in \mathcal{P}(\mathbb{N}) \) as follows: \( \vartheta(x)(n) = r_n \) for any \( n \in x \), and \( \vartheta(x)(n) = 0 \) for any other \( n \). Then \( x \Delta y \in \mathcal{I}_{\{r_n\}} \iff \vartheta(x) \mathcal{I}_1 \vartheta(y) \), as required.

To prove the other direction, it suffices to define a Borel reduction of \( \mathcal{I}_1 \restriction X \) to \( E_{\{r_n\}} \). We can associate a (generally, infinite) set \( s_{nk} \subseteq \mathbb{N} \) with any pair of \( n \) and \( k < 2^n \), so that the sets \( s_{nk} \) are pairwise disjoint and \( \sum_{j \in s_{nk}} r_j = 2^{-n} \). The map \( \vartheta(x) = \bigcup_n \bigcup_{k<2^n} x(n) s_{nk}, x \in X \), is the reduction required. \( \square \)

4.c Summables irreducible to density-0

The \( \leq_\mathbb{N} \)-independence of \( \mathcal{I}_1 \) and \( \_0 \), two best known "Banach" equivalence relations, is quite important. In one direction it is provided by (ii) of the next theorem. The other direction actually follows from Lemma 15.

Is there any example of Borel ideals \( \mathcal{I} \leq_\mathbb{N} \mathcal{J} \) which do not satisfy \( \mathcal{I} \leq_\Delta \mathcal{J} \)? Typically the reductions found to witness \( \mathcal{I} \leq_\mathbb{N} \mathcal{J} \) are \( \Delta \)-homomorphisms, and even better maps. The following lemma proves that Borel reduction yields \( \leq_\mathbb{N}^{+} \)-reduction in quite a representative case. Let us say that \( \mathcal{I} \leq_\mathbb{N}^{+} \mathcal{J} \) holds exponentially if there is a map \( i \mapsto w_i \) witnessing \( \mathcal{I} \leq_\mathbb{N}^{+} \mathcal{J} \) and in addition a sequence of natural numbers \( k_i \) with \( w_i \subseteq [k_i, k_{i+1}) \) and \( k_{i+1} \geq 2k_i \).

**Theorem 22.** Suppose that \( r_n \geq 0 \), \( r_n \to 0 \), \( \sum_{n} r_n = +\infty \). Then

\footnote{Thus we have pairwise disjoint finite non-empty sets \( w_k \subseteq \mathbb{N} \) (assuming \( \mathcal{I}, \mathcal{J} \) are ideals over \( \mathbb{N} \)) such that \( A \in \mathcal{I} \iff w_A = \bigcup_{k \in A} w_k \in \mathcal{J} \), and \( \max w_k < \min w_{k+1} \).}
(i) (Farah [5, 2.1]) If $\mathcal{I}$ is a Borel $P$-ideal and $\mathcal{I}_{\{r_n\}} \leq_{B} \mathcal{J}$ then we have $\mathcal{I}_{\{r_n\}} \leq_{RB}^{++} \mathcal{J}$ exponentially;

(ii) (Hjorth [13]) $\mathcal{I}_{\{r_n\}}$ is not Borel-reducible to $\mathcal{Z}_0$.

**Proof.** (i) Let a Borel $\vartheta : \mathcal{P}(N) \to \mathcal{P}(N)$ witness $\mathcal{I}_{\{r_n\}} \leq_{B} \mathcal{J}$. Let, according to Theorem 41, $\nu$ be a $\lambda$-s.c. submeasure on $N$ with $\mathcal{J} = \text{Exh}_\nu$. The construction makes use of stabilizers. Suppose that $n \in N$. If $u, v \subseteq [0, n)$ then $(u \cup X) \Delta (v \cup X) \in \mathcal{I}_{\{r_n\}}$ for any $X \subseteq [n, +\infty)$, hence, $\vartheta(u \cup X) \Delta \vartheta(v \cup X) \in \mathcal{J}$. It follows, by the choice of the submeasure $\nu$, that for any $\varepsilon > 0$ there are numbers $n' > k > n$ and a set $s \subseteq [n, n')$ such that

$$
\nu((\vartheta(u \cup s \cup X) \Delta \vartheta(v \cup s \cup X)) \cap [k, \infty)) < \varepsilon
$$

holds for all $u, v \subseteq [0, n)$ and all generic $X \subseteq [n', \infty)$.

This allows us to define an increasing sequence of natural numbers $0 = k_0 = a_0 < b_0 < k_1 < a_1 < b_1 < k_2 < \ldots$ and, for any $i$, a set $s_i \subseteq [b_i, a_i+1)$ such that, for all generic $X, Y \subseteq [a_i+1, \infty)$ and all $u, v \subseteq [0, b_i)$, we have

1. $\nu((\vartheta(u \cup s_i \cup X) \Delta \vartheta(v \cup s_i \cup X)) \cap [k_i+1, \infty)) < 2^{-i}$;
2. $\vartheta(u \cup s_i \cup X) \Delta \vartheta(u \cup s_i \cup Y) \cap [0, k_i+1) = \emptyset$;
3. any $Z \subseteq N$, satisfying $Z \cap [b_i, a_i+1) = s_i$ for infinitely many $i$, is generic;
4. $k_{i+1} \geq 2k_i$ for all $i$;

and in addition, under the assumptions on $\{r_n\}$,

5. there is a set $g_i \subseteq [a_i, b_i)$ such that $|r_i - \sum_{n \in g_i} r_n| < 2^{-i}$.

It follows from (5) that $A \mapsto g_A = \bigcup_{i \in A} g_i$ is a reduction of $\mathcal{I}_{\{r_n\}}$ to $\mathcal{I}_{\{r_n\}} \restriction N$, where $N = \bigcup_i [a_i, b_i)$. Let $S = \bigcup_i s_i$; note that $S \cap N = \emptyset$.

Put $\xi(Z) = \vartheta(Z \cup S) \Delta \vartheta(S)$ for any $Z \subseteq N$. Then, for any sets $X, Y \subseteq N$,

$$X \Delta Y \in \mathcal{I}_{\{r_n\}} \iff \vartheta(X \cup S) \Delta \vartheta(Y \cup S) \in \mathcal{J} \iff \xi(X) \Delta \xi(Y) \in \mathcal{J},$$

thus $\xi$ reduces $\mathcal{I}_{\{r_n\}} \restriction N$ to $\mathcal{J}$. Now put $w_i = \xi(g_i) \cap [k_i, k_{i+1})$ and $w_A = \bigcup_{i \in A} w_i$. We assert that the map $i \mapsto w_i$ proves $\mathcal{I}_{\{r_n\}} \leq_{RB}^{++} \mathcal{J}$. In view of the above, it remains to show that $\xi(g_A) \Delta w_A \in \mathcal{J}$ for any $A \in \mathcal{P}(N)$.

As $\mathcal{J} = \text{Exh}_\nu$, it suffices to demonstrate that $\nu(w_i \Delta (\xi(g_A) \cap [k_i, k_{i+1}))) < 2^{-i}$ for all $i \in A$ while $\nu(\xi(g_A) \cap [k_i, k_{i+1}))) < 2^{-i}$ for $i \not\in A$. After dropping the common term $\vartheta(S)$, it suffices to check that

(a) $\nu((\vartheta(g_i \cup U) \Delta \vartheta(g_A \cup U)) \cap [k_i, k_{i+1})) < 2^{-i}$ for all $i \in A$ while

---

8 In the course of the proof, “generic” means Cohen-generic over a sufficiently large countable model of a big enough fragment of $\mathbf{ZFC}$. 

---
(b) \( \nu (((\partial(S) \Delta \partial(g_A \cup S)) \cap [k_i, k_{i+1}]) < 2^{-i} \) for \( i \notin A \).

Note that, as any set of the form \( X \cup S \), where \( S \subseteq N \), is generic by (3). It follows, by (2), that we can assume, in (a) and (b), that \( A \subseteq [0, i] \), i.e., resp. \( \max A = i \) and \( \max A < i \). We can finally apply (1), with \( u = A \cup \bigcup_{j < i} s_j \), \( X = \bigcup_{j > i} s_j \), and \( v = u_i \cup \bigcup_{j < i} s_j \) if \( i \in A \) while \( v = \bigcup_{j < i} s_j \) if \( i \notin A \).

(ii) Otherwise \( \mathcal{I}_{\{\tau\}} \leq_{\text{HH}} \mathcal{I}_0 \) exponentially by (i). Let this be witnessed by \( i \mapsto w_i \) and a sequence of numbers \( k_i \), so that \( k_{i+1} \geq 2k_i \) and \( w_i \subseteq [k_i, k_{i+1}] \). If \( d_i = \#(w_i) \rightarrow 0 \) then easily \( \bigcup w_i \in \mathcal{I}_0 \) by the choice of \( \{k_i\} \). Otherwise there is a set \( A \in \mathcal{I}_{\{\tau\}} \) such that \( d_i > \varepsilon \) for all \( i \in A \) and one and the same \( \varepsilon > 0 \), so that \( w_A = \bigcup_{i \in A} w_i \notin \mathcal{I}_0 \). In both cases we have a contradiction with the assumption that the map \( i \mapsto w_i \) witnesses \( \mathcal{I}_{\{\tau\}} \leq_{\text{HH}} \mathcal{I}_0 \). \qed

**Question 23.** Farah [5] points out that Theorem 22(i) also holds for \( 0 \times \text{Fin} \) (instead of \( \mathcal{I}_{\{\tau\}} \)) and asks for which other ideals it is true. \qed

### 4.4 The family \( \ell^p \)

It follows from the next theorem that Borel reducibility between equivalence relations \( \ell^p \), \( 1 \leq p < \infty \), is fully determined by the value of \( p \).

**Theorem 24** (Dougherty – Hjorth [3]). If \( 1 \leq p < q < \infty \) then \( \ell^p \prec_B \ell^q \).

**Proof.** **Part 1:** show that \( \ell^q \nleq_B \ell^p \).

By Lemma 19, it suffices to prove that \( \ell^q \nleq_B \ell^p \). Suppose, on the contrary, that \( \partial : X \rightarrow X \) is a Borel reduction of \( \ell^q \) to \( \ell^p \). Arguing as in the proof of Theorem 22, we can reduce the general case to the case where there exist increasing sequences of numbers \( 0 = j(0) < j(1) < j(2) < \ldots \) and \( 0 = a_0 < a_1 < a_2 < \ldots \) and a map \( \tau : Y \rightarrow X \), where \( Y = \prod_{n=0}^\infty X_{j(n)} \), which reduces \( \ell^q \) to \( \ell^p \) and has the form \( \tau(x) = \prod_{n \in \mathbb{N}} t^x(n) \), where \( t^x_n \in \prod_{k=a_n}^{a_{n+1}} X_k \) for any \( x \in X_{j_n} \). (See Definition 18.)

**Case 1:** there are \( c > 0 \) and a number \( N \) such that \( \|\tau_n - \tau_n^0\|_p \geq c \) for all \( n \geq N \). Since \( p < q \), there is a non-decreasing sequence of natural numbers \( i_n \leq j_n \), \( n = 0, 1, 2, \ldots \), such that \( \sum_n 2^{q(i_n - j_n)} \) diverges but \( \sum_n 2^{q(i_n - j_n)} \) converges. (Hint: \( i_n \approx j_n - p^{-1} \log_2 n \).

Now consider any \( n \geq N \). As \( \|\tau_n - \tau_n^0\|_p \geq c \) and because \( \|\tau\|_p \) is a norm, there exists a pair of rationals \( u(n) < v(n) \) in \( X_{j_n} \) with \( v(n) - u(n) = 2^{i_n - j_n} \) and \( \|\tau_n^0 - u\|_p \geq c \). In addition, put \( u(n) = v(n) = 0 \) for \( n < N \). Then the \( \ell^p \)-distance between the infinite sequences \( u = \{u(n)\}_{n \in \mathbb{N}} \) and \( v = \{v(n)\}_{n \in \mathbb{N}} \) is equal to \( \sum_{n=N}^\infty 2^{q(i_n - j_n)} \) < 0, infinite, while the \( \ell^q \)-distance between \( \tau(u) \) and \( \tau(v) \) is non-smaller than \( \sum_{n=N}^\infty C^p 2^{q(i_n - j_n)} = \infty \). But this contradicts the assumption that \( \tau \) is a reduction.

**Case 2:** otherwise. Then there is a strictly increasing sequence \( n_0 < n_1 < n_2 < \ldots \) with \( \|\tau_{n_k} - \tau_{n_k}^0\|_p \leq 2^{-k} \) for all \( k \). Let now \( x \in Y \) be the constant 0.
while \( y \in \mathbb{Y} \) be defined by \( y(n_k) = 1, \forall k \) and \( y(n) = 0 \) for all other \( n \). Then \( x \in \ell^p \) if \( \tau(x) \in \ell^p \) holds, contradiction.

**Part 2:** show that \( \ell^p \leq B \ell^q \).

It suffices to prove that \( \ell^p \leq [0, 1]^n \leq B \ell^q \) (Lemma 19). We w.l.o.g. assume that \( q < 2p \); any bigger \( q \) can be approached in several steps. For \( \bar{x} = (x, y) \in \mathbb{R}^2 \), let \( \|x\|_h = (x^h + y^h)^{1/h} \).

**Lemma 25.** For any \( \frac{1}{2} < \alpha < 1 \) there is a continuous map \( K_\alpha : [0, 1] \to [0, 1]^2 \) and positive real numbers \( m < M \) such that for all \( x < y \) in \( [0, 1] \) we have \( m(y - x)\alpha \leq \|K_\alpha(y) - K_\alpha(x)\|_2 \leq M(y - x)\alpha \).

**Proof (Lemma).** The construction of such a map \( K \) can be easier described in terms of fractal geometry rather than by an analytic expression. Let \( r = 4^{-\alpha} \), so that \( \frac{1}{2} < r < \frac{1}{2} \) and \( \alpha = -\log_4 r \). Starting with the segment \( [(0, 0), (1, 0)] \) of the horizontal axis of the Cartesian plane, we replace it by four smaller segments of length \( r \) each (thin lines on Fig. 2, left). Each of them we replace by four segments of length \( r^2 \) (thin lines on Fig. 2, right). And so on, infinitely many steps. The resulting curve \( K \) is parametrized by giving the vertices of the polygons values equal to multiples of \( 4^{-n} \), \( n \) being the number of the polygon. For instance, the vertices of the left polygon on Fig. 2 are given values \( 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \).

![Diagram](image.png)

**Pic. 2:** \( r = \frac{1}{3} \), left: step 1, right: step 2

Note that the curve \( K : [0, 1] \to [0, 1]^2 \), approximated by the polygons, is bounded by certain triangles built on the sides of the polygons. For instance, the whole curve lies inside the triangle bounded by dotted lines in Fig. 2, left. (The dotted line that follows the basic side \( [(0, 0), (1, 0)] \) of the triangle is drawn slightly below its true position.) Further, the parts \( 0 \leq t \leq \frac{1}{4} \) and \( \frac{1}{2} \leq t \leq \frac{3}{2} \) of the curve lie inside the triangles bounded by (slightly different) dotted lines in Fig. 2, right. And so on. Let us call those triangles bounding triangles.

To prove the inequality of the lemma, consider any pair of reals \( x < y \in [0, 1] \).

Let \( n \) be the least number such that \( x, y \) belong to non-adjacent intervals, resp., \( \left[ \frac{2j - 1}{4^n}, \frac{2j}{4^n} \right) \) and \( \left[ \frac{2j - 1}{4^n}, \frac{2j}{4^n} \right) \), with \( j > i + 1 \). Then \( 4^{-n} \leq |y - x| \leq 8 \cdot 4^{-n} \).

The points \( K(x) \) and \( K(y) \) then belong to one and the same side or adjacent sides of the \( n - 1 \)-th polygon. Let \( C \) be a common vertex of these sides. It is quite clear geometrically that the euclidean distances from \( K(x) \) and \( K(y) \) to \( C \) do not exceed \( r^{n-1} \) (the length of the side), thus \( \|K(x) - K(y)\|_2 \leq 2r^{n-1} \).
Estimation from below needs more work. The points \( K(x) \), \( K(y) \) belong to the bounding triangles built on the segments, resp., \([K\left(\frac{1}{n}\right), K\left(\frac{1}{n}\right)]\) and \([K\left(\frac{1}{n}\right), K\left(\frac{1}{n}\right)]\), and obviously \( i + 1 < j \leq i + 8 \), so that there exist at most six bounding triangles between these two. Note that adjacent bounding triangles meet each other at only two possible angles (that depend on \( r \) but not on \( n \)), and taking it as geometrically evident that non-adjacent bounding triangles are disjoint, we conclude that there is a constant \( c > 0 \) (that depends on \( r \) but not on \( n \)) such that the distance between two non-adjacent bounding triangles of rank \( n \), having at most 6 bounding triangles of rank \( n \) between them, does not exceed \( c \cdot r^9 \). In particular, \( \|K(x) - K(y)\|_2 \geq c \cdot r^9 \). Combining this with the inequalities above, we conclude that \( m(y - x) \leq \|K(y) - K(x)\|_2 \leq M(y - x)^\alpha \), where \( m = \frac{c}{n} \) and \( M = \frac{2}{n} \) (and \( \alpha = -\log_4 r \)). \( \square \) (Lemma)

Coming back to the theorem, let \( \alpha = p/q \), and let \( K_\alpha \) be as in the lemma. Let \( x = \langle x_0, x_1, x_2, \ldots \rangle \in [0,1]^N \). Then \( K_\alpha(x_i) = \langle x_i', x_i'' \rangle \in [0,1]^2 \). We put \( \vartheta(x) = \langle x_0', x_0'', x_1', x_1'', x_2', x_2'', \ldots \rangle \). Prove that \( \vartheta \) reduces \( \ell^p \) \([0,1]^N \) to \( \ell^q \).

Let \( x = \{x_i\}_{i \in \mathbb{N}} \) and \( y = \{y_i\}_{i \in \mathbb{N}} \) belong to \([0,1]^N \); we have to prove that \( x \in \ell^p \) iff \( \vartheta(x) \in \ell^q \vartheta(y) \). To simplify the picture note the following:

\[
2^{-1/2} \|w\|_2 \leq \max\{w', w''\} \leq \|w\|_q \leq \|w\|_1 \leq 2\|w\|_2
\]

for any \( w = \langle w', w'' \rangle \in \mathbb{R}^2 \). The task takes the following form:

\[
\sum_i (x_i - y_i)^p < \infty \iff \sum_i \|K_\alpha(x_i) - K_\alpha(y_i)\|_2^q < \infty.
\]

Furthermore, by the choice of \( K_\alpha \), this converts to

\[
\sum_i (x_i - y_i)^p < \infty \iff \sum_i (x_i - y_i)^{pq} < \infty,
\]

which holds because \( pq = p \). \( \square \) (Theorem 24)

4.e \( \ell^\infty \): maximal \( K_\sigma \)

Recall that \( K_\sigma \) denotes the class of all \( \sigma \)-compact sets in Polish spaces. Easy computations show that this class contains, among others, the equivalence relations \( \mathcal{E}_1 \), \( \mathcal{E}_\infty \), \( \ell^p \), \( 1 \leq p \leq \infty \), considered as sets of pairs in corresponding Polish spaces. Note that if \( \mathcal{E} \) a \( K_\sigma \) equivalence on a Polish space \( \mathcal{X} \) then \( \mathcal{X} \) is \( K_\sigma \) as well since projections of compact sets are compact. Thus \( K_\sigma \) ERs on Polish spaces is one and the same as \( \Sigma^0_2 \) ERs on \( K_\sigma \) Polish spaces.

Theorem 26. Any \( K_\sigma \) equivalence relation on a Polish space, in particular, \( \mathcal{E}_1 \), \( \mathcal{E}_\infty \), \( \ell^p \), is Borel reducible to \( \ell^\infty \). \( ^9 \)

\( ^9 \) The result for \( \ell^p \) is due to Su Gao [11]. He defines \( d_\rho(x, s) = (\sum_{k=0}^{n-1} |x(k) - s(k)|^\rho)^{1/\rho} \) for any \( x \in \mathbb{R}^n \) and \( s \in \mathbb{Q}^<_\omega \) (a finite sequence of rationals). Easily the \( \ell^\rho \)-distance \( \langle \sum_{k=0}^{n-1} |x(k) - y(k)|^\rho \rangle^{1/\rho} \) between any pair of \( x, y \in \mathbb{R}^n \) is finite iff there is a constant \( C \) such that \( |d_\rho(x, s) - d_\rho(y, s)| < C \) for all \( s \in \mathbb{Q}^<_\omega \). This yields a reduction required.
4 “ELEMENTARY” STUFF

Proof (from Rosendal [38]). Let \( \mathcal{A} \) be the set of all \( \subseteq \)-increasing sequences \( A = \{ A_n \}_{n \in \mathbb{N}} \) of subsets of \( \mathbb{N} \) — a closed subset of the Polish space \( \mathcal{P}(\mathbb{N})^\mathbb{N} \). Define an ER \( \mathcal{H} \) on \( \mathcal{A} \) by

\[
\{ A_n \} \cap \{ B_m \} \text{ iff } \exists N \forall m \ (A_m \subseteq B_{N+m} \land B_m \subseteq A_{N+m}).
\]

Claim 1: \( \mathcal{H} \leq_n \mathcal{E}^\infty \). This is easy. Given a sequence \( A = \{ A_n \}_{n \in \mathbb{N}} \), define \( \vartheta(A) \in \mathbb{N}^{N \times \mathbb{N}} \) by \( \vartheta(A)(n,k) \) to be the least \( j \leq k \) such that \( n \in A_j \), or \( \vartheta(A)(n,k) = k \) whenever \( n \notin A_k \). Then \( \{ A_n \} \cap \{ B_m \} \) iff there is \( N \) such that \( |\vartheta(A)(n,k) - \vartheta(B)(n,k)| \leq N \) for all \( n, k \).

Claim 2: any \( K_\sigma \) equivalence \( \mathcal{E} \) on a Polish space \( \mathcal{X} \) is Borel reducible to \( \mathcal{H} \). As a \( K_\sigma \) set, \( \mathcal{E} \) has the form \( \mathcal{E} = \bigcup_n E_n \), where each \( E_n \) is a compact subset of \( \mathcal{X} \times \mathcal{X} \) (not necessarily an ER) and \( E_n \subseteq E_{n+1} \). We can w.l.o.g. assume that each \( E_n \) is reflexive and symmetric on its domain \( D_n = \text{dom} E_n = \text{ran} E_n \) (a compact set), in particular, \( x \in D_n \implies (x,x) \in E_n \). Define \( P_0 = E_0 \) and

\[
P_{n+1} = P_n \cup E_{n+1} \cup P_n^{(2)}, \quad \text{where } P_n^{(2)} = \{ (x,y) : \exists z ((x,z) \in P_n \land (z,y) \in P_n) \},
\]

by induction. Thus all \( P_n \) are still compact subsets of \( \mathcal{X} \times \mathcal{X} \), moreover, of \( \mathcal{E} \) since \( \mathcal{E} \) an equivalence relation, and \( E_n \subseteq P_n \subseteq P_{n+1} \), therefore \( \mathcal{E} = \bigcup_n P_n \).

Let \( \{ U_k : k \in \mathbb{N} \} \) be a basis for the topology of \( \mathcal{X} \). Put, for any \( x \in \mathcal{X} \), \( \vartheta_n(x) = \{ k : U_k \cap R_n(x) \neq \emptyset \} \), where \( R_n(x) = \{ y : (x,y) \in E_n \} \). Then obviously \( \vartheta_n(x) \subseteq \vartheta_{n+1}(x) \), and hence \( \vartheta(x) = \{ \vartheta_n(x) \}_{n \in \mathbb{N}} \in \mathcal{A} \). Then \( \vartheta \) reduces \( \mathcal{E} \) to \( \mathcal{H} \).

Indeed if \( x \mathcal{E} y \) then \( (x,y) \in P_n \) for some \( n \), and for all \( m \) and \( z \in \mathcal{X} \) we have \( (x,z) \in R_m \implies (y,z) \in R_{1+\max\{m,n\}} \). In other words, \( R_m(x) \subseteq R_{1+\max\{m,n\}}(y) \) and hence \( \vartheta_m(x) \subseteq \vartheta_{1+\max\{m,n\}}(y) \) hold for all \( m \). Similarly, for some \( n' \) we have \( \vartheta_n(y) \subseteq \vartheta_{1+\max\{m,n'\}}(y) \), \( \forall m \). Thus \( \vartheta(x) \mathcal{H} \vartheta(y) \).

Conversely, suppose that \( \vartheta(x) \mathcal{H} \vartheta(y) \), thus, for some \( N \), we have \( R_m(x) \subseteq R_{N+m}(y) \) and \( R_m(y) \subseteq R_{N+m}(x) \) for all \( m \) and \( y \). Taking \( m \) big enough for \( P_m \) to contain \( (x,x) \), we obtain \( x \in R_{N+m}(y) \), so that immediately \( x \mathcal{E} y \). \( \square \)
5 Smooth ERs and the first dichotomy

This Section is mainly related to the node \( c = D(2^N) \) in the diagram on page 16. After a few rather simple results on smooth ERs which admit a Borel transversal, we show that countable, and sometimes even continual unions of smooth ERs are smooth. In the end, we prove the 1st dichotomy theorem.

5.a Smooth and below

An important subspecies of smooth ERs consists of those having a Borel transversal: a set with exactly one element in every equivalence class.

**Lemma 27.** (i) Any Borel ER that has a Borel transversal is smooth;

(ii) any Borel finite (with finite classes) ER admits a Borel transversal;

(iii) any Borel countable smooth ER admits a Borel transversal;

(iv) any Borel ER \( E \) on a Polish space \( X \), such that every \( E \)-class is closed and the saturation \([\mathcal{O}]_E\) of every open set \( \mathcal{O} \subseteq X \) is Borel, admits a Borel transversal, hence, is smooth. 10

(v) \( E_0 \) is not smooth.

**Proof.** (i) Let \( T \) be a Borel transversal for \( E \). The map \( \vartheta(x) = \text{"the only element of } T \text{ } E\text{-equivalent to } x\text{"} \) reduces \( E \) to \( D(T) \). 11

(ii) Consider the set of the \( <\)-least elements of \( E \)-classes, where \( < \) is a fixed Borel linear order on the domain of \( E \).

(iii) Use Countable-to-1 Uniformization.

(iv) Since any uncountable Polish space is a continuous image of \( N^N \), we can assume that \( E \) is a ER on \( N^N \). Then, for any \( x \in N^N \), \([x]_E\) is a closed subset of \( N^N \), naturally identified with a tree, say, \( T_x \subseteq N^{<\omega} \). Let \( \vartheta(x) \) denote the leftmost branch of \( T_x \). Then \( x \in E \vartheta(x) \) and \( x \in E y \implies \vartheta(x) = \vartheta(y) \), so that it remains to show that \( Z = \{ \vartheta(x) : x \in N^N \} \) is Borel. Note that

\[
\forall z \in Z \iff \forall m \forall s, t \in N^m (s <_{\text{lex}} t \land z \in \mathcal{O}_t \implies [z]_E \cap \mathcal{O}_t = \emptyset),
\]

where \( <_{\text{lex}} \) is the lexicographical order on \( N^m \) and \( \mathcal{O}_s = \{ x \in N^N : s \subseteq x \} \). However \( [x]_E \cap \mathcal{O}_t = \emptyset \iff x \notin [\mathcal{O}_t]_E \) and \([\mathcal{O}_t]_E \) is Borel for any \( t \).

(v) Otherwise \( E_0 \) has a Borel transversal \( T \), (by (iii)), which is a contradiction, see Example 11(ii). \( \square \)

---

10 Srivastava [44] proved the result for ERs with \( G_\delta \) classes, which is the best possible as \( E_0 \) is a Borel ER, whose classes are \( F_\sigma \) and saturations of open sets are even open, but without any Borel transversal. See also [26, 18.20 iv]).

11 To see that a smooth ER does not necessarily have a Borel transversal take a closed set \( P \subseteq N^N \times N^N \) with \( \text{dom } P = N^N \), not uniformizable by a Borel set, and let \( \langle x, y \rangle \in E \langle x', y' \rangle \) iff both \( \langle x, y \rangle \) and \( \langle x', y' \rangle \) belong to \( P \) and \( x = x' \).
5. b Assembling smooth equivalence relations

If $E$ and $F$ are smooth ERs on disjoint sets, resp., $X$ and $Y$, then easily $E \cup F$ is a smooth ER on $X \cup Y$. The question becomes less clear when we have a Borel ER $E$ on a Polish space $X \cup Y$ such that both $E \mid X$ and $E \mid Y$ are smooth but the sets $X, Y$ not necessarily $E$-invariant in $X \cup Y$ if even disjoint; is $E$ smooth? We answer this in the positive, even in the case of countable unions.

**Theorem 28.** Let $E$ be a Borel ER on a Borel set $X = \bigcup_k X_k$, with all $X_k$ also Borel. Suppose that each $E \mid X_k$ is smooth. Then $E$ is smooth.

**Proof.** First consider the case of a union $X = Y \cup Z$ of just two Borel sets, so that a Borel ER $E$ is smooth on both $Y$ and $Z$. We can assume that $Y \cap Z = \emptyset$. Let the smoothness be witnessed by Borel reductions $f : Y \to Q$ and $g : Z \to R$, with $Q, R$ being disjoint Borel sets. The set

$$F = \{(q, r) : \exists y \in Y \exists z \in Z \ (f(y) = q \land g(z) = r \land y \in E(z))\} \subseteq Q \times R$$

is a partial $\Sigma^1_1$ map $Q \to R$. Let $G : Q \to R$ be any Borel map with $F \subseteq G$, and $H : R \to Q$ be any Borel map with $F^{-1} \subseteq H$. Then $\Phi = G \cap H^{-1}$ is a $1-1$ Borel partial map $P \to Q$ with $F \subseteq \Phi$. Now the $\Pi^1_1$ set

$$P = \{(q, r) \in \Phi : \forall y \in Y \forall z \in Z \ (f(y) = q \land g(z) = r \implies y \in E(z))\},$$

satisfies $F \subseteq P \subseteq \Phi$, hence, there is a Borel function $\Psi$ with $F \subseteq \Psi \subseteq P$. The sets $A = \text{dom} \Psi$ and $B = \text{ran} \Psi$ are Borel subsets of resp. $Q, R$, and it follows from the construction that $\Psi \cap (\text{dom} F \times \text{ran} F) = F$. Finally, put

$$D = \Psi \cup \{(q, q) : q \in Q \setminus A\} \cup \{(r, r) : r \in R \setminus B\},$$

then, for any $y \in Y$ there is unique $h(y) = (q, r) \in D$ with $q = f(y)$, correspondingly, for any $z \in Z$ there is unique $h(z) = (q, r) \in D$ with $r = g(z)$, and if $y \in E(z)$ then $h(y) = h(z) = (f(y), g(z))$, hence, $h$ witnesses that $E$ is smooth.

As for the general case, we can now assume that $X_k \subseteq X_{k+1}$ for all $k$. Then there are disjoint Borel sets $W_k$ and Borel maps $f_k : X_k \to W_k$ which witness that $E \mid X_k$ are smooth ERs. Let $R_k = \text{ran} f_k$ (a $\Sigma^1_1$ set) and

$$F_k = \{(a, b) \in R_k \times R_{k+1} : \exists x \in X_k \ (f_k(x) = a \land f_{k+1}(x) = b)\},$$

this is a $\Sigma^1_1$ set and a $1-1$ map $R_k \to R_{k+1}$. For each $k$ there is a Borel $1-1$ map $G_k$ with $F_k \subseteq G_k$. Let $A_k = \text{dom} G_k$ and $B_k = \text{ran} G_k = B_k$ these are Borel sets with $R_k \subseteq A_k$. We can assume that $B_k \subseteq A_{k+1}$. (Otherwise $G_k$ can be constructed in an analogous way.)

\[\text{5. b Assembling smooth equivalence relations}\]

\[\text{5. b Assembling smooth equivalence relations}\]
reduced in a certain iterative manner to achieve this property.) Then, for any \( k \) and \( b \in A_k \) there is the least \( n = n(b) \leq k \) such that the application

\[
h(b) = G_n^{-1}(G_{n+1}^{-1}(G_{n+2}^{-1}(\ldots G_{k-1}^{-1}(b)\ldots)))
\]

is possible, for instance, \( n(b) = k \) and \( h(b) = b \) whenever \( b \in A_k \setminus B_{k-1} \). Then, \( h(f_k(x)) = h(f_{k+1}(x)) \) holds for any \( x \in X_k \) because \( F_k \subseteq G_k \), so that the map \( g(x) = h(f_k(x)) \) for \( x \in X_k \setminus X_{k-1} \) witnesses the smoothness of \( E \). \( \square \)

5. c The 1st dichotomy theorem.

The following result is known as the 1st dichotomy theorem.

**Theorem 29** (Silver [40]). Any \( \Pi^1_1 \) ER \( E \) on \( \mathbb{N}^\mathbb{N} \) either has at most countably many equivalence classes or admits a perfect set of pairwise \( E \)-inequivalent reals, in other words, either \( E \leq_b D(\mathbb{N}) \) or \( D(\mathbb{N}^\mathbb{N}) \leq_b E \).

**Proof.** 13 As usual, we can suppose that \( E \) is a lightface \( \Pi^1_1 \) relation.

*Case 1:* any \( x \in \mathbb{N}^\mathbb{N} \) belongs to a \( \Delta^1_1 \) \( E \)-equivalent set \( X \) (i.e., all elements of \( X \) are \( E \)-equivalent to each other, in other words, the saturation \( \lbrack X \rbrack_E \) is an equivalence class). Then \( E \) has at most countably many equivalence classes.

*Case 2:* otherwise. Then the set \( H \) of all \( x \), which do not belong to a \( \Delta^1_1 \) pairwise \( E \)-equivalent set (the domain of nontriviality), is non-empty.

**Claim 29.1.** \( H \) is \( \Sigma^1_1 \). Any \( \Sigma^1_1 \) set \( \emptyset \neq X \subseteq H \) is not pairwise \( E \)-equivalent.

**Proof.** \( x \in H \) iff for any \( e \in \mathbb{N} : \text{ if } e \text{ codes a } \Delta^1_1 \text{ set, say, } W_e \subseteq \mathbb{N}^\mathbb{N} \text{ and } x \in W_e \text{ then, } W_e \text{ is not } E \)-equivalent. The "if" part of this characterization is \( \Pi^1_1 \) while the "then" part is \( \Sigma^1_1 \), by \( \Delta^1_1 \) Enumeration (see §A.1).

If \( X \neq \emptyset \) is a pairwise \( E \)-equivalent \( \Sigma^1_1 \) set then \( B = \bigcap_{x \in X} [x]_E \) is a \( \Pi^1_1 \) \( E \)-equivalence class and \( X \subseteq B \). By Separation, there is a \( \Delta^1_1 \) set \( C \) with \( X \subseteq C \subseteq B \). Then, if \( X \subseteq H \) then \( C \subseteq H \) is a \( \Delta^1_1 \) pairwise \( E \)-equivalent set, a contradiction to the definition of \( H \).

\( \square \) (Claim)

Let us fix a countable transitive model \( M \) of a big enough fragment of ZFC, and an elementary submodel of the universe w. r. t. all analytic formulas 14. Consider \( P = \{ X \subseteq \mathbb{N}^\mathbb{N} : X \text{ is non-empty and } \Sigma^1_1 \} \) as a forcing to extend \( M \) (smaller sets are stronger conditions), the Gandy – Harrington forcing. We have \( P \not\subseteq \mathbb{N} \), of course, but clearly \( P \) can be adequately coded in \( M \), say, via a universal \( \Sigma^1_1 \) set.

---

13 We present a forcing proof of Miller [36], with some simplifications. See [32] for another proof, based on the Gandy – Harrington topology. In fact both proofs involve essentially the same combinatorics.

14 For instance, \( M \) models ZC and, in addition, Replacement for \( \Sigma^1_{100} \varepsilon \)-formulas and the first one million of instances of Replacement overall. Being an elementary submodel is useful to guarantee that relations like the inclusion orders of \( C_x \) and \( C_0 \) are absolute for \( M \).
5 SMOOTHERS AND THE FIRST DICHOTOMY

Corollary 29.2 (from Theorem 85). If \( G \subseteq \mathbb{P} \) is a \( \mathbb{P} \)-generic, over \( \mathcal{M} \), set, then \( \bigcap G \) contains a single real, denoted \( x_G \). \( \square \)

Reals of the form \( x_G \), \( G \) as in the Corollary, are called \( \mathbb{P} \)-generic (over \( \mathcal{M} \)). Let \( \mathcal{R} \) be the name for \( x_G \). Then any \( A \in \mathbb{P} \) forces that \( \mathcal{R} \in A \).

Let \( \mathbb{P}^2 \) consist of all \( \langle X \times Y \rangle \), \( X, Y \in \mathbb{P} \). It follows from the above by the product forcing lemmas that any \( \mathbb{P}^2 \)-generic, over \( \mathcal{M} \), set \( G \subseteq \mathbb{P}^2 \) produces a pair of reals (a \( \mathbb{P}^2 \)-generic pair), say, \( x^G_{\text{left}} \) and \( x^G_{\text{right}} \), so that \( \langle x^G_{\text{left}}, x^G_{\text{right}} \rangle \in W \) for any \( W \in G \). Let \( x^G_{\text{left}} \) and \( x^G_{\text{right}} \) be their names.

Lemma 29.3. \( H \times H \) \( \mathbb{P}^2 \)-forces \( \mathcal{R}_{\text{left}} \vDash \mathcal{R}_{\text{right}}. \)

Proof. Otherwise a “condition” \( X \times Y \subseteq \mathbb{P}^2 \) with \( X \cup Y \subseteq H \) \( \mathbb{P}^2 \)-forces \( \mathcal{R}_{\text{left}} \vDash \mathcal{R}_{\text{right}} \), so that any \( \mathbb{P}^2 \)-generic pair \( \langle x, y \rangle \in X \times Y \) satisfies \( x \vDash \mathcal{E} y \). By the product forcing lemmas for any pair of \( \mathbb{P} \)-generic \( x', x'' \in X \) there is \( y \in Y \) such that both \( \langle x, y \rangle \) and \( \langle x', y \rangle \) are \( \mathbb{P}^2 \)-generic pairs, hence, we have

\((*)\) If \( x', x'' \in X \) are \( \mathbb{P} \)-generic over \( \mathcal{M} \) then \( \mathcal{R} \vDash x' \vDash x'' \).

The set \( \mathbb{P}_2 \) of all non-empty \( \Sigma^1_1 \) subset of \( \mathbb{N}^n \) is just a copy of \( \mathbb{P} \) (not of \( \mathbb{P}^2 \! \)!) as a forcing, in particular, if \( G \subseteq \mathbb{P}_2 \) is \( \mathbb{P}_2 \)-generic over \( \mathcal{M} \) then there is a unique pair of reals (\( \mathbb{P}_2 \)-generic pair) \( \langle x^G_{\text{left}}, x^G_{\text{right}} \rangle \) which belongs to every \( W \) in \( G \), and in this case, both \( x^G_{\text{left}} \) and \( x^G_{\text{right}} \) are \( \mathbb{P} \)-generic, because if \( G \subseteq \mathbb{P}_2 \) is \( \mathbb{P}_2 \)-generic then the sets \( G' \) and \( G'' \) of all projections of sets \( W \in G \) to resp. 1st and 2nd co-ordinate, are easily \( \mathbb{P} \)-generic. Now let \( G \subseteq \mathbb{P}_2 \) be a \( \mathbb{P}_2 \)-generic set, over \( \mathcal{M} \), containing the \( \Sigma^1_1 \) set \( P = X^2 \vDash \mathcal{E} \). (Note that \( P \neq \emptyset \) by Lemma 29.1.) Then \( \langle x^G_{\text{left}}, x^G_{\text{right}} \rangle \in P \), hence, \( x^G_{\text{left}} \vDash x^G_{\text{right}} \), however, as we observed, both \( x^G_{\text{left}} \) and \( x^G_{\text{right}} \) are \( \mathbb{P} \)-generic elements of \( X \) (because \( P \subseteq X \times X \)), which contradicts \((*)\). \( \square \) (Lemma 29.3)

Fix enumerations \( \{ \mathcal{D}(n) \}_{n \in \mathbb{N}} \) and \( \{ \mathcal{D}^2(n) \}_{n \in \mathbb{N}} \) of all dense subsets of resp. \( \mathbb{P} \) and \( \mathbb{P}^2 \) which are coded in \( \mathcal{M} \). Then there is a system \( \{ X_u \}_{u \in 2^{<\omega}} \) of sets \( X_u \), satisfying

(i) \( X_u \in \mathbb{P} \), moreover, \( X_A \subseteq H \) and \( X_u \in \mathcal{D}(n) \) whenever \( u \in 2^n \);

(ii) \( X_u \cap \subseteq X_u \) for all \( u \in 2^{<\omega} \) and \( i = 0, 1 \);

(iii) if \( u \neq v \in 2^n \) then \( X_u \times X_v \notin \mathcal{D}^2(n) \).

It follows from (i) that, for any \( a \in 2^\mathbb{N} \), the set \( \{ X_{a|m} : m \in \mathbb{N} \} \) is \( \mathbb{P} \)-generic over \( \mathcal{M} \), hence, \( \bigcap_m X_{a|m} \) is a singleton, say, \( x_a \), by Corollary 29.2. Moreover the map \( a \mapsto x_a \) is continuous as diameters of \( X_u \) converge to 0 uniformly with \( 1 \mapsto u \mapsto 0 \), by (i). In addition, by (iii) and Lemma 29.3, \( x_a \vDash x_b \) whenever \( a \neq b \), in particular, \( x_a \neq x_b \), hence, we have a perfect \( \mathcal{E} \)-inequivalent set \( Y = \{ x_a : a \in 2^\mathbb{N} \} \).

\( \square \) (Theorem 29)
6 Hyperfinite and countable ERs

This Section is mainly devoted to the node \( E_0 \) in the diagram on page 16. Together with the 2nd dichotomy theorem, we present some other properties of \( E_0 \), the ideal \( \text{Fin} \), and hyperfinite (Borel) equivalence relations. This class of equivalence relations is a very interesting object of study even aside of pure descriptive set theory. Papers \([2, 19]\) give a comprehensive account of most basic results, with further references.

After a rather simple theorem which shows that \( \text{Fin} \) is the least ideal in the sense of \( \leq^+ \), \( \leq^\ast \), \( \leq_\text{B} \), we prove the “Glimm–Effros”, or second, dichotomy which asserts that \( E_0 = E_{\text{Fin}} \) is the \( \leq_\text{B} \)-least among all non-smooth Borel ERs. Finally, we present a characterization, in terms of the existence of transversals, of those Borel sets \( X \) for which \( E_0 \upharpoonright X \) is smooth.

6.1 \text{Fin is the least!}

The proof of the following useful result is based on a short argument involved in many other results. A somewhat more pedestrian version of the argument was used in several proofs in Section 4.

Theorem 30.  
(i) \([20, 34, 45]\) If \( \mathcal{I} \) is a (nontrivial) ideal on \( \mathbb{N} \), with the Baire property in the topology of \( \mathcal{P}(\mathbb{N}) \), then \( \text{Fin} \leq^+ \mathbb{R} \) and \( \leq_\text{B} \mathcal{I} \);  
(ii) however \( D(2^\mathbb{N}) \prec B E_0 \) strictly, thus \( D(2^\mathbb{N}) \) is not \( \sim_\text{n} \)-equivalent to an equivalence relation of the form \( E_0 \);  
(iii) if \( \mathcal{I} \leq^+_\text{RB} \mathcal{J} \) are Borel ideals, and there is an infinite set \( Z \subseteq \text{dom} \mathcal{J} \) such that \( \mathcal{I} \upharpoonright Z = \mathcal{P}_{\text{fin}}(Z) \), then \( \mathcal{J} \leq_\text{BB} \mathcal{I} \).

Proof. (i) First of all \( \mathcal{I} \) must be meager in \( \mathcal{P}(\mathbb{N}) \). (Otherwise \( \mathcal{I} \) would be comeager somewhere, easily leading to contradiction.) Thus, all \( X \subseteq N \) “generic” (over a certain countable family of dense open subsets of \( \mathcal{P}(\mathbb{N}) \)) do not belong to \( \mathcal{I} \). Now it suffices to define non-empty finite sets \( w_i \subseteq N \) with \( \max w_i < \min w_{i+1} \) such that any union of infinitely many of them is “generic”. Clearly the following observation yields the result: if \( D \) is an open dense subset of \( \mathcal{P}(\mathbb{N}) \) and \( n \in \mathbb{N} \) then there is \( m > n \) and a set \( u \subseteq [n, m] \) with \( m, n \in u \) such that any \( x \in \mathcal{P}(\mathbb{N}) \) satisfying \( x \cap [n, m] = u \) belongs to \( D \).

Thus we have \( \text{Fin} \leq^+ \mathcal{I} \). To derive \( \text{Fin} \leq^\ast \mathcal{I} \) cover each \( w_k \) by a finite set \( u_k \) such that \( \bigcup_{k \in \mathbb{N}} u_k = N \) and still \( u_k \cap u_l = \emptyset \) for \( k \neq l \).

(ii) That \( D(2^\mathbb{N}) \leq B E_0 \) is witnessed by any perfect set \( X \subseteq 2^\mathbb{N} \) which is a partial transversal for \( E_0 \) (i.e., any \( x \neq y \) in \( X \) are \( E_0 \)-inequivalent). On the other hand, \( D(2^\mathbb{N}) \) is smooth but \( E_0 \) is non-smooth by Lemma 27(v).

(iii) Assume w.l.o.g. that \( \mathcal{I}, \mathcal{J} \) are ideals over \( \mathbb{N} \). Let pairwise disjoint finite sets \( w_k \subseteq N \) witness \( \leq^\ast \mathcal{J} \) \( \mathcal{I} \). Put \( Z' = N \setminus Z \), \( X = \bigcup_{k \in Z} w_k \), and \( Y = \bigcup_{k \in Z'} w_k \). The reduction via \( \{w_k\} \) reduces \( \mathcal{P}_{\text{fin}}(Z) \) to \( \mathcal{I} \upharpoonright X \) and \( \mathcal{J} \upharpoonright Z' \).
6 HYPERFINITE AND COUNTABLE ERS

to \( \mathcal{F} \upharpoonright Y \). Keeping the latter, replace the former by a \( \preceq_{\text{fin}} \)-like reduction of \( \mathcal{F}_{1n}(z) \) to \( \mathcal{F} \upharpoonright Y' \), where \( Y' = N \setminus Y \), which exists by Theorem 30. \( \square \)

Despite of Theorem 30, \( E_0 = E_{\text{fin}} \) is not the \( \preceq_{\text{fin}} \)-least among Borel ERs.

Thus, \( D(2^\aleph_1) \) is not a ER generated by a Borel ideal, even modulo \( \sim_{\text{B}} \).

6.b Countable equivalence relations

This class of equivalence relations, essentially bigger than hyperfinite (modulo \( \leq_{\text{fin}} \)), is a subject of ongoing intense study. Yet we can only present here the following important theorem and a few more results below, leaving [19, 10, 30] as basic references in this domain.

Theorem 31 ([8, Thm 1], [2, 1.8]). Any Borel countable ER \( E \) on a Polish space \( \mathcal{X} \):

(i) is induced by a Polish action of a countable group \( G \) on \( \mathcal{X} \);

(ii) satisfies \( E \leq_{\text{B}} E_{\infty} = E(F_2, 2) \), where \( F_2 \) is the free group with two generators and \( E(F_2, 2) \) is the ER induced by the shift action of \( F_2 \) on \( 2^{F_2} \).

Proof. (i) We w.l.o.g. assume that \( \mathcal{X} = 2^\aleph_1 \). According to Countable-to-1 Enumeration (in a relativized version, if necessary, see Remark 82), there is a sequence of Borel maps \( f_n : 2^\aleph_1 \to 2^\aleph_1 \) such that \( [a]_E = \{f_n(a) : n \in \mathbb{N}\} \) for each \( a \in 2^\aleph_1 \). Put \( \Gamma'_n = \{\langle a, f_n(a) \rangle : a \in \mathbb{N}\} \) (the graph of \( f_n \)) and \( \Gamma_n = \Gamma'_n \cup \bigcup_{k<n} \Gamma'_k \). The sets \( P_{nk} = \Gamma_n \cap \Gamma_{k}^{-1} \) form a partition of (the graph of) \( E \) onto countably many Borel injective sets. Further define \( \Delta = \{\langle a, a \rangle : a \in 2^\aleph_1\} \) and let \( \{D_m\}_{m \in \mathbb{N}} \) be an enumeration of all non-empty sets of the form \( P_{nk} \setminus \Delta \). Intersecting the sets \( D_m \) with the rectangles of the form

\[
R_s = \{\langle a, b \rangle \in 2^{\aleph_1} \times 2^{\aleph_1} : s^0 \subseteq a \wedge s^1 \subseteq b\} \quad \text{and} \quad R_s^{-1},
\]

we reduce the general case to the case when \( \text{dom} D_m \cap \text{ran} D_m = \emptyset \), \( \forall m \).

Now, for any \( m \) define \( h_m(a) = b \) whenever either \( \langle a, b \rangle \in D_m \) or \( \langle a, b \rangle \in D_m^{-1} \), or \( a = b \not\in \text{dom} D_m \cup \text{ran} D_m \). Clearly \( h_m \) is a Borel bijection \( 2^{\aleph_1} \rightarrow 2^{\aleph_1} \). Thus \( \{h_m\}_{m \in \mathbb{N}} \) is a family of Borel automorphisms of \( 2^{\aleph_1} \) such that \( [a]_E = \{h_m(a) : m \in \mathbb{N}\} \). It does not take much effort to expand this system to a Borel action of \( F_\omega \), the free group with \( \aleph_0 \) generators, on \( 2^{\aleph_1} \), whose induced equivalence relation is \( E \).

(ii) First of all, by (i), \( E \leq_{\text{B}} R \), where \( R \) is induced by a Borel action \( \cdot \) of \( F_\omega \) on \( 2^{\aleph_1} \). The map \( \vartheta(a) = \{g^{-1} \cdot a \}_{g \in F_\omega}, \ a \in 2^{\aleph_1} \), is a Borel reduction of \( R \) to \( E(F_\omega, 2^{\aleph_1}) \). If now \( F_\omega \) is a subgroup of a countable group \( H \) then \( E(F_\omega, 2^{\aleph_1}) \leq_{\text{B}} E(H, 2^{\aleph_1}) \) by means of the map sending any \( \{a_g\}_{g \in F_\omega} \) to \( \{b_h\}_{h \in H} \), where \( b_g = a_g \) for \( g \in F_\omega \) and \( b_h \) equal to any fixed \( b' \in 2^{\aleph_1} \) for \( h \in H \setminus F_\omega \). As \( F_\omega \) admits a homomorphism into \( F_2 \) we conclude that \( E \leq_{\text{B}} E(F_2, 2^{\aleph_1}) \).

\(^{15}\text{Why ?}\)
6 HYPERFINITE AND COUNTABLE ERS

It remains to transform \( \mathcal{E}(F_2, 2^\mathbb{N}) \) to \( \mathcal{E}(F_2, 2) \). The inequality \( \mathcal{E}(F_2, 2^\mathbb{N}) \leq \mathcal{E}(F_2, 2^{2^\mathbb{N}}) \) is clear. Further \( \mathcal{E}(F_2, 2^{2^\mathbb{N}}) \leq \mathcal{E}(F_2 \times \mathbb{Z}, 3) \), by means of the map sending any \( \{ a_g \}_{g \in F_2} (a_g \in 2^{2^\mathbb{N}}) \) to \( \{ b_{gj} \}_{g \in F_2, j \in \mathbb{Z}} \), where \( b_{gj} = a_g(j) \) for \( j \neq 0 \) and \( b_{g0} = 2 \). Further, for any \( G \), \( \mathcal{E}(G, 3) \leq \mathcal{E}(G \times \mathbb{Z}, 2) \) by means of the map sending any \( \{ a_g \}_{g \in G} (a_g = 0, 1, 2) \) to \( \{ b_{gi} \}_{g \in G, i \in \mathbb{Z}_2} \), where

\[
    b_{gi} = \begin{cases} 
    0, & \text{if } a_g = 0 \text{ or } a_g = 1 \text{ and } i = 0, \\
    1, & \text{if } a_g = 2 \text{ or } a_g = 1 \text{ and } i = 1.
    \end{cases}
\]

Thus \( \mathcal{E}(F_2, 2^\mathbb{N}) \leq \mathcal{E}(F_2 \times \mathbb{Z} \times \mathbb{Z}_2, 2) \). However, \( F_2 \times \mathbb{Z} \times \mathbb{Z}_2 \) admits a homomorphism into \( F_\omega \), and then into \( F_2 \) (see above), so that \( \mathcal{E}(F_2, 2^\mathbb{N}) \leq \mathcal{E}(F_2, 2) \), as required. \(\square\)

6.c Hyperfinite equivalence relations

All Borel finite ERs are smooth (see §5.a), accordingly, all hyperfinite ERs are hypersmooth. On the other hand, any finite or hyperfinite equivalence relation is countable, of course. It follows from the next theorem that, conversely, every hypersmooth countable ER is hyperfinite. (But there exist countable non-hypersmooth ERs, for instance, \( \mathcal{E}_\infty \), which are not hyperfinite.)

The theorem also shows that \( \mathcal{E}_0 \) is a universal hyperfinite ER. (To see that \( \mathcal{E}_0 \) is hyperfinite, let \( x \mathcal{F}_n y \iff x \Delta y \subseteq [0, n) \) for \( x, y \subseteq \mathbb{N} \).)

**Theorem 32** (Theorems 5.1 and, partially, 7.1 in [2] and 12.1(ii) in [19]). *The following are equivalent for a Borel ER \( \mathcal{E} \) on a Polish space \( X \):

(i) \( \mathcal{E} \leq \mathcal{E}_0 \) and \( \mathcal{E} \) is countable;

(ii) \( \mathcal{E} \) is hyperfinite;

(iii) \( \mathcal{E} \) is hypersmooth and countable;

(iv) there is a Borel set \( X \subseteq \mathcal{P}(\mathbb{N})^\mathbb{N} \) such that \( \mathcal{E}_1 \upharpoonright X \) is a countable ER and \( \mathcal{E} \) is isomorphic, via a Borel bijection of \( X \) onto \( X \), to \( \mathcal{E}_1 \upharpoonright X \);

(v) \( \mathcal{E} \) is induced by a Borel action of \( \mathbb{Z} \), the additive group of the integers.

(vi) there exists a pair of Borel ERs \( F, R \) of type 2 such that \( \mathcal{E} = F \cup R \). \(16\)

**Proof.** (ii) \( \Rightarrow \) (iii) and (i) \( \Rightarrow \) (iv).

(iii) \( \Rightarrow \) (iv). Let \( \mathcal{E} = \bigcup_n \mathcal{F}_n \) be a countable and hypersmooth ER on a space \( X \), all \( \mathcal{F}_n \) being smooth (and countable), and \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1}, \forall n \). We may assume that \( X = \mathcal{P}(\mathbb{N}) \) and \( \mathcal{F}_0 = D(\mathcal{P}(\mathbb{N})) \). Let \( T_n \subseteq X \) be a Borel transversal for \( \mathcal{F}_n \) (recall Lemma 27(iii)). Now let \( \vartheta_n(x) \) be the only element of \( T_n \) with

\(16\) An equivalence relation \( F \) is of type \( n \) if any \( F \)-class contains at most \( n \) elements. \( F \cup R \) denotes the least ER which includes \( F \cup R \).
\( v \mathcal{F}_n \partial_n(x) \). Then \( x \mapsto \{ \partial_n(x) \}_{n \in \mathbb{N}} \) is a 1–1 Borel map \( \mathcal{X} \to \mathcal{P}(\mathbb{N})^\omega \) and \( x \in y \iff \partial(x) \mathcal{E}_1 \partial(y) \). Take \( X \) to be the image of \( \mathcal{X} \).

(iv) \( \implies \) (v). Let \( X \) be as indicated. For any \( \mathbb{N} \)-sequence \( x \) and \( n \in \mathbb{N} \), let 
\[ x \upharpoonright_{>n} = x \upharpoonright_{(n, \infty)} \] 
be as indicated. Then it follows from the relativization of Countable-to-1 Projection and Countable-to-1 Enumeration that for any \( n \) the set \( X \upharpoonright_{>n} = \{ x \upharpoonright_{>n} \mid x \in X \} \) is Borel and there is a countable family of Borel functions \( g^n_i : X \upharpoonright_{>n} \to X \) for all \( i \in \mathbb{N} \), such that the set \( X_\xi = \{ x \in X : x \upharpoonright_{>n} = \xi \} \) is equal to \( \{ g^n_i(\xi) : i \in \mathbb{N} \} \) for any \( \xi \in X \upharpoonright_{>n} \), hence, \( \{ g^n_i(\xi)(n) : i \in \mathbb{N} \} = \{ x(n) : x \in X_\xi \} \).

For any \( x \in \mathcal{P}(\mathbb{N})^\omega \) let \( \varphi(x) = \{ \varphi_n(x) \}_{n \in \mathbb{N}} \), where \( \varphi_n(x) \) is the least number \( i \) such that \( x(n) = f_i^n(x)(n) \); thus, \( \varphi(x) \in \mathbb{N}^\mathbb{N} \). Let \( \mu(x) \) be the sequence
\[ \varphi_0(x), \varphi_0'(x), \varphi_1(x) + 1, \varphi_1'(x) + 1, \ldots, \varphi_n(x) + n, \varphi_n'(x) + n, \ldots, \]
where \( \varphi_n'(x) = \max_{k \leq n} \varphi_k(x) \). Easily if \( x \neq y \in X \) satisfy \( x \mathcal{E}_1 y \), i.e., \( x \upharpoonright_{>n} = y \upharpoonright_{>n} \) for some \( n \), then \( \varphi(x) \upharpoonright_{>n} = \varphi(y) \upharpoonright_{>n} \) but \( \varphi(x) \neq \varphi(y) \), \( \mu(x) \neq \mu(y) \), and \( \mu(x) \upharpoonright_{>m} = \mu(y) \upharpoonright_{>m} \) for some \( m \geq n \).

Let \( <_{\text{lex}} \) be the anti-lexicographical partial order on \( \mathbb{N}^\mathbb{N} \), i.e., \( a <_{\text{lex}} b \) iff there is \( n \) such that \( a \upharpoonright_{>n} = b \upharpoonright_{>n} \) and \( a(n) < b(n) \). For \( x, y \in X \) define \( x <_0 y \) iff \( \mu(x) <_{\text{lex}} \mu(y) \). It follows from the above that \( <_0 \) linearly orders every \( \mathcal{E}_1 \)-class \( [x]_{\mathcal{E}_1} \cap X \) of \( x \in X \). Moreover, it follows from the definition of \( \mu(x) \) that any \( <_{\text{lex}} \)-interval between some \( \mu(x) <_{\text{lex}} \mu(y) \) contains only finitely many elements of the form \( \mu(z) \). (For \( \varphi \) this would not be true.) We conclude that any class \( [x]_{\mathcal{E}_1} \cap X \), \( x \in X \), is linearly ordered by \( <_0 \) similarly to a subset of \( \mathbb{Z} \), the integers. That \( <_0 \) can be converted to a required Borel action of \( \mathbb{Z} \) on \( X \) is rather easy (however the \( \mathcal{E}_1 \)-classes in \( X \) ordered similarly to \( \mathbb{N} \), the inverse of \( \mathbb{N} \), or finite, should be treated separately).

(v) \( \implies \) (ii). Assume w.l.o.g. that \( \mathcal{X} = 2^\mathbb{N} \). An increasing sequence of ERs \( \mathcal{F}_n \) whose union is \( \mathcal{E} \) is defined separately on each \( \mathcal{E} \)-class \( C \) by “integrate” into Borel ERs \( \mathcal{F}_n \) defined on the whole of \( 2^\mathbb{N} \) because the action allows to replace quantifiers over a \( \mathcal{E} \)-class \( C \) by quantifiers over \( \mathbb{Z} \).

Let \( C \) be any \( \mathcal{E} \)-class of \( x \in X \). Note that if an element \( x_C \in C \) can be chosen in some Borel-definable way then we can define \( x \mathcal{F}_n y \) iff there exist integers \( j, k \in \mathbb{Z} \) with \( |j| \leq n, |k| \leq n \), and \( x = j \cdot x_C, y = k \cdot x_C \). This applies, for instance, when \( C \) is finite, thus, we can assume that \( C \) is infinite. Let \( <_{\text{lex}} \) be the lexicographical ordering of \( 2^\mathbb{N} \), and \( <_{\text{act}} \) be the partial order induced by the action, i.e., \( x <_{\text{act}} y \) iff \( y = j \cdot x, j > 0 \). By the same reason we can assume that neither of \( a = \inf_{<_{\text{int}}} C \) and \( b = \sup_{<_{\text{int}}} C \) belongs to \( C \). Let \( C_n \) be the set of all \( x \in C \) with \( x \upharpoonright_{>n} \neq a \upharpoonright_{>n} \) and \( x \upharpoonright_{>n} \neq b \upharpoonright_{>n} \). Define \( x \mathcal{F}_n y \) iff \( x, y \) belong to one and the same \( <_{\text{lex}} \)-interval in \( C \) lying entirely within \( C_n \), or just \( x = y \).

In our assumptions, any \( \mathcal{F}_n \) has finite classes, and for any two \( x, y \in C \) there is \( n \) with \( x \mathcal{F}_n y \).

(v) \( \implies \) (i). This is more complicated. A preliminary step is to show that \( \mathcal{E} \leq_n \mathcal{E}(\mathbb{Z}, 2^\mathbb{N}) \), where \( \mathcal{E}(\mathbb{Z}, 2^\mathbb{N}) \) is the orbit equivalence induced by the shift action of

\[ E \leq_n \mathcal{E}(\mathbb{Z}, 2^\mathbb{N}) \]
Z on \((2^N)^Z\): \(k \cdot \{x_j\}_{j \in Z} = \{x_{j-k}\}_{j \in Z}\) for \(k \in Z\). Assuming w.l.o.g. that \(E\) is an ER on \(2^N\), we obtain a Borel reduction of \(E\) to \(E(Z, 2^N)\) by \(\vartheta(x) = \{j \cdot x\}_{j \in Z}\), where \(\vartheta\) is a Borel action of \(Z\) on \(2^N\) which induces \(E\). Then Theorem 7.1 in [2] proves that \(E(Z, 2^N) \leq B E_0\).

(vi) \(\Rightarrow\) (v). Let \(E = F \lor R\), where \(F, R\) are of type 2. For any \(x \in X\) (the domain of \(E\)), if \([x]_F\) contains another element \(y \neq x\) then call \(y\) the left, resp., right neighbour of \(x\) if \(y < x\), resp., \(y > x\), where \(<\) is a fixed Borel linear ordering of \(X\). If the class \([x]_R\) also contains another element, say, \(z\), call it the neighbour of \(x\) of the opposite side w.r.t. \(y\). The neighbour relation linearly orders any \(E\)-class similarly to a subset of \(Z\), which easily leads to (v).

(v) \(\Rightarrow\) (vi). The authors of [19] present a short proof which refers to several difficult theorems on hyperfinite ERs. Here we give an elementary proof.

Let \(E\) be induced by a Borel action of \(Z\). We are going to define \(F\) and \(R\) on any \(E\)-class \(C = [x]_E\). If we can choose an element \(x_C \in C\) in some uniform Borel-definable way then a rather easy construction is possible, which we leave to the reader. This applies, for instance, when \(C\) is finite, hence, let us assume that \(C\) is infinite. Let \(\prec_{\text{act}}\) be the linear order on \(C\), induced by the action of \(Z\); it is similar to \(Z\). Let \(\prec_{\text{lex}}\) be the lexicographical ordering of \(2^N = \text{dom } E\).

Our goal is to define \(F\) on \(C\) so that every \(F\)-class contains exactly two (distinct) elements. The ensuing definition of \(R\) is then rather simple. (First, order pairs \(\{x, y\}\) of elements of \(C\) in accordance with the \(\prec_{\text{act}}\)-lexicographical ordering of pairs \((\max \\prec_{\text{act}} \{x, y\}, \min \\prec_{\text{act}} \{x, y\})\), this is still similar to \(Z\). Now, if \(\{x, y\}\) and \(\{x', y'\}\) are two \(F\)-classes, the latter being the next to the former in the sense just defined, and \(x \prec_{\text{act}} y, x' \prec_{\text{act}} y'\), then define \(y \lor R x'\).

Suppose that \(W \subseteq C\). An element \(z \in W\) is \(lmin\) (locally minimal) in \(W\) if it is \(\prec_{\text{lex}}\)-smaller than both of its \(\prec_{\text{act}}\)-neighbours in \(W\). Put \(W_{lmin} = \{z \in W: z \text{ is } lmin \text{ in } W\}\). If \(C_{lmin}\) is not unbounded in \(C\) in both directions then an appropriate choice of \(x_C \in C\) is possible. (Take the \(\prec_{\text{act}}\)-least or \(\prec_{\text{act}}\)-largest point in \(C_{lmin}\), or if \(C_{lmin} = \emptyset\), so that, for instance, \(\prec_{\text{act}}\) and \(\prec_{\text{lex}}\) coincide on \(C\), we can choose something like a \(\prec_{\text{lex}}\)-middlest element of \(C\).) Thus, we can assume that \(C_{lmin}\) is unbounded in \(C\) in both directions.

Let a \(lmin\)-interval be any \(\prec_{\text{act}}\)-semi-interval \([x, x']\) between two consecutive elements \(x \prec_{\text{act}} x'\) of \(C_{lmin}\). Let \([x, x'] = \{x_0, x_1, \ldots, x_{m-1}\}\) be the enumeration in the \(\prec_{\text{act}}\)-increasing order \((x_0 = x)\). Define \(x_{2k} \lor \varsigma x_{2k+1}\) whenever \(2k + 1 < m\). If \(m\) is odd then \(x_{m-1}\) remains unmatched. Let \(C^1\) be the set of all unmatched elements. Now, the nontrivial case is when \(C^1\) is unbounded in \(C\) in both directions. We define \(C^1_{lmin}\) as above, and repeat the same construction, extending \(F\) to a part of \(C^1\), with, perhaps, a remainder \(C^2 \subseteq C^1\) where \(F\) remains undefined. Et cetera.

Thus, we define a decreasing sequence \(C = C^0 \supseteq C^1 \supseteq C^2 \supseteq \ldots\) of subsets of \(C\), and the equivalence relation \(F\) on each difference \(C^n \setminus C^{n+1}\) whose classes contain exactly two points each, and the nontrivial case is when every \(C^n\) is \(\prec_{\text{act}}\)-
unbounded in $C$ in both directions. (Otherwise there is an appropriate choice of $x_C \in C$.) If $C^\infty = \bigcap_n C^n = \emptyset$ then $F$ is defined on $C$ and we are done. If $C^\infty = \{ x \}$ is a singleton then $x_C = x$ chooses an element in $C$. Finally, $C^\infty$ cannot contain two different elements as otherwise one of $C^n$ would contain two $\prec_{\text{act}}$-neighbours $x \prec_{\text{act}} y$ which survive in $C^{n+1}$, which is easily impossible.  

6.d Non-hyperfinite countable equivalence relations

It follows from Theorem 32(i),(ii) that hyperfinite equivalence relations form an initial segment, in the sense of $\leq_B$, among all countable equivalence relations. Let us show that not all countable equivalence relations are hyperfinite.

**Theorem 33.** The equivalence relation $E_\infty$ is not hyperfinite.

**Proof.** A clean elementary proof is given in [41].

6.e Assembling hyperfinite equivalence relations

The following theorem shows that, similarly to the case of smooths ERs (Thm 28), hyperfinite ones possess a certain form of countable additivity.

**Theorem 34.** Let $E$ be a Borel ER on a Borel set $X = \bigcup_k X_k$, with all $X_k$ also Borel. Suppose that $E \restriction X_k \leq_B E_0$ for each $k$. Then $E \leq_B E_0$.

**Proof.** We consider only the case when $X_k \subseteq X_{k+1}$ for all $k$ (the result will be used below only for this particular case), the general case needs to consider separately the two-sets case, as in Theorem 28, which we leave to the reader.

There are disjoint Borel sets $B_k \subseteq P(n)$ and Borel maps $f_k : X_k \to B_k$ which witness that $E \restriction X_k \leq_B E_0$. We shall assume that the sets $B_k$ are $E_0$-incompatible in the sense that if $k \neq n$ then $a E_0 b$ does not hold for any $a \in B_k$ and $b \in B_n$. Let $R_k = \text{ran } f_k$ (a $\Sigma^1_1$ subset of $B_k$). Then

$$F_k = \{ \langle a, b \rangle \in R_k \times R_{k+1} : \exists x \in X_k \ (f_k(x) = a \wedge f_{k+1}(x) = b) \} ,$$

is a $\Sigma^1_1$ set, 1–1 modulo $E_0$ in the sense that if $\langle a, b \rangle$ and $\langle a', b' \rangle$ belong to $F_k$ then $a E_0 a' \iff b E_0 b'$. As “to be 1–1 modulo $E_0$” is a $\Pi^1_1$ property in the codes (of $\Sigma^1_1$ subsets of $P(n)^2$), there is, by Reflection, a $\Delta^1_1$ set $F'_k$ with $F_k \subseteq F'_k \subseteq B_k \times B_{k+1}$ and still 1–1 modulo $E_0$. The following $\Delta^1_1$ set

$$G_k = \{ \langle a', b' \rangle : \exists \langle a, b \rangle \in F'_k \ (a E_0 a' \wedge b E_0 b') \}$$

is still 1–1 modulo $E_0$, hence, both “vertical” and “horizontal” cross-sections of $G_k$ are countable, thus, $A_k = \text{dom } G_k$ and $B_k = \text{ran } G_k$ are $E_0$-invariant Borel sets (and $R_k = \text{dom } F_k \subseteq A_k$), and there are Borel maps $h_k : B_k \to A_k$ such that $\langle h_k(b), b \rangle \in G_k$ whenever $b \in B_k$. It follows still from the “1–1 modulo $E_0$” property that if $b \in B_k$ and $b' E_0 b$ then $b' \in B_k$ and $h_k(b) E_0 h_k(b')$.
7  THE 2ND DICHOTOMY

We can assume that \( B_{k+1} \subseteq A_k \) for all \( k \). Then, for any \( k \) and \( b \in A_k \) there is the least \( n = n(b) \leq k \) such that the application

\[
h(b) = h_n(h_{n+1}(h_{n+2}(\ldots h_{k-1}(b)\ldots)))
\]

is possible, for instance, \( n(b) = k \) and \( h(b) = b \) whenever \( b \in A_k \setminus B_{k-1} \). As in the proof of Theorem 28, the map \( g(x) = h(f_k(x)) \) for \( x \in X_k \setminus X_{k-1} \) witnesses \( E \subseteq B E_0 \).

\[ \square \]

7  The 2nd dichotomy

The following result is known as 2nd, or “Glimm–Effros”, dichotomy.

**Theorem 35** (Harrington, Kechriv, Louveau [12]). If \( E \) is a Borel ER then either \( E \) is smooth or \( E_0 \subseteq C E \).

7.a  The Gandy – Harrington closure

Beginning the proof of Theorem 35 (it will be completed in §7.d), we suppose, as usual, that \( E \) is a lightface \( \Delta^1_1 \) ER on \( \mathbb{N}^N \). Consider an auxiliary ER \( x \in \mathbb{E} \) iff \( x, y \in \mathbb{N}^N \) belong to the same \( E \)-invariant \( \Delta^1_1 \) sets. (A set \( X \) is \( E \)-invariant iff \( X = [X]_E \).) Easily \( E \subseteq \bar{E} \). To see that \( \bar{E} \) is the closure of \( E \) in the Gandy – Harrington topology, prove

**Lemma 35.2.** If \( F \) is a \( \Sigma^1_1 \) ER on \( \mathbb{N}^N \), and \( X,Y \subseteq \mathbb{N}^N \) are disjoint \( F \)-invariant \( \Sigma^1_1 \) sets, then there is an \( F \)-invariant \( \Delta^1_1 \) set \( X' \) separating \( X \) from \( Y \).

**Proof.** By Separation, for any \( \Sigma^1_1 \) set \( A \) with \( A \cap Y = \emptyset \) there is a \( \Delta^1_1 \) set \( A' \) with \( A \subseteq A' \) and \( A' \cap Y = \emptyset \) — note that then \( [A']_F \cap Y = \emptyset \) because \( Y \) is \( F \)-invariant. It follows that that there is a sequence \( X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq \ldots \), where \( A'_i \) are \( \Delta^1_1 \) sets, accordingly, \( A_{i+1} = [A'_i]_F \) are \( \Sigma^1_1 \) sets, and \( A_i \cap Y = \emptyset \). Then \( X' = \bigcup_n A_n = \bigcup_n A'_n \) and is an \( F \)-invariant Borel set which separates \( X \) from \( Y \). To make \( X' \) \( \Delta^1_1 \) we have to maintain the choice of sets \( A_n \) effectively.

Let \( U \subseteq \mathbb{N} \times \mathbb{N}^N \) be a “good” universal \( \Sigma^1_1 \) set (see §A.c). Then there is a recursive \( h : \mathbb{N} \to \mathbb{N} \) such that \( [U_n]_F = U_{h(n)} \) for each \( n \). Moreover, applying Lemma 83 (to the complement of \( U \) as a “good” universal \( \Pi^1_1 \) set, and with a code for \( Y \) fixed), we obtain a pair of recursive functions \( f, g : \mathbb{N} \to \mathbb{N} \) such that for any \( n \), if \( U_n \cap Y = \emptyset \) then \( U_{f(n)} \) and \( U_{g(n)} \) are complementary sets (hence, either of them is \( \Delta^1_1 \)) containing, resp., \( U_n \) and \( Y \). A suitable iteration of \( h \) and \( f,g \) allows us to define a sequence \( X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq \ldots \) as above effectively enough for the union of those sets to be \( \Delta^1_1 \).

\( \square \) (Lemma)

**Lemma 35.3.** \( \bar{E} \) is a \( \Sigma^1_1 \) relation.
7 THE 2ND DICHOTOMY

**Proof.** Let $C \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}$ be as in $\Delta^1_1$ Enumeration (§A.c). The formula $\text{inv}(e)$ saying that $e \in C$ and $W_e = W'_e$ is $E$-invariant, i.e.,

$$e \in C \land \forall a, b \ (a \in W_e \land b \notin W'_e \implies a \not\in b)$$

is obviously $\Pi^1_1$, however $x \not\in \mathbb{E}$ iff

$$\forall e (\text{inv}(e) \implies (x \in W_e \implies y \in W'_e) \land (y \in W_e \implies x \in W'_e)) \quad \square \text{(Lemma)}$$

Let us return to the proof of the theorem. We have two cases.

**Case 1:** $E = \overline{E}$, i.e., $E$ is Gandy–Harrington closed.

**Lemma 35.4.** If $E = \overline{E}$ then there is a $\Delta^1_1$ reduction of $E$ to $D(2^\mathbb{N})$.

**Proof.** Let $C \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}$ be as in the $\Delta^1_1$ Enumeration of §A.c. By Kreisel Selection there is a $\Delta^1_1$ function $\varphi : X^2 \to C$ such that $W_e(x, y) = W'_e(x, y)$ is a $E$-invariant $\Delta^1_1$ set containing $x$ but not $y$ whenever $x, y \in X$ are $E$-inequivalent. Then $R = \text{ran} \varphi$ is a $\Sigma^1_1$ subset of $C$, hence, by Separation, there is a $\Delta^1_1$ set $N$ with $R \subseteq N \subseteq C$. The map $\varphi(x) = \{n \in N : x \in D_n\}$ is a $\Delta^1_1$ reduction of $E$ to $D(2^\mathbb{N})$. □ (Lemma and Case 1)

**Case 2:** $E \not\subseteq \overline{E}$. Then the $\Sigma^1_1$ set $H = \{x : [x]_E \not\subseteq [x]_{\overline{E}}\}$ (the union of all $\overline{E}$-classes containing more than one $E$-class) is non-empty.

**Lemma 35.5.** If $X \subseteq H$ is a $\Sigma^1_1$ set then $E \not\subseteq \overline{E}$ on $X$.

**Proof.** Suppose that $E \restriction X = \overline{E} \restriction X$. Then $E = \overline{E}$ on $Y = [X]_E$ as well. (If $y, y' \in Y$ then there are $x, x' \in X$ such that $x \not\in y$ and $x' \not\in y'$, so that if $y \not\in x'$ then $y \not\in x'$ by transitivity, hence, $x \not\in x'$, and $y \not\in x'$ again by transitivity.) It follows that $E = \overline{E}$ on an even bigger set, $Z = [X]_{\overline{E}}$. (Otherwise the $\Sigma^1_1$ set $Y' = Z \setminus Y = \{z : \exists x \in X : (xy \land z \not\in x)\}$ is non-empty and $E$-invariant, together with $Y$, hence by Lemma 35.2 there is a $E$-invariant $\Delta^1_1$ set $B$ with $Y \subseteq B$ and $Y' \cap B = \emptyset$, which implies that no point in $Y'$ is $E$-equivalent to a point in $Y'$, contradiction.) Then by definition $Z \cap H = \emptyset$. □ (Lemma)

**Lemma 35.6.** If $A, B \subseteq H$ are non-empty $\Sigma^1_1$ sets with $A \not\subseteq B$ then there exist non-empty disjoint $\Sigma^1_1$ sets $A' \subseteq A$ and $B' \subseteq B$ still satisfying $A' \not\subseteq B'$.

**Proof.** We assert that there are points $a \in A$ and $b \in B$ with $a \neq b$ and $a \not\in b$.

(Otherwise $E$ is the equality on $X = A \cup B$. Prove that then $E = \overline{E}$ on $X$, a contradiction to Lemma 35.5. Take any $x \neq y$ in $X$. Let $U$ be a clopen set containing $x$ but not $y$. Then $A = [U \cap X]_E$ and $C = [X \setminus U]_E$ are two disjoint $E$-invariant $\Sigma^1_1$ sets containing resp. $x$, $y$. Then $x \not\in y$ fails by Lemma 35.2.)

Thus let $a, b$ be as indicated. Let $U$ be a clopen set containing $a$ but not $b$. Put $A' = A \cap U \cap [U^C]_E$ and $B' = B \cap U \cap \overline{U}_E$. □ (Lemma)
7. **THE 2ND DICHOTOMY**

7.b **Restricted product forcing**

Recall that forcing notions $\mathbb{P}$ and $\mathbb{P}_2$ were introduced in §5.c. In continuation of the proof of Theorem 35 (Case 2), let $\mathbb{P}^2\restriction E$ be the collection of all sets of the form $X \times Y$, where $X, Y \subseteq \mathbb{N}^n$ are non-empty $\Sigma^1_1$ sets and $X \in Y$ (which means here that $[X]_E = [Y]_E$). Easily $\mathbb{P}_2 \subseteq \mathbb{P}^2\restriction E \subseteq \mathbb{P}^2$. The forcing $\mathbb{P}^2\restriction E$ is not really a product, yet if $X \times Z \in \mathbb{P}^2\restriction E$ and $0 \neq X' \subseteq X$ is $\Sigma^1_1$ then $Z' = Z \cap [X']_E$ is $\Sigma^1_1$ and $X' \times Z' \in \mathbb{P}^2\restriction E$. It follows that any $\mathbb{P}^2\restriction E$-generic set $G \subseteq \mathbb{P}^2\restriction E$ produces a pair of $\mathbb{P}$-generic sets $G_{\text{left}} = \{ \text{dom } P : P \in G \}$ and $G_{\text{right}} = \{ \text{ran } P : P \in G \}$, hence, produces a pair of $\mathbb{P}$-generic reals $x_{\text{left}}^G$ and $x_{\text{right}}^G$, whose names will be $\dot{x}_{\text{left}}$ and $\dot{x}_{\text{right}}$.

**Lemma 35.2.** In the sense of the forcing $\mathbb{P}^2\restriction E$, any $P = X \times Z \in \mathbb{P}^2\restriction E$ forces $\langle \dot{x}_{\text{left}}, \dot{x}_{\text{right}} \rangle \in P$ and forces $\dot{x}_{\text{left}} \in \dot{x}_{\text{right}}$, but $H \times H$ forces $\dot{x}_{\text{left}} \notin \dot{x}_{\text{right}}$.

**Proof.** To see that $\dot{x}_{\text{left}} \in \dot{x}_{\text{right}}$ is forced suppose otherwise. Then, by the definition of $\mathbb{P}$, there is a condition $P = X \times Z \in \mathbb{P}^2\restriction E$ and an $E$-invariant $\Delta^1_1$ set $B$ such that $P$ forces $\dot{x}_{\text{left}} \in B$ but $\dot{x}_{\text{right}} \notin B$. Then easily $X \subseteq B$ but $Z \cap B = \emptyset$, a contradiction with $[X]_E = [Z]_E$.

To see that $H \times H$ forces $\dot{x}_{\text{left}} \notin \dot{x}_{\text{right}}$ suppose towards the contrary that some $P = X \times Z \in \mathbb{P}^2\restriction E$ with $X \cup Z \subseteq H$ forces $\dot{x}_{\text{left}} \in \dot{x}_{\text{right}}$, thus,

1. $x \in z$ holds for every $\mathbb{P}^2\restriction E$-generic pair $\langle x, z \rangle \in P$.

**Claim 35.3.** If $x, y \in X$ are $\mathbb{P}$-generic over $\mathbb{M}$, and $x \in y$, then $x \in y$.

**Proof.** We assert that

2. $x \subseteq A \iff y \subseteq A$ holds for each $E$-invariant $\Sigma^1_1$ set $A$.

Indeed, if, say, $x \subseteq A$ but $y \not\subseteq A$ then by the genericity of $y$ there is a $\Sigma^1_1$ set $C$ with $y \subseteq C$ and $A \cap C = \emptyset$. As $A$ is $E$-invariant, Lemma 35.2 yields an $E$-invariant $\Delta^1_1$ set $B$ such that $C \subseteq B$ but $A \cap B = \emptyset$. Then $x \not\subseteq B$ but $y \in B$, a contradiction to $x \in y$.

Let $\{ \mathcal{D}_n \}_{n \in \mathbb{N}}$ be an enumeration of all dense subsets of $\mathbb{P}^2\restriction E$ which are coded in $\mathbb{M}$. We define two sequences $P_0 \supseteq P_1 \supseteq \ldots$ and $Q_0 \supseteq Q_1 \supseteq \ldots$ of conditions $P_n = X_n \times Z_n$ and $Q_n = Y_n \times Z_n$ in $\mathbb{P}^2\restriction E$, so that $P_0 = Q_0 = P$, $x \in X_n$ and $y \in Y_n$ for any $n$, and finally $P_n, Q_n \in \mathcal{D}_{n-1}$ for $n \geq 1$. If this is done then we have a real $z$ (the only element of $\bigcap_n Z_n$) such that both $\langle x, z \rangle$ and $\langle y, z \rangle$ are $\mathbb{P}^2\restriction E$-generic, hence, $x \in z$ and $y \in z$ by (1), hence, $x \in y$.

Suppose that $P_n$ and $Q_n$ have been defined. As $x$ is generic, there is (we leave details for the reader) a condition $P' = A \times C \in \mathcal{D}_n$ and $P_n \supseteq P'$ such that $x \in A$. Let $B = Y_n \cap [A]_E$ : then $y \in B$ by (2), and easily $[B]_E = [C]_E = [A]_E$.

\footnote{Over a countable model $\mathbb{M}$ chosen in accordance with the requirements in Footnote 14.}
(as \([X_n]_E = [Z_n]_E = [Y_n]_E\), thus, \(B \times C \in P^2 \upharpoonright E\), so there is a condition \(Q' = V \times W \in D_n\) and \(B \times C \subseteq Q_n\) such that \(y \in V\). Put \(Y_{n+1} = V, Z_{n+1} = W,\) and \(X_{n+1} = A \cap [W]_E\). \(\square\) (Claim)

It follows that \(E = E\) on \(X\). (Otherwise \(S = \{\langle x, y \rangle \in X^2 : x E y \land x E y\} \) is a non-empty \(\Sigma_1\) set, and any \(P_2\)-generic pair \(\langle x, y \rangle \in S\) implies a contradiction to Claim 35.3. Recall that \(P_2\) = all non-empty \(\Sigma_1\) subsets of \((\mathbb{N}^N)^2\).) But this implies \(X \cap H = \emptyset\) by Lemma 35.5, contradiction. \(\square\) (Lemma 35.2)

### 7.3 Splitting system

Let us fix enumerations \(\{D(n)\}_{n \in \mathbb{N}}, \{D_2(n)\}_{n \in \mathbb{N}}, \{D^2(n)\}_{n \in \mathbb{N}}\) of all dense subsets of resp. \(P, P_2, P^2 \upharpoonright E\), which belong to \(\mathcal{M}\); we assume that \(D(n+1) \subseteq D(n), D_2(n+1) \subseteq D_2(n),\) and \(D^2(n+1) \subseteq D^2(n).\) If \(u, v \in 2^m\) (binary sequences of length \(m\)) have the form \(u = 0^k \upharpoonright 0^w\) and \(v = 0^k \upharpoonright 1^w\) for some \(k < m\) and \(w \in 2^{m-k-1}\) then we call \(\langle u, v \rangle\) a crucial pair. It can be proved, e.g., by induction on \(m\), that \(2^m\) is a connected tree (i.e., a connected graph without cycles) of crucial pairs, with sequences beginning with \(1\) as the endpoints of the graph. We define a system of sets \(X_u\) \((u \in 2^{<\omega})\) and \(R_{uv}, \langle u, v \rangle\) being a crucial pair, so that the following conditions are satisfied:

(i) \(X_u \in P\), moreover, \(X_u \subseteq H,\) and \(X_u \in D(n)\) for any \(u \in 2^n\);

(ii) \(X_u \upharpoonright i \subseteq X_u\) for all \(u\) and \(i\);

(iii) \(R_{uv} \in P_2,\) moreover, \(R_{uv} \in D_2(n)\) for any crucial pair \(\langle u, v \rangle\) in \(2^n\);

(iv) \(R_{uv} \subseteq E\) and \(X_u \cap R_{uv} \subseteq X_v\) for any crucial pair \(\langle u, v \rangle\) in \(2^n\);

(v) \(R_{u \upharpoonright i, v \upharpoonright i} \subseteq R_{uv}\);

(vi) if \(u, v \in 2^n\) and \(u(n - 1) \neq v(n - 1)\) then \(X_u \times X_v \in D^2(n)\) and also \(X_u \cap X_v = \emptyset\).

Note that (iv) implies that \(X_u \subseteq X_v\) for any crucial pair \(\langle u, v \rangle\), hence, also for any pair in \(2^n\) because any \(u, v \in 2^n\) are connected by a unique chain of crucial pairs. It follows that \(X_u \times X_v \in P^2 \upharpoonright E\) for any pair of \(u, v \in 2^n,\) for any \(n\).

Assume that such a system has been defined. Then for any \(a \in 2^\mathbb{N}\) the sequence \(\{X_{a\upharpoonright n}\}_{n \in \mathbb{N}}\) is \(\mathcal{F}\)-generic over \(\mathcal{M}\), hence, \(\bigcap_{n} X_{a\upharpoonright n} = \{x_a\}\), where \(x_a\) is \(\mathcal{F}\)-generic, and the map \(a \mapsto x_a\) is continuous since diameters of \(X_u\) converge to 0 uniformly with \(1h u \rightarrow 0\) by (i), and is \(1 - 1\) by the last condition of (vi).

Let \(a, b \in 2^\mathbb{N}\). If \(a \mathcal{E}_0 b\) then, by (vii), \(\langle x_a, x_b \rangle\) is a \(P^2 \upharpoonright E\)-generic pair, hence, \(x_a \in x_b\) by Lemma 35.2. Now suppose that \(a \mathcal{E}_0 b\), prove that then \(x_a \subseteq x_b\). We can suppose that \(a = w^0 \upharpoonright 0^c\) and \(b = w^0 \upharpoonright 0^c\), where \(w \in 2^{<\omega}\) and \(c \in 2^\mathbb{N}\) (indeed if \(a \mathcal{E}_0 b\) then \(a, b\) can be connected by a finite chain of such special pairs). Then \(\langle x_a, x_b \rangle\) is \(P_2\)-generic, actually, the only member of the intersection
7 THE 2ND DICHOTOMY

\[ \cap_n R^{w^\bot c(n), w^\bot 1(c\mid n)} \] by (iii) and (iv), in particular, \( x_a \models x_b \) because we have \( R_{uv} \subseteq E \) for all \( u, v \).

Thus we have a continuous \( 1 - 1 \) reduction of \( E_0 \) to \( E \).

\( \square \) (Case 2 in Theorem 35 modulo the construction)

7.d Construction of a splitting system

Let \( X_A \) be any member of \( \mathcal{D}(0) \) satisfying \( X_A \subseteq H \). Now suppose that \( X_s \) and \( R_{st} \) have been defined for all \( s \in 2^n \) and all crucial pairs in \( 2^n \), and extend the construction on \( 2^{n+1} \). Temporarily, define \( X_s \cap t = X_s \) and \( R_s \cap t = R_{st} : 
\text{this leaves } R_{0^n \cap 0, 0^n \cap 1} \text{ still undefined, so we put } R_{0^n \cap 0, 0^n \cap 1} = E \cap X_{0^n} \times X_{0^n}. 
\text{Note that the such defined system of sets } X_u \text{ and relations } R_{uv} \text{ at level } n + 1 
\text{satisfies all requirements of (i) – (vi) except for the requirement of membership in the dense sets – say in this case that the system is “coherent”. It remains to }
\text{produce a still “coherent” system of smaller sets and relations which also satisfies the membership in the dense sets. This will be achieved in several steps.}

**Step 1:** achieve that \( X_u \in \mathcal{D}(n + 1) \) for any \( u \in 2^{n+1} \). Take any particular \( u_0 \in 2^{n+1} \). There is, by the density, \( X'_{u_0} \in \mathcal{D}(n + 1) \) and \( \subseteq X_{u_0} \). Suppose that \( (u_0, v) \) is a crucial pair. Put \( R'_{u_0, v} = \{ (x, y) \in R_{u_0, v} : x \in X'_{u_0} \} \) and \( X'_v = \text{ran} R'_{u_0, v} \). This shows how the change spreads along the whole set \( 2^{n+1} \) viewed as the tree of crucial pairs. Finally we obtain a coherent system with the additional requirement that \( X'_{u_0} \in \mathcal{D}(n + 1) \). Do this consecutively for all \( u_0 \in 2^{n+1} \). The total result – we re-define it as still \( X_u \) and \( R_{uv} \) – is a “coherent” system with \( X_u \in \mathcal{D}(n + 1) \) for all \( u \). Note that still \( X_{0^n \cap 0} = X_{0^n \cap 1} \) and

\[ R_{0^n \cap 0, 0^n \cap 1} = E \cap (X_{0^n \cap 0} \times X_{0^n \cap 1}). \] (*)

**Step 2:** achieve that \( X_s \cap 0 \times X_s \cap 1 \in \mathcal{D}(n + 1) \) for all \( s, t \in 2^{n+1} \). Consider a pair of \( u_0 = s_0 \cap 0 \) and \( v_0 = t_0 \cap 1 \) in \( 2^{n+1} \). By the density there is a set \( X'_{u_0} \times X'_{v_0} \in \mathcal{D}(n + 1) \) and \( \subseteq X_{u_0} \times X_{v_0} \). By definition we have \( X'_{u_0} \models X'_{v_0} \) but, due to Lemma 35.6 we can maintain that \( X'_{u_0} \cap X'_{v_0} = \emptyset \). The two “shockwaves”, from the restrictions at \( u_0 \) and \( v_0 \), as in Step 1, meet only at the pair \( 0^m \cap 0, 0^m \cap 1 \) where the new sets satisfy \( X'_{u_0} \cap 0, X'_{v_0} \cap 1 \) just because \( E \)-equivalence is everywhere kept and preserved through the changes. Now, in view of (*), we can define \( R'_{0^n \cap 0, 0^n \cap 1} = E \cap (X'_{0^n \cap 0} \times X'_{0^n \cap 1}) \), preserving (⋆) as well. All pairs considered, we will be left with a coherent system of sets and relations, re-denoted as \( X_u \) and \( R_{uv} \), which satisfies the \( \mathcal{D}(n + 1) \)-requirements in (i) and (vi).

**Step 3:** achieve that \( R_{uv} \in \mathcal{D}(n + 1) \) for any crucial pair at level \( n + 1 \), and also that \( X_{p^n \cap 0} \cap X_{p^n \cap 1} = \emptyset \). Consider any crucial pair \( (u_0, v_0) \). If this is not \( (0^n \cap 0, 0^n \cap 1) \) then let \( R'_{u_0 v_0} \subseteq R_{u_0 v_0} \) be any set in \( \mathcal{D}(n + 1) \). If this is \( u_0 = 0^n \cap 0 \) and \( v_0 = 0^n \cap 1 \) then first we choose (Lemma 35.6) disjoint non-empty \( \Sigma^1_1 \) sets \( U \subseteq X_{0^n \cap 0} \) and \( V \subseteq X_{0^n \cap 1} \) still with \( U \cap V \), and only then a set \( R'_{u_0 v_0} \subseteq E \cap (U \times V) \) which belongs to \( \mathcal{D}(n + 1) \). In both cases, put \( X'_{u_0} = \text{dom} R'_{u_0 v_0} \).
and $X'_{v_0} = \text{ran} R'_{u_0}$. It remains to spread the changes, along the chain of crucial pairs, to the left of $u_0$ and to the right of $v_0$, exactly as in Case 1. Executing such a reduction for all crucial pairs $(u_0, v_0)$ at level $n + 1$ one by one, we end up with a system of sets fully satisfying (i) – (vi).

$\square$ (Theorem 35)

7.e A forcing notion associated with $E_0$

We here consider the forcing notion $P_{E_0/D(2^N)}$ (see §3.e), that will be denoted by $P_{E_0}$ below. Thus by definition $P_{E_0}$ consists of all Borel sets $X \subseteq 2^N$ such that $E_0 \upharpoonright X$ is non-smooth while the related ideal $\mathcal{I}_{E_0} = \mathcal{I}_{E_0/D(2^N)}$ consists of all Borel sets $X \subseteq 2^N$ such that $E_0 \upharpoonright X$ is smooth.

**Lemma 36.** (i) $\mathcal{I}_{E_0}$ is a $\sigma$-additive ideal. Let $X \subseteq 2^N$ be a Borel set.

(ii) $X$ belongs to $P_{E_0}$ iff $E_0 \subseteq E_0 \upharpoonright X$ (by a continuous injection).

(iii) $X$ belongs to $\mathcal{I}_{E_0}$ iff $E_0 \upharpoonright X$ admits a Borel transversal.

**Proof.** (i) immediately follows from Theorem 28. In (ii), if $X \in P_{E_0}$ then $E_0 \subseteq E_0 \upharpoonright X$ by Theorem 35, while if $E_0 \subseteq E_0 \upharpoonright X$ then $E_0 \upharpoonright X$ is not smooth since $E_0$ itself is not smooth by Lemma 27(v). In (iii), if $E_0 \upharpoonright X$ admits a Borel transversal then it is smooth by Lemma 27(i) and hence $X$ belongs to $\mathcal{I}_{E_0}$. To prove the converse apply Lemma 27(iii). $\square$

Note that any $X \in P_{E_0}$ contains a closed subset $Y \subseteq X$ also in $P_{E_0}$ by Theorem 35. (Apply the theorem for $E = E_0 \upharpoonright X$. As $E_0 \upharpoonright X$ is not smooth, we have $E_0 \subseteq E_0 \upharpoonright X$, by a continuous reduction $\vartheta$. Take as $Y$ the full image of $\vartheta$. $Y$ is compact, hence closed.) Such sets $Y$ can be chosen in a special family.

**Definition 37 (Zapletal [47]).** Suppose that two binary sequences $u^0_n \neq u^1_n \in 2^{<\omega}$ of equal length $1h u^0_n = 1h u^1_n \geq 1$ are chosen for each $n$, together with one more sequence $u_0 \in 2^{<\omega}$. Define $\vartheta(a) = u_0 \wedge a^{(0)} \wedge a^{(1)} \ldots$ for any $a \in 2^N$. Easily $\vartheta$ is a continuous injection $2^N \to 2^N$, $Y = \text{ran} \vartheta$ is a closed set in $2^N$, $\vartheta$ witnesses $E_0 \subseteq E_0 \upharpoonright Y$, and hence $Y \in P_{E_0}$.

Let $P'_{E_0}$ denote the collection of all sets $Y$ definable in such a form. $\square$

**Theorem 38 (Zapletal [47]).** $P'_{E_0}$ is a dense subset of $P_{E_0}$: for any $X \in P_{E_0}$ there exists $Y \in P_{E_0}$, $Y \subseteq X$. In addition, $P_{E_0}$ forces that the “old” continuum remains uncountable.

**Proof.** The proof employs splitting technique for the forcing $P_{E_0}$. This technique somewhat differs from the splittings used in the proof of Theorem 35. First of all, as mentioned above, we can consider only closed sets in $P_{E_0}$, that enables us to replace the Gandy – Harrington stuff by a simple compactness argument. Second, the equivalence relation considered has the form $E_0 \upharpoonright X$. 


7 THE 2ND DICHOTOMY

For any sequences \( r, w \in 2^{<\omega} \) with \( 1h r \leq 1h w \), define \( rw \in 2^{<\omega} \) (the \( r \)-shift of \( w \)) so that \( 1h rw = 1h w \) and \( (rw)(k) = 1 - w(k) \) whenever \( k < 1h r \) and \( r(k) = 1 \), and \( (rw)(k) = w(k) \) otherwise. Clearly \( r(rw) = w \). Similarly define \( ra \in 2^{\mathbb{N}} \) for \( a \in 2^{\mathbb{N}} \), and \( rX = \{ ra : a \in X \} \) for any set \( X \subseteq 2^{\mathbb{N}} \).

We are going to define sequences \( u \in 2^{<\omega} \) and \( u_n \neq u_n^1 \in 2^{<\omega} \) (\( n \in \mathbb{N} \)) such that \( 1h u_n^0 = 1h u_n^1 \), as in Definition 37, and also a system of closed sets \( X_s \in \mathbb{P}_{E_0} \) (\( s \in 2^{<\omega} \)) satisfying the following:

(i) \( X_\Lambda \subseteq X \) and \( X_s \cap X \subseteq X_s \);

(ii) \( X_s \subseteq \mathcal{E}_w \), where \( w_s = u_0 \land u_0^0 \land u_1^0 \land \ldots \land u_{k-1}^0 \in 2^{<\omega} \), \( k = 1h s \), and \( \mathcal{E}_w = \{ a \in 2^{\mathbb{N}} : w \in a \} \) for \( w \in 2^{<\omega} \);

(iii) if \( s, t \in 2^n \) for some \( n \) then \( X_t = w_t w_s X_s \).

Then define the map \( \vartheta \) as in Definition 37. The set \( Y = \text{ran} \vartheta = \bigcap_s \bigcup_{s \in 2^n} X_s \subseteq X \) belongs to \( \mathbb{P}_{E_0'} \), proving the density claim of the theorem.

Step 0. We put \( X_\Lambda = X \) and let \( u_0 \in 2^{<\omega} \) be the largest sequence such that \( X_\Lambda \subseteq \mathcal{E}_u_0 \). Let \( \ell_0 = 1h u_0 \).

Step 1. Here we define \( u_i^0 \) and \( X_{(i)} \) for \( i = 0, 1 \). Let \( R \) be the set of all sequences \( r \in 2^{<\omega} \) containing at least one term equal to 1 (and hence \( ra \neq a \) for any \( a \)). Consider the union \( Z = \bigcup_{r \in R} \bigcup_{s \subseteq 2^n} Z_{rs} \), where \( Z_{rs} = \{ a \in X_\Lambda : ra \in X_\Lambda \} \) for each \( Z_r \) closed. The difference \( D = X_\Lambda \setminus Z \) is pairwise \( E_0 \)-inequivalent, hence \( D \in \mathcal{F}_{E_0} \) by Lemma 36. Thus at least one of \( Z_r \), \( r \in R \), belongs to \( \mathbb{P}_{E_0} \) by Lemma 36. Let \( r_1 \) be any \( r \in R \) of this sort. Put \( \ell_1 = 1h r_1 \); clearly \( 1h u_0 = \ell_0 < \ell_1 \) and \( r_1 \upharpoonright \ell_0 \) consists only of terms equal to 0.

There is a sequence \( w_{(0)} \in 2^{<\omega} \) such that \( 1h w_{(0)} = \ell_1 \) and the set \( X_{(0)} = Z_{r_1} \cap \mathcal{E}_{w_{(0)}} \) still belongs to \( \mathbb{P}_{E_0} \). Put \( w_{(1)} = r_1 w_{(0)} \). Then the set \( X_{(1)} = r_1 X_{(0)} = \{ r_1 a : a \in X_{(0)} \} = Z_{r_1} \cap \mathcal{E}_{w_{(1)}} \) belongs to \( \mathbb{P}_{E_0} \) together with \( X_{(0)} \). Note that \( u_0 \subseteq w_{(i)} \), and hence there exist sequences \( u_0^0 \neq u_0^1 \in 2^{<\omega} \) of length \( \ell_1 - \ell_0 \) such that \( w_{(0)} = u_0 \land u_0^0 \) and \( w_{(1)} = u_0 \land u_0^1 \). It follows from the construction that \( w_{(0)} w_{(1)} = r_1 \), therefore \( X_{(1)} = w_{(0)} w_{(1)} X_{(1)} \), and (iii) holds.

Step 2. Here we define \( u_i^0 \) for \( i = 0, 1 \) and \( X_s \) for \( s \in 2^{<\omega} \) with \( 1h s = 2 \).

Once again there is a sequence \( r_2 \in R \) such that the (closed) set \( Z_{r_2} = \{ a \in X_{(0)} : ra \in X_{(0)} \} \) still belongs to \( \mathbb{P}_{E_0} \). Put \( \ell_2 = 1h r_2 \); then \( 1h r_1 = \ell_1 < \ell_2 \) and \( r_2 \upharpoonright \ell_1 \) consists only of terms equal to 0. Once again there is a sequence \( w_{(0)} \in 2^{<\omega} \) such that \( 1h w_{(0),0} = \ell_2 \) and the set \( X_{(0),0} = Z_{r_2} \cap \mathcal{E}_{w_{(0),0}} \) belongs to \( \mathbb{P}_{E_0} \). Put \( w_{(0),1} = r_2 w_{(0)} \). Then the set \( X_{(0),1} = r_2 X_{(0),0} = Z_{r_2} \cap \mathcal{E}_{w_{(0),1}} \) belongs to \( \mathbb{P}_{E_0} \) together with \( X_{(0),0} \). Also, put \( w_{(1),0} = r_1 w_{(0),0} \) and \( X_{(1),0} = r_1 X_{(0),0} = Z_{r_2} \cap \mathcal{E}_{w_{(1),0}} \) for \( i = 0, 1 \); these sets also belong to \( \mathbb{P}_{E_0} \). As for (iii) at this level, take, for instance, \( s = (0, 1) \) and \( t = (1, 0) \). By definition \( X_{(1),1} = r_1 X_{(0),0} = r_2 r_1 X_{(1),0} \), on the other hand, \( w_{(1),0} = r_2 r_1 w_{(0),1} \), too.

Finally, there exist sequences \( u_0^1 \neq u_1^1 \in 2^{<\omega} \) of length \( \ell_2 - \ell_1 \) such that \( w_{(i,j)} = u_0^j \lor u_1^j \lor u_1^j \) for \( i, j = 0, 1 \).
Steps \( \geq 3 \). Et cetera. The construction results in a system of sets and sequences satisfying requirements (i), (ii), (iii), as required.

To prove the additional claim of the theorem, the splitting construction has to be modified so that for any \( n \) the sets \( X_s, \ s \in 2^n \), belong to the \( n \)-th dense subset of \( \mathbb{P}_{E_0} \), in the sense of a given countable sequence of dense sets. \( \square \)

We observe that \( \mathbb{P}_{E_0} \) as a forcing is somewhat closer to Silver rather than Sacks forcing. The property of minimality of the generic real, common to both Sacks and Silver, holds for \( \mathbb{P}_{E_0} \) as well, the proof resembles known arguments, but in addition the following is applied: if \( X \in \mathbb{P}_{E_0} \) and \( f : X \to 2^\mathbb{N} \) is a Borel \( E_0 \)-invariant map (that is, \( x E_0 y \implies f(x) = f(y) \)) then \( f \) is constant on a set \( Y \in \mathbb{P}_{E_0}, \ Y \subseteq X. \)

\section{Ideal \( \mathcal{I}_1 \) and \( P \)-ideals}

By definition the ideal \( \text{Fin} \times 0 = \mathcal{I}_1 \) consists of all sets \( x \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N}) \) such that all, except for finitely many, cross-sections \( (x)_n = \{ k : (n, k) \in x \} \) are empty.

\subsection{Ideals below \( \mathcal{I}_1 \)}

It turns out that there exist only three different ideals Borel reducible to \( \mathcal{I}_1 \), they are \( \text{Fin} \), the disjoint sum \( \text{Fin} \oplus \mathcal{P}(\mathbb{N}) \), and \( \mathcal{I}_1 \) itself.

\begin{definition}
An ideal \( \mathcal{I} \) is a \textit{trivial variation} of \( \mathcal{J} \) if there is an infinite set \( D \) such that \( I \upharpoonright D \cong \mathcal{J} \) \( \uparrow \) while \( \mathcal{I} \upharpoonright \complement D = \mathcal{P}(\complement D) \). (The last condition is equivalent to \( \mathcal{I} = \{ x : x \cap D \in \mathcal{J} \upharpoonright D \}. \) \( \square \)
\end{definition}

\begin{theorem}[Kechris \cite{27}]
If \( \mathcal{I} \leq_{\text{b}} \mathcal{I}_1 \) is a Borel (nontrivial) ideal on \( \mathbb{N} \) then either \( \mathcal{I} \cong \mathcal{I}_1 \) or \( \mathcal{I} \) is a trivial variation of \( \text{Fin} \).
\end{theorem}

\begin{exercise}
Prove that any trivial variation of \( \mathcal{I}_1 \) is isomorphic to \( \mathcal{I}_1 \) while any trivial variation of \( \text{Fin} \) is isomorphic either to \( \text{Fin} \) or to the disjoint sum \( \text{Fin} \oplus \mathcal{P}(\mathbb{N}) \), e.g., realized in the form of \( \{ x \subseteq \mathbb{N} : x \cap \text{odd} \in \text{Fin} \} \). \( \square \)
\end{exercise}

\begin{proof} (Theorem).
We begin with another version of the method used in the proof of Theorem 30. Suppose that \( \{ \mathcal{B}_k \}_{k \in \mathbb{N}} \) is a fixed system of Borel subsets of \( \mathcal{P}(\mathbb{N}) \). (It will be specified later.) Then there exists an increasing sequence of integers \( 0 = n_0 < n_1 < n_2 < \ldots \) and sets \( s_k \subseteq [n_k, n_{k+1}) \) such that

\begin{enumerate}
\item \( \text{Suppose, for the sake of brevity, that } X = 2^\mathbb{N}. \) For any \( n \), the set \( Y_n = \{ a : f(a)(n) = 0 \} \) is Borel and \( E_0 \)-invariant. It follows that \( Y_n \) is either meager or comeager. Put \( b(n) = 0 \) if \( Y_n \) is comeager. Then \( D = \{ a : f(a) = b \} \) is comeager. A splitting construction as in the proof of Theorem 38 yields a set \( Y \in \mathbb{P}_{E_0}, \ Y \subseteq D. \)
\item Recall that \( \mathcal{I} \cong \mathcal{J} \) means isomorphism via a bijection between the underlying sets.
\end{enumerate}
\end{proof}
(1) any \( x \subseteq \mathbb{N} \) with \( \forall k \in \mathbb{N} \) \( x \cap [n_k, n_{k+1}] = s_k \) is “generic” \(^{20}\);

(2) if \( k' \geq k \) and \( u \subseteq [0, n_{k'}] \) then \( u \cup s_{k'} \) decides \( \mathcal{B}_k \) in the sense that either any “generic” \( x \in \mathcal{P}(\mathbb{N}) \) with \( x \cap [0, n_{k'+1}] = u \cup s_{k'} \) belongs to \( \mathcal{B}_k \) or any “generic” \( x \) with \( x \cap [0, n_{k'+1}] = u \cup s_{k'} \) does not belong to \( \mathcal{B}_k \).

Now put \( \mathcal{D}_0 = \{ x \cup S_1 : x \subseteq Z_0 \} \) and \( \mathcal{D}_1 = \{ x \cup S_0 : x \subseteq Z_1 \} \), where

\[
S_0 = \bigcup_k s_{2k} \subseteq Z_0 = \bigcup_k [n_{2k}, n_{2k+1}], \quad S_1 = \bigcup_k s_{2k+1} \subseteq Z_1 = \bigcup_k [n_{2k+1}, n_{2k+2}].
\]

Clearly any \( x \in \mathcal{D}_0 \cup \mathcal{D}_1 \) is “generic” by (1), hence, by (2),

(3) each \( \mathcal{B}_k \) is clopen on both \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \).

As \( \mathcal{I} \leq_H \mathcal{A}_1 \), it follows from Lemma 1 and the trivial fact that \( \mathcal{I}_1 \oplus \mathcal{I}_1 \cong \mathcal{I}_1 \) that there exists a continuous reduction \( \vartheta : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N} \times \mathbb{N}) \) of \( \mathcal{I} \) to \( \mathcal{A}_1 \). Thus \( \mathcal{E}_\mathcal{I} \) is the union of an increasing sequence of (topologically) closed ERs \( R_m \subseteq \mathcal{E}_\mathcal{I} \) just because \( \mathcal{I}_1 \) admits such a form. We now require that \( \{ \mathcal{B}_k \} \) includes all sets \( \mathcal{B}_m^s = \{ x \in \mathcal{P}(\mathbb{N}) : x \subseteq [0, l] \} \) \( x \in \mathcal{R}_m (\Delta s) \). Then by (3) and the compactness of \( \mathcal{D}_l \) for any \( l \) there is \( m(l) \geq l \) satisfying

(4) \( \forall x \in \mathcal{D}_0 \cup \mathcal{D}_1 \forall s \subseteq [0, l] \left( x \in \mathcal{R}_m (\Delta s) \right) \).

To prove the theorem it suffices to obtain a sequence \( x_0 \subseteq x_1 \subseteq x_2 \subseteq \ldots \) of sets \( x_k \in \mathcal{I} \) with \( \mathcal{I} = \bigcup_n \mathcal{P}(x_n) \) : that in this case \( \mathcal{I} \) is as required is an easy exercise. As any topologically closed ideal is easily \( \mathcal{P}(x) \) for some \( x \in \mathbb{N} \), it suffices to show that \( \mathcal{I} \) is a union of a countable sequence of closed subideals. It suffices to demonstrate this fact separately for \( \mathcal{I} \mid Z_0 \) and \( \mathcal{I} \mid Z_1 \). Prove that \( \mathcal{I} \mid Z_0 \) is a countable union of closed subideals, ending the proof of the theorem.

If \( m \in \mathbb{N} \) and \( s \subseteq u \subseteq Z_0 \) are finite then let

\[
I_{us}^m = \{ A \subseteq Z_0 : \forall x \in \mathcal{D}_0 (x \cap u = s \implies (x \cup (A \setminus u))R_m x) \}.
\]

Lemma 40.2. Sets \( I_{us}^m \) are closed topologically and under \( \cup \), and \( I_{us}^m \subseteq \mathcal{I} \).

Proof. \( I_{us}^m \) are topologically closed because so are \( R_m \).

Suppose that \( A, B \in I_{us}^m \). To prove that \( A \cup B \in I_{us}^m \), let \( x \in \mathcal{D}_0 \) satisfy \( x \cap u = s \). Then \( x' = x \cup (A \setminus u) \in \mathcal{D}_0 \) satisfies \( x' \cap u = s \), too, hence, as \( B \in \mathcal{I}_us \), we have \( (x' \cup (B \setminus u))R_m x' \), thus, \( (x \cup ((A \cup B) \setminus u))R_m x' \). However \( x' \subseteq \mathcal{R}_m \) because \( A \in I_{us}^m \). It remains to recall that \( R_m \) is a ER.

To prove that any \( A \in I_{us}^m \) belongs to \( \mathcal{I} \) take \( x = s \cup S_1 \). Then we have \( x \cup (A \setminus u) \subseteq \mathcal{R}_m x \), thus, \( A \in \mathcal{I} \) as \( s \) is finite and \( R_m \subseteq \mathcal{E}_{\mathcal{I}} \). \( \square \) (Lemma)

Lemma 40.3. \( \mathcal{I} \mid Z_0 = \bigcup_{m,u,s} I_{us}^m \).

\(^{20}\) We mean, Cohen generic over a certain fixed countable transitive model \( \mathcal{M} \) of a big enough fragment of \( \text{ZFC} \), which contains Borel codes for all sets \( \mathcal{B}_k \).
8 IDEAL $\mathcal{I}_1$ AND P-IDEALS

**Proof.** Let $A \in \mathcal{I}$, $A \subseteq Z_0$. The sets $Q_m = \{x \in \mathcal{D}_0 : (x \cup A) R_m x\}$ are closed and satisfy $\mathcal{D}_0 = \bigcup_m Q_m$. It follows that one of them has a non-empty interior in $\mathcal{D}_0$, thus, there exist finite sets $s \subseteq u \subseteq Z_0$ and some $m_0$ with

$$\forall x \in \mathcal{D}_0 \ (x \cap u = s \implies (x \cup A) R_{m_0} x).$$

This is not exactly what we need, however, by (4), there exists a number $m = \max\{m_0, m(\sup u)\}$ big enough for

$$\forall x \in \mathcal{D}_0 : (x \cup A) R_m (x \cup (A \setminus u)).$$

It follows that $A \in I_{su}^m$, as required. $\square$ (Lemma)

Let $J^m_{su}$ be the hereditary hull of $I_{su}^m$ (all subsets of sets in $I_{su}^m$). It follows from Lemma 40.2 that any $J^m_{su}$ is a topologically closed subideal of $\mathcal{I} | Z_0$, however, $\mathcal{I} | Z_0$ is the union of those ideals by Lemma 40.3, as required. $\square$

8.b $\mathcal{I}_1$ and P-ideals

Thus $\mathcal{I}_1$ is a $\leq^*_n$-minimal ideal over $\text{Fin}$ : we have $\text{Fin} <^*_n \mathcal{I}_1$ and the $<^*_n$-interval $(\text{Fin}, \mathcal{I}_1)$ is empty. Although $\mathcal{I}_1$ is not the least over $\text{Fin}$, still it turns out that $\mathcal{I}_1$ is the least among all Borel ideals which are not P-ideals.

The next theorem is of great importance for the whole theory of Borel ideals.

**Theorem 41** (Solecki [42, 43]). The following families of ideals on $\mathbb{N}$ coincide:

(i) ideals of the form $\text{Exh}_\varphi$, where $\varphi$ is a l.s.c. submeasure on $\mathbb{N}$;

(ii) polishable ideals.

(iii) analytic P-ideals;

(iv) analytic ideals $\mathcal{I}$ with $\mathcal{I}_1 \not\subseteq_{RB} \mathcal{I}$;

(v) analytic ideals $\mathcal{I}$ such that all countable unions of $\mathcal{I}$-small sets are $\mathcal{I}$-small, where a set $X \subseteq \mathcal{P}(\mathbb{N})$ is $\mathcal{I}$-small if there is $A \in \mathcal{I}$ such that $X \upharpoonright A = \{x \cap A : x \in X\} \subseteq \mathcal{P}(A)$ is meager in $\mathcal{P}(A)$.

It follows that all analytic P-ideals actually belong to $\Pi^0_3$, just because any ideal of type (i) is easily $\Pi^0_3$.

**Proof.** The formal scheme of the proof is: (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (i). The hard part will be (v) $\implies$ (i), the rest is rather elementary but tricky in some points. The elementary part of the proof is organized so that the proofs that (i) $\iff$ (ii) and (iii) $\iff$ (iv) $\iff$ (v) and that the first group implies the second, are obtained independently of the hard part.

(i) $\implies$ (ii) If $\varphi(\{n\}) > 0$ for all $n$ then the required metric on $\mathcal{I} = \text{Exh}_\varphi$ can be defined by $d_\varphi(x, y) = \varphi(x \Delta y)$. Then any set $U \subseteq \mathcal{I}$ open in the sense
of the ordinary topology (the one inherited from $\mathcal{P}(\mathbb{N})$) is $d_2$-open, while any $d_2$-open set is Borel in the ordinary sense. In the general case we assemble the required metric of $d_2$ on the domain $\{n: \varphi(\{|n\}) > 0\}$ and the ordinary Polish metric on $\mathcal{P}(\mathbb{N})$ on the complementary domain.

(ii) $\implies$ (i) Let $\tau$ be a Polish group topology on $\mathcal{I}$, generated by a $\Delta$-invariant compatible metric $d$. It can be shown (Solecki [43, p. 60]) that $\varphi(x) = \sup_{y \in \mathcal{I}, y \subseteq x} d(\emptyset, x)$ is a l.s.c. submeasure with $\mathcal{I} = \text{Exh}_\varphi$. The key observation is that for any $x \in \mathcal{I}$ the sequence $\{x \cap [0, n]\}_{n \in \mathbb{N}}$ $d$-converges to $x$ by the last statement of Lemma 7, which implies both that $\varphi$ is l.s.c. (because the supremum above can be restricted to finite sets $y$) and that $\mathcal{I} = \text{Exh}_\varphi$ (where the inclusion $\subseteq$ needs another “identity map” argument).

(i) $\implies$ (iii) That any $\mathcal{I} = \text{Exh}_\varphi$, $\varphi$ being l.s.c., is a P-ideal, is an easy exercise: if $x_1, x_2, x_3, \ldots \in \mathcal{I}$ then define an increasing sequence of numbers $n_i \in x_i$ with $\varphi(x \cap [n_i, \infty)) \leq 2^{-n}$ and put $x = \bigcup_i (x \cap [n_i, \infty))$.

Any of (iii), (i), (ii), (v) $\implies$ (iv) This is because $\mathcal{I}_1$ easily does not satisfy any of the four properties indicated. For the formal purpose to complete the proof of Theorem 41, we need here only the implication (iii) $\implies$ (iv).

(iv) $\implies$ (v) Suppose that sets $X_n \subseteq \mathcal{P}(\mathbb{N})$ are $\mathcal{I}$-small, so that $X_n | A_n$ is meager in $\mathcal{P}(A_n)$ for some $A_n \in \mathcal{I}$, but $X = \bigcap_n X_n$ is not $\mathcal{I}$-small, and prove $\mathcal{I}_1 \leq_{RB} \mathcal{I}$. Arguing as in the proof of Theorem 30, we use the meagerness to find, for any $n$, a sequence of pairwise disjoint non-empty finite sets $w_k^n \subseteq x_n$, $k \in \mathbb{N}$, and subsets $w_k^n \subseteq w_k^n$, such that

(a) if $x \subseteq \mathbb{N}$ and $\exists^\infty k (x \cap w_k^n = w_k^n)$ then $x \notin X_n$.

Dropping some sets $w_k^n$ away and reenumerating the rest, we can strengthen the disjointness to the following: $w_k^n \cap w_k^m = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w_k^n = w_k^m = w_k^n = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w_k^n = w_k^n = w_k^n = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w_k^n = w_k^n = w_k^n = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w_k^n = w_k^n = w_k^n = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w_k^n = w_k^n = w_k^n = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w_k^n = w_k^n = w_k^n = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w_k^n = w_k^n = w_k^n = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w_k^n = w_k^n = w_k^n = \emptyset$ unless both $n = m$ and $k = l$.

Hence $x \subseteq \mathbb{N}$, hence, $\in \mathcal{I}$, for any $i$;

(c) if a set $Z \subseteq \mathbb{N} \times \mathbb{N}$ does not belong to $\mathcal{I}_1$, i.e., $\exists^\infty k (\langle i, j \rangle \in Z)$, then $\forall n \exists^\infty k (w_k^n \subseteq w_k^n)$, where $w_k^n = \bigcup_{(i, j) \in K} w_k^n$.

We assert that the map $\langle i, j \rangle \mapsto w_k^n$ witnesses $\mathcal{I}_1 \leq_{RB} \mathcal{I}$. (Then a simple argument, as in the proof of Theorem 30, gives $\mathcal{I}_1 \leq_{RB} \mathcal{I}$.)

Indeed if $Z \subseteq \mathbb{N} \times \mathbb{N}$ belongs to $\mathcal{I}_1$ then $w_k^n \subseteq \mathcal{I}$ by (b). Suppose that $Z \notin \mathcal{I}_1$. It suffices to show that $X_n | w_k^n$ is meager in $\mathcal{P}(w_k^n)$ for any $n$. Note that by (c) the set $K = \{k: w_k^n \subseteq w_k^n\}$ is infinite and in fact $w_k^n x_n = \bigcup_{k \in K} w_k^n$. Therefore, any $x \subseteq w_k^n$ satisfying $x \cap w_k^n = w_k^n$ for infinitely many $k \in K$, does not belong to $X_n$ by (a). Now the meagerness of $X_n | w_k^n$ is clear.

(v) $\implies$ (iii) This also is quite easy: if a sequence of sets $Z_n \in \mathcal{I}$ witnesses that $\mathcal{I}$ is not a P-ideal, then the union of $\mathcal{I}$-small sets $\mathcal{P}(Z_n)$ is not $\mathcal{I}$-small.
8. c The hard part

We prove (v) $\implies$ (i), the hard part of Theorem 41. A couple of definitions before the key lemma.

- Let $C(\mathcal{I})$ be the collection of all hereditary (i.e., $y \subseteq x \in K \implies y \in K$) compact $\mathcal{I}$-large sets $K \subseteq \mathcal{P}(\mathbb{N})$.
- Given sets $A, B \subseteq \mathcal{P}(\mathbb{N})$, let $A + B = \{x \cup y : x \in A \land y \in B\}$.

Lemma 42. Assuming that $\mathcal{I}$ is of type (v), there is a countable sequence of sets $K_m \in C(\mathcal{I})$ such that for any set $K \in C(\mathcal{I})$ there are $m, n$ with $K_m + K_n \subseteq K$.

Proof. Fix a continuous map $f : \mathbb{N}^\omega \xrightarrow{\text{onto}} \mathcal{I}$. For any $s \in \mathbb{N}^\omega$, we define

$$N_s = \{a \in \mathbb{N}^\omega : s \subseteq a\} \quad \text{and} \quad B_s = f^n N_s \quad \text{(the $f$-image of $N_s$).}$$

Consider the set $T = \{s : B_s \text{ is } \mathcal{I}\text{-large}\}$. As $\mathcal{I}$ itself is clearly $\mathcal{I}$-large, $A \in T$. On the other hand, the assumption (v) easily implies that $T$ has no endpoints and no isolated branches, hence, $P = \{a \in \mathbb{N}^\omega : \forall n (a \upharpoonright n \in T)\}$ is a perfect set. Moreover, $A_s = f^n(P \cap N_s)$ is $\mathcal{I}$-large for any $s \in T$ because $B_s \setminus A_s$ is a countable union of $\mathcal{I}$-small sets.

Now consider any set $K \in C(\mathcal{I})$. By definition, if $x, y \in \mathcal{I}$ then $Z = x \cup y \in \mathcal{I}$, thus, $K \upharpoonright Z$ is not meager in $\mathcal{P}(Z)$, hence, by the compactness, $K \upharpoonright Z$ includes a basic nbhd of $\mathcal{P}(Z)$, hence, by the hereditariness, there is a number $n$ such that $Z \cap [n, \infty) \in K$. We conclude that $P^2 = \bigcup_n Q_n$, where each $Q_n = \{a, b \in P^2 : (f(a) \cup f(b)) \cap [n, \infty) \in K\}$ is closed in $P$ because so is $K$ and $f$ is continuous. Thus, there are $s, t \in T$ such that $P^2 \cap (N_s \times N_t) \subseteq Q_n$, in other words, $(A_s + A_t) \cap [n, \infty) \subseteq K$, hence, $(A_s + A_t) \cap [n, \infty) \subseteq K$, where

$$\big\{K_m\big\}, \text{ all sets of the form } K_m = \overline{A_s \upharpoonright n}. \quad \square$$

Using the fact that $C(\mathcal{I})$ is a filter (as easy exercise which makes main use if the hereditariness), we can define (still in the assumption that $\mathcal{I}$ is of type (v)) a $\subseteq$-decreasing sequence of sets $K_n \in C(\mathcal{I})$ such that

1. for any $K \in C(\mathcal{I})$ there is $n$ with $K_n \subseteq K$,

and $K_{n+1} + K_{n+1} \subseteq K_n$ for any $n$. Taking any other term of the sequence, we can sharpen the latter requirement to

2. for any $n : K_{n+1} + K_{n+1} + K_{n+1} \subseteq K_n$.

This is the starting point for the construction of a l.s.c. submeasure $\varphi$ with $\mathcal{I} = \text{Exh}_\varphi$. Assuming that, in addition, $K_0 = \mathcal{P}(\mathbb{N})$, let, for any $x \in \mathcal{P}_{\text{fin}}(\mathbb{N})$,

$$\varphi_1(x) = \inf \{2^{-n} : x \in K_n\} \quad \text{and} \quad \varphi_2(x) = \inf \{\sum_{i=1}^m \varphi_1(x_i) : m \geq 1 \land x_i \in \mathcal{P}_{\text{fin}}(\mathbb{N}) \land x \subseteq \bigcup_{i=1}^m x_i\}.$$
9 EQUIVALENCE RELATION $E_1$

Then set $\varphi(x) = \sup_n \varphi_2(x \cap [0,n))$ for any $x \subseteq \mathbb{N}$. A routine verification shows that $\varphi$ submeasure and that $\mathcal{I} = \text{Exh}_\varphi$. (See Solecki [43]. To check that any $x \in \text{Exh}_\varphi$ belongs to $\mathcal{I}$ we use the following observation: $x \in \mathcal{I}$ iff for any $K \in C(\mathcal{I})$ there is $n$ such that $x \cap [0,n) \subseteq K$.)

\[ \square \text{ (Theorem 41)} \]

**Corollary 43.** Suppose that $\mathcal{I}$ is an analytic $P$-ideal. Then any ideal $\mathcal{I} \leq_B \mathcal{I}$ is an analytic $P$-ideal, too.

**Proof.** Use equivalence (iv) $\iff$ (iii) of the theorem. (The result can be obtained via a more direct argument, of course.) \[ \square \]

9 Equivalence relation $E_1$

The ideal $\mathcal{I}_1$ naturally defines the ER $E_1 = E_{\mathcal{I}_1}$ on $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ so that $x E_1 y$ iff $x \Delta y \in \mathcal{I}_1$. We can as well consider $E_1$ as an ER on $\mathcal{P}(\kappa)$ for any uncountable Polish space $\mathbb{X}$, defined as $x E_1 y$ iff $x(k) = y(k)$ for all but finite $k$.

9.a $E_1$ and hypersmoothness

The following notation will be rather useful in our study of subsets of $\mathcal{P}(\mathbb{N})^\kappa$ or $(2^\kappa)^\mathbb{N}$. If $x$ is a function defined on $\mathbb{N}$ then, for any $n$, let

\[ x \downarrow_n = x \downarrow [0,n) \quad x \downarrow_n = x \downarrow [0,n] \quad x \downarrow_n = x \downarrow (n, \infty) \quad x \uparrow_n = x \uparrow [n, \infty) . \]

For any set $X$ of $\mathbb{N}$-sequences, let $X \downarrow_n = \{ x \downarrow_n : x \in X \}$, and similarly for $\leq, \geq, \geq$. If $\xi \in X \downarrow_n$ then let $S_X(\xi) = \{ x(n) : x \in X \land x \uparrow_n = \xi \}$.

Recall that a hypersmooth ER is a countable increasing union of Borel smooth ERs. The following lemma shows that $E_1$ is universal in this class.

**Lemma 44.** For a Borel ER $E$ to be hypersmooth it is necessary and sufficient that $E \leq_B E_1$.

**Proof.** Let $\mathbb{X}$ be the domain of $E$. Assume that $E$ is hypersmooth, i.e., $E = \bigcup_n E_n$, where $x E_n y$ iff $\vartheta_n(x) = \vartheta_n(y)$, each $\vartheta_n : \mathbb{X} \to \mathcal{P}(\mathbb{N})$ is Borel, and $E_n \subseteq E_{n+1}$, $\forall n$. Then $\vartheta_n(x) = \{ \vartheta_n(x) \}_{n \in \mathbb{N}}$ witnesses $E \leq_B E_1$. Conversely, if $\vartheta : \mathbb{X} \to \mathcal{P}(\mathbb{N})^\kappa$ is a Borel reduction of $E$ to $E_1$ then the sequence of ERs $x E_n y$ iff $\vartheta(x) \uparrow_n = \vartheta(y) \uparrow_n$ witnesses that $E$ is hypersmooth. \[ \square \]

This Subsection contains a couple of results which describe the relationships between hypersmooth and countable ERs. The following result is given in [29] with a reference to earlier papers.

**Lemma 45.** (i) $E_1$ is not essentially countable, i.e., there is no Borel countable (that is, with at most countable classes) ER $E$ such that $E_1 \leq_B E$. 


9 EQUIVALENCE RELATION $E_1$

(ii) $E_0 \leq_B E_1$, in other words, $\text{Fin} \leq_B \mathcal{I}$.

Proof. (i) (A version of the argument in [29], 1.4 and 1.5.) Let $\mathbb{X}$ be the domain of $E$, and $\vartheta: \mathcal{P}(\mathbb{N})^N \to \mathbb{X}$ a Borel map satisfying $x E_1 y \iff \vartheta(x) \equiv D \vartheta(y)$. Then $\vartheta$ is continuous on a dense $G_\delta$ set $D \subseteq \mathcal{P}(\mathbb{N})^N$. We begin with a few definitions. Let “generic” mean Cohen generic over a certain fixed countable transitive model $\mathcal{M}$ of a big enough fragment of $\text{ZFC}$, which contains codes for $D, \vartheta \upharpoonright D, \mathbb{X}$.

We are going to define, for any $k$, a pair of $x_k \neq y_k \in \mathcal{P}(\mathbb{N})$, a number $\ell(k)$ and a tuple $\zeta_k \in \mathcal{P}(\mathbb{N})^{\ell(k)}$ such that

1. both $x = \langle x_0 \rangle^{\zeta_0} \langle x_1 \rangle^{\zeta_1} \ldots$ and $y = \langle y_0 \rangle^{\zeta_0} \langle y_1 \rangle^{\zeta_1} \ldots$ are “generic” elements of $\mathcal{P}(\mathbb{N})^N$;
2. for any $k$, $\zeta_{\leq k} = \langle x_0, y_0 \rangle^{\zeta_0} \langle x_1, y_1 \rangle^{\zeta_1} \ldots \langle x_k, y_k \rangle^{\zeta_k}$ is “generic”, hence, so are $\xi_{\leq k} = \langle x_0 \rangle^{\zeta_0} \ldots \langle x_k \rangle^{\zeta_k}$ and $\eta_{\leq k} = \langle y_0 \rangle^{\zeta_0} \ldots \langle y_k \rangle^{\zeta_k}$;
3. for any $k$ and any $z \in \mathcal{P}(\mathbb{N})^N$ such that $\zeta_{\leq k}^\downarrow z$ is “generic” we have $\vartheta(\xi_{\leq k}^\downarrow z) = \vartheta(\eta_{\leq k}^\downarrow z)$.

If this is done then we can choose, using (2), a point $z_{(k)} \in \mathcal{P}(\mathbb{N})^N$ for any $k$ so that $\zeta_{\leq k}^\downarrow z_{(k)} \in \mathcal{P}(\mathbb{N})^N$ is “generic”, hence, by (3), for $x_{(k)} = \xi_{\leq k}^\downarrow z_{(k)}$ and $y_{(k)} = \eta_{\leq k}^\downarrow z_{(k)}$ we have $\vartheta(x_{(k)}) = \vartheta(y_{(k)})$. Note that $x_{(k)} \to x$ and $y_{(k)} \to y$, and on the other hand, all of $x_{(k)}, x, y_{(k)}, y$ belong to $D$ because all are “generic”. It follows that $\vartheta(x) = \vartheta(y)$ by the choice of $D$. However obviously $\neg x E_1 y$, so that $\vartheta$ is not a reduction, as required.

To define $x_0, y_0, \zeta_0$ note that, by an ordinary splitting argument, there is a set $X \subseteq \mathcal{P}(\mathbb{N})$ of cardinality $\mathfrak{c}$ and $z \in \mathcal{P}(\mathbb{N})^N$ such that $\langle a, b \rangle^\downarrow z$ is “generic” for any two $a \neq b \in X$. In particular, all $\langle a \rangle^\downarrow z, a \in X$, are “generic”. But all of them are pairwise $E_1$-equivalent, hence, $\vartheta$ sends all of them into one and the same F-class, which is a countable set by the choice of $F$. It follows that there is a pair of $a \neq b$ in $X$ such that $\vartheta(\langle a \rangle^\downarrow z) \neq \vartheta(\langle b \rangle^\downarrow z)$. This equality is a property of the “generic” object $\langle a, b \rangle^\downarrow z$, hence, it is forced in the sense that there is a number $\ell$ such that $\vartheta(\langle a \rangle^\downarrow z') \neq \vartheta(\langle b \rangle^\downarrow z')$ whenever $\langle a, b \rangle^\downarrow z'$ is “generic” with $z' \upharpoonright \ell = z$ $\upharpoonright \ell$. Put $x_0 = a, y_0 = b, \zeta_0 = z \upharpoonright \ell$.

The induction step is carried out by the same argument.

(ii) That $E_0 \leq_B E_1$ is witnessed by the map $f(x) = \{\langle 0, n \rangle : n \in x\}$.

While $E_1$ is not countable, the conjunction of hypersmoothness and countability characterizes the essentially more primitive class of hyperfinite ERs.

9.b The 3rd dichotomy

The following major result is called the 3rd dichotomy theorem.

Theorem 46 (Kechris and Louveau [29]). Suppose that $E$ is a Borel ER on some Polish space, and $E \leq_B E_1$. Then either $E \leq_B E_0$ or $E_1 \leq_B E$.
9 EQUIVALENCE RELATION $E_1$

Proof. Starting the proof, we may assume that $E$ is a $\Delta^1_1$ ER on $\mathcal{P}(\mathbb{N})$, and that there is a reduction $\rho$ of $E$ to $E_1$, of class $\Delta^1_1$. Then $R = \text{ran} \, \rho$ is a $\Sigma^1_1$ subset of $\mathcal{P}(\mathbb{N})^\mathbb{N}$. The idea behind the proof is to show that the set $R$ is either small enough for $E_1 \upharpoonright R$ to be Borel reducible to $E_0$, or otherwise it is big enough to contain a closed subset $X$ such that $E_1 \upharpoonright X$ is Borel isomorphic to $E_1$.

Relations $<$ and $\preceq$ will denote the inverse order relations on $\mathbb{N}$, i.e., $m \preceq n$ iff $n \preceq m$, and $m < n$ iff $n < m$. If $x \in \mathcal{P}(\mathbb{N})^\mathbb{N}$ then $x \upharpoonright \leq n$ denotes the restriction of $x$ (a function defined on $\mathbb{N}$) on the domain $\preceq n$, i.e., $[n, \infty)$. If $X \subseteq \mathcal{P}(\mathbb{N})^\mathbb{N}$ then let $X \upharpoonright \leq n = \{x \upharpoonright \leq n : x \in X\}$. Define $x \upharpoonright \prec n$ and $X \upharpoonright \prec n$ similarly. In particular, $\mathcal{P}(\mathbb{N})^\mathbb{N} \upharpoonright \leq n = \mathcal{P}(\mathbb{N})^\mathbb{N} \upharpoonright \leq n = \mathcal{P}(\mathbb{N})^[n,\infty).

For a sequence $x \in \mathcal{P}(\mathbb{N})^\mathbb{N}$, let $\text{dep} \, x$ (the depth of $x$) be the number (finite or infinite) of elements of the set $\nabla(x) = \{ j \prec n : x(j) \not\in \Delta^1_1(x \upharpoonright \prec j)\}$. The formula $\text{dep} \, x \geq d$ (of two variables, $d$ running over $\mathbb{N} \cup \{\infty\}$) is obviously $\Sigma^1_1$.

We have two cases:

**Case 1:** all $x \in R = \text{ran} \, \rho$ satisfy $\text{dep} \, x < \infty$.

**Case 2:** there exist $x \in R$ with $\text{dep} \, x = \infty$.

Case 1 is the easier case. First of all we observe that $R$, a $\Sigma^1_1$ set, is a subset of the $\Pi^1_1$ set $Z = \{x : \text{dep} \, x < \infty\}$, hence, there is a $\Delta^1_1$ set $Y$ with $\text{ran} \, \rho \subseteq Y \subseteq Z$. The following lemma ends the argument.

**Lemma 46.1.** Suppose that $X \subseteq \mathcal{P}(\mathbb{N})^\mathbb{N}$ is a $\Delta^1_1$ set and any $x \in X$ satisfies $\text{dep} \, x < \infty$. Then $E_1 \upharpoonright X \preceq E_0$.

**Proof.** By the choice of $X$ for any $x \in X$ there is a number $n$ such that $\forall m \preceq n \, (x(m) \in \Delta^1_1(x \upharpoonright \leq m))$. As the relation between $x$ and $n$ here is clearly $\Pi^1_1$, the “Kreisel selection” theorem yields a $\Delta^1_1$ map $\nu : X \to \mathbb{N}$ such that $x(m) \in \Delta^1_1(x \upharpoonright \leq m)$ holds whenever $x \in X$ and $m \preceq \nu(x)$. Now define, for each $x \in X$, $\vartheta(x) \in \mathcal{P}(\mathbb{N})^\mathbb{N}$ as follows: $\vartheta(x) \upharpoonright \leq \nu(x) = x \upharpoonright \leq \nu(x)$ but $\vartheta(x)(j) = \emptyset$ for all $j \prec \nu(x)$. Note that $x \in E_1 \vartheta(x)$ for any $x \in X$.

The other important thing is that $\text{ran} \, \vartheta \subseteq Z = \{x \in \mathcal{P}(\mathbb{N})^\mathbb{N} : \text{dep} \, x = 0\}$, where $Z$ is a $\Pi^1_1$ set, hence, there is a $\Delta^1_1$ set $Y$ with $\text{ran} \, \vartheta \subseteq Y \subseteq Z$. In particular $\vartheta$ reduces $E_1 \upharpoonright X$ to $E_1 \upharpoonright Y$. We observe that $E_1 \upharpoonright Y$ is a countable ER: any $E_1$-class in $\mathcal{P}(\mathbb{N})^\mathbb{N}$ intersects $Y$ by an at most countable set (as so is the property of $Z$, a bigger set). Thus, $E_1 \upharpoonright Y$ is hyperfinite by Theorem 32. □

### 9.2 Case 2

Since $\text{dep} \, x = \infty$ is a $\Sigma^1_1$ formula, it suffices to show that for any non-empty $\Sigma^1_1$ set $R \subseteq \mathcal{P}(\mathbb{N})^\mathbb{N}$ with $\forall x \in R \, (\text{dep} \, x = \infty)$ we have a $\Delta^1_1$ subset $X \subseteq R$ with $E_1 \preceq X \subseteq E_1$. Fix a set $R$, as indicated, for the course of the proof. The subset $X$ of $R$ will be defined with the help of splitting construction developed in [23] for the study of "ill"founded Sacks iterations.
9 EQUVALENCE RELATION $E_1$

We shall define a map $\varphi : \mathbb{N} \to \mathbb{N}$, which assumes infinitely many values and assumes each its value infinitely many times (but $\text{ran} \varphi$ may be a proper subset of $\mathbb{N}$), and, for each $u \in 2^{<\omega}$, a non-empty $\Sigma^1_1$ subset $X_u \subseteq R$, which satisfy a quite long list of properties. First of all, if $\varphi$ is already defined at least on $[0, n)$ and $u \neq v \in 2^{<\omega}$ then $\nu_{\varphi}(u, v) = \min_{\varphi} \{ \varphi(k) : k < n \land u(k) \neq v(k) \}$. (Note that the minimum is taken in the sense of $\preceq$, hence, it is $\max$ in the sense of $\leq$, the usual order). Separately, put $\varphi[u, u] = -1$ for any $u$.

Now we give the list of requirements.

(i) if $\varphi(n) \notin \{ \varphi(k) : k < n \}$ then $\varphi(n) < \varphi(k)$ for any $k < n$;

(ii) every $X_u$ is a non-empty $\Sigma^1_1$ subset of $R$;

(iii) if $u \in 2^n$, $x \in X_u$, and $k < n$, then $\varphi(k) \in \nabla(x)$;

(iv) if $u, v \in 2^n$ then $X_u \upharpoonright_{\varphi[u, v]} = X_v \upharpoonright_{\varphi[u, v]}$;

(v) if $u, v \in 2^n$ then $X_u \upharpoonright_{\varphi[u, v]} \cap X_v \upharpoonright_{\varphi[u, v]} = \emptyset$;

(vi) $X_{u \wedge i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

(vii) $\max_{u \in 2^n} \text{diam} X_u \to 0$ as $n \to \infty$ (a reasonable Polish metric on $\mathcal{P}(\mathbb{N})^\mathbb{N}$ is assumed to be fixed);

(viii) a certain condition, in terms of the Choquet game, which connects each $X_{u \wedge i}$ with $X_u$ so that, as a consequence, $\bigcap_n X_{a|n} \neq \emptyset$ for any $a \in 2^\mathbb{N}$.

Let us demonstrate how such a system of sets and a function $\varphi$ accomplish Case 2. According to (vii) and (viii), for any $a \in 2^\mathbb{N}$ the intersection $\bigcap_n X_{a|n}$ contains a single point, let it be $F(a)$, and $F$ is continuous and $1 - 1$.

Put $J = \text{ran} \varphi = \{ j_m : m \in \mathbb{N} \}$, in the $<$-increasing order; $J \subseteq \mathbb{N}$ is infinite.

Let $n \in \mathbb{N}$. Then $\varphi(n) = j_m$ for some (unique) $m$ : we put $\psi(n) = m$. Thus $\psi : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ and the preimage $\psi^{-1}(m) = \varphi^{-1}(j_m)$ is an infinite subset of $\mathbb{N}$ for any $m$. This allows us to define a parallel system of sets $Y_u$, $u \in 2^{<\omega}$, as follows. Put $Y_A = \mathcal{P}(\mathbb{N})^\mathbb{N}$. Suppose that $Y_u$ has been defined, $u \in 2^n$. Put $j = \varphi(n) = j_{\psi(n)}$. Let $K$ be the number of all indices $k < n$ still satisfying $\varphi(k) = j$, perhaps $K = 0$. Put $Y_{u \wedge i} = \{ x \in Y_u : x(j)(K) = i \}$ for $i = 0, 1$.

Each of $Y_u$ is clearly a basic clopen set in $\mathcal{P}(\mathbb{N})^\mathbb{N}$, and one easily verifies that conditions (i) - (vii), except for (iii), are satisfied for the sets $Y_u$ (instead of $X_u$) and the map $\psi$ (instead of $\varphi$), in particular, for any $a \in 2^\mathbb{N}$, $\bigcap_n Y_{a|n} = \{ G(a) \}$ is a singleton, and the map $G$ is continuous and $1 - 1$. (We can, of course, define $G$ explicitly: $G(a)(m)(l) = a(n)$, where $n \in \mathbb{N}$ is chosen so that $\psi(n) = m$ and there is exactly $l$ numbers $k < n$ with $\psi(k) = m$.) Note finally that $\{ G(a) : a \in 2^n \} = \mathcal{P}(\mathbb{N})^\mathbb{N}$ since by definition $Y_{u \wedge 1} \cup Y_{u \wedge 0} = Y_u$ for all $u$.

We conclude that the map $\vartheta(x) = F(G^{-1}(x))$ is a continuous bijection (hence, in this case, a homeomorphism by compactness) $\mathcal{P}(\mathbb{N})^\mathbb{N} \xrightarrow{\text{onto}} X$. We
further assert that \( \vartheta \) satisfying the following: for each \( y, y' \in \mathcal{P}(\mathbb{N})^N \) and \( m \),

\[
y \mid \preceq_m = y' \mid \preceq_m \quad \text{iff} \quad \vartheta(y) \mid \preceq_{jm} = \vartheta(y') \mid \preceq_{jm} .
\] (*)

Indeed, let \( y = G(a) \) and \( x = F(a) = \vartheta(y) \), and similarly \( y' = G(a') \) and \( x' = F(a') = \vartheta(y') \), where \( a, a' \in 2^\mathbb{N} \). Suppose that \( y \mid \preceq_m = y' \mid \preceq_m \). According to (v) for \( \psi \) and the sets \( Y_u \), we then have \( m \prec \nu_\psi(a \mid n, a' \mid n) \) for any \( n \).

It follows, by the definition of \( \psi \), that \( j_m \prec \nu_\varphi(a \mid n, a' \mid n) \) for any \( n \), hence, \( X_{(a|n)} \mid \preceq_{jm} = X_{(a|n)} \mid \preceq_{jm} \) for all \( n \) by (iv). Assuming now that Polish metrics on all spaces \( \mathcal{P}(\mathbb{N})^{\preceq_j} \) are chosen so that \( \text{diam} Z \geq \text{diam} (Z \mid \preceq_j) \) for all \( Z \subseteq \mathcal{P}(\mathbb{N}) \) and \( j \), we easily obtain that \( x \mid \preceq_{jm} = x' \mid \preceq_{jm} \), \( i.e. \), the right-hand side of (*)

The inverse implication in (*) is proved similarly.

Thus we have (*) but this means that \( \vartheta \) is a continuous reduction of \( E_1 \) to \( E_1 \upharpoonright X \), thus, \( E_1 \leq_R E_1 \upharpoonright X \), as required.

\( \Box \) (Theorem 46 modulo the construction (i) – (viii))

9.d The construction

Recall that \( R \subseteq \mathcal{P}(\mathbb{N})^N \) is a fixed non-empty \( \Sigma^1_1 \) set such that \( \text{dep} x = \infty \) for each \( x \in R \). Set \( X_\lambda = R \).

Now suppose that the sets \( X_u \subseteq R \) with \( u \in 2^n \) have been defined and satisfy the applicable part of (i) – (viii).

Step 1. Our 1st task is to choose \( \varphi(n) \). Let \( \{ j_1 < ... < j_m \} = \{ \varphi(k) : k < n \} \).

For any \( 1 \leq p \leq m \), let \( N_p \) be the number of all \( k < n \) with \( \varphi(k) = j_p \).

Case 1a. If some numbers \( N_p \) are \( < m \) then choose \( \varphi(n) \) among \( j_p \) with the least \( N_p \), and among them the least one.

Case 1b: \( N_p \geq m \) (then actually \( N_p = m \)) for all \( p \leq m \). It follows from our assumptions, in particular (iv), that \( X_u \mid \preceq_{jm} = X_v \mid \preceq_{jm} \) for all \( u, v \in 2^n \). Let \( Y = X_u \mid \preceq_{jm} \) for any such \( u \). Take any \( y \in Y \). Then \( \nabla(y) \) is infinite, hence, there is some \( j \in \nabla(y) \) with \( j \prec j_m \). Put \( \varphi(n) = j \).

We have something else to do in this case. Let \( X'_u = \{ x \in X_u : j \in \nabla(y) \} \) for any \( u \in 2^m \). Then we easily have \( X'_u = \{ x \in X_u : x \mid \preceq_{jm} \in \nabla(y) \} \), where \( Y' = \{ y \in Y : j \in \nabla(y) \} \) is a non-empty \( \Sigma^1_1 \) set, so that the sets \( X'_u \subseteq X_u \) are non-empty \( \Sigma^1_1 \). Moreover, as \( j_m \) is the \( \preceq \)-least in \( \{ \varphi(k) : k < n \} \), we can easily show that the system of sets \( X'_u \) still satisfies (iv). This allows us to assume, without any loss of generality, that, in Case 1b, \( X'_u = X_u \) for all \( u \), or, in other words, that any \( x \in X_u \) for any \( u \in 2^n \) satisfies \( j = \varphi(n) \in \nabla(x) \). (This is true in Case 1a, of course, because then \( \varphi(n) = \varphi(k) \) for some \( k < n \).)

Note that this manner to choose \( \varphi(n) \) implies (i) and also implies that \( \varphi \) takes infinitely many values and takes each its value infinitely many times.

The construction of the construction requires the following
Lemma 46.2. If $u_0 \in 2^n$ and $X' \subseteq X_{u_0}$ is a non-empty $\Sigma_1$ set then there is a system of $\Sigma_1$ sets $\emptyset \neq X'_u \subseteq X_u$ with $X'_{u_0} = X'$, which still satisfies (iv).

Proof. For any $u \in 2^n$, let $X'_u = \{ x \in X_u : x \restriction \leq n(u) \in X' \restriction \leq n(u) \}$, where $n(u) = \nu_\varphi[u, u_0]$. In particular, this gives $X'_{u_0} = X'$, because $\nu_\varphi[u_0, u_0] = -1$. The sets $X'_u$ are as required, via a routine verification.

Step 2. First of all put $j = \varphi(n)$ and $Y_u = X_u \restriction _j$. (All $Y_u$ are equal to $Y$ in Case 1b, but the argument pretends to make no difference between 1a and 1b). Take any $u_1 \in 2^n$. By the construction any element $x \in X_{u_1}$ satisfies $j \in \nabla(x)$, so that $x(j) \not\in \Delta(x \restriction j)$. As $X_{u_1}$ is a $\Sigma_1$ set, it follows that $\{ x'(j) : x' \in X_{u_1} \land x' \restriction _j = x \restriction _j \}$ is not a singleton, in fact is uncountable. It follows that there is a number $l_{u_1}$ having the property that the $\Sigma_1$ set

$$Y'_{u_1} = \{ y \in Y_{u_1} : \exists x, x' \in X_{u_1} (x' \restriction _j = x \restriction _j = y \land l_{u_1} \in x(j) \land l_{u_1} \not\in x'(j)) \}$$

is non-empty. We now put $X' = \{ x \in X_{u_1} : x \restriction _j \in Y'_{u_1} \}$ and define $\Sigma_1$ sets $\emptyset \neq X' \subseteq X_u$ as in the lemma, in particular, $X'_{u_1} = X'$, $X'_{u_1} \restriction _j = Y'_{u_1}$, still (iv) is satisfied, and in addition

$$\forall y \in X'_{u_1} \exists x, x' \in X_{u_1} (x' \restriction _j = x \restriction _j = y \land l_{u_1} \in x(j) \land l_{u_1} \not\in x'(j)) \quad (1)$$

Now take some other $u_2 \in 2^n$. Let $\nu = \nu_\varphi[u_1, u_2]$. If $j < \nu$ then $X_{u_1} \restriction _j = X_{u_2} \restriction _j$, so that we already have, for $l_{u_2} = l_{u_1}$, that

$$\forall y \in X'_{u_2} \exists x, x' \in X_{u_2} (x' \restriction _j = x \restriction _j = y \land l_{u_2} \in x(j) \land l_{u_2} \not\in x'(j)) \quad (2)$$

and can pass to some $u_3 \in 2^n$. Suppose that $\nu \leq j$. Now things are somewhat nastier. As above there is a number $l_{u_2}$ such that

$$Y''_{u_2} = \{ y \in Y_{u_2} : \exists x, x' \in X_{u_2} (x' \restriction _j = x \restriction _j = y \land l_{u_2} \in x(j) \land l_{u_2} \not\in x'(j)) \}$$

is a non-empty $\Sigma_1$ set, thus, we can define $X'' = \{ x \in X_{u_1} : x \restriction _j \in Y''_{u_2} \}$ and maintain the construction of Lemma 46.2, getting non-empty $\Sigma_1$ sets $X'' \subseteq X'_u$ still satisfying (iv) and $X''_{u_2} = X''$, therefore, we still have (2) for the set $X''_{u_2}$.

Yet it is most important in this case that (1) is preserved, i.e., it still holds for the set $X''_{u_1}$ instead of $X'_{u_1}$! Why is this? Indeed, according to the construction in the proof of Lemma 46.2, we have $X''_{u_1} = \{ x \in X'_{u_1} : x \restriction _{\leq \nu} \in X'' \restriction _{\leq \nu} \}$. Thus, although, in principle, $X''_{u_1}$ is smaller than $X'_{u_1}$, for any $y \in X''_{u_1} \restriction _j$ we have

$$\{ x \in X''_{u_1} : x \restriction _j = y \} = \{ x \in X'_{u_1} : x \restriction _j = y \},$$

simply because now we assume that $\nu \leq j$. This implies that (1) still holds.

Iterating this construction so that each $u \in 2^n$ is eventually encountered, we obtain, in the end, a system of non-empty $\Sigma_1$ sets, let us call them “new” $X_u$, but they are subsets of the “original” $X_u$, still satisfying (iv), still satisfying that
$\varphi(n) \in \nabla(x)$ for each $x \in \bigcap_{u \in 2^n} X_u$, and, in addition, for any $u \in 2^n$ there is a number $l_u$ such that $j < \nu_{\varphi}[u,v] \implies l_u = l_v$ and

$$\forall y \in X_u \mid \varphi \exists x, x' \in X_u (\varphi_x \mid \varphi_j = x \mid \varphi_j = y \wedge l_u \in x(j) \wedge l_u \notin x'(j)).$$  

(*)

**Step 3.** We define the $(n+1)$-th level of sets by $X_{u \uplus 0} = \{x \in X_u : l_u \in x(j)\}$ and $X_{u \uplus 1} = \{x \in X_u : l_u \notin x(j)\}$ for all $u \in 2^n$, where still $j = \varphi(n)$. It follows from (*) that all these $\Sigma^1_1$ sets are non-empty.

**Lemma 46.3.** The just defined system of sets $X_s$, $s \in 2^{n+1}$, satisfies (iv), (v).

**Proof.** Let $s = u \uplus 0$ and $t = v \uplus 0$ belong to $2^{n+1}$, so that $u, v \in 2^n$ and $i, i' \in \{0, 1\}$. Let $\nu = \nu_{\varphi}[u,v]$ and $\nu' = \nu_{\varphi}[s,t]$.

**Case 3a:** $\nu \equiv j = \varphi(n)$. Then easily $\nu = \nu'$, so that (v) immediately follows from (iv) at level $n$ for $X_u$ and $X_v$. As for (iv), we have $X_x \mid \varphi_u = X_u \mid \varphi_u$ (because by definition $X_x \mid \varphi_j = X_u \mid \varphi_j$), and similarly $X_t \mid \varphi_v = X_u \mid \varphi_v$, therefore $X_t \mid \varphi_v' = X_x \mid \varphi_v'$ since $X_u \mid \varphi_v = X_u \mid \varphi_v$ by (iv) at level $n$.

**Case 3b:** $j < \nu$ and $i = i'$. Then still $\nu = \nu'$, thus we have (v). Further, $X_x \mid \varphi_u = X_u \mid \varphi_u$ by (iv) at level $n$, hence, $X_u \mid \varphi_j = X_u \mid \varphi_j$, hence, $l_u = l_v$ (see above). Now, assuming that, say, $i = i' = 1$ and $l_u = l_v = l$, we conclude that

$$X_x \mid \varphi_u = \{y \in X_u \mid \varphi_u : l \in y(j)\} = \{y \in X_v \mid \varphi_u : l \in y(j)\} = X_t \mid \varphi_u'.$$

**Case 3c:** $j < \nu$ and $i \neq i'$, say, $i = 0$ and $i' = 1$. Now $\nu' = j$. Yet by definition $X_x \mid \varphi_j = X_u \mid \varphi_j$ and $X_t \mid \varphi_j = X_v \mid \varphi_j$, so it remains to apply (iv) for level $n$. As for (v), note that by definition $l \notin x(j)$ for any $x \in X_s = X_{u \uplus 0}$ while $l \in x(j)$ for any $x \in X_t = X_{v \uplus 1}$, where $l = l_u = l_v$. $\Box$ (Lemma)

**Step 4.** In addition to (iv) and (v), we already have (i), (ii), (iii), (vi) at level $n+1$. To achieve the remaining properties (vii) and (viii), it suffices to consider, one by one, all elements $s \in 2^{n+1}$, finding, at each such a substep, a non-empty $\Sigma^1_1$ subset of $X_s$ which is consistent with the requirements of (vii) and (viii) (for instance, for (vii), just take its diameter is $\leq 2^{-n}$), and then reducing all other sets $X_t$ by Lemma 46.2 at level $n+1$.

$\Box$ (Construction and Theorem 46)

**9.5 Above $E_1$**

Recall that an embedding is a $1-1$ reduction, and an invariant embedding is an embedding $\vartheta$ such that its range is an invariant set, see Subsection 1.d above.

**Theorem 47** (Kechris and Louveau [29]). Suppose that $E_1 \leq_B F$, where $F$ is an analytic ER on a Polish space $\mathcal{Y}$. Then both $E_1 \subseteq C F$ and $E_1 \subseteq B F$. 

9 EQUVALENE RELATION $E_1$

Proof. To prove the first statement, let $\preccurlyeq$ be the inverted order on $\mathbb{N}$, i.e., $m \preccurlyeq n$ iff $n \leq m$. Let $\mathcal{P}$ be the collection of all sets $P \subseteq \mathcal{P}(\mathbb{N})^\mathbb{N}$ such that there is a continuous $1 - 1$ map $\eta: \mathcal{P}(\mathbb{N})^\mathbb{N} \rightarrow P$ such that we have

$$x |_{\preccurlyeq n} = y |_{\preccurlyeq n} \iff \eta(x) |_{\preccurlyeq n} = \eta(y) |_{\preccurlyeq n}$$

for all $n$ and $x, y \in \mathcal{P}(\mathbb{N})^\mathbb{N}$, where $x |_{\preccurlyeq n} = \{x_i\}_{i \leq n}$ for any $x = \{x_i\} \in \mathcal{P}(\mathbb{N})^\mathbb{N}$. Clearly any such a map is a continuous embedding of $E_1$ into itself.

This set $\mathcal{P}$ is a forcing notion to extend the universe by a sequence of reals $x_i$ so that each $x_n$ is Sacks-generic over $\{x_i\}_{i \leq n}$, an example of iterated Sacks extensions with an ill-founded “skeleton” of iteration, which we defined in [23]. Here, the “skeleton” is $\mathbb{N}$ with the inverted order $\preccurlyeq$.

The method of [23] contains a study of continuous and Borel functions on sets in $\mathcal{P}$. In particular it is shown there that Borel maps admit the following cofinal classification on sets in $\mathcal{P}$: if $\mathcal{Y}$ is Polish, $P' \in \mathcal{P}$, and $\vartheta: P' \rightarrow \mathbb{Y}$ is Borel then there is a set $P \in \mathcal{P}$, $P \subseteq P'$, on which $\vartheta$ is continuous, and either a constant or, for some $n$, $1 - 1$ on $P |_{\preccurlyeq n}$ in the sense that,

$$x \in P \implies \vartheta(x) = \vartheta(y).$$

We apply this to a Borel map $\vartheta: \mathcal{P}(\mathbb{N})^\mathbb{N} \rightarrow \mathcal{Y}$ which reduces $E_1$ to $F$. We begin with $P' = \mathcal{P}(\mathbb{N})^\mathbb{N}$ and find a set $P \in \mathcal{P}$ as indicated. Since $\vartheta$ cannot be a constant on $P$ (indeed, any $P \in \mathcal{P}$ contains many pairwise $E_1$-inequivalent elements), we have (*) for some $n$. In other words, there is a $1 - 1$ continuous map $f: P |_{\preccurlyeq n} \rightarrow \mathcal{Y}$ (where $P |_{\preccurlyeq n} = \{x |_{\preccurlyeq n} : x \in P\}$) such that $\vartheta(x) = f(x |_{\preccurlyeq n})$ for all $x \in P$. Now, let $x = \{x_i\}_{i \in \mathbb{N}} \in \mathcal{P}(\mathbb{N})^\mathbb{N}$. Define $\zeta(x) = z = \{z_i\}_{i \in \mathbb{N}}$ so that $z_i = 0$ for $i < n$ and $z_{n+i} = x_i$ for all $i$. Finally set $\vartheta'(x) = f(\eta(\zeta(x)) |_{\preccurlyeq n})$ for all $x \in \mathcal{P}(\mathbb{N})^\mathbb{N}$: this is a continuous embedding of $E_1$ in $F$.

Now we prove the second claim. We can assume that $\mathcal{Y} = \mathcal{P}(\mathbb{N})$ and that $\vartheta: \mathcal{P}(\mathbb{N})^\mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$ is already a continuous embedding $E_1$ into $F$. Let $Y = \text{ran } \vartheta$ and $Z = [Y]_F$. Normally $Y, Z$ are analytic, but in this case they are even Borel. Indeed $Z$ is the projection of $P = \{(z, x) : z \in F \vartheta(x)\}$, a Borel subset of $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})^\mathbb{N}$ whose all cross-sections are $E_1$-equivalence classes, i.e., $\sigma$-compact sets. It is known that in this case $Z$ is Borel and, moreover, there is a Borel map $f: Z \rightarrow \mathcal{P}(\mathbb{N})^\mathbb{N}$ such that $f(z) \in E_1 x$ whenever $z \in F \vartheta(x)$.

We can convert $f$ to a $1 - 1$ map $\vartheta: \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})^\mathbb{N}$ with the same properties: $g(z) |_{\preccurlyeq n} = f(z) |_{\preccurlyeq n}$ for $n \geq 1$, but $g(z) |_0 = z$. Then $f: \mathcal{P}(\mathbb{N})^\mathbb{N} \rightarrow Z \subseteq \mathcal{P}(\mathbb{N})$ and $g: Z \rightarrow \mathcal{P}(\mathbb{N})^\mathbb{N}$ are Borel $1 - 1$ maps ($\vartheta$ is even continuous, but this does not matter now), and, for any $x \in \mathcal{P}(\mathbb{N})$, $\vartheta$ maps $[x]_{E_1}$ into $[\vartheta(x)]_F \subseteq Z$, and $g$ maps $[\vartheta(x)]_F$ back into $[x]_{E_1}$. It remains to apply the construction from the Cantor–Bendixson theorem, to get a Borel embedding, say, $F$ of $E_1$ into $F$ with $\text{ran } F = Z$, i.e., an invariant embedding. $\square$

The following theorem shows that orbit equivalence relations of Polish group actions cannot reduce $E_1$.
Theorem 48 (Kechris and Louveau [29]). Suppose that \( G \) is a Polish group and \( \mathcal{X} \) is a Borel \( G \)-space. Then \( E_1 \) is not Borel reducible to \( E_1^G \).

Proof. Towards the contrary, let \( \vartheta : \mathcal{P}(\mathbb{N})^\mathbb{N} \rightarrow \mathcal{X} \) be a Borel reduction of \( E_1 \) to \( E \). We can assume, by Theorem 47, that \( \vartheta \) is in fact an invariant embedding, i.e., \( 1 - 1 \) and \( Y = \text{ran} \vartheta \) is an \( E \)-invariant set. Define, for \( g \in G \) and \( x \in \mathcal{P}(\mathbb{N})^\mathbb{N} \), \( g \cdot x = \vartheta^{-1}(g \cdot \vartheta(x)) \). Then this is a Borel action of \( G \) on \( \mathcal{P}(\mathbb{N})^\mathbb{N} \) such that the induced relation \( E_1^G \) coincides with \( E_1 \).

Let us fix \( x \in \mathcal{P}(\mathbb{N})^\mathbb{N} \).

Consider any \( y = \{y_n\}_n \in [x]_{E_1} \). Then \( [x]_{E_1} = \bigcup_n C_n(y) \), where each set \( C_n(y) = \{y' \in \mathcal{P}(\mathbb{N})^\mathbb{N} : \forall m \geq n (y_m = y'_m) \} \) is Borel (even compact). It follows that \( G = \bigcup_n G_n(y) \), where each \( G_n(y) = \{g \in G : g(x) \in C_n(y) \} \) is Borel. Thus, as \( G \) is Polish, there is a number \( n \) such that \( G_n(y) \) is not meager in \( G \) (then this will hold for all \( n' \geq n \), of course). Let \( n(y) \) be the least such an \( n \).

We assert that for any \( n \) the set \( Y_n(x) = \{ y \in [x]_{E_1} : y(x) = n \} \) is at most countable. Indeed suppose that \( Y_n(x) \) is not countable. Note that if \( y_1 \) and \( y_2 \) in \( [x]_{E_1} \) have different restrictions \( y_i \upharpoonright [n, \infty) \) then the sets \( C_n(y_1) \) and \( C_n(y_2) \) are disjoint, therefore, the sets \( G_n(y_1) \) and \( G_n(y_2) \) are disjoint, so we would have uncountably many pairwise disjoint non-meager sets in \( G \), contradiction. Thus all sets \( Y_n(x) \) are countable.

It is most important that \( Y_n(x) \) depends on \( [x]_{E_1} \) rather than \( x \) itself, more exactly, if \( x' \in [x]_{E_1} \) then \( Y_n(x) = Y_n(x') \) : this is because any set \( G_n(y) \) in the sense of \( x' \) is just a shift, within \( G \), of \( G_n(y) \) in the sense of \( x \). Therefore, putting \( Y(x) = \bigcup_n \{ \tilde{u} : u \in Y_n(x) \} \), where, for \( u \in \mathcal{P}(\mathbb{N})^{[n, \infty)} \), \( \tilde{u} \in \mathcal{P}(\mathbb{N})^\mathbb{N} \) is defined by \( \tilde{u} \upharpoonright [n, \infty) = u \) and \( \tilde{u}(k) = 0 \) for \( k < n \), we have the set \( Y = \bigcup_{x \in \mathcal{P}(\mathbb{N})^\mathbb{N}} Y(x) \) with the property that \( Y \cap [x]_{E_1} \) is non-empty and at most countable for any \( x \in \mathcal{P}(\mathbb{N})^\mathbb{N} \).

The other important fact is that the relation \( y \in Y(x) \) is Borel: this is because it is assembled from Borel relations via the Vaught quantifier “there exists nonmeager-many”, known to preserve the Borelness. It follows that

\[
Y = \{ y : \exists x \ (y \in Y_x) \} = \{ y : \forall x \ (x \in [y]_{E_1} \implies y \in Y(x)) \}
\]

is a Borel subset of \( \mathcal{P}(\mathbb{N})^\mathbb{N} \). By the uniformization theorem for Borel sets with countable sections, there is a Borel map \( f \) defined on \( \mathcal{P}(\mathbb{N})^\mathbb{N} \) so that \( f(x) \in Y(x) \) for any \( x \), which implies \( E_1 \leq_B E_1 \upharpoonright Y \). On the other hand, \( E_1 \upharpoonright Y \) is a countable ER by the above, which is a contradiction to Lemma 45. \qed

10 Actions of the infinite symmetric group

This Section is connected with the next one (on turbulence). We concentrate on a main result in this area, due to Hjorth, that turbulent ERs are not reducible to those induced by actions of \( S_\infty \). In particular, we shall prove the following:
10 ACTIONS OF THE INFINITE SYMMETRIC GROUP

I. Lopez-Escobar: any invariant Borel set of countable models is the truth domain of a formula of $\mathcal{L}_{\omega_1 \omega}$.

II. Any orbit ER of a Polish action of a closed subgroup of $S_{\infty}$ is classifiable by countable structures (up to isomorphism).

III. Any ER, classifiable by countable structures, is Borel reducible to isomorphism of countable ordered graphs.

IV. Any Borel ER, classifiable by countable structures, is Borel reducible to one of ERs $T_\xi$.

V. Any ER, classifiable by countable structures and induced by a Polish action (of a Polish group), is Borel reducible to one of ERs $T_\xi$ on a comeager set.

VI. Any “turbulent” ER $E$ is generically $T_\xi$-ergodic for any $\xi < \omega_1$, in particular, $E$ is not Borel reducible to $T_\xi$.

VII. Any “turbulent” ER is not classifiable by countable structures: a corollary of VI and V.

VIII. A generalization of VII: any “turbulent” ER is not Borel reducible to a ER which can be obtained from $D(\mathbb{N})$ using operations defined in §3.c.

Scott’s analysis, involved in proofs of IV and V, appears only in a rather mild and self-contained version.

10.a Infinite symmetric group $S_{\infty}$

Let $S_{\infty}$ be the group of all permutations ($i.e., 1$–$1$ maps $\mathbb{N} \twoheadrightarrow \mathbb{N}$) of $\mathbb{N}$, with the superposition as the group operation. Clearly $S_{\infty}$ is a $\mathbf{G}_\delta$ subset of $\mathbb{N}^\mathbb{N}$, hence, a Polish group. A compatible complete metric on $S_{\infty}$ can be defined by $D(x, y) = d(x, y) + d(x^{-1}, y^{-1})$, where $d$ is the ordinary complete metric of $\mathbb{N}^\mathbb{N}$, i.e., $d(x, y) = 2^{-m-1}$, where $m$ is the least such that $x(m) \neq y(m)$. Yet $S_{\infty}$ admits no compatible left-invariant complete metric [1, 1.5].

For instance isomorphism relations of various kinds of countable structures are orbit ERs induced by $S_{\infty}$. Indeed, suppose that $\mathcal{L} = \{R_i\}_{i \in I}$ is a countable relational language, i.e., $0 < \text{card } I \leq \aleph_0$ and each $R_i$ is an $m_i$-ary relational symbol. We put $\text{Mod}_{\mathcal{L}} = \prod_{i \in I} \mathcal{P}(\mathbb{N})$, the space of (coded) $\mathcal{L}$-structures on $\mathbb{N}$. The logic action $j_{\mathcal{L}}$ of $S_{\infty}$ on $\text{Mod}_{\mathcal{L}}$ is defined as follows: if $x = \{x_i\}_{i \in I} \in \text{Mod}_{\mathcal{L}}$ and $y = j_{\mathcal{L}}(g, x) = g \cdot x = \{y_i\}_{i \in I} \in \text{Mod}_{\mathcal{L}}$, where we have

$$\langle k_1, \ldots, k_{m_i} \rangle \in x_i \iff \langle g(k_1), \ldots, g(k_{m_i}) \rangle \in y_i$$

for all $i \in I$ and $\langle k_1, \ldots, k_{m_i} \rangle \in \mathbb{N}^{m_i}$. Then $\langle \text{Mod}_{\mathcal{L}}, j_{\mathcal{L}} \rangle$ is a Polish $S_{\infty}$-space and $j_{\mathcal{L}}$-orbits in $\text{Mod}_{\mathcal{L}}$ are exactly the isomorphism classes of $\mathcal{L}$-structures, which is a reason to denote the associated equivalence relation $E_{j_{\mathcal{L}}}^{\text{Mod}_{\mathcal{L}}}$ as $\equiv_{\mathcal{L}}$.

$^{21}$ $X_{\mathcal{L}}$ is often used to denote $\text{Mod}_{\mathcal{L}}$. 

Proof of $S_{\infty}$ not clt ?
If $G$ is a subgroup of $S_{\infty}$ then $j_{\mathcal{L}}$ restricted to $G$ is still an action of $G$ on $\text{Mod}_{\mathcal{L}}$, whose orbit $\text{ER}$ will be denoted by $\cong_{G_{\mathcal{L}}}$, i.e., $x \cong_{G_{\mathcal{L}}} y$ iff $\exists g \in G : g \cdot x = y$.

10.b Borel invariant sets

A set $M \subseteq \text{Mod}_{\mathcal{L}}$ is invariant if $[M]_{\cong_{\mathcal{L}}} = M$. There is a convenient characterization of Borel invariant sets, in terms of $\mathcal{L}_{\omega_1 \omega}$, an infinitary extension of $\mathcal{L}$, by countable conjunctions and disjunctions. To be more exact,

1) any $R_i(v_0, \ldots, v_{m_i-1})$ is an atomic formula of $\mathcal{L}_{\omega_1 \omega}$ (all $v_i$ being variables over $\mathbb{N}$ and $m_i$ is the arity of $R_i$), and propositional connectives and quantifiers $\exists, \forall$ can be applied as usual;

2) if $\varphi_i$, $i \in \mathbb{N}$, are formulas of $\mathcal{L}_{\omega_1 \omega}$ whose free variables are among a finite list $v_0, \ldots, v_n$ then $\bigvee_i \varphi_i$ and $\bigwedge_i \varphi_i$ are formulas of $\mathcal{L}_{\omega_1 \omega}$.

If $x \in \text{Mod}_{\mathcal{L}}$, $\varphi(v_1, \ldots, v_n)$ is a formula of $\mathcal{L}_{\omega_1 \omega}$, and $i_1, \ldots, i_n \in \mathbb{N}$, then $x \models \varphi(i_1, \ldots, i_n)$ means that $\varphi(i_1, \ldots, i_n)$ is satisfied on $x$, in the usual sense that involves transfinite induction on the “depth” of $\varphi$, see [26, 16.C].

**Theorem 49** (Lopez-Escobar, see [26, 16.8]). A set $M \subseteq \text{Mod}_{\mathcal{L}}$ is invariant and Borel iff $M = \{ x \in \text{Mod}_{\mathcal{L}} : x \models \varphi \}$ for a closed formula $\varphi$ of $\mathcal{L}_{\omega_1 \omega}$.

**Proof.** To prove the nontrivial direction let $M \subseteq \text{Mod}_{\mathcal{L}}$ be invariant and Borel. Put $B_s = \{ g \in S_{\infty} : s \subseteq g \}$ for any injective $s \in \mathbb{N}^{<\omega}$ (i.e., $s_i \neq s_j$ for $i \neq j$), this is a clopen subset of $S_{\infty}$ (in the Polish topology of $S_{\infty}$ inherited from $\mathbb{N}^\mathbb{N}$). If $A \subseteq S_{\infty}$ then let $s \models A(\bar{y})$ mean that the set $B_s \cap A$ is co-meager in $B_s$, i.e., $g \in A$ holds for a.a. $g \in S_{\infty}$ with $s \subseteq g$. The proof consists of two parts:

(i) $M = \{ x \in \text{Mod}_{\mathcal{L}} : \Lambda \models \bar{y} \cdot x \in M \}$ (where $g \cdot x = j_{\mathcal{L}}(g, x)$, see above);

(ii) For any Borel $M \subseteq \text{Mod}_{\mathcal{L}}$ and any $n$ there is a formula $\psi^n_M(v_0, \ldots, v_{n-1})$ of $\mathcal{L}_{\omega_1 \omega}$ such that we have, for every $x \in \text{Mod}_{\mathcal{L}}$ and every injective $s \in \mathbb{N}^n$:

$x \models \psi^n_M(s_0, \ldots, s_{n-1})$ iff $s \models \bar{y} \models \bar{y}^{-1}. x \in M$.

(i) is clear: since $M$ is invariant, we have $g \cdot x \in M$ for all $x \in M$ and $g \in S_{\infty}$, on the other hand, if $g \cdot x \in M$ for at least one $g \in S_{\infty}$ then $x \in M$.

To prove (ii) we argue by induction on the Borel complexity of $M$. Suppose, for the sake of simplicity, that $\mathcal{L}$ contains a single binary predicate, say, $R(\cdot, \cdot)$; then $\text{Mod}_{\mathcal{L}} = \mathcal{P}(\mathbb{N})^2$. If $M = \{ x \in \mathbb{N}^2 : (k, l) \notin x \}$ for some $k, l \in \mathbb{N}$ then take

$$\forall u_0 \ldots \forall u_m \left( \bigwedge_{i \leq m} (u_i \neq u_j) \land \bigwedge_{i \leq m} (u_i = v_i) \implies -R(u_k, u_1) \right),$$

where $m = \max\{l, k, n\}$, as $\varphi_M^n(v_0, \ldots, v_{n-1})$. Further, take

$$\bigwedge_{k \leq n} \forall u_0 \ldots \forall u_{k-1} \bigvee_{m \geq k} \exists w_0 \ldots \exists w_{m-1} \left( \bigwedge_{i \leq k} (u_i \neq u_j) \land \bigwedge_{i \leq k} (u_i = v_i) \implies \bigwedge_{i \leq k} (w_i \neq w_j) \land \bigwedge_{i \leq k} (w_i = v_i) \land \varphi_M^n(w_0, \ldots, w_{m-1}) \right)$$

as $\varphi_M^n(v_0, \ldots, v_{n-1})$. Finally, if $M = \bigcap_j M_j$ then we take $\bigwedge_j \varphi_M^n_v(v_0, \ldots, v_{n-1})$ as $\varphi_M^n(v_0, \ldots, v_{n-1})$. □ (Theorem 49)
10.c ERs classifiable by countable structures

The classifiability by countable structures means that we can associate, in a Borel way, a countable \( \mathcal{L} \)-structure, say, \( \vartheta(x) \) with any point \( x \in X = \text{dom} \mathcal{E} \) so that \( x \mathcal{E} y \) iff \( \vartheta(x) \) and \( \vartheta(y) \) are isomorphic.

**Definition 50** (Hjorth [15, 2.38]). An ER \( \mathcal{E} \) is **classifiable by countable structures** if there is a countable relational language \( \mathcal{L} \) such that \( \mathcal{E} \leq_B \cong \mathcal{L} \).

**Remark 51.** Any \( \mathcal{E} \) classifiable by countable structures is \( \Sigma_1^1 \), of course, and many of them are Borel. The equivalence relations \( T_2, \ E_3, \) all countable Borel ERs (see the diagram on page 16) are classifiable by countable structures, but \( E_1, \ E_2, \) Tsirelson ERs are not.

**Theorem 52** (Becker and Kechris [1]). Any orbit ER of a Polish action of a closed subgroup of \( S_\infty \) is classifiable by countable structures.

Thus all orbit ERs of Polish actions of \( S_\infty \) and its closed subgroups are Borel reducible to a very special kind of actions of \( S_\infty \).

**Proof.** First show that any orbit ER of a Polish action of \( S_\infty \) itself is classifiable by countable structures. Hjorth’s simplified argument [15, 6.19] is as follows. Let \( X \) be a Polish \( S_\infty \)-space with basis \( \{ U_i \}_{i \in \mathbb{N}} \), and let \( \mathcal{L} \) be the language with relations \( R_{ik} \) where each \( R_{ik} \) has arity \( k \). If \( x \in X \) then define \( \vartheta(x) \in \text{Mod}_X \) by stipulation that \( \vartheta(x) \models R_{ik}(s_0, ..., s_{k-1}) \) iff 1) \( s_i \neq s_j \) whenever \( i < j < k \), and 2) \( \forall g \in B_s \langle g^{-1} \cdot x \in U_i \rangle \), where \( B_s = \{ g \in S_\infty : s \subset g \} \) and \( s = (s_0, ..., s_{k-1}) \in \mathbb{N}^k \). Then \( \vartheta \) reduces \( \mathcal{E} \) to \( \cong \).

To accomplish the proof of the theorem, it remains to apply the following result (an immediate corollary of Theorem 2.3.5b in [1]):

**Proposition 52.1.** If \( G \) is a closed subgroup of a Polish group \( H \) and \( X \) is a Polish \( G \)-space then there is a Polish \( H \)-space \( Y \) such that \( \mathcal{E}^G_X \leq_B \mathcal{E}^H_Y \).

**Proof.** Hjorth [15, 7.18] outlines a proof as follows. Let \( Y = X \times H \); define \( \langle x, h \rangle \approx \langle x', h' \rangle \) if \( x' = g \cdot x \) and \( h' = gh \) for some \( g \in G \), and consider the quotient space \( Y = Y/\approx \) with the topology induced by the Polish topology of \( Y \) via the surjection \( \langle x, h \rangle \mapsto [(x, h)]_\approx \), on which \( H \) acts by \( h' \cdot [(x, h)]_\approx = [(x, hh'^{-1})]_\approx \). Obviously \( \mathcal{E}^G_X \leq_B \mathcal{E}^H_Y \) via the map \( x \mapsto [(x, 1)]_\approx \), hence, it remains to prove that \( Y \) is a Polish \( H \)-space, which is not really elementary — we refer the reader to [15, 7.18] or [1, 2.3.5b].

To bypass 52.1 in the proof of Theorem 52, we can use a characterization of all closed subgroups of \( S_\infty \). Let \( \mathcal{L} \) be a language as above, and \( x \in \text{Mod}_\mathcal{L} \).

Define \( \text{Aut}_x = \{ g \in S_\infty : g \cdot x = x \} \) : the group of all automorphisms of \( x \).

**Proposition 52.2** (see [1, 1.5]). \( G \subseteq S_\infty \) is a closed subgroup of \( S_\infty \) iff there is an \( \mathcal{L} \)-structure \( x \in \text{Mod}_\mathcal{L} \) of a countable language \( \mathcal{L} \), such that \( G = \text{Aut}_x \).
Proof. For the nontrivial direction, let \( G \) be a closed subgroup of \( S_\infty \). For any \( n \geq 1 \), let \( I_n \) be the set of all \( G \)-orbits in \( \mathbb{N}^n \), i.e., equivalence classes of the ER \( s \sim t \) iff \( \exists g \in G \ (t = g \circ s) \), thus, \( I_n \) is an at most countable subset of \( \mathcal{P}(\mathbb{N}^n) \). Let \( I = \bigcup_n I_n \), and, for any \( i \in I_n \), let \( R_i \) be an \( n \)-ary relational symbol, and \( \mathcal{L} = \{ R_i \}_{i \in I} \). Let \( x \in \text{Mod}_{\mathcal{L}}' \) be defined as follows: if \( i \in I_n \) then \( x \models R_i(k_0, ..., k_{n-1}) \) iff \( \langle k_0, ..., k_{n-1} \rangle \in i \). Then \( G = \text{Aut}_x \), actually, if \( G \) is not necessarily closed subgroup then \( \text{Aut}_x = \overline{G} \). \( \square \) (Proposition)

Now come back to Theorem 52. The same argument as in the beginning of the proof shows that any orbit ER of a Polish action of \( G \), a closed subgroup of \( S_\infty \), is \( \leq_n \cong_{\mathcal{L}}^G \) for an appropriate countable language \( \mathcal{L} \). Yet, by 52.2, \( G = \text{Aut}_{y_0} \) where \( y_0 \in \text{Mod}_{\mathcal{L}}' \) and \( \mathcal{L}' \) is a countable language disjoint from \( \mathcal{L} \). The map \( x \mapsto \langle x, y_0 \rangle \) witnesses that \( \cong_{\mathcal{L}}^G \leq_n \cong_{\mathcal{L} \cup \mathcal{L}'} \). \( \square \) (Theorem 52)

10.d Reduction to countable graphs

It could be expected that the more complicated a language \( \mathcal{L} \) is accordingly the more complicated isomorphism equivalence relation \( \cong_{\mathcal{L}} \) it produces. However this is not the case. Let \( \mathcal{G} \) be the language of (oriented binary) graphs, i.e., \( \mathcal{G} \) contains a single binary predicate, say \( R(\cdot, \cdot) \).

Theorem 53. If \( \mathcal{L} \) is a countable relational language then \( \cong_{\mathcal{L}} \leq_n \cong_{\mathcal{G}} \). Therefore, an ER \( \mathcal{E} \) is classifiable by countable structures iff \( \mathcal{E} \leq_n \cong_{\mathcal{G}} \). In other words, a single binary relation can code structures of any countable language.

Becker and Kechris [1, 6.1.4] outline a proof based on coding in terms of lattices, unlike the following argument, yet it may in fact involve the same idea.

Proof. Let \( \text{HF}(\mathbb{N}) \) be the set of all hereditarily finite sets over the set \( \mathbb{N} \) considered as the set of atoms, and \( \varepsilon \) be the associated “membership” (any \( n \in \mathbb{N} \) has no \( \varepsilon \)-elements, \( \{0, 1\} \) is different from 2, etc.). Let \( \cong_{\text{HF}(\mathbb{N})} \) be the HF(\( \mathbb{N} \)) version of \( \cong_{\mathcal{G}} \), i.e., if \( P, Q \subseteq \text{HF}(\mathbb{N})^2 \) then \( P \cong_{\text{HF}(\mathbb{N})} Q \) means that there is a bijection \( b \) of \( \text{HF}(\mathbb{N}) \) such that \( Q = b \cdot P = \{(b(s), b(t)) : (s, t) \in P\} \). Obviously \( (\cong_{\mathcal{G}}) \sim_n (\cong_{\text{HF}(\mathbb{N})}) \), thus, we have to prove that \( \cong_{\mathcal{L}} \leq_n \cong_{\text{HF}(\mathbb{N})} \) for any \( \mathcal{L} \).

An action of \( S_\infty \) on \( \text{HF}(\mathbb{N}) \) is defined as follows. If \( g \in S_\infty \) then \( g \circ n = g(n) \) for any \( n \in \mathbb{N} \), and, by \( \varepsilon \)-induction, \( g \circ \{a_1, ..., a_n\} = \{g \circ a_1, ..., g \circ a_n\} \) for all \( a_1, ..., a_n \in \text{HF}(\mathbb{N}) \). Clearly the map \( a \mapsto g \circ a \ (a \in \text{HF}(\mathbb{N})) \) is an \( \varepsilon \)-isomorphism of \( \text{HF}(\mathbb{N}) \), for any fixed \( g \in S_\infty \).

Lemma 53.1. Suppose that \( X, Y \subseteq \text{HF}(\mathbb{N}) \) are \( \varepsilon \)-transitive subsets of \( \text{HF}(\mathbb{N}) \), the sets \( \mathbb{N} \setminus X \) and \( \mathbb{N} \setminus Y \) are infinite, and \( \varepsilon \restriction X \cong_{\text{HF}(\mathbb{N})} \varepsilon \restriction Y \). Then there is \( f \in S_\infty \) such that \( Y = f \circ X = \{f \circ s : s \in X\} \).
10 ACTIONS OF THE INFINITE SYMMETRIC GROUP

Proof. It follows from the assumption $\varepsilon \upharpoonright X \cong_{\text{HF}(\mathbb{N})} \varepsilon \upharpoonright Y$ that there is an $\varepsilon$-isomorphism $\pi : X \overset{\text{onto}}{\rightarrow} Y$. Easily $\pi \upharpoonright (X \cap \mathbb{N})$ is a bijection of $X_0 = X \cap \mathbb{N}$ onto $Y_0 = Y \cap \mathbb{N}$, hence, there is $f \in S_\infty$ such that $f \upharpoonright X_0 = \pi \upharpoonright X_0$, and then we have $f \circ s = \pi(s)$ for any $s \in X$. □ (Lemma)

Coming back to the proof of Theorem 53, we first show that $\cong_{\mathcal{G}(m)} \leq_{\text{B}} \cong_{\text{HF}(\mathbb{N})}$ for any $m \geq 3$, where $\mathcal{G}(m)$ is the language with a single $m$-ary predicate. Note that $(i_1, \ldots, i_m) \in \text{HF}(\mathbb{N})$ whenever $i_1, \ldots, i_m \in \mathbb{N}$.

Put $\Theta(x) = \{\vartheta(s) : s \in x\}$ for every element $x \in \text{Mod}_{\mathcal{G}(m)} = \mathcal{P}(\mathbb{N})^m$, where $\vartheta(s) = \text{TC}_\varepsilon(\{2i_1, \ldots, 2i_m\})$ for each $s = (i_1, \ldots, i_m) \in \mathbb{N}_m$, and finally, for $X \subseteq \text{HF}(\mathbb{N})$, $\text{TC}_\varepsilon(X)$ is the least $\varepsilon$-transitive set $T \subseteq \text{HF}(\mathbb{N})$ with $X \subseteq T$. It easily follows from Lemma 53 that $x \cong_{\mathcal{G}(m)} y$ iff $\varepsilon \upharpoonright \Theta(x) \cong_{\text{HF}(\mathbb{N})} \varepsilon \upharpoonright \Theta(y)$. This ends the proof of $\cong_{\mathcal{G}(m)} \leq_{\text{B}} \cong_{\text{HF}(\mathbb{N})}$.

It remains to show that $\cong_{\mathcal{L}'} \leq_{\text{B}} \cong_{\text{HF}(\mathbb{N})}$, where $\mathcal{L}'$ is the language with infinitely many binary predicates. In this case $\text{Mod}_{\mathcal{L}'} = \mathcal{P}(\mathbb{N}^2)^\mathbb{N}$, so that we can assume that every $x \in \text{Mod}_{\mathcal{L}'}$ has the form $x = \{x_n\}_{n \geq 1}$, with $x_n \subseteq (\mathbb{N} \setminus \{0\})^2$ for all $n$. Let $\Theta(x) = \{s_n(k, l) : n \geq 1 \land \langle k, l \rangle \in x_n\}$ for any such $x$, where $s_n(k, l) = \text{TC}_\varepsilon(\{\ldots, \langle k, l \rangle, \ldots\}, 0)$, with $n + 2$ pairs of brackets $\{} , \}$.

Then $\Theta$ is a continuous reduction of $\cong_{\mathcal{L}'}$ to $\cong_{\text{HF}(\mathbb{N})}$. □ (Theorem)

10.e Borel countably classified ERs: reduction to $T_\xi$

Equivalence relations $T_\xi$ of §3.c offer a perfect calibration tool for those Borel ERs which admit classification by countable structures. First of all,

Proposition 54. Every $T_\xi$ admits classification by countable structures.

Proof. $T_0$, the equality on $\mathbb{N}$, is the orbit ER of the action of $S_\infty$ by $g \cdot x = x$ for all $g, x$. The operation $(o2)$ of §3.c (countable disjoint union) easily preserves the property of being Borel reducible to an orbit ER of continuous action of $S_\infty$.

Now consider operation $(o5)$ of countable power. Suppose that a ER $\mathcal{E}$ on a Polish space $\mathcal{X}$ is Borel reducible to $\mathcal{F}$, the orbit relation of a continuous action of $S_\infty$ on some Polish $\mathcal{Y}$. Let $D$ be the set of all points $x = \{x_k\}_{k \in \mathbb{N}} \in \mathcal{X}^\mathbb{N}$ such that either $x_k \in \mathcal{E} x_l$ whenever $k \neq l$, or there is $m$ such that $x_k \in \mathcal{E} x_l$ iff $m$ divides $|k - l|$. Then $\mathcal{E}^\infty \leq_{\text{B}} (\mathcal{E}^\infty \upharpoonright D)$ (via a Borel map $\vartheta : \mathcal{X}^\mathbb{N} \rightarrow D$ such that $x \mathcal{E}^\infty \vartheta(x)$ for all $x$). On the other hand, obviously $(\mathcal{E}^\infty \upharpoonright D) \leq_{\text{B}} \mathcal{F}'$, where, for $y, y' \in \mathcal{Y}^\mathbb{N}$, $y \mathcal{F}' y'$ means that there is $f \in S_\infty$ such that $y_k \mathcal{F} y'_{f(k)}$ for all $k$. Finally, $\mathcal{F}'$ is the orbit ER of a continuous action of $S_\infty \times S_\infty^\mathbb{N}$, which can be realized as a closed subgroup of $S_\infty$, so it remains to apply Theorem 52.1. □

The relations $T_\alpha$ are known in different versions, which reflect the same idea of coding sets of $\alpha$-th cumulative level over $\mathbb{N}$, as, e.g., in [18, §1], where results similar to Proposition 54 are obtained in much more precise form.
**Theorem 55.** If $E$ is a Borel ER classifiable by countable structures then $E \leq_B T_\xi$ for some $\xi < \omega_1$.

**Proof.** The proof (a version of the proof in [9]) is based on Scott’s analysis. Define, by induction on $\alpha < \omega_1$, a family of Borel ERs $\equiv^\alpha$ on $\mathbb{N}^{<\omega} \times \mathcal{P}(\mathbb{N}^2)$:

* $A \equiv^\alpha_{st} B$ means $\langle s, A \rangle \equiv^\alpha \langle t, B \rangle$;

thus, all $\equiv^\alpha_{st}$ ($s, t \in \mathbb{N}^{<\omega}$) are binary relations on $\mathcal{P}(\mathbb{N}^2)$, and among them all relations $\equiv^\alpha_{ss}$ are ERs;

- $A \equiv^0_{st} B$ iff $A(s_i, s_j) \iff B(t_i, t_j)$ for all $i, j < \text{lh } s = \text{lh } t$;
- $A \equiv^{\alpha+1}_{st} B$ iff $\forall k \exists l (A \equiv^\alpha_{s_k, t^l} B)$ and $\forall l \exists k (A \equiv^\alpha_{s^k, t^l} B)$;
- if $\lambda < \omega_1$ is limit then: $A \equiv^\lambda_{st} B$ iff $A \equiv^\alpha_{st} B$ for all $\alpha < \lambda$.

Easily $\equiv^\beta \subseteq \equiv^\alpha$ whenever $\alpha < \beta$.

Recall that, for $A, B \subseteq \mathbb{N}^2$, $A \equiv^\alpha_{st} B$ means that there is $f \in S_\infty$ with $A(k, l) \iff B(f(k), f(l))$ for all $k, l$. Then we have $\equiv^\alpha_{st} \subseteq \bigcap_{\alpha < \omega_1} \equiv^\alpha_{\Lambda \alpha}$ by induction on $\alpha$ (in fact $= \subset$ rather than $\subseteq$, see below), where $\Lambda$ is the empty sequence. Call a set $P \subseteq \mathcal{P}(\mathbb{N}^2) \times \mathcal{P}(\mathbb{N}^2)$ unbounded if $P \cap \equiv^\alpha_{\Lambda \alpha} \neq \emptyset$ for all $\alpha < \omega_1$.

**Lemma 55.1.** Any unbounded $\Sigma^1_1$ set $P$ contains $\langle A, B \rangle \in P$ with $A \equiv^\alpha_{st} B$.

It follows that $A \equiv^\alpha_{st} B$ iff $A \equiv^\alpha_{\Lambda \alpha} B$ for all $\alpha < \omega_1$ (take $P = \{\langle A, B \rangle\}$).

**Proof.** Since $P$ is $\Sigma^1_1$, there is a continuous map $F : \mathbb{N}^\mathbb{N} \twoheadrightarrow P$. For $u \in \mathbb{N}^{<\omega}$, let $P_u = \{F(a) : u \subseteq a \in \mathbb{N}^\mathbb{N}\}$. There is a number $n_0$ such that $P_{\langle n_0 \rangle}$ is still unbounded. Let $k_0 = 0$. By a simple cofinality argument, there is $l_0$ such that $P_{\langle n_0 \rangle}$ is still unbounded over $\langle k_0, \langle l_0 \rangle \rangle$ in the sense that there is no ordinal $\alpha < \omega_1$ such that $P_{\langle n_0 \rangle} \cap \equiv^\alpha_{\langle k_0, \langle l_0 \rangle \rangle} = \emptyset$. Following this idea, we can define infinite sequences of numbers $n_k, k_k, l_k$ such that both $\{m^k\}_{m \in \mathbb{N}}$ and $\{l^k\}_{m \in \mathbb{N}}$ are permutations of $\mathbb{N}$ and, for any $m$, the set $P_{\langle n_0, \ldots, n_m \rangle}$ is still unbounded over $\langle k_0, \ldots, k_m \rangle, \langle l_0, \ldots, l_m \rangle$ in the same sense. Note that $a = \{n^k\}_{m \in \mathbb{N}} \in \mathbb{N}$ and $F(a) = \langle A, B \rangle \in P$ (both $A, B$ are subsets of $\mathbb{N}^2$).

Prove that the map $f(k^m) = l^m$ witnesses $A \equiv^\alpha_{st} B$, i.e., $A(k^j, k^i) \iff B(l^j, l^i)$ for all $j, i$. Take $m > \max\{j, i\}$ big enough for the following: if $\langle A', B' \rangle \in P_{\langle n_0, \ldots, n_m \rangle}$ then $A'_{k^j, k^i}$ iff $B'(l^j, l^i)$, and similarly $B(l^j, l^i) \iff B'(l^j, l^i)$. By the construction, there is a pair $\langle A', B' \rangle \in P_{\langle n_0, \ldots, n_m \rangle}$ with $A' \equiv^0_{\langle k_0, \ldots, k_m, l_0, \ldots, l_m \rangle} B'$, in particular, $A'_{k^j, k^i}$ iff $B'(l^j, l^i)$, as required. $\Box$ (Lemma)

**Corollary 55.2** (See, e.g., Friedman [9]). If $E$ is a Borel ER and $E \leq_B \equiv_{st}$ then $E \leq_B \equiv^\alpha_{\Lambda \alpha}$ for some $\alpha < \omega_1$. 
11 TURBULENT GROUP ACTIONS

Proof. Let \( \vartheta \) be a Borel reduction of \( E \) to \( \approx_g \). Then \( \{(\vartheta(x), \vartheta(y)) : x \not\approx_g y\} \) is a \( \Sigma_1^1 \) subset of \( \mathcal{P}(\mathbb{N}^2) \times \mathcal{P}(\mathbb{N}^2) \) which does not intersect \( \approx_g \), hence, it is bounded by Lemma 55.1. Take an ordinal \( \alpha < \omega \) which witnesses the boundedness. \( \square \)

Now, if \( E \) is a Borel ER classifiable by countable structures then \( E \leq_B \approx_g \) by Theorem 53, hence, it remains to establish the following:

Proposition 55.3. Any ER \( \equiv^\alpha \) is Borel reducible to some \( T_\xi \).

Proof. We have \( \equiv^0 \leq_B T_0 \) since \( \equiv^0 \) has countably many equivalence classes, all of which are clopen sets. To carry out the step \( \alpha \mapsto \alpha + 1 \) note that the map \( \langle s, A \rangle \mapsto \{(s^k, A)\}_{k \in \mathbb{N}} \) is a Borel reduction of \( \equiv^{\alpha+1} \) to \( (\equiv^\alpha)^\infty \). To carry out the limit step, let \( \lambda = \{\alpha_n : n \in \mathbb{N}\} \) be a limit ordinal, and \( R = \bigvee_{n \in \mathbb{N}} \equiv^{\alpha_n} \), i.e., \( R \) is a ER on \( \mathbb{N} \times \mathbb{N}^{<\omega} \times \mathcal{P}(\mathbb{N}^2) \) defined so that \( \langle m, s, A \rangle \in R \langle n, t, B \rangle \) iff \( m = n \) and \( A \equiv_{s,m}^t B \). However the map \( \langle s, A \rangle \mapsto \{\langle m, s, A \rangle\}_{m \in \mathbb{N}} \) is a Borel reduction of \( \equiv^\lambda \) to \( R^\infty \). \( \square \) (Proposition)

\( \square \) (Theorem 55)

11 Turbulent group actions

This is an entirely different class of orbit ERs, disjoint with those which admit classification by countable structures.

11a Local orbits and turbulence

Suppose that a group \( G \) acts on a space \( \mathcal{X} \). If \( G \subseteq G \) and \( X \subseteq \mathcal{X} \) then let

\[ R^X_G = \{(x, y) \in X^2 : \exists g \in G (x = g \cdot y)\} \]

and let \( \sim^X_G \) denote the ER-hull of \( R^X_G \), i.e., the \( \subseteq \)-least ER on \( X \) such that \( x R^X_G y \implies x \sim^X_G y \). In particular \( \sim^X_G = E^X_G \), but generally we have \( \sim^X_G \not\subseteq E^X_G | X \).

Finally, define \( \mathcal{O}(x, X, G) = [x]_{E^X_G} \) for \( x \in X \) - the local orbit of \( x \). In particular, \( [x]_G = [x]_{E^X_G} = \mathcal{O}(x, X, G) \), the full \( G \)-orbit of \( x \in X \).

Definition 56 (This particular version taken from Kechris [28, § 8]). Suppose that \( \mathcal{X} \) is a Polish space and \( G \) is a Polish group acting on \( \mathcal{X} \) continuously.

(t1) A point \( x \in \mathcal{X} \) is turbulent if for any open non-empty set \( X \subseteq \mathcal{X} \) containing \( x \) and any nbhd \( G \subseteq G \) (not necessarily a subgroup) of \( 1_G \), the local orbit \( \mathcal{O}(x, X, G) \) is somewhere dense (i.e., not a nowhere dense set) in \( \mathcal{X} \).

(t2) An orbit \( [x]_G \) is turbulent if \( x \) is such (then all \( y \in [x]_G \) are turbulent).
The action (of $G$ on $\mathcal{X}$) is generically\(^{22}\), or gen. turbulent and $\mathcal{X}$ is a gen. turbulent Polish $G$-space, if the union of all dense, turbulent, and meager orbits $[x]_G$ is comeager.  \hfill $\square$

Our proof of the following theorem, based on ideas in [15, § 3.2], [28, § 12], [9], is designed so that only quite common tools of descriptive set theory are involved. It will also be shown that “turbulent” ERs are not reducible actually to a much bigger family of ERs than orbit ERs of Polish actions of $S_\infty$.

**Theorem 57** (Hjorth [15]). Suppose that $G$ is a Polish group, $\mathcal{X}$ is a gen. turbulent Polish $G$-space. Then $E^X_G$ is not BM reducible\(^ {23}\) to a Polish action of $S_\infty$, hence, not classifiable by countable structures.

We begin the proof with two rather simple technical results.

**Lemma 57.1.** In the assumptions of the theorem, suppose that $\emptyset \neq X \subseteq \mathcal{X}$ is an open set, $G \subseteq \mathcal{G}$ is a nbhd of $1_G$, and $\mathcal{O}(x,X,G)$ is dense in $X$ for $X$-comeager many $x \in X$. Let $U, U' \subseteq X$ be non-empty open and $D \subseteq X$ comeager in $X$. Then there exist points $x \in D \cap U$ and $x' \in D \cap U'$ with $x \sim_G X x'$.

**Proof.** Under our assumptions there exist points $x_0 \in U$ and $x'_0 \in U'$ with $x_0 \sim_{G^X} x'_0$, i.e., there are elements $g_1, \ldots, g_n \in G \cup G^{-1}$ such that $x'_0 = g_n \cdot g_{n-1} \cdot \ldots \cdot g_1 \cdot x_0$ and in addition $g_k \cdot \ldots \cdot g_1 \cdot x_0 \in X$ for all $k \leq n$. Since the action is continuous, there is a nbhd $U_0 \subseteq U$ of $x_0$ such that $g_k \cdot \ldots \cdot g_1 \cdot x \in X$ for all $k$ and $g_n \cdot g_{n-1} \cdot \ldots \cdot g_1 \cdot x \in U'$ for all $x \in U_0$. Since $D$ is comeager, easily there is $x \in U_0 \cap D$ such that $x' = g_n \cdot g_{n-1} \cdot \ldots \cdot g_1 \cdot x \in U' \cap D$.  \hfill $\square$ (Lemma)

**Lemma 57.2.** In the assumptions of the theorem, for any open non-empty $U \subseteq \mathcal{X}$ and $G \subseteq \mathcal{G}$ with $1_G \in G$ there is an open non-empty $U' \subseteq U$ such that the local orbit $\mathcal{O}(x,U',G)$ is dense in $U'$ for $U'$-comeager many $x \in U'$.

**Proof.** Let $\text{INT} \mathcal{X}$ be the interior of the closure of $X$. If $x \in U$ and $\mathcal{O}(x,U,G)$ is somewhere dense (in $U$) then the set $U_x = U \cap \text{INT} \mathcal{O}(x,U,G) \subseteq U$ is open and $\sim_{G^X}$-invariant (an observation made, e.g., in [28, proof of 8.4]), moreover, $\mathcal{O}(x,U,G) \subseteq U_x$, hence, $\mathcal{O}(x,U,G) = \mathcal{O}(x,U_x,G)$. It follows from the invariance that the sets $U_x$ are pairwise disjoint, and it follows from the turbulence that the union of them is dense in $U$. Take any non-empty $U_x$ as $U'$.  \hfill $\square$ (Lemma)

11.b  Ergodicity

The non-reducibility in Theorem 57 will be established in a special stronger form. Let $E, F$ be ERs on Polish spaces resp. $\mathcal{X}, \mathcal{Y}$. A map $\vartheta : \mathcal{X} \to \mathcal{Y}$ is

---

\(^{22}\) In this research direction, “generically”, or, in our abbreviation, “gen.” (property) intends to mean that (property) holds on a comeager domain.

\(^{23}\) Reducible via a Baire measurable function. This is weaker than Borel reducibility, of course.
11 TURBULENT GROUP ACTIONS

- **E, F-invariant** if \( x \in E \iff \vartheta(x) \in F \) for all \( x, y \in X \);
- **gen. E, F-invariant** if \( x \in E \iff \vartheta(x) \in F \) holds for all \( x, y \) in a comeager subset of \( X \);
- **gen. F-constant** if \( \vartheta(x) \in F \) for all \( x, y \) in a comeager subset of \( X \).

Finally, following Hjorth and Kechris, say that \( E \) is **gen. F-ergodic** if every BM \( E, F \)-invariant map is gen. F-constant.

**Proposition 57.2.** \( E \) is gen. F-ergodic if and only if every Borel gen. \( E, F \)-invariant map is gen. F-constant.

**Proof.** Let \( E, F \) live in resp. \( X, \mathcal{Y} \). Suppose that \( \vartheta : X \to \mathcal{Y} \) is a Borel gen. \( E, F \)-invariant map. There is a Borel comeager set \( D \subseteq X \) on which \( \vartheta \) is \( E, F \)-invariant. Then we can extend \( \vartheta \mid D \) to a BM map \( \vartheta' : X \to \mathcal{Y} \) which is still (everywhere) \( E, F \)-invariant. This proves implication \( \implies \) of the lemma. To prove the opposite implication, let \( \vartheta : X \to \mathcal{Y} \) be a BM \( E, F \)-invariant map. Then \( \vartheta \mid D \) is Borel for a suitable comeager Borel set \( D \subseteq X \). Let \( \vartheta' \) be any Borel extension of \( \vartheta \mid D \) to the whole \( X \).

**Proposition 57.3.** Suppose that \( E \) is gen. F-ergodic and does not have a comeager equivalence class. Then \( E \) is not Borel reducible to \( F \).

This is exactly how the non-reducibility is often established. 24 Our proof of Theorem 57 is of this type. It consists of two parts 25:

**Lemma 57.4.** If \( G \) is a Polish group, \( X \) a Polish \( G \)-space, and \( E^X_G \) is BM reducible to a Polish action of \( S_\infty \), then there is a comeager \( G \)-set \( D \subseteq X \) such that \( E^X_G \mid D \) is Borel reducible to one of ERs \( T_\xi \).

In other words, any ER, BM reducible to a Polish action of \( S_\infty \), is “generically” Borel reducible to one of \( T_\xi \). Note that any ER Borel reducible, in proper sense, to one of \( T_\xi \), is Borel.

**Lemma 57.5.** Any ER induced by a gen. turbulent Polish action is gen. \( T_\xi \)-ergodic for every \( \xi \).

\[ \square \] (Theorem 57 modulo 57.4 and 57.5)

24 Yet there are cases when \( E \) is neither F-ergodic nor Borel reducible to \( F \), for instance, among the ERs of the form \( \ell^p \).

25 There are slightly different ways to the same goal. Hjorth [15, 3.18] proves outright and with different technique that any gen. turbulent ER is gen. ergodic w.r.t. any Polish action of \( S_\infty \). Kechris [28, § 12] proves that 1) any gen. \( T_2 \)-ergodic ER is gen. ergodic w.r.t. any Polish action of \( S_\infty \), and 2) any turbulent ER is gen. \( T_2 \)-ergodic.
11. c “Generical” reduction of countably classified ERs to $T_\Xi$

Here, we prove Lemma 57.4. Suppose that $G$ is a Polish group, $K$ a Polish $G$-space, and the orbit ER $E = E_G^K$ is BM reducible to a Polish action of $S_\infty$. Then, according to Theorems 52 and 53, there is a BM reduction $\rho : K \to \mathcal{P}(\mathbb{N}^2)$ of $E$ to $\cong_G$, the isomorphism of binary relations on $\mathbb{N}$. The remainder of the argument borrows notation from the proof of Theorem 55.

There is a dense $G_\delta$ set $D_0 \subseteq K$ such that $\vartheta = \rho \upharpoonright D_0$ is continuous on $D_0$. By definition, we have $x \in \mathbb{E} y \implies \vartheta(x) \cong_G \vartheta(y)$ and $x \not\in \mathbb{E} y \implies \vartheta(x) \not\cong_G \vartheta(y)$ for all $x, y \in D_0$. We are mostly interested in the second implication, and the aim is to find a $G_\delta$ dense set $D \subseteq D_0$ such that, for some $\alpha < \omega_1$, we have

\[ (*) \text{ implication } x \in \mathbb{E} y \implies \vartheta(x) \not\cong_\alpha \vartheta(y) \text{ holds for all } x, y \in D. \]

(Recall that $A \not\cong_G B$ iff $\exists \alpha < \omega_1 A \not\cong_\alpha B$, see a remark after Lemma 55.1.)

To find such an $\alpha$ we apply a Cohen forcing argument. Let us fix a countable transitive model $M$ of ZFHC, i.e., ZFC minus the Power Set axiom but plus the axiom: “every set belongs to HC = \{x : x is hereditarily countable\}”.

We shall assume that $K$ is coded in $M$ in the sense that there is a set $D_K \in M$ which is a dense (countable) subset of $K$, and $d_K \upharpoonright D_K$ (the distance function of $K$ restricted to $D_K$) also belongs to $M$. Further, $G$, the action, $D_0$, $\vartheta$ are also assumed to be coded in $M$ in a similar sense. In this assumption, in particular, the notion of a Cohen generic, over $M$, point of $K$, or of $G$, makes sense, in particular, the set $D$ of all Cohen generic, over $M$, points of $K$ is a dense $G_\delta$ subset of $K$ and $D \subseteq D_0$. We are going to prove that $D$ fulfills $(*)$.

Suppose that $x, y \in D$, and $\langle x, y \rangle$ is a Cohen generic, pair over $M$. If $x \in \mathbb{E}^K y$ is false then we have $\vartheta(x) \not\equiv_G \vartheta(y)$, moreover, this fact holds in $M[x, y]$ by the Mostowski absoluteness, hence, arguing in $M[x, y]$ (which is still a model of ZFHC) we find an ordinal $\alpha \in \text{Ord}^{M[x, y]} = \text{Ord}^{M[x, y]}$ with $\vartheta(x) \not\equiv_\alpha \vartheta(y)$. Moreover, since the Cohen forcing satisfies ccc, there is an ordinal $\alpha \in M$ such that we have $\vartheta(x) \not\equiv_\alpha \vartheta(y)$ for every Cohen generic, over $M$, pair $\langle x, y \rangle \in D^2$ such that $x \in \mathbb{E}^K y$ is false. It remains to show that this also holds when $x, y \in D$ (are generic separately, but) do not form a pair, Cohen generic over $M$.

Let $g \in G$ be Cohen generic over $M[x, y]$. 26 Then $x' = g \cdot x$ is easily Cohen generic over $M[x, y]$ (because the action is continuous), furthermore, $x' \in \mathbb{E}^K x$, hence, $x' \in \mathbb{E}^K y$ fails. Yet $y$ is generic over $M$ and $x'$ is generic over $M[y]$, thus, $\langle x', y \rangle$ is Cohen generic over $M$, hence, we have $\vartheta(x') \not\equiv_\alpha \vartheta(y)$ by the choice of $\alpha$. On the other hand, $\vartheta(x) \equiv_\alpha \vartheta(x')$ holds because $x' \in \mathbb{E}^K x$, thus, we finally obtain $\vartheta(x') \not\equiv_\alpha \vartheta(y)$, as required.

\[ \Box \text{ (Lemma 57.4)} \]

26 In this case, we cannot, generally speaking, define $M[x, y]$ as a generic extension of $M$, hence, let $M[x, y]$ be any (countable transitive) model of ZFHC containing $x, y$, and all sets in $M$. It is not really harmful here that $M[x, y]$ can contain more ordinals than $M$. 

11. Ergodicity of turbulent actions w.r.t. \( T_\xi \)

Here, we prove Lemma 57.5. The proof involves a somewhat stronger property than gen. ergodicity in \( \S 11.b \). Suppose that \( F \) is an ER on a Polish space \( X \).

- An action of \( G \) on \( X \) and the induced equivalence relation \( E^X_G \) are hereditarily generically (h. gen., for brevity) \( F \)-ergodic if \( E^X_G \) is generically \( F \)-ergodic whenever \( X \subseteq X \) is a non-empty open set, \( G \subseteq G \) is a non-empty open set containing \( 1_G \), and the local orbit \( \mathcal{O}(x, X, G) \) is dense in \( X \) for comeager (in \( X \)) many \( x \in X \).

This obviously implies gen. \( F \)-ergodicity of \( E^X_G \) provided the action is gen. turbulent. Therefore, Lemma 57.5 is a corollary of the following theorem:

**Theorem 58.** Let \( X \) be a gen. turbulent Polish \( G \)-space. Suppose that an ER \( F \) belongs to \( \mathcal{F}_0 \), the least collection of ERs containing \( D(N) \) (the equality on \( N \)) and closed under the operations \((o1)-(o5)\) of \( \S 3.c \). Then \( E^X_G \) is h. gen. \( F \)-ergodic, in particular, is not Borel reducible to \( F \).

**Remark 58.1.** Due to the other creative operation, the Fubini product, \( \mathcal{F}_0 \) contains a lot of ERs very different from \( T_\xi \), among them some Borel ERs which do not admit classification by countable structures, e.g., all ERs of the form \( E_{\mathcal{I}} \), where \( \mathcal{I} \) is one of Fréchet ideals, indecomposable ideals, or Weiss ideals of \( \S 2.F \). (In fact it is not so easy to show that ideals of the two last families produce ERs in \( \mathcal{F}_0 \).) In particular, it follows that no gen. turbulent ER is Borel reducible to a Fréchet, or indecomposable, or Weiss ideal. \( \square \)

Our proof of Theorem 58 goes on by induction on the number of applications of the basic operations, in several following subsections.

Right now, we begin with the initial step: prove that, under the assumptions of the theorem, \( E^X_G \) is h. gen. \( D(N) \)-ergodic. Suppose that \( X \subseteq X \) and \( G \subseteq G \) are non-empty open sets, \( 1_G \in G \), and \( \mathcal{O}(x, X, G) \) is dense in \( X \) for \( X \)-comeager many \( x \in X \), and prove that \( \sim^X_G \) is generically \( D(N) \)-ergodic.

Consider, accordingly with Proposition 57.2, a Borel gen. \( \sim^X_G \), \( D(N) \)-invariant map \( \vartheta : X \to N \). Suppose, on the contrary, that \( \vartheta \) is not gen. \( D(N) \)-constant. Then there exist two open non-empty sets \( U_1, U_2 \subseteq X \), two numbers \( \ell_1 \neq \ell_2 \), and a comeager set \( D \subseteq X \) such that \( \vartheta(x) = \ell_1 \) for all \( x \in D \cap U_1 \), \( \vartheta(x) = \ell_2 \) for all \( x \in D \cap U_2 \), and \( \vartheta | D \) is “strictly” \( \sim^X_G \), \( D(N) \)-invariant. Lemma 57.1 yields a pair of points \( x_1 \in U_1 \cap D \) and \( x_2 \in U_2 \cap D \) with \( x_1 \sim^X_G x_2 \), contradiction.

11.e Inductive step of countable power

To carry out this step in the proof of Theorem 58, suppose that

- \( X \) is a gen. turbulent Polish \( G \)-space, \( F \) is a Borel ER on a Polish space \( Y \), and the action of \( G \) on \( X \) is h. gen. \( F \)-ergodic,
and prove that the action is h-gen. $F^\infty$-ergodic. Fix a nonempty open set $X_0 \subseteq \mathcal{X}$ and a nbhd $G_0$ of 1$_G$ in $\mathcal{G}$, such that $\partial(x, X_0, G_0)$ is dense in $X_0$ for $X_0$-comeager many $x \in X_0$. Consider, accordingly to Proposition 57.2, a Borel function $\vartheta : X_0 \to Y^N$, $\sim_{G_0}^X$, $F^\infty$-invariant on a dense $G_0$ set $D_0 \subseteq X_0$, so that

$$x \sim_{G_0}^X x' \implies \forall k \exists l \left( \vartheta_k(x) \neq \vartheta_l(x') \right) : \text{for all } x, x' \in D_0,$$

where $\vartheta_k(x) = \vartheta(x)(k)$, $\vartheta_k : X_0 \to Y$, and prove that $\vartheta$ is gen. $F^\infty$-constant.

Below, let $C_X$ be the Cohen forcing for $\mathcal{X}$, which consists of rational balls with centers in a fixed dense countable subset of $\mathcal{X}$, and let $C_G$ be the Cohen forcing for $\mathcal{G}$ defined similarly (the dense subset is assumed to be a subgroup). Smaller sets are stronger conditions. Let us fix a countable transitive model $\mathcal{M}$ of $\text{ZFHC}$ (see above), which contains all relevant objects or their codes, in particular, codes of the topologies of $\mathcal{X}$, $\mathcal{G}$, $\mathcal{Y}$ and the Borel map $\vartheta$.

**Claim 58.2.** Suppose that $(x, g) \in \mathcal{X} \times \mathcal{G}$ is $C_X \times C_G$-generic over $\mathcal{M}$. Then $g \cdot x$ is $C_x$-generic over $\mathcal{M}$. (Because the action is continuous.)

Coming back to the theorem, fix $k \in \mathbb{N}$. Consider an open non-empty $U' \subseteq U_0$. By the invariance of $\vartheta$ and Claim 58.2 there are conditions $U' \subseteq C_X$, $U' \subseteq U$, and $Q \subseteq C_G$, $Q \subseteq G_0$, and a number $l$, such that $\vartheta_k(x) \neq \vartheta_l(g \cdot x)$ holds for any $C_X \times C_G$-generic over $\mathcal{M}$ pair $(x, g) \in U' \times Q$. As $Q$ is open, there is $g_0 \in Q \cap \mathcal{M}$ and a nbhd $G \subseteq G_0$ of 1$_G$ such that $g_0 G \subseteq Q$.

**Claim 58.3** (The key point of the turbulence). If $x, x' \in U'$ are $C_X$-generic over $\mathcal{M}$ and $x \sim_{G} U' x'$ then we have $\vartheta_k(x) \neq \vartheta_k(x')$.

**Proof.** We argue by induction on $n(x, x') = \{ \text{the least number } n \text{ such that there exist } g_1, \ldots, g_n \in G, \text{ satisfying} \}

$$x' = g_n \cdot g_{n-1} \cdots \cdot g_1 \cdot x, \text{ and } g_k \cdots g_1 \cdot x \in U' \text{ for all } k \leq n.$$

Suppose that $n(x, x') = 1$, thus, $x = h \cdot x'$ for some $h \in G$. Take any $C_G$-generic, over $\mathcal{M}[x, x']$ (see Footnote 26) element $g \in Q \cup Q^{-1}$, close enough to $g_0$ for $g' = gh^{-1}$ to belong to $Q$. Then $g$ is $C_G$-generic over $\mathcal{M}[x]$, hence, $(x, g)$ is $C_X \times C_G$-generic over $\mathcal{M}$ by the product forcing theorem. Therefore $\vartheta_k(x) \neq \vartheta_l(g \cdot x)$. Moreover, $g'$ also is $C_G$-generic over $\mathcal{M}[x']$, so that $\vartheta_k(x') \neq \vartheta_l(g' \cdot x')$ by the same argument. Yet we have $g' \cdot x' = gh^{-1} \cdot (h \cdot x) = g \cdot x$.

As for the inductive step, suppose that (*) holds for some $n \geq 2$. Take a $C_G$-generic, over $\mathcal{M}[x]$, element $g_1' \in G$ close enough to $g_1$ for $g_2 = g_2 g_1 g_1^{-1}$ to belong to $G$ and for $x^* = g_1' \cdot x$ to belong to $U'$. Note that $x^*$ is $C_X$-generic over $\mathcal{M}$ (product forcing) and $n(x^*, x') \leq n - 1$ because $g_2' \cdot x^* = g_2 \cdot g_1 \cdot x^*$. □ (Claim)

To summarize, we have shown that for any $k$ and any open $\emptyset \neq U \subseteq U_0$ there exist: an open set $\emptyset \neq U' \subseteq U$, and an open $G \subseteq G_0$ with 1$G \subseteq G$, such
that $\vartheta_k(x)$ is gen. $\sim_{U'}$, F-invariant on $U'$. We can also assume that the orbit $\mathcal{O}(x, U', G)$ is dense in $U'$ for $U'$-comeager many $x \in U'$, by Lemma 57.2. Then, by the h. gen. F-ergodicity, $\vartheta_k$ is gen. F-constant on $U'$, that is, there is a dense $G_\delta$ set $D' \subseteq U'$ and $y' \in \mathbb{Y}$ such that $\vartheta_k(x) F y'$ for all $x \in D'$.

We conclude that there exist: an $U_0$-comeager set $D \subseteq U_0$, and a countable set $Y = \{ y_j : j \in \mathbb{N} \} \subseteq \mathbb{Y}$ such that, for any $k$ and for any $x \in D$ there is $j$ with $\vartheta_k(x) F y_j$. Let $\eta(x) = \bigcup_{k \in \mathbb{N}} \{ j : \vartheta_k(x) F y_j \}$. Then, for any pair $x, x' \in D$, $\vartheta(x) F^\infty \vartheta(x')$ iff $\eta(x) = \eta(x')$, so that, by the invariance of $\vartheta$, we have:

$$x \sim_{U_0} x' \implies \eta(x) = \eta(x') : \text{ for all } x, x' \in D.$$ 

It remains to show that $\eta$ is a constant on a comeager subset of $D$.

Suppose, on the contrary, that there exist two non-empty open sets $U_1, U_2 \subseteq U_0$, a number $j \in \mathbb{N}$, and a comeager set $D' \subseteq D$ such that $j \in \eta(x_1)$ and $j \not\in \eta(x_2)$ for all $x_1 \in D' \cap U_1$ and $x_2 \in D' \cap U_2$. Now Lemma 57.1 yields a contradiction to (*), as in the end of §11.d.

$\square$ (Inductive step of countable power in Theorem 58)

### 11.f Inductive step of the Fubini product

To carry out this step in the proof of Theorem 58, suppose that

- $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space, for any $k$, $F_k$ be a Borel ER on a Polish space $\mathbb{Y}_k$, the action of $G$ on $\mathbb{X}$ is h. gen. $F_k$-ergodic for any $k$, and $F = \prod_k F_k / \text{Fin}$ is, accordingly, a Borel ER on $\mathbb{Y} = \prod_k \mathbb{Y}_k$,

and prove that the action is h. gen. F-ergodic.

Fix a nonempty open set $U_0 \subseteq \mathbb{X}$ and a nbhd $G_0$ of $1_G$ in $G$, such that $U_0$-comeager many orbits $\mathcal{O}(x, U_0, G_0)$ with $x \in U_0$ are dense in $U_0$. Consider a Borel function $\vartheta : U_0 \to \mathbb{Y}$, $\sim_{U_0} F_k$-F-invariant on a dense $G_\delta$ set $D_0 \subseteq U_0$, i.e.,

$$x \sim_{U_0} y \implies \exists k_0 \forall k \geq k_0 \left( \vartheta_k(x) F_k \vartheta_k(y) \right) : \text{ for all } x, y \in D_0,$$

where $\vartheta_k(x) = \vartheta(x)(k)$, and prove that $\vartheta$ is gen. $F$-constant.

Consider an open non-empty set $U \subseteq U_0$. By the invariance of $\vartheta$ and Claim 58.2 there are conditions $U' \subseteq \mathbb{C}_\mathbb{X}$, $U' \subseteq U$, and $Q \subseteq \mathbb{C}_G$, $Q \subseteq G_0$, and a number $k_0$, such that $\vartheta_k(x) F_k \vartheta_k(g \cdot x)$ holds for all $k \geq k_0$ and for any $C_\mathbb{X} \times C_\mathbb{G}$-generic over $\mathcal{M}$ pair $(x, g)$ of $x \in U'$ and $g \in Q$. As $Q$ is open, there is $g_0 \in Q \cap \mathcal{M}$ and a symmetric nbhd $G \subseteq G_0$ of $1_G$ such that $g_0 G \subseteq Q$.

**Claim 58.2.** If $k \geq k_0$ and points $x, y \in U'$ are $C_\mathbb{X}$-generic over $\mathcal{M}$ and $x \sim_{U'} y$ then $\vartheta_k(x) F_k \vartheta_k(y)$. (Similarly to Claim 58.3) $\square$
11 TURBULENT GROUP ACTIONS

Thus, for any open non-empty \( U \subseteq U_0 \) there exist: a number \( k_0 \), an open non-empty \( U' \subseteq U \), and a nbhd \( G \subseteq G_0 \) of \( 1_G \), such that \( \vartheta_k(x) \) is gen. \( \sim_{U'}^G, F_k \)-invariant on \( U' \) for all \( k \geq k_0 \). We can assume that \( U' \)-comeager many orbits \( \vartheta(x, U', G) \) are dense in \( U' \), by Lemma 57.2. Now, by the h. gen. \( F_k \)-ergodicity, any \( \vartheta_k \) with \( k \geq k_0 \) is gen. \( F_k \)-constant on such a set \( U' \), hence, \( \vartheta \) itself is gen. \( F \)-constant on \( U' \) since \( F = \prod_k F_k / \operatorname{Fin} \). It remains to show that these constants are \( F \)-equivalent to each other.

Suppose, on the contrary, that there exist two non-empty open sets \( U_1, U_2 \subseteq U_0 \) and a pair of \( y F y' \) in \( \mathcal{Y} \) such that \( \vartheta(x) F y \) and \( \vartheta(x') F y' \) for comeager many \( x \in U_1 \) and \( x' \in U_2 \). Contradiction follows as in the end of §11.e.

\( \square \) (Inductive step of Fubini product in Theorem 58)

11.g Other inductive steps

Here, we accomplish the proof of Theorem 58, by carrying out induction steps, related to operations (o1), (o2), (o3) of §3.c.

Countable union. Suppose that \( F_1, F_2, F_3, \ldots \) are Borel ERs on a Polish space \( \mathcal{Y} \), and \( F = \bigcup_k F_k \) is still a ER, and the Polish and gen. turbulent action of \( G \) on \( \mathcal{X} \) is h. gen. \( F_k \)-ergodic for any \( k \), and prove that it remains h. gen. \( F \)-ergodic.

Fix a nonempty open set \( U_0 \subseteq \mathcal{X} \) and a nbhd \( G_0 \) of \( 1_G \) in \( G \), such that \( U_0 \)-comeager many orbits \( \vartheta(x, U_0, G_0) \) with \( x \in U_0 \) are dense in \( U_0 \). Consider a Borel function \( \vartheta : U_0 \to \mathcal{Y}, \sim_{G_0}^U, F \)-invariant on a dense \( G_δ \) set \( D_0 \subseteq U_0 \). It follows from the invariance that for any open \( \emptyset \neq U \subseteq U_0 \) there exist: a number \( k \) and open non-empty sets \( U' \subseteq U \) and \( Q \subseteq G_0 \) such that \( \vartheta(x) F_k \vartheta(g \cdot x) \) holds for any \( C_\mathcal{X} \times C_G \)-generic, over \( \mathcal{M} \), pair \( \langle x, g \rangle \in U' \times Q \). We can find, as above, \( g_0 \in Q \cap \mathcal{M} \) and a nbhd \( G \subseteq G_0 \) of \( 1_G \) such that \( g_0 G \subseteq Q \). Similarly to Claims 58.3 and 58.2, we have \( \vartheta(x) F_k \vartheta(x') \) for any pair of \( C_\mathcal{X} \)-generic, over \( \mathcal{M} \), elements \( x, x' \in U' \), satisfying \( x \sim_{U'} x' \). It follows, by the ergodicity, that \( \vartheta \) is \( F_k \)-constant, hence, \( F \)-constant, on a comeager subset of \( U' \). It remains to show that these \( F \)-constants are \( F \)-equivalent to each other, which is demonstrated exactly as in the end of §11.e.

Disjoint union. Let \( F_k \) be Borel ERs on Polish spaces \( \mathcal{Y}_k, k = 0, 1, 2, \ldots \). By definition, \( \bigvee_k F_k = \bigcup_k F'_k \), where each \( F'_k \) is a Borel ER defined on the space \( \mathcal{Y} = \bigcup_k \{ k \} \times \mathcal{Y}_k \) as follows: \( \langle l, y \rangle F'_k \langle l', y' \rangle \) iff either \( l = l' \) and \( y = y' \) or \( l = l' = k \) and \( y \not\in F_k y' \).

Countable product. Let \( F_k \) be ERs on a Polish spaces \( \mathcal{Y}_k \). Then \( F = \prod_k F_k \) is a ER on the space \( \mathcal{Y} = \prod_k \mathcal{Y}_k \). For any map \( \vartheta : \mathcal{X} \to \mathcal{Y} \), to be gen. \( E, F \)-invariant (where \( E \) is any ER on \( \mathcal{X} \)) it is necessary and sufficient that every co-ordinate map \( \vartheta_k(x) = \vartheta(x)(k) \) is gen. \( E, F_k \)-invariant. This allows to easily accomplish this induction step.
11 TURBULENT GROUP ACTIONS

□ (Theorem 58, Lemma 57.5, Theorem 57)

11.h An application to the shift actions of ideals

Say that a Borel ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is special if there is a sequence of reals $r_n > 0$ with $\{r_n\} \to 0$, such that $\mathcal{I}_{\{r_n\}} \subseteq \mathcal{I}$. Nontrivial in the next theorem means: containing no cofinite sets.

**Theorem 59.** Suppose that $\mathcal{I}$ is a nontrivial Borel special ideal, and $\mathcal{F}$ belongs to the family $\mathcal{F}_0$ of Theorem 58. Then $\mathcal{E}_\mathcal{I}$ is generically $\mathcal{F}$-ergodic, hence, is not Borel reducible to $\mathcal{F}$.

**Proof.** The “hence” statement follows because by the nontriviality all $\mathcal{E}_\mathcal{I}$-equivalence classes are meager subsets of $\mathcal{P}(\mathbb{N})$.

As $\mathcal{I}$ is special, let $\{r_k\} \to 0$ be a sequence of positive reals such that $\mathcal{I}_{\{r_k\}} \subseteq \mathcal{I}$. It obviously suffices to prove that $\mathcal{E}_{\{r_k\}} = \mathcal{E}_{\{r_n\}}$ is generically $\mathcal{F}$-ergodic. Further, by Theorem 58, it suffices to prove that the shift action of $\mathcal{I}_{\{r_n\}}$ on $\mathcal{P}(\mathbb{N})$ is Polish and gen. turbulent.

The ideal $\mathcal{I}_{\{r_n\}}$ is easily a $\mathcal{P}$-ideal, hence, a polishable group (with $\Delta$ as the operation). For instance, $\mathcal{I}_{\{r_n\}}$ is a Polish group in the topology generated by the metric $d_{\{r_n\}}(a, b) = \varphi_{\{r_n\}}(a \Delta b)$ on $\mathcal{I}_{\{r_n\}}$, where

- $\varphi_{\{r_n\}}(x) = \sum_{n \in x} r_n$ for $x \in \mathcal{P}(\mathbb{N})$, so that $\mathcal{I}_{\{r_n\}} = \{x : \varphi_{\{r_n\}}(x) < +\infty\}$.

The shift action of $\mathcal{I}_{\{r_n\}}$ by $x \cdot y = x \Delta y$ on $\mathcal{P}(\mathbb{N})$ (considered in the product topology; $\mathcal{P}(\mathbb{N})$ is here identified with $2^{\mathbb{N}}$) is then continuous. It remains to verify the turbulence.

Let $x \in \mathcal{P}(\mathbb{N})$. The orbit $[x]_{\mathcal{I}_{\{r_n\}}} = \mathcal{I}_{\{r_n\}} \Delta x$ is easily dense and meager, hence, it suffices to prove that $x$ is a turbulent point of the action. Consider an open set $X \subseteq \mathcal{P}(\mathbb{N})$ containing $x$, and a $d_{\{r_n\}}$-hbhd $G$ of $\emptyset$ (the neutral element of $\mathcal{I}_{\{r_n\}}$); we may assume that, for some $k$, $X = \{y \in \mathcal{P}(\mathbb{N}) : y \cap [0, k) = u\}$, where $u = x \cap [0, k)$, and $G = \{g \in \mathcal{I}_{\{r_n\}} : \varphi(g) < \varepsilon\}$ for some $\varepsilon > 0$. Prove that the local orbit $\mathcal{O}(x, X, G)$ is somewhere dense in $X$.

Let $l \geq k$ be big enough for $r_n < \varepsilon$ for all $n \geq l$. Put $v = x \cap [0, l)$ and prove that $\mathcal{O}(x, X, G)$ is dense in $Y = \{y \in \mathcal{P}(\mathbb{N}) : y \cap [0, l) = v\}$. Consider an open set $Z = \{z \in Y : z \cap [l, j) = w\}$, where $j \geq l$, $w \subseteq [l, j)$. Let $z$ be the only element of $Z$ with $z \cap [j, +\infty) = x \cap [j, +\infty)$, thus, $x \Delta z = \{l_1, ..., l_m\} \subseteq [l, j)$. Each $g_i = \{l_i\}$ belongs to $G$ by the choice of $l$ (indeed, $l_i \geq l$). Moreover, easily $x_i = g_i \Delta g_{i-1} \Delta ... \Delta g_1 \Delta x = \{l_1, ..., l_i\} \Delta x$ belongs to $X$ for any $i = 1, ..., m$, and $x_m = z$, thus, $z \in \mathcal{O}(x, X, G)$, as required.

The next corollary returns us to the discussion in the end of §3.b.

**Corollary 60.** The equivalence relations $\mathcal{E}_0$ and $\mathcal{E}_2$ are not Borel reducible to any ideal $\mathcal{F}$ in the family $\mathcal{F}_0$ of Theorem 58, in particular, are not Borel reducible to $T_2$. □
Proof. According to lemmas 20 and 21, it suffices to prove that the ideals $\mathcal{Z}_0$ (density 0) and $\mathcal{I}_{\{1/n\}}$ are special. The latter is special by definition. As for the former, see ??? (that $\mathcal{I}_{\{1/n\}} \subseteq \mathcal{Z}_0$). \qed
12 Ideal $\mathcal{I}_3$ and the equivalence relation $E_3$

The ideal $0 \times \text{Fin}$ is traditionally denoted by $\mathcal{I}_3$. It consists of all sets $x \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ such that all cross-sections $(x)_n = \{k : \langle n, k \rangle \in x\}$ are finite. It defines the ER $E_3 = E_{\mathcal{I}_3}$ on $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ by $x E_3 y$ iff $x \Delta y \in \mathcal{I}_3$. But we rather consider $E_3$ as an ER on $\mathcal{P}(\mathbb{N})^\mathbb{N}$ defined by $x E_3 y$ iff $x(n) E_0 y(n)$ for all $n$: here $x, y$ belong to $\mathcal{P}(\mathbb{N})^\mathbb{N}$.

12.a Ideals below $\mathcal{I}_3$

Lemma 61. $\text{Fin} \subseteq \mathcal{I}_3$. $\mathcal{I}_3$ and $\mathcal{I}_1$ are $\subseteq$-incomparable.

Proof. To see that $\text{Fin} \subseteq \mathcal{I}_3$ take $\vartheta(x) = \{\langle n, 0 \rangle : n \in x\}$. That $\mathcal{I}_3 \subseteq \mathcal{I}_1$ can be shown as follows: otherwise by Theorem 40 $\mathcal{I}_3$ would be isomorphic either to one of $\text{Fin}, \mathcal{I}_1$, or to a trivial variation of $\text{Fin}$, which can be easily shown to be not the case. To see that $\mathcal{I}_1 \subseteq \mathcal{I}_3$ recall that $\mathcal{I}_3 = 0 \times \text{Fin}$ is of the form $\text{Exh}_\psi$ for a l.s.c. submeasure $\psi$ (Example 5) and apply Theorem 41.

The following theorem is analogous to Theorem 40, yet the method of its proof is absolutely different.

Theorem 62 (Kechris [27]). If $\mathcal{I} \subseteq \mathcal{I}_3$ is a Borel (nontrivial) ideal on $\mathbb{N}$ then either $\mathcal{I} \cong \mathcal{I}_3$ or $\mathcal{I}$ is a trivial variation of $\text{Fin}$.

Proof. First of all we make use of Theorem 41: $\mathcal{I}_1 \subseteq \mathcal{I}_3$ according to Lemma 61, therefore, $\mathcal{I} = \text{Exh}_\varphi$ for a l.s.c submeasure $\varphi$ on $\mathbb{N}$. We can w.l.o.g. suppose that $\varphi(x) \leq 1$ for any $x \in \mathcal{P}(\mathbb{N})$. Now put $U_n = \{k : \varphi(\{k\}) \leq \frac{1}{n}\}$.

We assert that $\lim_{n \to \infty} \varphi(U_n) = 0$. Indeed, otherwise $\varphi(U_n) > \varepsilon$ for some $\varepsilon > 0$ and all $n$. As $\varphi$ is l.s.c. we can choose a sequence of numbers $n_1 < n_2 < n_3 < \ldots$ and for any $l$ a finite set $w_l \subseteq U_{n_l} \setminus U_{n_{l+1}}$ with $\varphi(w_l) > \varepsilon$. Then $W = \bigcup_l w_l \notin \mathcal{I}$ and obviously $\{\varphi(\{k\})\}_{k \in W} \to 0$. Note that the Borel ideal $\mathcal{I} = \mathcal{I} \upharpoonright W$ satisfies $\mathcal{I} \subseteq \mathcal{I}_3$ (via the identity map), because $W \notin \mathcal{I}$. On the other hand, $\mathcal{I}$ is isomorphic to a special ideal (see §11.1) via the order preserving bijection of $W$ onto $\mathbb{N}$. It follows from Theorem 59 that $E_\mathcal{I}$ is not Borel reducible to any equivalence relation in $\mathcal{F}_0$, hence, neither is $E_\mathcal{I}$. But $E_{\mathcal{I}_3} = E_3$ obviously belongs to $\mathcal{F}_0$, which is a contradiction because $\mathcal{I} \subseteq \mathcal{I}_3$.

Thus $\varphi(U_n) \to 0$. Then clearly a set $x \in \mathcal{P}(\mathbb{N})$ belongs to $\mathcal{I}$ iff $x \cap (U_n \setminus U_{n+1})$ is finite for any $m$, which easily implies that $\mathcal{I}$ is as required.

12.b Assembling equivalence relations

The next theorem, similar to a couple of results above, will be used in the proof of a dichotomy theorem related to $E_3$. 

←− check the proof

←− give ref
Theorem 63. Suppose that \( \mathcal{X}, \mathcal{Y} \) are Polish spaces, \( P \subseteq \mathcal{X} \times \mathcal{Y} \) is a Borel set, \( E \) is a Borel ER on \( P \), and \( G \) is a countable group acting on \( \mathcal{X} \) in a Borel way so that \( (x, y) \in E(x', y') \) implies \( x \in E(x') \). Finally, assume that \( E \upharpoonright P(x) \) is smooth for each \( x \in \mathcal{X} \), where \( P(x) = \{ (x', y) : x' = x \} \). Then \( E \) is Borel-reducible to a Borel action of \( G \).

Proof. We can assume that \( \mathcal{X} = \mathcal{Y} = 2^\mathbb{N} \) and both \( P \) and \( E \) are \( \Delta^1_1 \). We can also assume that the action of \( G \) (a countable group) is \( \Delta^1_1 \). Then clearly \( x \in E_0^2 \leftrightarrow \Delta^1_1(x) = \Delta^1_1(x') \). Define \( P^*(x) = \bigcup_{a \in G} P(a \cdot x) \) for \( x \in \mathcal{X} \).

Claim 63.1. Suppose that \( \langle x, y \rangle \) and \( \langle x', y' \rangle \) belong to \( P \) and \( x \in E_0^2 \). Then \( \langle x, y \rangle \in E \langle x', y' \rangle \) iff the equivalence \( \langle x, y \rangle \in U \iff \langle x', y' \rangle \in U \) holds for any \( E \upharpoonright P^*(x) \)-invariant \( \Delta^1_1(x) \) set \( U \subseteq P^*(x) \).

Proof. Note that \( E \upharpoonright P^*(x) \) is still smooth by Theorem 28 because \( G \) is countable. In addition \( E \upharpoonright P^*(x) \) is \( \Delta^1_1(x) \). This observation yields the result, because otherwise, \( i.e., \) if the ER, defined on \( P^*(x) \) by intersections with \( E \upharpoonright P^*(x) \)-invariant \( \Delta^1_1(x) \) sets, is coarser than \( E \upharpoonright P^*(x) \), then it is known from the proof of the 2nd dichotomy theorem (Theorem 35) that we would have \( E_0 \leq B \equiv E \upharpoonright P^*(x) \), a contradiction with the smoothness. \( \square \) (Claim)

For any \( x \in \mathcal{X} \) let \( E(x) \) be the set of all \( e \in \mathbb{N} \) which code a \( \Delta^1_0(x) \) subset of \( P \), and, for \( e \in E(x) \), let \( W^e_x \) be the \( \Delta^1_0(x) \) subset of \( P \) coded by \( e \). It is known that \( \{ \langle x, e \rangle : e \in E(x) \} \) is \( \Pi^1_1 \).) Let \( \text{inv}(x, e) \) be the formula \( x \in \mathcal{X} \land e \in E(x) \land W^e_x \subseteq P^*(x) \land W^e_x \) is \( E \upharpoonright P^*(x) \)-invariant.

Corollary 63.2. Let \( \langle x, y \rangle, \langle x', y' \rangle \) be as in Claim 63.1. Then \( \langle x, y \rangle \in E \langle x', y' \rangle \) iff \( \langle x, y \rangle \in W^e_x \iff \langle x', y' \rangle \in W^e_x \) holds for any \( e \) with \( \text{inv}(x, e) \). \( \square \)

Implication \( \iff \) of the “iff” in this Corollary can be considered as a property of the \( \Pi^1_1 \) set \( C = \{ \langle x, e \rangle : \text{inv}(x, e) \} \), \( i.e., \) the property that

- for all pairs \( \langle x, y \rangle \) and \( \langle x', y' \rangle \) in \( P \) with \( x \in E_0^2 \) \( x' \), we have:
  
  if \( \forall \langle x, e \rangle \in C \langle x, y \rangle \in W^e_x \iff \langle x', y' \rangle \in W^e_x \) then \( \langle x, y \rangle \in E \langle x', y' \rangle \).

This is easily a \( \Pi^1_1 \) property in the codes, hence, by the \( \Pi^1_1 \) Reflection, there is a \( \Delta^1_1 \) set \( B \subseteq C \) satisfying the same property, that is, we have

Corollary 63.3. Let \( \langle x, y \rangle, \langle x', y' \rangle \) be as in Claim 63.1. Then \( \langle x, y \rangle \in E \langle x', y' \rangle \) iff \( \langle x, y \rangle \in W^e_x \iff \langle x', y' \rangle \in W^e_x \) holds for any \( e \) with \( \langle x, e \rangle \in B \). \( \square \)

To continue the proof of the theorem, define, for any \( \langle x, y \rangle \in P \),

\[
D_{xy} = \{ \langle a, e \rangle : a \in G \land \langle a \cdot x, e \rangle \in B \land \langle x, y \rangle \in W^e_{a \cdot x} \}.
\]

Clearly \( \langle x, y \rangle \mapsto D_{xy} \) is a \( \Delta^1_1 \) map \( P \to \mathcal{P}(G \times \mathbb{N}) \).

If \( D \subseteq G \times \mathbb{N} \) and \( b \in G \) then put \( b \circ D = \{ \langle ab^{-1}, e \rangle : \langle a, e \rangle \in D \} \).
Claim 63.4. Suppose that \( \langle x, y \rangle \) and \( \langle x', y' \rangle \) belong to \( P \), \( b \in \mathcal{G} \), and \( x' = b \cdot x \). Then \( \langle x, y \rangle \in \mathcal{E} \iff b \circ D_{xy} = D_{x'y'} \).

**Proof.** Assume that \( b \circ D_{xy} = D_{x'y'} \). According to Corollary 63.3, to prove \( \langle x, y \rangle \in \mathcal{E} \langle x', y' \rangle \) it suffices to show that \( \langle x, y \rangle \in W^b_x \iff \langle x', y' \rangle \in W^b_{x'} = W^b_x \), as required. Conversely, let \( \langle x, y \rangle \in \mathcal{E} \langle x', y' \rangle \). If \( \langle a, e \rangle \in D_{xy} \) then \( \langle a \cdot x, e \rangle \in D_{x'y'} \), hence, \( \langle x', y' \rangle \in W^a_{x' \cdot x} \), too, because the set \( W^a_{x' \cdot x} \) is invariant and \( \langle x, y \rangle \in \mathcal{E} \langle x', y' \rangle \). Yet \( a \cdot x = ab^{-1} \cdot x' \), therefore, by definition, \( \langle ab^{-1}, e \rangle \in D_{x'y'} \). The same argument can be carried out in the opposite direction, so that \( \langle a, e \rangle \in D_{xy} \iff \langle ab^{-1}, e \rangle \in D_{x'y'} \), that means \( b \circ D_{xy} = D_{x'y'} \). \( \square \) (Claim)

To end the proof of the theorem, consider \( Z = \mathcal{X} \times \mathcal{P}(\mathcal{G} \times \mathbb{N}) \), a Polish space. Define a Borel action \( b \cdot \langle x, D \rangle = \langle b \cdot x, b \circ D \rangle \) of \( \mathcal{G} \) on \( Z \). We assert that \( \vartheta(x, y) = \langle x, D_{xy} \rangle \) is a Borel reduction of \( \mathcal{E} \restriction P \) to the action \( \mathcal{E} \). Indeed, let \( \langle x, y \rangle \) and \( \langle x', y' \rangle \) belong to \( P \). Suppose that \( \langle x, y \rangle \in \mathcal{E} \langle x', y' \rangle \). Then \( x \in \mathcal{E}^x_{x'} \), so that \( x' = b \cdot x \) for some \( b \in \mathcal{G} \). Moreover, \( b \circ D_{xy} = D_{x'y'} \) by Claim 63.4, hence, \( \vartheta(x', y') = b \circ \vartheta(x, y) \). Let, conversely, \( \vartheta(x', y') = b \circ \vartheta(x, y) \), so that \( x' = b \cdot x \) and \( D_{x'y'} = b \circ D_{xy} \). Then \( \langle x, y \rangle \in \mathcal{E} \langle x', y' \rangle \) by Claim 63.4, as required. \( \square \)

12.c The 6th dichotomy

**Theorem 64 (Hjorth and Kechris [16, 17]).** If \( \mathcal{E} \leq_b \mathcal{E}_3 \) is a Borel ER then either \( \mathcal{E} \leq_n \mathcal{E}_0 \) or \( \mathcal{E} \sim_n \mathcal{E}_3 \).

**Proof.** (a modification of the proof in [17]). We may assume that \( \mathcal{E} \) is a \( \Delta_1 \)
ER on a recursively presented Polish space \( \mathcal{X} \), and there is a \( \Delta_1 \) reduction \( \vartheta: \mathcal{X} \to \mathcal{P}(\mathcal{N})^\mathcal{N} \) of \( \mathcal{E} \) to \( \mathcal{E}_3 \). Let \( Q = \text{ran} \vartheta \), a \( \Sigma_1 \) subset of \( \mathcal{P}(\mathcal{N})^\mathcal{N} \).

For \( x, y \in \mathcal{P}(\mathcal{N})^\mathcal{N} \) and \( n \in \mathcal{N} \), define \( x \equiv_n y \) iff \( x \in Q \) and \( x \equiv_n y \) iff \( x \equiv_n y \) (the latter requirement means \( x_k = y_k \) for all \( k < n \)). For \( n, k, p \in \mathcal{N} \) put

\[ A_{nkp} = \{ A \subseteq \mathcal{P}(\mathcal{N})^\mathcal{N} : A \text{ is } \Sigma_1 \wedge \forall x, y \in A (x \equiv_n y \Rightarrow x_k \Delta y_k \subseteq \{0, p\}) \} \]

**Claim 64.1.** If \( A \in A_{nkp} \) then there is a \( \Delta_1 \) set \( B \in A_{nkp} \) with \( A \subseteq B \).

**Proof.** (Reflection)

\( \square \) (Claim)

Put \( A_{nkp} = \bigcup\{ A : A \in A_{nkp} \} \) and \( \hat{A} = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} \bigcup_{p \in \mathbb{N}} A_{nkp} \)

**Case 1:** \( Q \subseteq \hat{A} \) \hspace{1cm} **Case 2:** otherwise.

---

\(^{27}\) Hjorth and Kechris [17] define \( A_{nkp} \) with \( \forall x, y \in Q \) instead of \( \forall x, y \in A \). Let us use \( \mathcal{A}_{nkp} \) to denote their version, thus, \( \mathcal{A}_{nkp} \subseteq \mathcal{A}_{nkp} \). However if Case 1 holds in the sense of \( \mathcal{A}_{nkp} \) then it also holds in the sense of \( \mathcal{A}_{nkp} \) because \( A \in \mathcal{A}_{nkp} \) if \( A \cap Q \in \mathcal{A}_{nkp} \).
12. Case 1

We are going to prove that in this case \( E \leq_R E_0 \).

As easily \( \hat{A} \) is \( \Pi^1_1 \) by Claim 64.1 and a standard computation, there is a \( \Delta^1_1 \) set \( R \) such that \( Q \subseteq R \subseteq \hat{A} \). Thus, for \( E \leq_R E_0 \) it suffices now to prove

**Lemma 65.** \( E_3 \upharpoonright R \leq_R E_0 \) for any \( \Delta^1_1 \) set \( R \subseteq \hat{A} \).

**Proof.** By Kreisel Selection there exists a \( \Delta^1_1 \) map \( \nu : R \to \mathbb{N} \) such that

\[
\forall k \geq \nu(x) \exists p \exists B \in \mathcal{A}_{\nu(x), k, p} \ (x \in B \in \Delta^1_1)
\]

for any \( x \in R \). Let \( R_n = \{ x \in R : \nu(x) \leq n \} \), these are increasing \( \Delta^1_1 \) subsets of \( R \), and \( R = \bigcup_n R_n \). According to Theorem 34, it suffices to prove that \( E_3 \upharpoonright R_n \leq_R E_0 \) for any \( n \). Thus let us fix \( n \). By definition we have

\[
\forall x \in R_n \forall k \geq n \exists p \exists B \in \mathcal{A}_{n, k, p} \ (x \in B \in \Delta^1_1).
\]

Recall that \( C \) is the least class of sets containing all open sets and closed under the \( \Lambda \)-operation and the complement. A map \( f \) is called \( C \)-measurable iff all \( f \)-preimages of open sets belong to \( C \).

**Claim 65.1.** For any \( n \) there is a \( C \)-measurable map \( f : R_n \to \mathcal{P}(\mathbb{N})^N \) such that \( f(x) = f(y) \equiv_n x \) whenever \( x, y \in R_n \) satisfy \( x \equiv_n y \).

**Proof.** Let \( E \subseteq \mathbb{N} \) be the \( \Pi^1_1 \) set of all codes of \( \Delta^1_1 \) subsets of \( \mathcal{P}(\mathbb{N})^N \), and let \( W_e \subseteq \mathcal{P}(\mathbb{N})^N \) be the \( \Delta^1_1 \) set coded by \( e \in E \). We have, by (*)

\[
\forall x \in R_n \forall k \geq n \exists p \exists e \in E \ (x \in W_e \in \mathcal{A}_{n, k, p}),
\]

and an ordinary application of the Kreisel selection yields a pair of \( \Delta^1_1 \) maps \( \pi, \varepsilon : R_n \times \mathbb{N} \to \mathbb{N} \) such that \( \varepsilon(x, k) \in \mathcal{P}(\mathbb{N}) \) and \( x \in W_{\varepsilon(x, k)} \in \mathcal{A}_{n, k, \pi(x, k)} \) hold whenever \( x \in R_n \) and \( k \geq n \). Let \( \tilde{\pi}(x, k) \) and \( \tilde{\varepsilon}(x, k) \) to be the least, in the sense of any fixed recursive \( \omega \)-long wellordering of \( \mathbb{N} \times \mathbb{N} \), of all possible pairs \( \pi(x', k) \) and \( \varepsilon(x', k) \) with \( x' \in R_n \cap [x]_{\equiv_n} \). Then \( \tilde{\pi} \) and \( \tilde{\varepsilon} \) are \( \equiv_n \)-invariant in the 1st argument. In addition, we have \( \tilde{W}_{\tilde{\varepsilon}(x, k)} \in \mathcal{A}_{n, k, \tilde{\pi}(x, k)} \) and the set \( Z_{xk} = R_n \cap [x]_{\equiv_n} \cap W_{\tilde{\varepsilon}(x, k)} \) is nonempty, whenever \( x \in R_n \) and \( k \geq n \).

Let \( x \in R_n \). For any \( k \geq n \), the set \( Y_{xk} = \{ y_k : y \in Z_{xk} \} \subseteq \mathcal{P}(\mathbb{N}) \) is finite (and nonempty) by the definition of \( \mathcal{A}_{n, k, p} \), thus, let \( f_k(x) \) be the least member of \( Y_{xk} \) in the sense of the lexicographical order of \( \mathcal{P}(\mathbb{N}) \). Define \( f(x) \in \mathcal{P}(\mathbb{N})^N \) so that \( f(x)_k = x_k \) for \( k < n \) and \( f(x)_k = f_k(x) \) for \( k \geq n \).

That \( f(x) = f(y) \) whenever \( x \equiv_n y \) follows from the invariance of \( \varepsilon \) and \( \pi \).

To see that \( f(x) \equiv_n x \) note that by definition \( f_k(x) \equiv x_k \) for \( k \geq n \); indeed, \( f_k(x) = y_k \) for some \( y \in [x]_{\equiv_n} \), but \( x \equiv_n y \) implies \( x_k \equiv y_k \) for all \( k \). Finally, the \( C \)-measurability needs a routine check. \( \Box \) (Claim)
For any $u \in \mathcal{P}(N)^n$ let $R_n(u) = \{ x \in R_n : x \restriction_n = u \}$.

Claim 65.2. If $u \in \mathcal{P}(N)^n$ then $E_3 \restriction R_n(u)$ is smooth.

Proof. As $E_3$ and $\equiv_n$ coincide on $R_n(u)$, the relation $E_3 \restriction R_n(u)$ is smooth via a $C$-measurable, hence, a Baire-measurable map. Suppose, towards the contrary, that it is not really smooth, i.e., via a Borel map. Then, by the 2-nd dichotomy theorem, we have $E_0 \subseteq B E_3 \restriction R_n(u)$, hence, $E_0$ turns out to be smooth via a Baire-measurable map, which is easily impossible. \hfill $\square$ (Claim)

To complete the proof of the lemma, let $G = \mathcal{R}_{1n}(N)^n$, acting on $X = \mathcal{P}(N)^n$ componentwise and by $\Delta$ at each of the $n$ co-ordinates, so that, for $u, v \in X$, we have $u G v$ iff $u_kE_0v_k$ for all $k < n$. Let us apply Theorem 63 with $G$ and $X$ as indicated, and $P = R_n$ and $E = E_3 \restriction R_n$, Claim 65.2 witnesses the principal requirement. We obtain: $E_3 \restriction R_n$ is Borel reducible to an ER induced by a Borel action of $G$. Yet $G$ is the increasing union of a countable sequence of its finite subgroups, hence, any ER induced by a Borel action of $G$ is hyperfinite, hence, Borel reducible to $E_0$. \hfill $\square$ (Lemma 65 and Case 1 in Theorem 64)

12.e Case 2

Then the $\Sigma_1^1$ set $H = Q \setminus \hat{A}$ is non-empty. Our idea will be to define a Borel subset $X$ of $H$ such that $E_3 \restriction X \sim B E_3$, the “or” case of Theorem 64.

By definition, $H = \bigcap_n \bigcup_k H_{nk}$, where $H_{nk} = H \setminus \bigcup_p A_{nkp}$. Note that

$$H_{nk} = \{ x \in H : \forall p \forall A \in \Delta_1^1 \ (x \in A \implies A \notin \mathcal{A}_{nkp}) \}$$

by Claim 64.1, and hence $H_{nk}$ is $\Sigma_1^1$ by rather elementary computation.

Let $b$ be any recursive bijection $\mathbb{N}^2 \ derechos \mathbb{N}$, increasing in each argument. Put $L(n) = \max\{ r : b(r, 0) \leq n \}$ - thus for any $\ell > L(n)$ we have $b(\ell, j) > n$, $\forall j$.

The splitting system used here will contain non-empty $\Sigma_1^1$ sets $X_s \subseteq \mathcal{P}(N)^N$, $s \in 2^{<\omega}$, numbers $k_m$, $m \in \mathbb{N}$, and elements $g_s \in \mathcal{P}(N)^N$, $s \in 2^{<\omega}$, satisfying the following requirements (i) – (vi):

(i) $X_s \subseteq H$, $X_s \smallsetminus \iota \subseteq X_s$, $\operatorname{diam} X_s \leq 2^{-2h_s}$, and a certain condition, in terms of the Choquet game, holds, connecting each $X_s \smallsetminus \iota$ with $X_s$ so that, as a consequence, $\bigcap_n X_{a|n} \neq \emptyset$ for any $a \in 2^\mathbb{N}$.

(ii) $0 < k_0 < k_1 < \ldots$ and $X_{0^{k_{i+1}}} \subseteq \bigcap_{r < L(n)} H_{r,k_r}$. \hfill 28

(iii) If $s \in 2^{n+1}$ then $g_s(i)$ is finite for all $i$ and is $= \emptyset$ for all $i > k_{L(n)}$; in addition, $g_{0^{n+1}}(i) = \emptyset$ for all $i$.

28 Recall that $0^m$ is a sequence of $m$ zeros.
13 SUMMABLE IDEALS

(iv) For any $s \in 2^{n+1}$, we have $\forall x \in X_{0^{n+1}} \exists y \in X_s (y \equiv_{kL(n)} g_s \cdot x)$; \footnote{For $g, x \in \mathcal{P}(N)^N$, $g \cdot x = y \in \mathcal{P}(N)^N$ is defined by $y(n) = g(n) \Delta x(n)$, $\forall n$.}

(v)

(vi) \hfill $\Box$ (Theorem 64)

13 Summable ideals

Farah [6, § 1.12] gives the following classification of summable ideals $\mathcal{I}_{\{r_n\}}$, based on the distribution of numbers $r_n$:

(S1) Atomic ideals: there is $\varepsilon > 0$ such that the set $A_\varepsilon = \{ n : r_n \geq \varepsilon \}$ is infinite and satisfies $\mu_{\{r_n\}}(C(A_\varepsilon)) < +\infty$. In this case $\mathcal{I}_{\{r_n\}} = \{ X : X \cap A_\varepsilon \in \text{Fin} \}$; Kecharis [27] called such ideals trivial variations of Fin.

(S2) Dense (summable) ideals: $r_n \to 0$.

(S3) There is a decreasing sequence of positive reals $\varepsilon_n \to 0$ such that all sets $D_n = A_{\varepsilon_n+1} \setminus A_{\varepsilon_n}$ are infinite.

(S4) Ideals of the form $\text{Fin} \oplus \text{dense}$: there is a real $\varepsilon > 0$ such that the set $A_\varepsilon$ is infinite, $\mu_{\{r_n\}}(C(A_\varepsilon)) = +\infty$, and $\lim_{n \to \infty} n \in C(A_\varepsilon), r_n = 0$.

In the sense of $\leq_B$, all ideals of types (S2), (S3), (S4) are equivalent to each other, and all ideals of type (S1) are equivalent to each other, so that we have just 2 summable ideals modulo $\sim_B$, namely, $\text{Fin}$ and $\mathcal{I}_{\{1/n\}}$. The structure under $\leq_{RB}$ or $\leq_{BE}$ is much more complicated (Farah ?).

13.a A useful lemma

Lemma 66 (Attributed to Kecharis in [13]). Suppose that $A, X$ are Borel sets, $E$ is a Borel ER on $A$, and $\rho : A \to X$ is a Borel map satisfying the following: first, the $\rho$-image of any $E$-class is at most countable, second, $\rho$-images of any different $E$-classes are disjoint. Then $E$ is an essentially countable ER.

Proof. The relation: $x \mathrel{R} y$ iff $x, y \in Y$ belong to the $\rho$-image of one and the same $E$-class in $A$, is a $\Sigma_1^1$ ER on the set $Y = \text{ran} \vartheta$, moreover,

$$ R \subseteq P = \{ (x, y) : \neg \exists a, b \in A (a \in E b \land x = \rho(a) \land y = \rho(b)) \}, $$

where $P$ is $\Pi_1^1$, hence, there is a Borel set $U$ with $R \subseteq U \subseteq P$, in particular, $U \cap (Y \times Y) = R$. As all $R$-equivalence classes are at most countable, we can assume that all cross-sections of $U$ are at most countable, too.
13 SUMMABLE IDEALS

Now it suffices to find a Borel ER $F$ with $R \subseteq F \subseteq U$. Say that a set $Z \subseteq X$ is "stable" if $U \cap (Z \times Z)$ is a ER, for example, $Y$ is "stable". We observe that the set $D_0 = \{ y : Y \cup \{ y \} \}$ is $\Pi^1_1$ and satisfies $Y \subseteq D_0$, hence, there is a Borel set $Z_1$ with $Y \subseteq Z_1 \subseteq D_0$. Similarly,

$$D_1 = \{ y' \in Z_1 : Y \cup \{ y, y' \} \} \text{ is "stable" for any } y \in Z_1 \}$$

is $\Pi^1_1$ and satisfies $Y \subseteq D_1$ by the definition of $Z_1$, so that there is a Borel set $Z_2$ with $Y \subseteq Z_2 \subseteq D_1$. Generally, we define

$$D_n = \{ y' \in Z_n : Y \cup \{ y_1, ..., y_n, y' \} \} \text{ is "stable" for all } y_1, ..., y_n \in Z_n \}$$

find that $Y \subseteq D_n$, and choose a Borel set $Z_n$ with $Y \subseteq Z_n \subseteq D_n$. Then, by the construction, $Y \subseteq Z = \bigcap_n Z_n$, and, for any finite $Z' \subseteq Z$, the set $Y \cup Z'$ is "stable", so that $Z$ itself is "stable", and we can take $F = U \cap (Z \times Z)$.

13.6 Under the summable ideal

Subsets of $\mathbb{N}$ will be systematically identified with their characteristic functions.

For $a, b \in 2^\mathbb{N}$ put $a \Delta b = \{ n : a(n) \neq b(n) \}$ (identified with the function $c(n) = 1$ iff $a(n) \neq b(n)$) and $\Sigma(a, b) = \sum_{n \in a \Delta b} \frac{1}{n+1}$. (This can be a nonnegative real or $+\infty$.) Generally, we define $\Sigma^m_k (a, b) = \sum_{n \in a \Delta b, k \leq n \leq m} \frac{1}{n+1}$, and accordingly $\Sigma^\infty_k (a, b) = \sum_{n \in a \Delta b, k \leq n < \infty} \frac{1}{n+1}$. Define $\Sigma(a) = \sum_{\{ n : a(n) = 1 \}} \frac{1}{n+1}$ and similarly $\Sigma^m_k (a)$ and $\Sigma^\infty_k (a)$.

Recall that the summable ideal is defined as

$$\mathcal{J}_{1/\mathbb{N}} = \{ a \in 2^\mathbb{N} : \Sigma(a) < +\infty \}.$$  

(The notation $\mathcal{J}_2$ and $\mathcal{J}_0$ is also used.) $E_{1/\mathbb{N}}$ will denote the associated Borel ER on $2^\mathbb{N}$, i.e., $a \sim b$ iff $\Sigma(a, b) < +\infty$.

**Theorem 67.** Let $E$ be a Borel ER on a Polish space $X$, and $E \leq_n E_{1/\mathbb{N}}$. Then either $E \sim_n E_{1/\mathbb{N}}$ or $E$ is essentially countable.

**Proof.** This is a long proof. Let $\vartheta : X \to 2^\mathbb{N}$ be a Borel reduction $E$ to $E_{1/\mathbb{N}}$. We can assume that $\vartheta$ is in fact continuous: indeed it is known that there is a stronger Polish topology on $X$ which makes $\vartheta$ continuous but does not add new Borel subsets of $X$. Now, as any Polish $X$ is a $1-1$ continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$, we can assume that $X = \mathbb{N}^\mathbb{N}$.

Finally, we can assume that $\vartheta$ is $\Delta^1_1$, not merely Borel.

If $a \in A \subseteq 2^\mathbb{N}$ and $q \in \mathbb{Q}^+$ then let $Gal^q_A(a)$ be the set of all $b \in A$ such that there is a finite chain $a = a_0, a_1, ..., a_n = b$ of reals $a_i \in A$ such that $\Sigma(a_i, a_{i+1}) < q$ for all $i$, the *q-galaxy of a in A*. 


13 SUMMABLE IDEALS

Definition 67.1. A set \( A \subseteq 2^\mathbb{N} \) is "\(q\)-grainy", where \( q \in \mathbb{Q}^+ \), iff \( \Sigma(a,b) < 1 \) for all \( a \in A \) and \( b \in \text{Gal}_q^*(a) \). A set \( A \) is "\(q\)-grainy" if it is \(q\)-grainy" for some \( q \in \mathbb{Q}^+ \). (In other words it is required that the galaxies are rather small.)

Claim 67.2. Any \(q\)-"grainy" \( \Sigma_1^n \) set \( A \subseteq 2^\mathbb{N} \) is covered by a \(q\)-"grainy" \( \Delta_1^n \) set.

Proof.\(^{30}\) The set \( D_0 = \{ b \in 2^\mathbb{N} : A \cup \{ b \} \text{ is } q\text{-"grainy"} \} \) is \( \Pi_1 \) and \( A \subseteq D_0 \), hence, there is a \( \Delta_1 \) set \( B_1 \) with \( A \subseteq B_1 \subseteq D_0 \). Note that \( A \cup \{ a \} \) is \(q\)-"grainy" for any \( a \in B_1 \). It follows that the \( \Pi_1 \) set

\[
D_1 = \{ b \in B_1 : A \cup \{ a, b \} \text{ is } q\text{-"grainy" for any } a \in B_1 \}
\]

still contains \( A \), hence, there is a \( \Delta_1 \) set \( B_2 \) with \( A \subseteq B_2 \subseteq D_1 \). Note that \( A \cup \{ a_1, a_2 \} \) is \(q\)-"grainy" for any \( a_1, a_2 \in B_2 \). In general, as soon as we have got a \( \Delta_1 \) set \( B_n \) with \( A \subseteq B_n \) and such that \( A \cup \{ a_1, ..., a_n \} \) is \(q\)-"grainy" for any \( a_1, ..., a_n \in B_n \), then the \( \Pi_1 \) set

\[
D_n = \{ b \in B_n : A \cup \{ a_1, ..., a_n, b \} \text{ is } q\text{-"grainy" for any } a_1, ..., a_n \in B_n \}
\]

contains \( A \), hence, there is a \( \Delta_1 \) set \( B_{n+1} \) with \( A \subseteq B_{n+1} \subseteq D_n \subseteq B_n \).

As usual in similar cases, the choice of the sets \( B_n \) can be made effective enough for the set \( B = \bigcap_n B_n \) to be still \( \Delta_1 \), not merely Borel. On the other hand, \( A \subseteq B \) and \( B \) is \(q\)-"grainy". \( \square \) (Claim)

Coming back to the proof of the theorem, let \( C \) be the union of all "grainy" \( \Delta_1 \) sets. An ordinary computation shows that \( C \) is \( \Pi_1 \). We have two cases.

**Case 1**: \( \text{ran } \vartheta \subseteq C \).

**Case 2**: otherwise.

13.c Case 1

We are going to prove that, in this case, \( E \) is essentially countable. First note that, by Separation, there is a \( \Delta_1 \) set \( H^* \subseteq 2^\mathbb{N} \) with \( \text{ran } \vartheta \subseteq H^* \subseteq C \).

Fix a standard enumeration \( \{ W_e \}_{e \in E} \) of all \( \Delta_1 \) subsets of \( 2^\mathbb{N} \), where, as usual, \( E \subseteq \mathbb{N} \) is a \( \Pi_1 \) set. By Kreisel Selection, there exist \( \Delta_1 \) functions \( a \mapsto e(a) \) and \( a \mapsto q(a) \), defined on \( H^* \), such that for any \( a \in H^* \) the \( \Delta_1 \) set \( W(a) = W_{e(a)} \) contains \( a \) and is \(q(a)\)-"grainy". The final point of our argument will be an application of Lemma 66, where \( \rho \) will be a derivate of the function \( G(a) = \text{Gal}^q_{W(a)}(a) \). We prove

Claim 67.2. If \( a \in H^* \) then \( \gamma_a = \{ G(b) : b \in [a]_{E(1/n)} \cap H^* \} \) is at most countable.

\(^{30}\) The result can be achieved as a routine application of a reflection principle, yet we would like to show how it works with a low level technique.
Proof. Otherwise there is a pair of \( e \in E \) and \( q \in \mathbb{Q}^+ \) and an uncountable set \( B \subseteq [a]_{E_{(1/n)}} \cap H^* \) such that \( q(b) = q \) and \( e(b) = e \) for any \( b \in B \) and \( G(b') \neq G(b) \) for any two different \( b, b' \in B \). Note that any \( G(b), b \in B \), is a \( q \)-galaxy in one and the same set \( W(a) = W(b) = W_e \), therefore, if \( b \neq b' \in B \) then \( b' \not\in G(b) \) and \( \Sigma(b, b') \geq q \). On the other hand, as \( B \subseteq [a]_{E_{(1/n)}} \), we have \( \Sigma(a, b) < +\infty \) for all \( b \in B \), hence, there is \( m \) and a still uncountable set \( B' \subseteq B \) such that \( \Sigma_m(a, b) < q/2 \) for all \( b \in B' \). Now take a pair of \( b \neq b' \in B' \) with \( b \upharpoonright [0, m) = b' \upharpoonright [0, m) : \) then \( \Sigma(b, b') < q \), contradiction. \( \square \) (Claim)

It follows that \( x \mapsto G(\vartheta(x)) \) maps any \( E \)-class into a countable set of galaxies \( G(a) \). To code the galaxies by single points, let \( S(a) = \bigcup_m \{ b \mid m : b \in G(a) \} \). Thus \( S(a) \subseteq 2^{<\omega} \) codes the Polish topological closure of the galaxy \( G(a) \).

Claim 67.3. If \( a, b \in H^* \) and \( \neg a \in E_{(1/n)} b \) then \( b \) does not belong to the (topological) closure of \( G(a) \), in particular, \( b \upharpoonright m \not\in S(a) \) for some \( m \).

Proof. Take \( m \) big enough for \( \Sigma^{m-1}(a, b) \geq 2 \). Then \( s = b \upharpoonright m \) does not belong to \( S(a) \) because any \( a' \in G(a) \) satisfies \( \Sigma(a, a') < 1 \). \( \square \) (Claim)

Elementary computation shows that the sets

\[
G = \{ (a, b) : a \in H^* \land b \in G(a) \} \quad \text{and} \quad S = \{ (a, s) : a \in H^* \land s \in S(a) \}.
\]

belong to \( \Sigma^1_1 \), but this is not enough to claim that \( a \mapsto S(a) \) is a Borel map. Yet we can change it appropriately to get a Borel map with similar properties. First of all define the following \( \Sigma^1_1 \) ER on \( H^* \):

\[
a \not\in b \quad \text{iff} \quad e(a) = e(b) \land q(a) = q(b) \land G(a) = G(b).
\]

(To see that \( F \) is \( \Sigma^1_1 \) note that here \( G(a) = G(b) \) is equivalent to \( b \in G(a) \), and that \( G \) is \( \Sigma^1_1 \).) It follows from Claim 67.3 and Kreisel Selection that there is a \( \Delta^1_1 \) function \( \mu : H^* \times H^* \to \mathbb{N} \) such that for any pair of \( a, b \in H^* \) with \( a \in E_{(1/n)} b \) we have \( b \upharpoonright \mu(a, b) \not\in S(a) \). Then the set

\[
R(a) = \{ b \mid \mu(a', b) : a', b \in H^* \land a \not\in b \} \subseteq 2^{<\omega}
\]

does not intersect \( S(a) \), for any \( a \in H^* \), hence, the \( \Sigma^1_1 \) set

\[
R = \{ (a, s) : a \in H^* \land s \in R(a) \}
\]

does not intersect \( S \). Note that by definition \( R \) is \( F \)-invariant w. r. t. the 1st argument, i.e., if \( a, a' \in H^* \) satisfy \( a \not\in a' \) then \( R(a) = R(a') \). It follows from Lemma 35.2 that there is a \( \Delta^1_1 \) set \( Q \subseteq H^* \times 2^{<\omega} \) with \( S \subseteq Q \) but \( R \cap Q = \emptyset \), \( F \)-invariant in the same sense. Then the map \( a \mapsto Q(a) = \{ s : Q(a, s) \} \) is \( \Delta^1_1 \).

Claim 67.4. Suppose that \( a, b \in H^* \). Then: \( a \not\in b \) implies \( Q(a) = Q(b) \) and \( a \in E_{(1/n)} b \) implies \( Q(a) \neq Q(b) \).
13 SUMMABLE IDEALS

Proof. The first statement holds just because \( Q \) is \( F \)-invariant. Now suppose that \( a \in E_{(1/n)} b \). Then by definition \( s = b \mid \mu(a, b) \in R(a) \), hence, \( s \notin Q(a) \). On the other hand, \( s \in S(b) \subseteq Q(b) \). □ (Claim)

Define \( \tau(x) = Q(\vartheta(x)) \) for \( x \in \mathbb{N}^\mathbb{N} \), so that \( \tau \) is a \( \Delta^1_1 \) map \( \mathbb{N}^\mathbb{N} \rightarrow \mathcal{P}(2^{<\omega}) \).

Claim 67.5. If \( x \in \mathbb{N}^\mathbb{N} \) then \( T_a = \{ \tau(y) : y \in [x]_E \} \) is at most countable.

Proof. Suppose that \( y, z \in [x]_E \). Then \( a = \vartheta(x) \), \( b = \vartheta(y) \), and \( c = \vartheta(z) \) belong to \( H^* \), and \( b, c \in [a]_{E_{(1/n)}} \). It follows from Claim 67.4 that if \( G(b) = G(c) \), \( e(b) = e(c) \), and \( q(b) = q(c) \), then \( Q(b) = Q(c) \). It remains to note that \( G \) takes only countably many values on \( H^* \cap [a]_{E_{(1/n)}} \) by Claim 67.2. □ (Claim)

Finally note that, if \( x \in E \ y \in \mathbb{N}^\mathbb{N} \) then \( \vartheta(x) \), \( \vartheta(y) \) belong to \( H^* \) and satisfy \( \vartheta(x) E_{(1/n)} \vartheta(y) \), hence, \( \tau(x) \neq \tau(y) \) by Claim 67.4. Thus, the Borel map \( \tau \) witnesses that the given ER \( E \) is essentially countable by Lemma 66.

13.d Case 2

Thus we suppose that the \( \Sigma^1_1 \) set \( B^* = \text{ran} \vartheta \land C \) is non-empty. Note that, by Claim 67.2, there is no non-empty \( \Sigma^1_1 \) “grainy” set \( A \subseteq B^* \).

Let \( \mathcal{B}_s = \{ a \in \mathbb{N}^\mathbb{N} : s \subseteq a \} \) for \( s \in 2^{\omega \omega} \) and \( \mathcal{N}_u = \{ x \in \mathbb{N}^\mathbb{N} : u \subseteq x \} \) for \( u \in \mathbb{N}^{<\omega} \) (basic open nbhds in \( 2^\mathbb{N} \) and \( \mathbb{N}^\mathbb{N} \)).

If \( A, B \subseteq \mathbb{N}^\mathbb{N} \) and \( m, k \in \mathbb{N} \), then \( A R_{ \geq k}^m B \) will mean that for any \( a \in A \) there is \( b \in B \) with \( \Sigma^\omega_k(a, b) < 2^{-m} \), and conversely, for any \( b \in B \) there is \( a \in A \) with \( \Sigma^\omega_k(a, b) < 2^{-m} \). This is not a ER, of course, yet the conjunction of \( A R_{ \geq k}^m B \) and \( B R_{ \geq k}^m C \) implies \( A R_{ \geq k}^{m+1} C \).

\( 0^n \) will denote the sequence of \( m \) zeros.

To prove that \( E_{(1/n)} \leq_B E \) in Case 2, we define an increasing sequence of natural numbers \( 0 = k_0 < k_1 < k_2 < ... \), and also objects \( A_s, g_s, v_s \) for any \( s \in 2^{\omega \omega} \), which satisfy the following list of requirements (i) – (viii).

(i) if \( s \in 2^m \) then \( g_s \in 2^{k_m} \), and \( s \subseteq t \implies g_s \subseteq g_t \);

(ii) \( \emptyset \neq A_s \subseteq B^* \land \mathcal{B}_{g_s}, A_s \in \Sigma^1_1 \), and \( s \subseteq t \implies A_t \subseteq A_s \).

(iii) if \( s \in 2^n \) then \( A_{0^n} R_{ \geq k_n}^{n+2} A_s \);

(iv) if \( s \in 2^m \), \( m < n \), \( s(m) = 0 \), then \( \Sigma^\omega_{k_m+1-1}(g_s, g_{0^m}) < 2^{-m-1} \);

(v) if \( s \in 2^m \), \( m < n \), \( s(m) = 1 \), then \( |\Sigma^\omega_{k_m+1-1}(g_s, g_{0^m}) - \frac{1}{m+1}| < 2^{-m-1} \);

(vi) if \( s, t \in 2^n \), \( m < n \), \( s(m) = t(m) \), then \( |\Sigma^\omega_{k_m+1-1}(g_s, g_t)| < 2^{-m} \);

(vii) if \( s \in 2^n \) then \( v_s \in \mathbb{N}^n \), and \( s \subseteq t \implies v_s \subseteq v_t \);

(viii) \( A_s \subseteq \{ a \in B^* : \vartheta^{-1}(a) \land \mathcal{N}_{v_s} \neq \emptyset \} \).
We can now accomplish Case 2 as follows. For any \( a \in 2^N \) define \( F(a) = \bigcup_n g_a | n \in 2^N \) (the only element satisfying \( g_a | n \subset F(a) \) for all \( n \)) and \( \rho(a) = \bigcup_n v_a | n \in N^N \). It follows, by (viii) and the continuity of \( \vartheta \), that \( F(a) = \vartheta(\rho(a)) \) for any \( a \in 2^N \). Thus the next claim proves that \( \rho \) is a Borel (in fact, here continuous) reduction \( E_{1/n} \) to \( E \) and ends Case 2.

**Claim 67.2.** The map \( F \) reduces \( E_{1/n} \) to \( E_{1/n} \), that is, the equivalence \( a E_{1/n} b \iff F(a) E_{1/n} F(b) \) holds for all \( a, b \in 2^N \).

**Proof.** By definition \( \Sigma(F(a), F(b)) = \lim_{n \to \infty} \Sigma^{|k_n - 1|}_{0}(g_a | n, g_b | n) \). However it follows from (iv), (v), (vi) that

\[
|\Sigma^{|k_n - 1|}_{0}(g_a | n, g_b | n) - \Sigma^{|n - 1|}_{0}(a \upharpoonright n, b \upharpoonright n)| \leq \sum_{m<n} 2^{-m} < 2.
\]

We conclude that \( |\Sigma(F(a), F(b)) - \Sigma(a, b)| \leq 2 \), as required. \( \square \) (Claim)

### 13.e Construction

The construction goes on by induction. To begin with we set \( k_0 = 0 \), \( g_A = A \) and \( A_\Lambda = B^* \). Suppose that, for some \( n \), we have the objects as required for all \( n' \leq n \), and extend the construction on the level \( n + 1 \).

As \( A_0^n \) is not “grainy” (see above), there is a pair of elements \( a^0, a^1 \in A_0^n \) such that \( |\Sigma(a^0, a^1) - \frac{1}{n+1}| < 2^{-n-2} \). Note that \( a^0 \upharpoonright k_n = a^1 \upharpoonright k_n \) by (i) and (ii), hence, there is \( k_{n+1} > k_n \) such that \( |\Sigma^{|k_{n+1} - 1|}_{k_n}(a^0, a^1) - \frac{1}{n+1}| < 2^{-n-2} \). According to (iii), for any \( s \in 2^n \) there exist \( b^0_s, b^1_s \in A_s \) such that and \( \Sigma^{|k_n - 1|}_{k_n}(a^i, b^i_s) < 2^{-n-2} \) for \( i = 0, 1 \); we can, of course, assume that \( b^0_s = a^i \). Moreover, the number \( k_{n+1} \) can be chosen big enough for the following to hold:

\[
\Sigma^{|k_{n+1} - 1|}_{k_n}(b_s, a^0) < 2^{-n-3} \quad \text{for all } s \in 2^n \text{ and } i = 0, 1. \tag{1}
\]

We let \( g_s \upharpoonright i = b_s \upharpoonright k_{n+1} \) for all \( s \upharpoonright i \in 2^{n+1} \). This definition preserves (i). To check (iv) for \( s' = s \upharpoonright 0 \in 2^{n+1} \) and \( m = n \), note that

\[
\Sigma^{|k_{n+1} - 1|}_{k_n}(g_{s'}(0^{n+1})) = \Sigma^{|k_{n+1} - 1|}_{k_n}(b_s, a^0) < 2^{-n-2}.
\]

To check (v) for \( s' = s \upharpoonright 1 \in 2^{n+1} \) and \( m = n \), note that

\[
|\Sigma^{|k_{n+1} - 1|}_{k_n}(g_{s'}(0^{n+1}) - \frac{1}{n+1}| \leq \Sigma^{|k_{n+1} - 1|}_{k_n}(b_s, a^1) + |\Sigma^{|k_{n+1} - 1|}_{k_n}(a^0, a^1) - \frac{1}{n+1}| < 2^{-n-1}.
\]

To fulfill (vii), choose, for any \( s \upharpoonright i \in 2^{n+1} \), a sequence \( v_s \upharpoonright i \in N^{n+1} \) so that \( v_s \subset v_s \upharpoonright i \) and there is \( \mathcal{N} v_s \upharpoonright i \cap \vartheta^{-1}(b_s) \neq \emptyset \).

Let us finally define the sets \( A_{s'} \subset A_s \), for all \( s' = s \upharpoonright i \in 2^{n+1} \) (so that \( s \in 2^n \) and \( i = 0, 1 \)). To fulfill (ii) and (viii), we begin with

\[
A'_{s \upharpoonright i} = \{ a \in A_s \cap g_{s \upharpoonright i} : \vartheta^{-1}(a) \cap \mathcal{N} v_s \upharpoonright i \neq \emptyset \}.
\]
This is a $\Sigma^1_1$ subset of $A_s$, containing $b''_s$. To fulfill (iii), we define $A_{0^n+1}$ to be the set of all $a \in A'_{0^n+1}$ such that
\[
\forall s' = s^\land i \in 2^{n+1} \exists b \in A'_{s'} (\Sigma^\infty_{k_{s+1}}(a,b) < 2^{-n-3}) ;
\]
this is still a $\Sigma^1_1$ set containing $b_0^{0^n} = a^0$ by (1). It remains to define, for any $s^\land i \neq 0^{n+1}$, $A_{s^\land i}$ to be the set of all $b \in A'_{s^\land i}$ such that
\[
\exists b \in A_{0^n+1} (\Sigma^\infty_{k_{s+1}}(a,b) < 2^{-n-3}) .
\]
This ends the definition for the level $n + 1$.

\[\square\] (Construction and Theorem 67)

14 $c_0$-equalities

Suppose that $\langle X_k ; d_k \rangle$ is a finite metric space for each $k \in \mathbb{N}$. Farah [7] defines an equivalence relation $D = D(\langle X_k ; d_k \rangle)$ on $\mathbb{X} = \prod_{k \in \mathbb{N}} X_k$ as follows:
\[
x \equiv y \iff \lim_{k \to \infty} d_k(x_k, y_k) = 0 .
\]

ERs of this form are called $c_0$-equalities. In addition, $D(\langle X_k ; d_k \rangle)$ is nontrivial if $\limsup_{k \to \infty} \text{diam}(X_k) > 0$ (otherwise $D(\langle X_k ; d_k \rangle)$ makes everything equivalent). Every $c_0$-equality is easily a Borel ER, more exactly, of class $\Pi^0_3$.

14.a Some examples and simple results

**Example 68.** (1) Let $X_k = \{0, 1\}$ with $d_k(0, 1) = 1$ for all $k$. Then clearly the relation $D(\langle X_k ; d_k \rangle)$ on $2^\mathbb{N} = \prod_{k \in \mathbb{N}} \{0, 1\}$ is just $E_0$.

(2) Let $X_{kl} = \{0, 1\}$ with $d_{kl}(0, 1) = k^{-1}$ for all $k, l \in \mathbb{N}$. Then the relation $D(\langle X_{kl} ; d_{kl} \rangle)$ on $2^{\mathbb{N} \times \mathbb{N}} = \prod_{k,l} \{0, 1\}$ is just $E_3 = E_0 \times \text{Fin}$.

(3) Generally, if $0 = n_0 < n_1 < n_2 < \ldots$ and $\varphi_i$ is a submeasure on $[n_i, n_{i+1})$, then let $X_i = \mathcal{P}([n_i, n_{i+1})]$ and $d_i(u, v) = \varphi_i(u \Delta v)$ for $u, v \subseteq [n_i, n_{i+1})$. Then $D(\langle X_i ; d_i \rangle)$ is clearly isomorphic to $E_\mathcal{F}$, where
\[
\mathcal{F} = \text{Exh}(\varphi) = \{ x \subseteq \mathbb{N} : \lim_{n \to \infty} \varphi(x \cap [n, \infty)) = 0 \}
\]

and $\varphi(x) = \sup_i \varphi_i(x \cap [n_i, n_{i+1}))$.

(4) Let $D_{\max} = D(\langle X_k ; d_k \rangle)$, where $X_k = \{0, 1\}$ and $d_k$ is the distance on $X_k$ inherited from $\mathbb{R}$.

**Proposition 69** (Farah [7] with a reference to Oliver). (i) $D_{\max} \sim_{\mathbb{N}} Z_0$;

(ii) if $D$ is a $c_0$-equality then $D \leq_{\mathbb{N}} D_{\max}$, moreover, $D \leq_{\Lambda} D_{\max}$.
Thus $D_{\text{max}}$ is a maximal, in a sense, among $c_0$-equalities.

**Proof.** (i) It is clear that $D_{\text{max}}$ is the same as $c_0 \upharpoonright \mathcal{X}$, where $\mathcal{X} \subseteq \mathbb{R}^n$ is defined as in the proof of Lemma 20, where it is also shown that $c_0 \sim_B c_0 \upharpoonright \mathcal{X}$.

(ii) To prove $D \leq_B D_{\text{max}}$, it suffices, by (i) and Lemma 20, to show that $D \leq_B c_0$. The proof is based on the following:

**Claim 69.1.** Any finite $n$-element metric space $\langle X ; d \rangle$ is isometric to an $n$-element subset of $\langle \mathbb{R}^n ; \rho_n \rangle$, where $\rho_n$ be the distance on $\mathbb{R}^n$ defined by $\rho_n(x,y) = \max_{i<n} |x_i - y_i|$. 

**Proof of the claim.** Let $X = \{x_1, \ldots, x_n\}$. It suffices to prove that for any $k \neq l$ there is a set of reals $\{r_1, \ldots, r_n\}$ such that $|r_k - r_l| = d(x_k, x_l)$ and

$$|r_i - r_j| \leq d_{ij} = d(x_i, x_j) \text{ for all } i, j.$$ 

We can assume that $k = 1$ and $l = n$.

**Step 1.** There is a least number $h_1 \geq 0$ such that (*) holds for the numbers $\{0, 0, \ldots, 0, h\}$ for any $0 \leq h \leq h_1$. Then, for some $k$, $1 \leq k < n$, we have $h_1 = d_{kn}$ exactly. Suppose that $k \neq 1$; then it can be assumed that $k = n - 1$.

**Step 2.** Similarly, there is a least number $h_2 \geq 0$ such that (*) holds for the numbers $\{0, 0, \ldots, 0, h, h_1 + h\}$ for any $0 \leq h \leq h_2$. Then, for some $k, \nu$, $1 \leq k < n - 1 \leq \nu \leq n$, we have $h_2 = d_{k\nu}$ exactly. Suppose that $k \neq 1$; then it can be assumed that $k = n - 2$.

**Step 3.** Similarly, there is a least number $h_3 \geq 0$ such that (*) holds for the numbers $\{0, 0, \ldots, 0, h, h_2 + h, h_1 + h_2 + h\}$ for any $0 \leq h \leq h_3$. Then again, for some $k, \nu$, $1 \leq k < n - 2 \leq \nu \leq n$, we have $h_3 = d_{k\nu}$ exactly. Suppose that $k \neq 1$; then it can be assumed that $k = n - 3$.

**Et cetera.**

This process ends, after a number $m$ ($m < n$) steps, in such a way that the index $k$ obtained at the final step is equal to 1. Then (*) holds for the numbers $\{0, 0, \ldots, 0, r_{n-m+1}, r_{n-m+2}, \ldots, r_n\}$, where $r_{n-m+j} = h_m + h_{m-1} + \cdots + h_{m-j+1}$ for each $j = 1, \ldots, m$. Moreover it follows from the construction that there is a decreasing sequence $n = k_0 > k_1 > k_2 > \cdots > k_\mu = 1$ ($\mu \leq m$) such that $r_{k_i} - r_{k_{i+1}} = d_{k_{i+1} k_i}$ exactly for any $i$. Then $d_{1n} \leq \sum_i (r_{k_i} - r_{k_{i+1}})$ by the triangle inequality. But the right-hand side is a part of the sum $r_n = h_1 + \cdots + h_m$, and hence $r_n \geq d_{1n}$. It follows that, cutting the construction at an appropriate step $m' \leq m$ (and taking an appropriate value of $h \leq h_{m'}$), we obtain a sequence of numbers $r_1 = 0 \leq r_2 \leq \cdots \leq r_{n-1} \leq r_n$ still satisfying (*) and satisfying $r_n = r_n - r_0 = d_{1n}$. This ends the proof. \(\square\) (Claim)
14 \textbf{c}_0\text{-EQUALITIES}

Now, to carry out the proof of $D \leq_B c_0$, suppose that $D = D(\langle X_k; d_k \rangle)$ is an equivalence relation on $\mathcal{X} = \prod_{k \in \mathbb{N}} X_k$, where each $\langle X_k; d_k \rangle$ is a finite metric space. Let $n_k$ be the number of elements in $X_k$. Let, by the claim, $\eta_k : X_k \to \mathbb{R}^{\alpha_k}$ be an isometric embedding of $\langle X_k; d_k \rangle$ into $\langle \mathbb{R}^{\alpha_k}; \rho_{n_k} \rangle$. The map $\vartheta(x) = \eta_0(x_0)^\lambda \eta_1(x_1)^\lambda \eta_2(x_2)^\lambda \ldots$ (from $\mathcal{X}$ to $\mathbb{R}^k$) reduces $D$ to $c_0$. \hfill $\square$

The structure of $c_0$-equalities tend to be connected more with the additive reducibility $\leq_A$ (see §1.d on $\leq_A$ and the associated relations $<_A$ and $\sim_A$) than with the general Borel reducibility. In particular, we have

**Lemma 70.** For any $c_0$-equality $D = D(\langle X_k; d_k \rangle)$, if $D'$ is a Borel ER on a set $\prod_k X'_k$ (with finite nonempty $X'_k$) and $D' \leq_A D$ then $D'$ is a $c_0$-equality.

**Proof.** Let a sequence $0 = n_0 < n_1 < n_2 < \ldots$ and a collection of maps $H_i : X'_i = \prod_{n_i \leq k < n_{i+1}} X_k$ witness $D' \leq_A D$. For $x', y' \in X'_i$ put

$$d'_i(x', y') = \max_{n_i \leq k < n_{i+1}} d_k(H_i(x'_k), H_i(y'_k)).$$

Then easily $D' = D(\langle X'_k; d'_k \rangle)$. \hfill $\square$

**Lemma 71** (Farah [7] with a reference to Hjorth). Every $c_0$-equality $D = D(\langle X_k; d_k \rangle)$ is induced by a continuous action of a Polish group.

(The domain $\mathcal{X} = \prod_k X_k$ of $D$ is considered with the product topology.)

**Proof.** (sketch) For any $k$ let $S_k$ be the (finite) group of all permutations of $X_k$, with the distance $\rho_k(s, t) = \max_{x \in X_k} d_k(s(x), t(x))$. Then

$$G = \{g \in \prod_k S_k : \lim_{k \to \infty} \rho_k(g e_k, e_k) = 0\},$$

where $e_k \in S_k$ is the identity,

is easily a subgroup of $\prod_k S_k$, moreover, the distance $d(g, h) = \sup_k \rho_k(g_k, h_k)$ converts $G$ into a Polish group, the natural action of which on $\mathcal{X}$ (i.e., $(g \cdot x)_k = g_k(x_k), \forall k$) is continuous and induces $D$. \hfill $\square$

### 14.b Classification

Recall that for a metric space $\langle A; d \rangle$, a rational $q > 0$, and $a \in A$, $\text{Gal}_A^q(a)$ is the set of all $b \in A$ which can be connected with $a$ by a finite chain $a = a_0, a_1, \ldots, a_n = b$ with $d(a_i, a_{i+1}) < q$ for all $i$. Farah defines, for $r > 0$,

$$\delta(r, A) = \inf \{q \in \mathbb{Q}^+ : \exists a \in A (\text{diam}(\text{Gal}_A^q(a)) \geq r)\}$$

(with the understanding that here $\inf \emptyset = +\infty$), and

$$\Delta(A) = \{d(a, b) : a \neq b \in A\},$$

so that $\text{diam} A = \sup(\Delta(A) \cup \{0\})$.

Now let $D = D(\langle X_k; d_k \rangle)$ be a $c_0$-equality on $\mathcal{X} = \prod_{k \in \mathbb{N}} X_k$. The basic properties of $D$ are determined by the following two conditions:
(co1) \( \liminf_{k \to \infty} \delta(r, X_k) = 0 \) for some \( r > 0 \).

\( \forall \varepsilon > 0 \exists \varepsilon' \in (0, \varepsilon) \exists \infty k \ (\Delta(X_k) \cap [\varepsilon', \varepsilon) \neq \emptyset). \)

Easily (co1) implies both the nontriviality of \( D = D((X_k ; d_k)) \) and (co2).

**Theorem 72** (Farah [7]). Let \( D = D((X_k ; d_k)) \) be a nontrivial \( c_0 \)-equality. Then

(i) If (co2), hence, (co1) fail then \( D \sim_A E_0 \), hence, \( D \sim B E_0 \);

(ii) If (co1) fails but (co2) holds then \( D \sim_A E_3 \), hence, \( D \sim_B E_3 \);

(iii) If (co1), hence, (co2) hold then \( E_0 \leq_A D \) and \( D_1 \leq_A D \) for a turbulent \( c_0 \)-equality \( D_1 \) satisfying \( E_3 \leq_A D_1 \).

**Proof.** (i) To show that \( E_0 \leq_A D \) note that, by the nontriviality of \( D \), there exist a number \( p > 0 \), an increasing sequence \( 0 = n_0 < n_1 < n_2 < \ldots \), and, for any \( i \), a pair of points \( x_{n_i}, y_{n_i} \in X_{n_i} \) with \( d_{n_i}(x_{n_i}, y_{n_i}) \geq p \). For \( n \) not of the form \( n_i \) fix an arbitrary \( x_n \in X_n \). Now, if \( a \in 2^k \), then define \( \vartheta(a) \in \Pi_k X_k \) so that \( \vartheta(a)_n = z_n \) for \( n \) not of the form \( n_i \), while \( \vartheta(a)_n = x_n \) or \( y_n \), if resp. \( a_i = 0 \) or \( a_i = 1 \). This map \( \vartheta \) witnesses \( E_0 \leq_A D \).

Now prove that \( D \leq_A E_0 \). As (co2) fails, there is \( \varepsilon > 0 \) such that for each \( \varepsilon' \) with \( 0 < \varepsilon' < \varepsilon \) we have only finitely many \( k \) with the property that \( \varepsilon' < d_k(\xi, \eta) < \varepsilon \) for some \( \xi, \eta \in X_k \). Let \( G_k \) be the (finite) set of all \( \frac{1}{k} \)-galaxies in \( X_k \), and let \( \vartheta : \mathcal{X} = \Pi_k X_k \to G = \Pi_k G_k \) be defined as follows: \( \vartheta(x)_k \) is that galaxy in \( G_k \) to which \( x_k \) belongs. Let \( E \) be the \( G \)-version of \( E_0 \), i.e., if \( g, h \in G \) then \( g Eh \) iff \( g_k = h_k \) for all but finite \( k \). As easily \( E \leq_A E_0 \), it suffices to demonstrate that \( D \leq_A E \) via \( \vartheta \). Suppose that \( x, y \in \mathcal{X} \) and \( \vartheta(x) E \vartheta(y) \) and prove \( x D y \) (the nontrivial direction). Let, on the contrary, \( x \not\sim y \), so that there is a number \( p > 0 \) with \( d_k(x_k, y_k) > p \) for infinitely many \( k \). We can assume that \( p < \frac{\varepsilon}{2} \). On the other hand, as \( \vartheta(x) E \vartheta(y) \), there is \( k_0 \) such that \( x_k \) and \( y_k \) belong to one and the same \( \frac{1}{k} \)-galaxy in \( X_k \) for all \( k > k_0 \). Then, for any \( k > k_0 \) with \( d_k(x_k, y_k) > p \) (i.e., for infinitely many values of \( k \)) there exists an element \( z_k \in X_k \) in the same galaxy such that \( p < d_k(x_k, z_k) < \varepsilon \), but this is a contradiction to the choice of \( \varepsilon \) (indeed, take \( \varepsilon' = p \)).

(ii) Let us show first that if (co2) holds then \( E_3 \leq_A D \) (independently of (co1)). It follows from (co2) that there exist: an infinite sequence \( \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots > 0 \), for any \( i \) an infinite set \( J_i \), and for any \( j \in J_i \) a pair of elements \( x_{ij}, y_{ij} \in X_j \) with \( d_j(x_{ij}, y_{ij}) \in [\varepsilon_{i+1}, \varepsilon_i) \). We may assume that the sets \( J_i \) are pairwise disjoint. Then the \( c_0 \)-equality \( D' = D((\{x_{ij}, y_{ij}\}; d_j)_{i \in \mathbb{N}, j \in J_i}) \) satisfies both \( D' \leq_A D \) and \( D' \cong E_3 \) (via a bijection between the underlying sets).

Now, assuming that, in addition, (co1) fails, we show that \( D \leq_A E_3 \). For all \( k, n \in \mathbb{N} \) let \( G_{kn} \) be the (finite) set of all \( \frac{1}{n} \)-galaxies in \( X_k \). For any \( x \in \mathcal{X} = \prod_i X_i \) define \( \vartheta(x) \in G = \prod_{k,n} G_{kn} \) so that \( \vartheta(x)_{kn} \) is that \( \frac{1}{n} \)-galaxy in
\[ G_{kn} \] to which \( x_k \) belongs (for all \( k, n \)). The ER \( E \) on \( G \), defined so that \( g \in E h \) iff \( \forall n \ (g_n = h_n) \) (\( g, h \in G \)) is easily \( \leq_\Lambda E_3 \), so it suffices to show that \( D \leq_\Lambda E \) via \( \vartheta \). Suppose that \( x, y \in X \) and \( \vartheta(x) \in \vartheta(y) \) and prove \( x \sim y \) (the nontrivial direction). Otherwise there is some \( r > 0 \) with \( d_k(x_k, y_k) > r \) for infinitely many \( k \). As \( (\text{co1}) \) fails for this \( r \), there is \( n \) big enough for \( \delta(r, X_k) > \frac{1}{n} \) to hold for almost all \( k \). Then, by the choice of \( r \), we have \( \vartheta(x)_k \neq \vartheta(y)_k \) for infinitely many \( k \), hence, \( \vartheta(x) \notin \vartheta(y) \), contradiction.

(iii) Fix \( r > 0 \) with \( \liminf_{k \to \infty} \delta(r, X_k) = 0 \). As for any increasing sequence \( n_0 < n_1 < n_2 < \ldots \), we have \( D(\langle X_{n_i}; d_{n_i} \rangle) \leq_\Lambda D \), it can be assumed that \( \lim_k \delta(r, X_k) = 0 \), and further that \( \delta(r, X_k) < \frac{1}{k} \) for all \( k \). Then each \( X_k \) contains a \( \frac{1}{k} \)-galaxy \( Y_k \subseteq X_k \) of \( \text{diam} Y_k \geq r \). As easily \( D(\langle Y_k; d_k \rangle) \leq_\Lambda D \), the following lemma suffices to prove (iii).

**Lemma 72.1.** Suppose that \( r > 0 \) and each \( X_k \) is a single \( \frac{1}{k} \)-galaxy in itself with \( \text{diam}(X_k) \geq r \). Then \( D = D(\langle X_k; d_k \rangle) \) is turbulent and \( E_3 \leq_\Lambda D \).

**Proof.** We know from the proof of (iii) above that \( E_3 \leq_\Lambda D \). Now prove that the natural action of the Polish group \( \mathbb{G} \) defined as in the proof of Lemma 71 is turbulent under the assumptions of the lemma.

That every \( D \)-class is dense in \( X = \bigprod_k X_k \) (with the product topology on \( X \)) is an easy exercise. To see that every \( D \)-class \([x]_D \) also is meager in \( X \), note that by the assumptions of the lemma any \( X_k \) contains a pair of elements \( x'_k, x''_k \) with \( d_k(x'_k, x''_k) \geq r \). Let \( y_k \) be one of \( x'_k, x''_k \) which is \( d_k \)-farther than \( \frac{1}{2} \) from \( x_k \). Now the set \( Z = \{ z \in X : \exists k \langle x_k \rangle \} \) is comeager in \( X \) and disjoint from \([x]_D \). It remains to prove that local orbits are somewhere dense.

Let \( G \) be an open nbhd of the identity in \( \mathbb{G} \) and \( \emptyset \neq X \subseteq X \) be open in \( X \).

We can assume that, for some \( n, G \), the \( \frac{1}{n} \)-ball around the identity in \( \mathbb{G} \), while \( X = \{ x \in X : \forall k < n (x_k = \xi_k) \} \), where elements \( \xi_k \in X_k, k < n, \) are fixed. It is enough to prove that all classes of the local orbit relation \( \sim_{X}^G \) are dense in \( X \).

Consider an open set \( Y = \{ y \in X : \forall k < m (y_k = \xi_k) \} \subseteq X \), where \( m > n \) and elements \( \xi_k \in X_k, n \leq k < m, \) are fixed in addition to the above.

Let \( x \in X \). Then \( x_k = \xi_k \) for \( k < n \). Let \( n \leq k < m \). The elements \( \xi_k \) and \( x_k \) belong to \( X_k \), which is a \( \frac{1}{k} \)-galaxy, therefore, there is a chain, of a length \( \ell(k) \), of elements of \( X_k \), which connects \( x_k \) and \( \xi_k \) so that every step within the chain has \( d_k \)-length \( < \frac{1}{k} \). Then there is a permutation \( g_k \) of \( X_k \) such that \( g_{k}^\ell(k)(x_k) = \xi_k \), \( g_k(\xi_k) = x_k \), and \( d_k(\xi, g_k(\xi)) < \frac{1}{k} \) for all \( \xi \in X_k \). Let \( g_k \) be the identity on \( X_k \) whenever \( k < n \) or \( k \geq m \). This defines an element \( g \in \mathbb{G} \) which obviously belongs to \( G \), moreover, \( X \) is \( g \)-invariant and \( g^\ell(x) \in U \), where \( \ell = \prod_{n \leq k < m} \ell(k) \), hence, \( x \sim_{X}^G g(x) \), as required.

\( \square \) (Lemma)

\( \square \) (Theorem 72)

**Remark 73.** Theorem 72 shows that any nontrivial \( c_0 \)-equality \( D \leq_\Lambda \)-contains a turbulent \( c_0 \)-equality \( D' \) with \( E_3 \leq_\Lambda D' \) (and the turbulence of \( D' \) holds, in
particular, via the natural action defined in the proof of Lemma 71), unless $D$ is
\[ \sim_A \] to $E_0$ or $E_3$, and that (co1) is necessary for the turbulence of $D$ itself and
sufficient for a turbulent $c_0$-equality $D' \leq_A D$ to exist. \hfill \Box

14. **LV-EQUALITIES**

By Farah, an **LV-equality** is a $c_0$-equality $D = D(<X_k; d_k>)$ satisfying

\[ (LV1) \quad \forall m \forall \varepsilon > 0 \forall n \forall x_0, \ldots, x_m \in X_k \quad d_k(x_0, x_m) \leq \max_{j<m} d_k(x_j, x_{j+1}) + \varepsilon. \]

In other words, the metrics involved are postulated to be “asymptotically close”
to ultrametrics. This sort of $c_0$-equalities was first considered by Louveau and
Velickovic [31]. The following simple fact is analogous to Lemma 70.

**Lemma 74.** For any LV-equality $D$, if $D'$ is a Borel ER on a set $\prod_k X'_k$ (with
finite nonempty $X'_k$) and $D' \leq_A D$ then $D'$ is an LV-equality. \hfill \Box

**Example 75** (Louveau and Velickovic [31]). We define $X_k = \{1, 2, \ldots, 2^k\}$ and
\[ d_k(m, n) = \log(|m - n| + 1)/k \] for $1 \leq m, n \leq 2^k$. \hfill \Box

**Theorem 76** (Essentially, Louveau and Velickovic [31]). Let $D = D(<X_k, d_k>)$
be a turbulent LV-equality. Then we can associate, with each infinite $A \subseteq \mathbb{N}$, a
$\nu$-equality $D_A \leq_A D$ such that for all $A, B \subseteq \mathbb{N}$ the following are equivalent:

(i) $A \subseteq^* B$ (i.e., $A \Delta B$ is finite);

(ii) $D_A \leq A D_B$;

(iii) $D_A \leq_{BM} D_B$ (i.e., via a Baire measurable reduction).

This theorem was the first major application of $c_0$-equalities. One of its corollaries is that there exist big families of mutually irreducible Borel ERs!

**Proof.** As $D$ is turbulent, the necessary turbulence condition (co1) of §14.b holds, moreover, as in the proof of Theorem 72 (case (iii)), we can assume that it
takes the following special form for some $r > 0$:

(1) Each $X_k$ is a single $\min\{r, \frac{1}{k+1}\}$-galaxy of $\text{diam}(X_k) \geq 4r$.

The intended transformations (reduction to a certain infinite subsequence of
spaces $<X_k; d_k>$, and then each $X_k$ to a suitable galaxy $Y_k \subseteq X_k$) preserve
(LV1), of course, moreover, going to subsequences once again, we can assume that (LV1) holds
in the following special form:

(2) $d_k(x_0, x_m) \leq \max_{i<m} d_k(x_i, x_{i+1}) + \frac{1}{k+1}$ whenever $x_0, \ldots, x_m \in X_k$, where
\[ m_k = 2 \prod_{j=0}^{\infty} \#(X_j). \]

We can derive the following important consequence:
(3) For any \( k \) there is a set \( Y_k \subseteq X_k \) of \( \#(Y_k) = m_k \) such that we have \( d_k(x, y) \geq r \) for all \( x \neq y \) in \( Y_k \).

To prove this note that by (1) there is a set \( \{x_0, ..., x_m\} \subseteq X_k \) such that \( d_k(x_0, x_m) \geq 4r \) but \( d_k(x_i, x_{i+1}) < r \) for all \( i \). We may assume that \( m \) is the least possible length of such a sequence \( \{x_i\} \). Now let us define a subsequence \( \{y_0, y_1, ..., y_n\} \) of \( \{x_i\} \), the number \( n \leq m \) will be specified in the course of the construction. Put \( y_0 = x_0 \). If \( y_j = x_{i(j)} \) has been defined, and there is \( l > i(j), l \leq m \), such that \( d_k(y_j, x_l) \geq r \), then let \( y_{j+1} = x_l \) for the least such \( l \), otherwise put \( n = j \) and stop the construction.

By definition \( d_k(y_j, y_{j+1}) \geq r \) for all \( j < n \), moreover, \( d_k(y_{j'}, y_{j'+1}) \geq r \) for any \( j' < j \) by the minimality of \( m \). Thus \( Y_k = \{y_j : j \leq n\} \) satisfies \( d_k(x, y) \geq r \) for all \( x \neq y \) in \( Y_k \). It remains to prove that \( n \geq m_k \). Indeed we have \( d_k(y_j, y_{j+1}) < 2r \) by the construction, hence, if \( n \leq m_k \) then we would have \( d_k(y_0, y_n) \leq 3r \) by (2), which implies \( d_k(y_n, x_m) \geq r \), a contradiction to the assumption that the construction stops with \( y_n \).

This said, we proceed to the proof of the theorem. First note that

**Lemma 76.1.** (iii) implies that (ii) holds at least for some (infinite) \( A' \subseteq A \).

**Proof.** A Borel reduction can be extracted from a Baire measurable one by a version of the “stabilizers” construction (see proofs of ... ) □ (Lemma 76.1)

Thus it remains only to show that (ii) implies (i), even simpler, that, for any disjoint infinite sets \( A, B \subseteq \mathbb{N} \), \( D_A \leq A \) \( D_B \) fails. Suppose, towards the contrary, that \( D_A \leq A \) \( D_B \) holds, and let this be witnessed by a reduction \( \Psi \) defined (as in §1.d) from an increasing sequence \( \min B = n_0 < n_1 < n_2 < ... \) of numbers \( n_i \in B \) and a collection of maps \( H_k : X_k \to \prod_{j \in n_i, n_{i+1}} \cap B X_j \), \( k \in A \). Let

\[
f_k(\delta) = \max_{\xi, \eta \in X_k, d_k(\xi, \eta) < \delta} \max_{j \in n_i, n_{i+1} \cap B} d_j(H_k(\xi)_j, H_k(\eta)_j),
\]

for \( k \in \mathbb{N} \) and \( \delta > 0 \) (with the understanding that \( \max \emptyset = 0 \) if applicable). Then \( f(\delta) = \sup_{k \in A} f_k(\delta) \) is a nondecreasing map \( \mathbb{R}^+ \to [0, \infty) \).

**Lemma 76.2.** \( \lim_{\delta \to 0} f(\delta) = 0 \).

**Proof.** Otherwise there is \( \varepsilon > 0 \) such that \( f(\delta) \geq \varepsilon \) for all \( \delta \). Then the numbers

\[
\mu_k = \min_{\xi, \eta \in X_k, \xi \neq \eta} d_k(\xi, \eta) \quad \text{(all of them are > 0)}
\]

must satisfy \( \inf_{k \in A} \mu_k = 0 \). This allows us to define a sequence \( k_0 < k_1 < k_2 < ... \) of numbers \( k_i \in A \), and, for any \( k_i \), a pair of \( \xi_i, \eta_i \in X_k \), with \( d_k(\xi_i, \eta_i) \to 0 \), and also \( j_i \in [n_{k_i}, n_{k_i+1}] \cap B \) such that \( d_{j_i}(H_k(\xi_i)_j, H_k(\eta_i)_j) \geq \varepsilon \). Let \( x, y \in \prod_{k \in A} X_k \) satisfy \( x_{k_i} = \xi_i \) and \( y_{k_i} = \eta_i \) for all \( i \) and \( x_k = y_k \) for all \( k \in A \) not of the form \( k_i \). Then easily \( x D_A y \) holds but \( \Psi(x) D_B \Psi(y) \) fails, which is a contradiction. □ (Lemma 76.2)
Let $k \in A$, and let $Y_k \subseteq X_k$ be as in (3). Then there exist elements $x_k \neq y_k$ in $Y_k$ such that $H_k(x_k) \mid k = H_k(y_k) \mid k$. By (1) there is a chain $x_k = \xi_0, \xi_1, ..., \xi_n = y_k$ of elements $\xi_i \in X_k$ with $d_k(z_i, z_{i+1}) \leq \frac{1}{k+1}$ for all $i < n$. Now $H_k(\xi_i) \in \prod_{j \in [n_i, n_{i+1})} X_j$ for each $i \leq n$. Let $j \in [n_i, n_{i+1}) \cap B$. If $j > k$ then the elements $y_j^i = H_k(\xi_i)_j$, $i \leq n$, satisfy $d_j(y_j^i, y_j^{i+1}) \leq f_k(\frac{1}{k+1})$. As clearly $n < m_j$, we conclude that $d_j(H_k(x_k)_j, H_k(y_k)_j) \leq f_k(\frac{1}{k+1}) + \frac{1}{k+1}$ by (2). If $j < k$ then simply $H_k(x_k)_j = H_k(y_k)_j$ by the choice of $x_k, y_k$. Thus totally

$$d_j(H_k(x_k)_j, H_k(y_k)_j) \leq f_k(\frac{1}{k+1}) + \frac{1}{k+1}$$

for all $j \in [n_i, n_{i+1}) \cap B$.

(4) $d_j(H_k(x_k)_j, H_k(y_k)_j) \leq f(\frac{1}{k+1}) + \frac{1}{k+1}$ for all $j \in [n_i, n_{i+1}) \cap B$.

(as $k \notin B$). Let $x = \{x_k\}_{k \in A}$ and $y = \{y_k\}_{k \in A}$, both are elements of $\prod_{k \in A} X_k$, and $x D_A y$ fails because $d_k(x_k, y_k) \geq r$ for all $k$. On the other hand, we have $\Psi(x) D_B \Psi(y)$ by (4), because $f(\delta) \to 0$ with $\delta \to 0$ by Lemma 76.2. This is a contradiction to the assumption that $\Psi$ reduces $D_A$ to $D_B$.

□ (Theorem 76)
15 \( \mathbf{T}_2 \) is not reducible to ...

This section contains a theorem saying that the ER \( \mathbf{T}_2 \) of equality of countable sets of the reals is not Borel reducible to ERs which belong to a family of pinned ERs, including, for instance, continuous actions of \( \text{c}_{1}1 \) groups and some ideals, not only Polishable, and is closed under the Fubini product modulo Fin. But the \emph{prima facie} definition of the family is based on a rather metamathematical property which we extracted from Hjorth [14].

Recall that \( \mathbf{T}_2 \) is defined on \((\mathbb{N}^\mathbb{N})^\mathbb{N}\) as follows: \( x \mathbf{T}_2 y \) iff \( \text{ran} \, x = \text{ran} \, y \).

Suppose that \( X \) is \( \Sigma^1_1 \) or \( \Pi^1_1 \) in the universe \( \mathcal{V} \), and an extension \( \mathcal{V}^+ \) of \( \mathcal{V} \) is considered. In this case, let \( X^\# \) denote what results by the definition of \( X \) applied in \( \mathcal{V}^+ \). There is no ambiguity here by Shoenfield, and easily \( X = X^\# \cap \mathcal{V} \).

15.a Pinned ERs do not reduce \( \mathbf{T}_2 \)

Fix a Polish space \( \mathcal{X} \) and let \( \{B_n\}_{n \in \mathbb{N}} \) be a base of its topology. By a \emph{Borel code} for \( \mathcal{X} \) we shall understand a pair \( p = (T, \mathcal{F}) \) of a wellfounded tree \( \emptyset \neq T = T_p \subseteq \text{Ord}^{<\omega} \) (then \( \Lambda \in T \)) and a map \( f : \text{Max} \, T \to \mathbb{N} \), where \( \text{Max} \, T \) is the set of all \( \subseteq \)-maximal elements of \( T \). We define \( B_p(t) \subseteq \mathbb{N}^\mathbb{N} \) for any \( t \in T \) by induction on the rank of \( t \) in \( T \), so that

- \( B_p(t) = B_f(t) \) for all \( t \in \text{Max} \, T \), and
- \( B_p(t) = \bigcup_{t^\xi \in T} B_p(t^\xi) \) for \( t \in T \setminus \text{Max} \, T \);
- finally, put \( B_p = B_p(\Lambda) \).

For a Borel code \( p = (T, \mathcal{F}) \), let \( \text{sup} \, p = \text{sup} \, T \) be the least ordinal \( \gamma \) with \( T \subseteq \gamma^{<\omega} \). A code \( p \) is \emph{countable} if \( \text{sup} \, p < \omega_1 \), in this case the coded set \( B_p \) is a Borel subset of \( \mathcal{X} \).

\textbf{Definition 77.} A \( \Sigma^1_1 \) ER \( E \) is \emph{pinned} if, for any (perhaps, uncountable) Borel code \( p \), if \( B_p \) is 2wise \( E^\# \)-equivalent in any generic extension of \( \mathcal{V} \) and non-empty in some generic extension of \( \mathcal{V} \), then there is a point \( x \in \text{dom} \, E \), “pinning” \( p \) in the sense that \( B_p \subseteq [x]_{E^\#} \) in any extension of \( \mathcal{V} \).

\textbf{Claim 77.1.} \( \mathbf{T}_2 \) is \emph{not} pinned.

\textbf{Proof.} Consider a Borel code \( p \) for the set \( \{x \in (\mathbb{N}^\mathbb{N})^\mathbb{N} : \text{ran} \, x = \mathbb{N}^\mathbb{N} \cap \mathcal{V} \} \), so that \( B_p \subseteq (\mathcal{V})^{<\omega} \). Then if of Definition 77 holds, actually, \( B_p \) is a \( \mathbf{T}_2 \)-equivalence class in any universe where it is non-empty, but then fails. \( \square \)

\textbf{Lemma 77.2.} If \( E, F \) are \( \Sigma^1_1 \) ERs, \( E \leq^B F \), and \( F \) is pinned, then so is \( E \).

\textbf{Proof.} Suppose that, in \( \mathcal{V} \), \( \vartheta : \mathcal{X} \to \mathcal{Y} \) is a Borel reduction of \( E \) to \( F \), where \( \mathcal{X} = \text{dom} \, E \) and \( \mathcal{Y} = \text{dom} \, F \). We can assume that \( \mathcal{X} \) and \( \mathcal{Y} \) are just two copies of
Let $\tau$ be a (countable) Borel code for $\emptyset$ as a subset of $\mathbb{X} \times \mathbb{Y}$. Let $p$ be a Borel code satisfying $\mathbf{if}$ of Definition 77. There is perhaps no Borel code $q$ such that $B_q = B_{r}''B_p$ everywhere, but still there is a code $q$ with $B_q \subseteq B_{r}''B_p$ and $B_q \neq \emptyset$ somewhere. Indeed, let, in $\mathbb{V}$, $\lambda = \text{card}(\sup p)$ and $\kappa = \lambda^+$ (the next cardinal). Consider the formula $A(p,r,y)$ saying:

- $y \in \mathbb{Y}$ and there is a forcing term $\tau \in L[p,r,y]$ such that the forcing $\text{COLL}(\mathbb{N},\lambda)$ forces $\tau[\mathcal{G}] \in B_p$ and $y = B_r(\tau[\mathcal{G}])$.

As it is known, there is a Borel code $q$ such that

$$B_q = \{ y : L_\kappa[p,r,y] \models A(p,r,y) \}$$

in any extension of $\mathbb{V}$. Then easily $B_q \subseteq B_{r}''B_p$, hence, $B_q$ is 2wise $\mathbb{F}^\#$-equivalent in any universe, in addition, $B_q$ is nonempty somewhere.

As $\mathbb{F}$ is pinned, there is, in $\mathbb{V}$, a point $y \in \mathbb{Y}$ such that $B_q \subseteq [y]_{\mathbb{F}^\#}$ holds, in particular, in $\text{COLL}(\mathbb{N},\lambda)$-generic extension $\mathbb{V}^+$ of $\mathbb{V}$, where $B_q \neq \emptyset$, hence, there is $x \in B_p \cap \mathbb{V}^+$ with $y \mathbb{F}^\# B_r(x)$. It follows, by Shoenfield, that $y \mathbb{F} \vartheta(x')$ for some $x' \in \mathbb{X}$ in $\mathbb{V}$. Thus $x \mathbb{E}^\# x'$, which implies that $x' \in \mathbb{V}$ pins $p$, as required. $\square$

15.b Fubini product of pinned ERs is pinned

Recall that the Fubini product $\mathbb{E} = \prod_{k \in \mathbb{N}} \mathbb{E}_k / \text{Fin}$ of ERs $\mathbb{E}_k$ on $\mathbb{N}^N$ modulo $\text{Fin}$ is an ER on $(\mathbb{N}^N)^N$ defined as follows: $x \mathbb{E} y$ if $x(k) \mathbb{E}_k y(k)$ for all but finite $k$.

**Proposition 78.** The family of all pinned $\Sigma^1_1$ ERs is closed under Fubini products modulo $\text{Fin}$.

**Proof.** Suppose that ERs $\mathbb{E}_k$ on $\mathbb{N}^N$ are pinned; prove that the Fubini product $\mathbb{E} = \prod_{k \in \mathbb{N}} \mathbb{E}_k / \text{Fin}$ is pinned. Define $x \mathbb{F}_k y$ iff $x(k) \mathbb{E}_k y(k) : \mathbb{F}_k$ are $\Sigma^1_1$ ERs on $(\mathbb{N}^N)^N$ and $x \mathbb{E} y$ iff $x \mathbb{F}_k y$ for almost all $k$.

**Claim 78.1.** Each $\mathbb{F}_k$ is pinned.

**Proof.** Consider a Borel code $p$ for a subset of $(\mathbb{N}^N)^N$, satisfying $\mathbf{if}$ of Definition 77 w.r.t. $\mathbb{F}_k$. By the same argument as in the proof of Lemma 77.2, there is a Borel code $q$ for a subset of $\mathbb{N}^N$, such that $B_q \neq \emptyset$ in some extension of $\mathbb{V}$ and $B_q \subseteq \{ x(k) : x \in B_p \}$ in any extension of $\mathbb{V}$, hence, $q$ satisfies $\mathbf{if}$ of Definition 77 w.r.t. $\mathbb{E}_k$. As $\mathbb{E}_k$ is pinned, there is $a \in \mathbb{N}^N$ such that $B_q \subseteq [a]_{\mathbb{E}_k^\#}$ in any extension, but then easily $B_p \subseteq [x]_{\mathbb{E}_k^\#}$ in any extension, where $x \in (\mathbb{N}^N)^N \cap \mathbb{V}$ has only to satisfy $x(k) = a$ for the given $k$. $\square$ (Claim)

In continuation of the proof of the proposition, consider a Borel code $p$ for a subset of $(\mathbb{N}^N)^N$, satisfying $\mathbf{if}$ of Definition 77 w.r.t. $\mathbb{E}$. Our plan is to find
another Borel code $\bar{p}$ with $B_p \subseteq B_p$ everywhere, which satisfies if of Definition 77 for almost all $E_k$. This involves a forcing by Borel codes.

Let, in $\mathbb{V}$, $\lambda = \text{sup} p$ and $\kappa = \lambda^+$. Thus, $\text{sup} p < \kappa$. Let $P$ be the set of all Borel codes $q \in \mathbb{V}$ for subsets of $(\mathbb{N}^N)^N$ such that $\text{sup} \ q < \kappa$ and $B_q \neq \emptyset$ in a generic extension of the universe $\mathbb{V}$. $P$ is considered as a forcing, with $q \leq p$ ($q$ is stronger) iff $B_q \subseteq B_p$ in all generic extensions of $\mathbb{V}$. It is known that $P$ forces a point of $(\mathbb{N}^N)^N$, so that $\bigcap_{q \in G} B_q = \{x_G\}$ for any $P$-generic, over $\mathbb{V}$, set $G \subseteq P$.

Let $x$ be the name of the generic element of $(\mathbb{N}^N)^N$.

By the choice of $p$, $\langle p, p \rangle \upharpoonright P \times P$-forces $\dot{x}_{\text{left}} \dot{E}^\# \dot{x}_{\text{right}}$, hence, there are codes $q, r \in P$ and a number $k_0$ such that $\langle q, r \rangle \upharpoonright P \times P$-forces $\dot{x}_{\text{left}} \dot{F}_k \dot{E}^\# \dot{x}_{\text{right}}$ for any $k \geq k_0$. By a standard argument, we have $x \in \dot{F}_{k_0}$ for all $k \geq k_0$ in any extension of $\mathbb{V}$ for any two $P$-generic, over $\mathbb{V}$, elements $x, y \in B_q$. We can straightforwardly define in $\mathbb{V}$ a Borel code $\bar{p}$ (perhaps, not a member of $P$!) such that, in any extension of $\mathbb{V}$, $B_{\bar{p}}$ is the set of all $P$-generic, over $\mathbb{V}$, elements of $B_q$. Then $\bar{p}$ satisfies if of Definition 77 w.r.t. any $F_k$ with $k \geq k_0$. Hence, by the above, there is, in $\mathbb{V}$, a sequence of points $x_k \in (\mathbb{N}^N)^N$ such that $B_{\bar{p}} \subseteq [x_k]_{\dot{E}^\#}$ in any generic extension of $\mathbb{V}$, for any $k \geq k_0$. Define $x \in (\mathbb{N}^N)^N \cap \mathbb{V}$ so that $x(k) = x_k(k)$ for any $k \geq k_0$, then, by the definition of $F_k$, we have $B_{\bar{p}} \subseteq [x]_{\dot{E}^\#}$ for all $k \geq k_0$ in any extension of $\mathbb{V}$. Yet $\bigcap_{k \geq k_0} [x]_{\dot{E}^\#} \subseteq [x]_{\dot{E}^\#}$. □ (Proposition)

15.c Complete left-invariant actions produce pinned ERs

Recall that a Polish group $G$ is complete left-invariant, c11 for brevity, if $G$ admits a compatible left-invariant complete metric. Then easily $G$ also admits a compatible right-invariant complete metric, which will be practically used.

**Theorem 79.** (Hjorth [14]) Suppose that $G$ is a Polish c11 group continuously acting on a Polish space $X$. Then $E_G^X$ is pinned, hence, $T_2$ is not Borel reducible to $E_G^X$.

**Proof.** Fix a Borel code $\hat{p}$ satisfying if of Definition 77 w.r.t. $E_G^X$. Let $\kappa$ be a cardinal in $\mathbb{V}$ satisfying $\text{sup} \ \hat{p} < \kappa$. Define forcing $P$ as above, thus, $P$ forces an element of $X$.

Let $\rho$ be a compatible right-invariant metric on $G$.

For any $\epsilon > 0$, let $G_\epsilon = \{g \in G : \rho(g, 1_G) < \epsilon\}$. Say that $q \in P$ is of size $\leq \epsilon$ if $\langle q, q \rangle \upharpoonright (P \times P)$-forces that there is $g \in G_\epsilon^\#$ with $\dot{x}_{\text{left}} = g \cdot \dot{x}_{\text{right}}$. In this case, in any generic extension of the universe, if $\langle x, y \rangle \in B_q \times B_q$ is a $(P \times P)$-generic pair then there is $g \in G_\epsilon^\#$ with $y = g \cdot x$.

**Lemma 79.1.** If $q \in P$, $q \leq \hat{p}$, and $\epsilon > 0$, then there exists a condition $r \in P$, $r \leq q$, of size $\leq \epsilon$.

**Proof.** Otherwise for any $r \in P$, $r \leq q$, there is a pair of conditions $r', r'' \in P$ stronger than $r$ and such that $\langle r', r'' \rangle \upharpoonright (P \times P)$-forces that there is no $g \in G_\epsilon^\#$
with $\mathcal{J}_{\text{left}} = g \cdot \mathcal{J}_{\text{right}}$. Applying, in a sufficiently generic extension $V^+$ of $V$, an ordinary splitting construction, we find a perfect set $X \subseteq B_q$ such that any pair $(x, y) \in X^2$ with $x \neq y$ is $(P \times P)$-generic, hence, there is no $g \in G^\#$ with $y = g \cdot x$. Fix $x_0 \in X$. As $X$ is a pairwise $E_6^\#$-equivalent set (together with $B_q$) we can associate, in $V^+$, with each $x \in X$, an element $g_x \in G^\#$ such that $x = g_x \cdot x_0$, and $g_x \notin G^\#$ by the above. Moreover, we have $g_y g_x^{-1} \cdot x = y$ for all $x, y \in X$, hence $g_y g_x^{-1} \notin G^\#$ whenever $x \neq y$, which implies $\rho(g_x, g_y) \geq \varepsilon$ by the right invariance. But this contradicts the separability of $G$. $\Box$ (Lemma)

It follows that there is, in $V$, a sequence of codes $q_n \in P$ such that $q_0 \leq \widehat{p}$, $q_{n+1} \leq q_n$, $q_n$ has size $\leq 2^{-n}$, and $B_{q_n}$ has $\chi$-diameter $\leq 2^{-n}$ for any $n$. The only limit point $x$ of the sequence of sets $B_{q_n}$ belongs to $V$, thus, it remains to show that $B_{\widehat{p}} \subseteq [x]_{(E_6^\#)^\#}$ in any extension $V^+$ of the universe $V$.

We can assume that $V^+$ is rich enough to contain, for any $n$, an element $x_n \in B_{q_n}$ such that each pair $(x_n, x_{n+1})$ is $(P \times P)$-generic (over $V$). Then $\lim_{n \to \infty} x_n = x$. Moreover, for any $n$, both $x_n$ and $x_{n+1}$ belong to $B_{q_n}$, hence, as $q_n$ has size $\leq 2^{-n+1}$, there is $g_{n+1} \in G^\#$ with $\rho(1, g) \leq 2^{-n}$ such that $x_{n+1} = g_{n+1} \cdot x_n$. Thus, $x_n = h_n \cdot x_0$, where $h_n = g_{n+1} \cdot g_n$. Note that $\rho(h_n, h_{n-1}) = \rho(g_n, 1_G) \leq 2^{-n-1}$ by the right-invariance of the metric, thus, $\{h_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $G^\#$. Let $h = \lim_{n \to \infty} h_n \in G^\#$ be its limit. As the action is continuous, we have $x = \lim_{n \to \infty} x_n = h \cdot x_0$. It follows that $x \in E_6^\# x_0$. However $x_0 \in B_{q_0} \subseteq B_{\widehat{p}}$, therefore, $B_{\widehat{p}} \subseteq [x]_{(E_6^\#)^\#}$, as required.

$\Box$ (Theorem 79)

15.d All $F^\sigma$ ideals are pinned

Let us say that a Borel ideal $\mathcal{J}$ is pinned if so is the induced ER $E_{\mathcal{J}}$. It immediately follows from Theorem 79 that any polishable ideal is pinned. Yet there are pinned ideals among non-polishable ones.

Theorem 80. Any $F^\sigma$ ideal $\mathcal{J} \subseteq P(\mathbb{N})$ is pinned.

Proof. We have $\mathcal{J} = \bigcup_n F_n$, where all sets $F_n \subseteq P(\mathbb{N})$ are closed. It can be assumed that $F_n \subseteq F_{n+1}$, moreover, since for any closed $F \subseteq P(\mathbb{N})$ the set $\Delta F = \{X \Delta Y : x, y \in F\}$ is also closed (by the compactness of $P(\mathbb{N})$), it can be assumed that $\Delta F_n \subseteq F_{n+1}$ for all $n$.

Let $\widehat{p}$ be a Borel code, for a subset of $P(\mathbb{N})$, satisfying $\widehat{p}$ of Definition 77 w.r.t. the induced ER $E_{\mathcal{J}}$ on $P(\mathbb{N})$, thus, $\widehat{p} \in P$, where $P$ is a forcing defined as in the proof of Proposition 78 (but now $P$ forces a subset of $P(\mathbb{N})$, of course). Obviously there exists a pair of conditions $q, r \in P$ with $q, r \leq \widehat{p}$, and a number $\nu \in \mathbb{N}$, such that $(q, r)$ forces that $(\mathcal{J}_{\text{left}}, \mathcal{J}_{\text{right}}) \in F_\nu^\#$. Then $(q, q)$ forces $\mathcal{J}_{\text{left}} \Delta \mathcal{J}_{\text{right}} \in F_{\nu+1}^\#$ because $\Delta F_\nu \subseteq F_{\nu+1}$. It follows that, in $V$, there is a sequence of numbers $i_0 < i_1 < i_2 < \ldots$, a sequence $q \geq p_0 \geq p_1 \geq p_2 \geq \ldots$ of codes in $P$, and, for any $n$, a set $u_n \subseteq [0, n)$, such that
15 $T_2$ IS NOT REDUCIBLE TO ...

(1) each $p_n$ $\mathbb{P}$-forces $\mathbf{1} \cap [0, n) = u_n$;

(2) any $\mathbb{P}$-generic, over $\mathcal{V}$, $x, y \in B_{p_n}$ satisfy $x \Delta y \in F_{\nu + 1}^\#$.

Let, in $\mathcal{V}$, $a = \bigcup_n u_n$, then $a \cap [0, n) = u_n$ for all $n$. Prove that $a$ pins $B_{\mathbf{p}}$, i.e., $B_{\mathbf{p}} \subseteq [a]_{E_{\mathbf{p}}}^\#$ in any extension of $\mathcal{V}$.

We can assume that, in the extension, for any $n$ there is a $\mathbb{P}$-generic, over $\mathcal{V}$, element $x_n \in B_{p_n}$. Then we have, by (2), $x_0 \Delta x_n \in F_{\nu + 1}^\#$ for any $n$, thus, $x_0 \Delta a \in F_{\nu + 1}^\#$ as well, because $\{x_n\} \to a$. We conclude that $x_0 \mathbf{E}_{\mathbf{p}}^\# a$, and $B_{\mathbf{p}} \subseteq [a]_{E_{\mathbf{p}}}^\#$, as required.

15.5 Another family of pinned ideals

We here present another family of pinned ideals. Suppose that $\{\varphi_i\}_{i \in \mathbb{N}}$ is a sequence of lower semicontinuous (l.s.c.) submeasures on $\mathbb{N}$. Define

$$\text{Exh}_{\{\varphi_i\}} = \{X \subseteq \mathbb{N} : \varphi_\infty(X) = 0\}, \text{ where } \varphi_\infty(X) = \limsup_{i \to \infty} \varphi_i(X).$$

the exhaustive ideal of the sequence of submeasures. By Solecki's Theorem 41 for any Borel $\mathbb{P}$-ideal $\mathcal{I}$ there is a single l.s.c. submeasure $\varphi$ such that $\mathcal{I} = \text{Exh}_{\{\varphi_i\}} = \text{Exh}_\varphi$, where $\varphi_i(x) = \varphi(x \cap [i, \infty))$, however, for example, the non-polishable ideal $\mathcal{I}_1 = \text{Fin} \times 0$ also is of the form $\text{Exh}_{\{\varphi_i\}}$, where for $x \subseteq \mathbb{N}^2$ we define $\varphi_i(x) = 0$ or 1 if resp. $x \subseteq$ or $\not\subseteq \{0, ..., n - 1\} \times \mathbb{N}$.

**Theorem 81.** Any ideal of the form $\text{Exh}_{\{\varphi_i\}}$ is pinned

**Proof.** Thus let $\mathcal{I} = \text{Exh}_{\{\varphi_i\}}$, all $\varphi_i$ being l.s.c. submeasures on $\mathbb{N}$. We can assume that the submeasures $\varphi_i$ decrease, i.e., $\varphi_{i+1}(x) \leq \varphi_i(x)$ for any $x$, for if not consider the l.s.c. submeasures $\varphi_i'(x) = \sup_{i \in \mathbb{N}} \varphi_i(x)$. Let $\widehat{p}$ be a Borel code, for a subset of $\mathcal{P}(\mathbb{N})$, satisfying $\mathbf{g}$ of Definition 77 w.r.t. the induced ER $E_{\mathcal{I}}$ on $\mathcal{P}(\mathbb{N})$, thus, $\widehat{p} \in \mathbb{P}$, where $\mathbb{P}$ is a forcing defined as in the proof of Proposition 78 ($\mathbb{P}$ forces a subset of $\mathcal{P}(\mathbb{N})$).

Using the same arguments as above, we see that for any $p \in \mathbb{P}$, $p \not\leq \widehat{p}$, and $n \in \mathbb{N}$, there are $i \geq n$ and codes $q, r \in \mathbb{P}$ with $q, r \not\leq p$, such that $(q, r) \mathbb{P} \times \mathbb{P}$-forces that $\varphi_i(\mathbf{1}_{\text{left}} \Delta \mathbf{1}_{\text{right}}) \leq 2^{-n-1}$, hence, any two $\mathbb{P}$-generic, over $\mathcal{V}$, elements $x, y \in B_q$ satisfy $\varphi_i(x \Delta y) \leq 2^{-n}$. It follows that, in $\mathcal{V}$, there is a sequence of numbers $i_0 < i_1 < i_2 < ...$, a sequence $\widehat{p} \succ p_0 > p_1 > p_2 > ...$ of codes in $\mathbb{P}$, and, for any $n$, a set $u_n \subseteq [0, n)$, such that

(1) each $p_n$ $\mathbb{P}$-forces $\mathbf{1} \cap [0, n) = u_n$;

(2) any $\mathbb{P}$-generic, over $\mathcal{V}$, $x, y \in B_{p_n}$ satisfy $\varphi_{i_n}(x \Delta y) \leq 2^{-n}$.

Let, in $\mathcal{V}$, $a = \bigcup_n u_n$, then $a \cap [0, n) = u_n$ for all $n$. Prove that $a$ pins $B_{\mathbf{p}}$, i.e., $B_{\mathbf{p}} \subseteq [a]_{E_{\mathbf{p}}}^\#$ in any extension of $\mathcal{V}$.
We can assume that, in the extension, for any \( n \) there is a \( \mathbb{P} \)-generic, over \( \mathcal{V} \), element \( x_n \in B_{p_n} \). Then we have, by (2), \( \varphi_n(x_n \Delta x_m) \leq 2^{-n} \) whenever \( n \leq m \). It follows that \( \varphi_n(x_n \Delta a) \leq 2^{-n} \), because \( a = \lim_{m \to \infty} x_m \) by (1). However we assume that the submeasures \( \varphi_j \) decrease, hence, \( \varphi_\infty(x_n \Delta a) \leq 2^{-n} \). On the other hand, \( \varphi_\infty(x_n \Delta x_0) = 0 \) because all elements of \( B_{p_0} \) are pairwise \( E_\# \)-equivalent. We conclude that \( \varphi_\infty(x_0 \Delta a) \leq 2^{-n} \) for any \( n \), in other words, \( \varphi_\infty(x_0 \Delta a) = 0 \), \( x_0 E_\# a \), and \( B_F \subseteq [a]_{E_\#} \), as required. \( \square \)

**Question 3.** Are all Borel ideals pinned? The expected answer “yes” would show that \( T_2 \) is not Borel reducible to any Borel ideal. Moreover, is any orbit ER of a Borel action of a Borel abelian group pinned? But even this would not fully cover Hjorth’s Theorem 79. \( \square \)

**Question 4 (Kechris).** If Question 3 answers in the positive, is it true that \( T_2 \) is the \( \leq_n \)-least non-pinned Borel ER? \( \square \)

[47]

16 Universal analytic ERs and reduction to ideals

A Technical introduction

A.a Notation

- \( \mathbb{N} = \{0, 1, 2, \ldots\} \) : natural numbers. \( \mathbb{N}^2 = \mathbb{N} \times \mathbb{N} \).
- \( \mathbb{N}^\mathbb{N} \) is the Baire space. If \( s \in \mathbb{N}^{<\omega} \) (a finite sequence of natural numbers) then \( \mathcal{B}_s(\mathbb{N}^\mathbb{N}) = \{ x \in \mathbb{N}^\mathbb{N} : s \subseteq x \} \), a basic clopen nbhd in \( \mathbb{N}^\mathbb{N} \).
- \( X \subseteq^* Y \) means that the difference \( X \setminus Y \) is finite.
- If a basic set \( A \) is fixed then \( \check{A} X = X \setminus A X = A \setminus X \) for any \( X \subseteq A \).
- If \( X \subseteq A \times B \) and \( a \in A \) then \( (X)_a = \{ b : (a, b) \in X \} \), a cross-section.
- \( \#X = \#(X) \) is the number of elements of a finite set \( X \).
- \( f\upharpoonright X = \{ f(x) : x \in X \cap \text{dom } f \} \), the \( f \)-image of \( X \).
- \( \Delta \) is the symmetric difference.
- \( \exists^\infty x \ldots \) means: “there exist infinitely many \( x \) such that ...”,
- \( \forall^\infty x \ldots \) means: “for all but finitely many \( x \), ... holds”.
• An ideal on a set $A$ is, as usual, any set $\emptyset \neq \mathcal{I} \subseteq \mathcal{P}(A)$, closed under $\cup$ and satisfying $x \in \mathcal{I} \Rightarrow y \in \mathcal{I}$ whenever $y \subseteq x \subseteq A$. Thus, any ideal contains $\emptyset$. We’ll usually consider only nontrivial ideals, i.e., those which contain all singletons \{a\} \subseteq A and do not contain $A$, i.e., $\mathcal{P}_{1n}(A) \subseteq \mathcal{I} \subsetneq \mathcal{P}(A)$.

• If $\mathcal{I}$ is an ideal on a set $A$ then let $E_\mathcal{I}$ be an equivalence relation (ER, for brevity) on $\mathcal{P}(A)$, defined as follows: $X E_\mathcal{I} Y$ iff $X \Delta Y \in \mathcal{I}$.

• If $E$ is an ER on a set $X$ then $[y]_E = \{x \in X : y E x\}$ for any $y \in X$ (the $E$-class of $x$) and $[Y]_E = \bigcup_{y \in Y} [y]_E$ (the $E$-saturation of $Y$) for $Y \subseteq X$. A set $Y \subseteq X$ is $E$-invariant if $[Y]_E = Y$.

• If $E$ is an ER on a set $X$ then a set $Y \subseteq X$ is pairwise $E$-equivalent, resp., pairwise $E$-inequivalent, if $x E y$, resp., $x E y$ holds for all $x \neq y$ in $Y$.

• If $X, Y$ are sets and $E$ any binary relation then $X E Y$ means that we have both $\forall x \in X \exists y \in Y (x E y)$ and $\forall y \in Y \exists x \in X (x E y)$.

A.b Descriptive set theory

A basic knowledge of Borel and projective hierarchy, both classical and effective, in the Baire space $\mathbb{N}^\mathbb{N}$ and other (recursively presented, in the effective case) Polish spaces, is assumed.

A map $f$ (between Borel sets in Polish spaces) is Borel iff its graph is a Borel set iff all $f$-preimages of open sets are Borel. A map $f$ is Baire measurable (BM, for brevity) iff all $f$-preimages of open sets are Baire measurable sets.

A.c Trivia of “effective” descriptive set theory

Apart of the very common knowledge, the whole instrumentarium of “effective” descriptive set theory employed in the study of reducibility of ideals and ERs, can be summarized in a rather short list of key “principles”. In those below, by a recursively presented Polish space one can understand any product space of the form $\mathbb{N}^m \times (\mathbb{N})^n$ without any harm for applications below, yet in fact this notion is much wider.

**Remark 82.** For the sake of brevity, the results below are formulated only for the “lightface” parameter-free classes $\Sigma_1^1$, $\Pi_1^1$, $\Delta_1^1$, but they remain true for $\Sigma_1^1(p)$, $\Pi_1^1(p)$, $\Delta_1^1(p)$ for any fixed real parameter $p$. □

**Reduction and Separation:** If $X, Y$ are $\Pi_1^1$ sets of a recursively presented Polish space then there disjoint $\Pi_1^1$ sets $X' \subseteq X$ and $Y' \subseteq Y$ with $X' \cup Y' = X \cup Y$. The sets $X'$, $Y'$ are said to reduce the pair $X, Y$. 


If \( X, Y \) are disjoint \( \Sigma^1_1 \) sets of a recursively presented Polish space then there is a \( \Delta^1_1 \) set \( Z \) with \( X \subseteq Z \) and \( Y \cap Z = \emptyset \). The set \( Z \) is said to separate the \( X \) from \( Y \).

**Countable-to-1 Projection:** If \( P \) is a \( \Delta^1_1 \) subset of the product \( \mathcal{X} \times \mathcal{Y} \) of two recursively presented Polish spaces and for any \( x \in \mathcal{X} \) the cross-section \( P_x = \{ y : P(x, y) \} \) is at most countable then \( \text{dom} \, P \) is a \( \Delta^1_1 \) set in \( \mathcal{X} \).

It follows that images of \( \Delta^1_1 \) sets via countable-to-1, in particular, 1-to-1 \( \Delta^1_1 \) maps are \( \Delta^1_1 \) sets, while images via arbitrary \( \Delta^1_1 \) maps are, generally, \( \Sigma^1_1 \).

**Countable-to-1 Enumeration:** If \( P, \mathcal{X}, \mathcal{Y} \) are as in Countable-to-1 Projection then there is a \( \Delta^1_1 \) map \( f : \text{dom} \, P \times \mathbb{N} \to \mathcal{Y} \) such that \( P_x = \{ f(x, n) : n \in \mathbb{N} \} \) for all \( x \in \text{dom} \, P \).

**Countable-to-1 Uniformization:** If \( P, \mathcal{X}, \mathcal{Y} \) are as in Countable-to-1 Projection then \( P \) can be uniformized by a \( \Delta^1_1 \) set.

**Kreisel Selection:** If \( \mathcal{X} \) is a recursively presented Polish space, \( P \subseteq \mathcal{X} \times \mathbb{N} \) is a \( \Pi^1_1 \) set, and \( X \subseteq \text{dom} \, P \) is a \( \Delta^1_1 \) set then there is a \( \Delta^1_1 \) function \( f : X \to \mathbb{N} \) such that \( \langle x, f(x) \rangle \in P \) for all \( x \in X \).

The proof is surprisingly simple. Let \( Q \subseteq P \) be a \( \Pi^1_1 \) set which uniformizes \( P \). For any \( x \in X \) let \( f(x) \) be the only \( n \) with \( \langle x, n \rangle \in Q \). Immediately, (the graph of) \( f \) is \( \Pi^1_1 \), however, as \( \text{ran} \, f \subseteq \mathbb{N} \), we have \( f(x) = n \iff \forall m \neq n \, (f(x) \neq m) \) whenever \( x \in X \), which demonstrates that \( f \) is \( \Sigma^1_1 \) as well.

**\( \Delta^1_1 \) Enumeration:** If \( \mathcal{X} \) is a recursively presented Polish space then there exist \( \Pi^1_1 \) sets \( C \subseteq \mathcal{Y} \times \mathbb{N} \) and \( W \subseteq \mathcal{X} \times \mathbb{N} \times \mathcal{X} \) and a \( \Sigma^1_1 \) set \( W' \subseteq \mathcal{X} \times \mathbb{N} \times \mathcal{X} \) such that \( W_e = W'_e \) for any \( e \in C \) and a set \( X \subseteq \mathcal{X} \) is \( \Delta^1_1 \) if there is \( e \in C \) such that \( X = W_e = W'_e \). (Here \( W_e = \{ x : W(y, e, x) \} \) and similarly \( W'_e \).

There is a generalization useful for relativised classes \( \Delta^1_1(y) \).

**Relativized \( \Delta^1_1 \) Enumeration:** If \( \mathcal{X}, \mathcal{Y} \) are recursively presented Polish spaces then there exist \( \Pi^1_1 \) sets \( C \subseteq \mathcal{Y} \times \mathbb{N} \) and \( W \subseteq \mathcal{Y} \times \mathbb{N} \times \mathcal{X} \) and a \( \Sigma^1_1 \) set \( W' \subseteq \mathcal{Y} \times \mathbb{N} \times \mathcal{X} \) such that \( W_{ye} = W'_{ye} \) for any \( \langle y, e \rangle \in C \) and, for any \( y \in \mathcal{Y} \), a set \( X \subseteq \mathcal{X} \) is \( \Delta^1_1(y) \) iff there is \( e \) such that \( \langle y, e \rangle \in C \) and \( X = W_{ye} = W'_{ye} \). (Here \( W_{ye} = \{ x : W(y, e, x) \} \) and similarly \( W'_{ye} \).

Suppose that \( \mathcal{X} \) is a recursively presented Polish space. A set \( U \subseteq \mathbb{N} \times \mathcal{X} \), is a a universal \( \Pi^1_1 \) set if for any \( \Pi^1_1 \) set \( X \subseteq \mathcal{X} \) there is an index \( n \) with \( X = U_n = \{ x : \langle n, x \rangle \in U \} \), and a a “good” universal \( \Pi^1_1 \) set if in addition for any other \( \Pi^1_1 \) set \( V \subseteq \mathbb{N} \times \mathcal{X} \) there is a recursive function \( f : \mathbb{N} \to \mathbb{N} \) such that \( V_n = U_{f(n)} \) for all \( n \).

The notions of universal and “good” universal \( \Sigma^1_1 \) sets are similar.
Universal Sets: For any recursively presented Polish space \( \mathcal{X} \) there exist a “good” universal \( \Pi^1_1 \) set \( U \subseteq \mathbb{N} \times \mathcal{X} \) and a “good” universal \( \Sigma^1_1 \) set \( V \subseteq \mathbb{N} \times \mathcal{X} \).

(In fact we can take \( V = (\mathbb{N} \times \mathcal{X}) \setminus U \).

If a “good” universal \( \Pi^1_1 \) set \( U \) is fixed then a collection \( \mathcal{A} \) of \( \Pi^1_1 \) sets \( X \subseteq \mathcal{X} \) is \( \Pi^1_1 \) in the codes if \( \{ n : U_n \in \mathcal{A} \} \) is a \( \Pi^1_1 \) set. Similarly, if a “good” universal \( \Sigma^1_1 \) set \( V \) is fixed then a collection \( \mathcal{A} \) of \( \Sigma^1_1 \) sets \( X \subseteq \mathcal{X} \) is \( \Pi^1_1 \) in the codes if \( \{ n : V_n \in \mathcal{A} \} \) is a \( \Pi^1_1 \) set. These notions quite obviously do not depend on the choice of “good” universal sets.

To show how “good” universal sets work, we prove:

**Proposition 83.** Let \( \mathcal{X} \) be a recursively presented Polish space and \( U \subseteq \mathbb{N} \times \mathcal{X} \) a “good” universal \( \Pi^1_1 \) set. Then for any pair of \( \Pi^1_1 \) sets \( V, W \subseteq \mathbb{N} \times \mathcal{X} \) there are recursive functions \( f, g : \mathbb{N} \to \mathbb{N} \) such that for any \( m, n \in \mathbb{N} \) the pair of cross-sections \( U_{f(m,n)}, U_{g(m,n)} \) reduces the pair \( V_m, W_n \).

**Proof.** Consider the following \( \Pi^1_1 \) sets in \( (\mathbb{N} \times \mathbb{N}) \times \mathcal{X} \):

\[
P = \{ \langle m, n, x \rangle : \langle m, x \rangle \in V \land n \in \mathbb{N} \}, \quad Q = \{ \langle m, n, x \rangle : \langle n, x \rangle \in W \land m \in \mathbb{N} \}.
\]

By Reduction, there is a pair of \( \Pi^1_1 \) sets \( P' \subseteq P \) and \( Q' \subseteq Q \) which reduce the given pair \( P, Q \). Accordingly, the pair \( P'_{mn}, Q'_{mn} \) reduces \( P_{mn}, Q_{mn} \) for any \( m, n \). Finally, by the “good” universality there are recursive functions \( f, g \) such that \( P'_{mn} = U_{f(m,n)} \) and \( Q'_{mn} = U_{g(m,n)} \) for all \( m, n \).

The following principle is less elementary than the results cited above, but it is very useful because it allows to “compress” some sophisticated arguments with multiple applications of Separation and Kreisel selection.

**Reflection:** Assume that \( \mathcal{X} \) is a recursively presented Polish space.

\( \Pi^1_1 \) form: Suppose that a collection \( \mathcal{A} \) of \( \Pi^1_1 \) sets \( X \subseteq \mathcal{X} \) is \( \Pi^1_1 \) in the codes.

(In the sense of a fixed “good” universal \( \Pi^1_1 \) set \( U \subseteq \mathbb{N} \times \mathcal{X} \).) Then for any \( X \in \mathcal{A} \) there is a \( \Delta^1_1 \) set \( Y \in \mathcal{A} \) with \( Y \subseteq X \).

\( \Sigma^1_1 \) form: Suppose that a collection \( \mathcal{A} \) of \( \Pi^1_1 \) sets \( X \subseteq \mathcal{X} \) is \( \Pi^1_1 \) in the codes.

Then for any \( X \in \mathcal{A} \) there is a \( \Delta^1_1 \) set \( Y \in \mathcal{A} \) with \( X \subseteq Y \).

One of (generally, irrelevant here) consequences of this principle is that the set of all codes of a properly \( \Pi^1_1 \) set or properly \( \Sigma^1_1 \) set is never \( \Pi^1_1 \).

**A.1** Polish-like families and the Gandy – Harrington topology

The following notion is similar to the Choquet property but somewhat more convenient to provide the nonemptiness of countable intersections of pointsets.
Definition 84. A family \( \mathcal{F} \) is \emph{Polish-like} if there exists a countable collection \( \{ \mathcal{D}_n : n \in \mathbb{N} \} \) of dense subsets \( \mathcal{D}_n \subseteq \mathcal{F} \) such that we have \( \bigcap_n F_n \neq \emptyset \) whenever \( F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots \) is a decreasing sequence of sets \( F_n \in \mathcal{F} \) which intersects every \( \mathcal{D}_n \). (Here, a set \( D \subseteq \mathcal{F} \) is dense if \( \forall F \in \mathcal{D} \exists D \in \mathcal{D} (D \subseteq F) \).

For instance if \( \mathcal{F} \) is a Polish space then the collection of all its non-empty closed sets is Polish-like, for take \( \mathcal{D}_n \) to be all closed sets of diameter \( \leq n^{-1} \).

Theorem 85 (Kanovei [22], Hjorth [13]). The collection \( \mathcal{F} \) of all non-empty \( \Sigma^1_1 \) subsets of \( \mathbb{N}^\mathbb{N} \) is Polish-like.

Proof. For any \( P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \) define \( \text{pr} \ P = \{ x : \exists y \ P(x,y) \} \) (the projection). If \( P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \) and \( s, t \in \mathbb{N}^{<\omega} \) then let \( P_{st} = \{ (x,y) \in P : s \subseteq x \wedge t \subseteq y \} \). Let \( \mathcal{D}(P,s,t) \) be the collection of all \( \Sigma^1_1 \) sets \( \emptyset \neq X \subseteq \mathbb{N}^\mathbb{N} \) such that either \( X \cap \text{pr} \ P_{st} = \emptyset \) or \( X \subseteq \text{pr} \ P_{s \cup t \cup j} \) for some \( i, j \). (Note that in the “or” case \( i \) is unique but \( j \) may be not unique.) Let \( \{ \mathcal{D}_n : n \in \mathbb{N} \} \) be an arbitrary enumeration of all sets of the form \( \mathcal{D}(P,s,t) \), where \( P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \) is \( \Pi^0_1 \). Note that in this case all sets of the form \( \text{pr} \ P_{st} \) are \( \Sigma^1_1 \) subsets of \( \mathbb{N}^\mathbb{N} \), therefore, \( \mathcal{D}(P,s,t) \) is easily a dense subset of \( \mathcal{F} \), so that all \( \mathcal{D}_n \subseteq \mathcal{F} \) are dense.

Now consider a decreasing sequence \( X_0 \supseteq X_1 \supseteq \ldots \) of non-empty \( \Sigma^1_1 \) sets \( X_k \subseteq \mathbb{N}^\mathbb{N} \), which intersects every \( \mathcal{D}_n \); prove that \( \bigcap_n X_n \neq \emptyset \). Call a set \( X \subseteq \mathbb{N}^\mathbb{N} \) \emph{positive} if there is \( n \) such that \( X_n \subseteq X \). For any \( n \), fix a \( \Pi^0_1 \) set \( P^n \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \) such that \( X_n = \text{pr} \ P^n \). For any \( s, t \in \mathbb{N}^{<\omega} \), if \( \text{pr} \ P^n_{st} \) is positive then, by the choice of the sequence of \( X_n \), there is a unique \( i \) and some \( j \) such that \( \text{pr} \ P^n_{s \cup t \cup j} \) is also positive. It follows that there is a unique \( x = x_n \in \mathbb{N}^\mathbb{N} \) and some \( y = y_n \in \mathbb{N}^\mathbb{N} \) (perhaps not unique) such that \( \text{pr} \ P^n_{x | k \cup y | k} \) is positive for any \( k \). As \( P^n \) is closed, we have \( P^n(x,y) \), hence, \( x_n = x \in X_n \).

It remains to show that \( x_m = x_n \) for \( m \neq n \). To see this note that if both \( P_{st} \) and \( Q_{s't'} \) are positive then either \( s \subseteq s' \) or \( s' \subseteq s \).

The collection of all non-empty \( \Sigma^1_1 \) subsets of \( \mathbb{N}^\mathbb{N} \) is a base of the \emph{Gandy–Harrington topology}, which has many remarkable applications in descriptive set theory. This topology is easily not Polish, even not metrizable at all, yet it shares the following important property of Polish topologies:

Corollary 86. The Gandy–Harrington topology is Baire, i.e., every comeager set is dense.

Proof. This can be proved using Choquet property of the topology, see [12], however, the Polish–likeliness (Theorem 85) also immediately yields the result.

Список литературы


