A weak dichotomy below $\mathsf{E}_1 \times \mathsf{E}_3$

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Abstract

We prove that if E is an equivalence relation Borel reducible to $\mathsf{E}_1 \times \mathsf{E}_3$ then either E is Borel reducible to the equality of countable sets of reals or E_1 is Borel reducible to E. The "either" case admits further strengthening.

Let $\mathbb{R} = 2^{\mathbb{N}}$. Recall that E_1 and E_3 are the equivalence relations defined on the set $\mathbb{R}^{\mathbb{N}}$ as follows:

$$x \,\mathsf{E}_1\, y \quad \text{iff} \quad \exists\, k_0\, \forall\, k \ge k_0\, (x(k) = y(k))\,;$$
$$x \,\mathsf{E}_3\, y \quad \text{iff} \quad \forall\, k\, (x(k) \,\mathsf{E}_0\, y(k))\,;$$

where E_0 is an equivalence relation defined on $\mathbb{R}$ so that

$$a \,\mathsf{E}_0\, b \quad \text{iff} \quad \exists\, n_0\, \forall\, n \ge n_0\, (a(n) = b(n))\,.$$

The equivalence E_3 is often denoted as $(\mathsf{E}_0)^\omega$.

Kechris and Louveau in [9] and Kechris and Hjorth in [3, 4] proved that any Borel equivalence relation E satisfying $\mathsf{E} <_{\rm B} \mathsf{E}_1$, resp., $\mathsf{E} <_{\rm B} \mathsf{E}_3$, also satisfies the non-strict $\mathsf{E} \le_{\rm B} \mathsf{E}_0$. Here $<_{\rm B}$ and $\le_{\rm B}$ are resp. strict and non-strict relations of Borel reducibility. Thus if E is an equivalence relation on a Borel set X [1] and F is an equivalence relation on a Borel set Y then $\mathsf{E} \le_{\rm B} \mathsf{F}$ means that there exists a Borel map $\vartheta : X \to Y$ such that

$$x \,\mathsf{E}\, x' \iff \vartheta(x) \,\mathsf{F}\, \vartheta(x')$$

holds for all $x, x' \in X$. Such a map ϑ is called a (Borel) *reduction* of E to F. If both $\mathsf{E} \le_{\rm B} \mathsf{F}$ and $\mathsf{F} \le_{\rm B} \mathsf{E}$ then they write $\mathsf{E} \sim_{\rm B} \mathsf{F}$ (Borel *bi-reducibility*), while $\mathsf{E} <_{\rm B} \mathsf{F}$ (strict reducibility) means that $\mathsf{E} \le_{\rm B} \mathsf{F}$ but not $\mathsf{F} \le_{\rm B} \mathsf{E}$. See the cited papers [3, 4] or *e.g.* [2, 8] on various aspects of Borel reducibility in set theory and mathematics in general.

The abovementioned results give a complete description of the $\le_{\rm B}$-structure of Borel equivalence relations below E_1 and below E_3. It is then a natural step

[1] We consider only Borel sets in Polish spaces.

to investigate the $\le_{\rm B}$-structure below E_{13}, where $\mathsf{E}_{13} = \mathsf{E}_1 \times \mathsf{E}_3$ is the product of E_1 and E_3, that is, an equivalence on $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ defined so that for any points $\langle x, \xi \rangle$ and $\langle y, \eta \rangle$ in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $\langle x, \xi \rangle \mathrel{\mathsf{E}_{13}} \langle y, \eta \rangle$ if and only if $x \mathrel{\mathsf{E}_1} y$ and $\xi \mathrel{\mathsf{E}_3} \eta$.

The intended result would be that the $\le_{\rm B}$-cone below E_{13} includes the cones determined separately by E_1 and E_3, together with the disjoint union of E_1 and E_3 (*i.e.*, the union of E_1 and E_3 defined on two disjoint copies of $\mathbb{R}^{\mathbb{N}}$), E_{13} itself, and nothing else. This is however a long shot. The following theorem, the main result of this note, can be considered as a small step in this direction.

Theorem 1. *Suppose that* E *is a Borel equivalence relation and* $\mathsf{E} \le_{\rm B} \mathsf{E}_{13}$. *Then* **either** E *is Borel reducible to* T_2 **or** $\mathsf{E}_1 \le_{\rm B} \mathsf{E}$.

Recall that the equivalence relation T_2, known as "the equality of countable sets of reals", is defined on $\mathbb{R}^{\mathbb{N}}$ so that $x \mathrel{\mathsf{T}_2} y$ iff $\{x(n) : n \in \mathbb{N}\} = \{y(n) : n \in \mathbb{N}\}$. It is known that $\mathsf{E}_3 <_{\rm B} \mathsf{T}_2$ strictly, and there exist many Borel equivalence relations E satisfying $\mathsf{E} <_{\rm B} \mathsf{T}_2$ but incomparable with E_3 : for instance non-hyperfinite Borel countable ones like E_∞. The two cases are incompatible because E_1 is known not to be Borel reducible to orbit equivalence relations of Polish actions (to which class T_2 belongs).

A rather elementary argument reduces Theorem 1 to the following:

Theorem 2. *Suppose that* $P_0 \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ *is a Borel set. Then* **either** *the equivalence* $\mathsf{E}_{13} \restriction P_0$ *is Borel reducible to* T_2 **or** $\mathsf{E}_1 \le_{\rm B} \mathsf{E}_{13} \restriction P_0$.

Indeed suppose that Z (a Borel set) is the domain of E, and $\vartheta : Z \to \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ is a Borel reduction of E to E_{13}. Let $f : Z \to 2^{\mathbb{N}} = \mathbb{R}$ be an arbitrary Borel injection. Define another reduction $\vartheta' : Z \to \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ as follows. Suppose that $z \in Z$ and $\vartheta(z) = \langle x, \xi \rangle \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. Put $\vartheta'(z) = \langle x', \xi \rangle$, where x', still a point in $\mathbb{R}^{\mathbb{N}}$, is related to x so that $x'(n) = x(n)$ for all $n \ge 1$ but $x'(0) = f(z)$. Then obviously $\vartheta(z)$ and $\vartheta'(z)$ are E_{13}-equivalent for all $z \in Z$, and hence ϑ' is still a Borel reduction of E to E_{13}. On the other hand, ϑ' is an injection (because so is f). It follows that its full image $P_0 = \mathtt{ran}\, \vartheta' = \{\vartheta'(z) : z \in Z\}$ is a Borel set in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, and $\mathsf{E} \sim_{\rm B} \mathsf{E}_{13} \restriction P_0$.

The remainder of the paper contains the proof of Theorem 2. The partition in two cases is described in Section 2. Naturally assuming that P_0 is a lightface $\varDelta^1_1$ set, Case 1 is essentially the case when for every element $\langle x, \xi \rangle \in P_0$ (note that x, ξ are points in $\mathbb{R}^{\mathbb{N}}$) and every n we have $x(n) = F(x \restriction_{>n}, \xi \restriction_{\leqslant k}, \xi \restriction_{>k})$ for some k, where F is a $\varDelta^1_1$ function E_3-invariant w.r.t. the 3rd argument. It easily follows that then the first projection of the equivalence class $[\langle x, \xi \rangle]_{\mathsf{E}_{13}} \cap P_0$ of every point $\langle x, \xi \rangle \in P_0$ is at most countable, leading to the **either** option of Theorem 2 in Section 4.

The results of theorems 1 and 2 in their **either** parts can hardly be viewed as satisfactory because one would expect it in the form: E is Borel reducible to E_3. Thus it is a challenging problem to replace T_2 by E_3 in the theorems. Attempts to improve the **either** option, so far rather insuccessful, lead us to the following theorem established in sections 5 and 6:

Theorem 3. *In the* **either** *case of Theorem 2 there exist a hyperfinite equivalence relation* G *on a Borel set* $P_0'' \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ *such that* $\mathsf{E}_{13} \restriction P_0$ *is Borel reducible to the conjunction of* G *and the equivalence relation* E_3 *acting on the 2nd factor of* $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. [2]

The equivalence G as in the theorem will be induced by a countable group $\mathbb{G}$ of homeomorphisms of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ preserving the second component. (That is, if $g \in \mathbb{G}$ and $g(x,\xi) = \langle y, \eta \rangle$ then $\eta = \xi$, but y generally speaking depends on both x and ξ.) And $\mathbb{G}$ happens to be even a *hyperfinite* group in the sense that it is equal to the union of an increasing chain of its finite subgroups. Recall that E_3 is induced by the product group $\mathbb{H} = \langle \mathscr{P}_{\mathtt{fin}}(\mathbb{N}) ; \Delta \rangle^{\mathbb{N}}$ naturally acting in this case on the second factor in the product $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. And there are further details here that will be presented in sections 5 and 6.

Case 2 is treated in Sections 7 through 12. The embedding of E_1 in $\mathsf{E}_{13} \restriction P_0$ is obtained by approximately the same splitting construction as the one introduced in [9] (in the version closer to [7]).

1 Preliminaries: extension of "invariant" functions

If E is an equivalence relation on a set X then, as usual, $[x]_{\mathsf{E}} = \{y \in X : y \mathrel{\mathsf{E}} x\}$ is the E-*class* of an element $x \in X$, and $[Y]_{\mathsf{E}} = \bigcup_{x \in Y} [x]_{\mathsf{E}}$ is the E-*saturation* of a set $Y \subseteq X$. A set $Y \subseteq X$ is E-*invariant* if $Y = [Y]_{\mathsf{E}}$.

The following "invariant" Separation theorem will be used below.

Proposition 4 (5.1 in [1])**.** *Assume that* E *is a* Δ^1_1 *equivalence relation on a* Δ^1_1 *set* $X \subseteq \mathbb{N}^{\mathbb{N}}$. *If* $A, C \subseteq X$ *are* Σ^1_1 *sets and* $[A]_{\mathsf{E}} \cap [C]_{\mathsf{E}} = \varnothing$ *then there exists an* E*-invariant* Δ^1_1 *set* $B \subseteq X$ *such that* $[A]_{\mathsf{E}} \subseteq B$ *and* $[C]_{\mathsf{E}} \cap B = \varnothing$. □

Suppose that f is a map defined on a set $Y \subseteq X$. Say that f is E-*invariant* if $f(x) = f(y)$ for all $x, y \in Y$ satisfying $x \mathrel{\mathsf{E}} y$.

Corollary 5. *Assume that* E *is a* Δ^1_1 *equivalence relation on a* Δ^1_1 *set* $A \subseteq \mathbb{N}^{\mathbb{N}}$, *and* $f : B \to \mathbb{N}^{\mathbb{N}}$ *is an* E*-invariant* Σ^1_1 *function defined on a* Σ^1_1 *set* $B \subseteq A$. *Then there exist an* E*-invariant* Δ^1_1 *function* $g : A \to \mathbb{N}^{\mathbb{N}}$ *such that* $f \subseteq g$.

Proof. It obviously suffices to define such a function on an E-invariant Δ^1_1 set Z such that $Y \subseteq Z \subseteq A$. (Indeed then define g to be just a constant on $A \smallsetminus Z$.) The set

$$P = \{\langle a, x \rangle \in A \times \mathbb{N}^{\mathbb{N}} : \forall b \left((b \in B \wedge a \mathrel{\mathsf{E}} b) \Longrightarrow x = f(b) \right) \}$$

[2] The conjunction as indicated is equal to the least equivalence relation F on P_0'' which includes G and satisfies $\xi \mathrel{\mathsf{E}_3} \eta \Longrightarrow \langle x, \xi \rangle \mathrel{\mathsf{F}} \langle y, \eta \rangle$ for all $\langle x, \xi \rangle$ and $\langle y, \eta \rangle$ in P_0''.

is $\varPi^1_1$ and $f \subseteq P$. Moreover P is F-invariant, where F is defined on $A \times \mathbb{N}^{\mathbb{N}}$ so that $\langle a, x\rangle \mathrel{\mathsf{F}} \langle a', y\rangle$ iff $a \mathrel{\mathsf{E}} a'$ and $x = y$. Obviously $[f]_{\mathsf{F}} \subseteq P$. Hence by Proposition 4 there exists an F-invariant $\varDelta^1_1$ set Q such that $f \subseteq Q \subseteq P$. The set

$$R = \{\langle a, x\rangle \in Q : \forall\, y\, (y \neq x \Longrightarrow \langle a, y\rangle \notin Q\}$$

is an F-invariant $\varPi^1_1$ set, and in fact a function, satisfying $f \subseteq R$. Applying Proposition 4 once again we end the proof. $\square$

2 An important population of $\boldsymbol{\Sigma}^1_1$ functions

Working with elements and subsets of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ as the domain of the equivalence relation E_{13}, we'll typically use letters x, y, z to denote points of the first copy of $\mathbb{R}^{\mathbb{N}}$ (where E_1 lives) and letters ξ, η, ζ to denote points of the second copy of $\mathbb{R}^{\mathbb{N}}$ (where E_3 lives). Recall that, for $P \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$,

$$\mathtt{dom}\, P = \{x : \exists\, \xi\, (\langle x, \xi\rangle \in P)\} \quad \text{and} \quad \mathtt{ran}\, P = \{\xi : \exists\, x\, (\langle x, \xi\rangle \in P)\}.$$

Points of $\mathbb{R} = 2^{\mathbb{N}}$ will be denoted by a, b, c.

Assume that $x \in \mathbb{R}^{\mathbb{N}}$. Let $x \restriction_{>n}$, resp., $x \restriction_{\geqslant n}$ denote the restriction of x (as a map $\mathbb{N} \to \mathbb{R}$) to the domain (n, ∞), resp., $[n, \infty)$. Thus $x \restriction_{>n} \in \mathbb{R}^{>n}$, where $>n$ means the interval (n, ∞), and $x \restriction_{\geqslant n} \in \mathbb{R}^{\geqslant n}$, where $\geqslant n$ means $[n, \infty)$. If $X \subseteq \mathbb{R}^{\mathbb{N}}$ then put $X \restriction_{>n} = \{x \restriction_{>n} : x \in X\}$ and $X \restriction_{\geqslant n} = \{x \restriction_{\geqslant n} : x \in X\}$.

The notation connected with $\restriction_{<n}$ and $\restriction_{\leqslant n}$ is understood similarly.

Let $\xi \equiv_k \eta$ mean that $\xi \mathrel{\mathsf{E}_3} \eta$ and $\xi \restriction_{<k} = \eta \restriction_{<k}$ (that is, $\xi(j) = \eta(j)$ for all $j < k$). This is a Borel equivalence on $\mathbb{R}^{\mathbb{N}}$. A set $U \subseteq \mathbb{R}^{\mathbb{N}}$ is $\equiv_k$-*invariant* if $U = [U]_{\equiv_k}$, where $[U]_{\equiv_k} = \bigcup_{\xi \in U} [\xi]_{\equiv_k}$.

Definition 6. Let $\mathscr{F}^k_n$ denote the set of all $\varSigma^1_1$ functions[3] $\varphi : U \to \mathbb{R}$, defined on a $\varSigma^1_1$ set $U = \mathtt{dom}\, \varphi \subseteq \mathbb{R}^{>n} \times \mathbb{R}^{\mathbb{N}}$, and $\equiv_k$-*invariant* in the sense that if $\langle y, \xi\rangle$ and $\langle y, \eta\rangle$ belong to U and $\xi \equiv_k \eta$ then $\varphi(y, \xi) = \varphi(y, \eta)$.

Let ${}^{\mathtt{T}}\mathscr{F}^k_n$ denote the set of all *total* functions in $\mathscr{F}^k_n$, that is, those defined on the whole set $\mathbb{R}^{>n} \times \mathbb{R}^{\mathbb{N}}$. $\square$

Lemma 7. *If $\varphi \in \mathscr{F}^k_n$ then there is a $\varDelta^1_1$ function $\psi \in {}^{\mathtt{T}}\mathscr{F}^k_n$ with $\varphi \subseteq \psi$.*

Proof. Apply Corollary 5. $\square$

Definition 8. Let us fix a suitable coding system $\{W^e\}_{e \in E}$ of all $\varDelta^1_1$ sets $W \subseteq \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \times \mathbb{R}$ (in particular for partial $\varDelta^1_1$ functions $\mathbb{R} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$), where $E \subseteq \mathbb{N}$ is a $\varPi^1_1$ set, such that there exist a $\varSigma^1_1$ relation $\boldsymbol{\Sigma}$ and a $\varPi^1_1$ relation $\boldsymbol{\Pi}$ satisfying

$$\langle b, \xi, a\rangle \in W^e \iff \boldsymbol{\Sigma}(e, b, a, \xi) \iff \boldsymbol{\Pi}(e, b, a, \xi) \qquad (1)$$

[3] A $\varSigma^1_1$ function is a function with a $\varSigma^1_1$ graph.

whenever $e \in E$ and $a, b \in \mathbb{R}$, $\xi \in \mathbb{R}^{\mathbb{N}}$.

Let us fix a Δ^1_1 sequence of homeomorphisms $H_n : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}^{\geqslant n}$. Put

$$\left.\begin{array}{rcl} W^e_n &=& \{\langle H_n(b), \xi, a\rangle : \langle b, \xi, a\rangle \in W^e\} \quad \text{for } e \in E \\ T &=& \{\langle e, k\rangle : e \in E \wedge W^e \text{ is a total and } \equiv_k\text{-invariant function}\} \end{array}\right\} \qquad (2)$$

Here the totality means that $\mathtt{dom}\, W^e = \mathbb{R} \times \mathbb{R}^{\mathbb{N}}$ while the invariance means that $W^e(b, \xi) = W^e(b, \eta)$ for all b, ξ, η satisfying $\xi \equiv_k \eta$. □

Note that if $\langle e, k\rangle \in T$ then, for any n, W^e_n is a function in ${}^{\mathtt{T}}\mathscr{F}^k_n$, and conversely, every function in ${}^{\mathtt{T}}\mathscr{F}^k_n$ has the form W^e_n for a suitable $e \in E$.

Proposition 9. *T is a Π^1_1 set.*

Proof. Standard evaluation based on the coding of Δ^1_1 sets. □

Corollary 10. *The sets*

$$\begin{array}{rcl} S^k_n &=& \{\langle x, \xi\rangle \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} : \exists\, \varphi \in \mathscr{F}^k_n\, (x(n) = \varphi(x{\restriction}_{>n}, \xi))\} \\ &=& \{\langle x, \xi\rangle \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} : \exists\, \varphi \in {}^{\mathtt{T}}\mathscr{F}^k_n\, (x(n) = \varphi(x{\restriction}_{>n}, \xi))\} \end{array}$$

belong to Π^1_1 uniformly on n, k. Therefore the set $\mathbf{S} = \bigcup_m \bigcap_{n \geq m} \bigcup_k S^k_n$ also belongs to Π^1_1.

Proof. The equality of the two definitions follows from Lemma 7. The definability follows from Proposition 9 by standard evaluation. □

Beginning **the proof of Theorem 2,** we can *w.l.o.g.* assume, as usual, that the Borel set P_0 in the theorem is a lightface Δ^1_1 set.

Case 1: $P_0 \subseteq \mathbf{S}$. We'll show that in this case $\mathsf{E}_{13} \restriction P_0$ is Borel reducible to T_2.

Case 2: $P_0 \smallsetminus \mathbf{S} \neq \varnothing$. We'll prove that then $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_{13} \restriction P_0$.

3 Case 1: simplification

From now on and until the end of Section 4 we work under the assumptions of Case 1. The general strategy is to prove that for any $\langle x, \xi\rangle \in P_0$ there exist at most countably many points $y \in \mathbb{R}^{\mathbb{N}}$ such that, for some η, $\langle y, \eta\rangle \in P_0$ and $\langle x, \xi\rangle \mathrel{\mathsf{E}_{13}} \langle y, \eta\rangle$, and that those points can be arranged in countable sequences in a certain controlled way.

Our first goal is to somewhat simplify the picture.

Lemma 11. *There exists a Δ^1_1 map $\mu : P_0 \to \mathbb{N}$ such that for any $\langle x, \xi\rangle \in P_0$ we have $\langle x, \xi\rangle \in \bigcap_{n \geq \mu(x,\xi)} \bigcup_k S^k_n$.*

Proof. Apply Kreisel Selection to the set

$$\{\langle\langle x,\xi\rangle,m\rangle \in P_0\times\mathbb{N} : \forall\, n\ge m\,\exists\, k\,(\langle x,\xi\rangle\in S^k_n)\}\,. \qquad \square$$

Let $\mathbf{0}=0^{\mathbb{N}}\in\mathbb{R}=2^{\mathbb{N}}$ be the constant 0: $\mathbf{0}(k)=0$, $\forall\, k$. For any $\langle x,\xi\rangle\in P_0$ put $f_\mu(x,\xi)=\mathbf{0}^{\mu(x,\xi)}{}^\wedge(x\restriction_{\geqslant\mu(x,\xi)})$: that is, we replace by $\mathbf{0}$ all values $x(n)$ with $n<\mu(x,\xi)$. Then $P'_0=\{\langle f_\mu(x,\xi),\xi\rangle : \langle x,\xi\rangle\in P_0\}$ is a Σ^1_1 set.

Put $\mathbf{S}'=\bigcap_n\bigcup_k S^k_n$ (a Π^1_1 set by Corollary 10).

Corollary 12. *There is a Δ^1_1 set P''_0 such that $P'_0\subseteq P''_0\subseteq\mathbf{S}'$. The map $\langle x,\xi\rangle\mapsto\langle f_\mu(x,\xi),\xi\rangle$ is a reduction of $\mathsf{E}_{13}\restriction P_0$ to $\mathsf{E}_{13}\restriction P''_0$.*

Proof. Obviously P'_0 is a subset of the Π^1_1 set $\mathbf{S}'$. It follows that there is a Δ^1_1 set P''_0 such that $P'_0\subseteq P''_0\subseteq\mathbf{S}'$. To prove the second claim note that $f_\mu(x,\xi)\,\mathsf{E}_1\,x$ for all $\langle x,\xi\rangle\in P_0$. $\square$

Let us fix a Δ^1_1 set P''_0 as indicated. By Corollary 12 to accomplish Case 1 it suffices to get a Borel reduction of $\mathsf{E}_{13}\restriction P''_0$ to T_2.

Lemma 13. *There exist: a Δ^1_1 sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ of natural numbers, and a Δ^1_1 system $\{F^i_n\}_{i,n\in\mathbb{N}}$ of functions $F^i_n\in{}^{\mathtt{T}}\!\mathscr{F}^{\kappa_i}_n$, such that for all $\langle x,\xi\rangle\in P''_0$ and $n\in\mathbb{N}$ there is $i\in\mathbb{N}$ satisfying $x(n)=F^i_n(x\restriction_{>n},\xi)$.*

Remark 14. Recall that by definition every function $F\in{}^{\mathtt{T}}\!\mathscr{F}^k_n$ is invariant in the sense that if $\langle x,\xi\rangle$ and $\langle x,\eta\rangle$ belong to $\mathbb{R}^{>n}\times\mathbb{R}^{\mathbb{N}}$, $\xi\restriction_{<k}=\eta\restriction_{<k}$, and $\xi\,\mathsf{E}_3\,\eta$, then $\varphi(x,\xi)=\varphi(x,\eta)$. This allows us to sometimes use the notation like $F^i_n(x\restriction_{>n},\xi\restriction_{<k},\xi\restriction_{\geqslant k})$, where $k=\kappa_i$, instead of $F^i_n(x\restriction_{>n},\xi)$, with the understanding that $F^i_n(x\restriction_{>n},\xi\restriction_{<k},\xi\restriction_{\geqslant k})$ is E_3-invariant in the 3rd argument.

In these terms, the final equality of the lemma can be re-written as $x(n)=F^i_n(x\restriction_{>n},\xi\restriction_{<k},\xi\restriction_{\geqslant k})$, where $k=\kappa_i$. $\square$

Proof (lemma)**.** By definition $P''_0\subseteq\mathbf{S}'$ means that for any $\langle x,\xi\rangle\in P''_0$ and n there exists k such that $\langle x,\xi\rangle\in S^k_n$. The formula $\langle x,\xi\rangle\in S^k_n$ takes the form

$$\exists\,\varphi\in{}^{\mathtt{T}}\!\mathscr{F}^k_n\;(x(n)=\varphi(x\restriction_{>n},\xi)),$$

and further the form $\exists\,\langle e,k\rangle\in T\;(x(n)=W^e_n(x\restriction_{>n},\xi))$. It follows that the Π^1_1 set

$$Z=\{\langle\langle x,\xi,n\rangle,\langle e,k\rangle\rangle\in(P_0\times\mathbb{N})\times T : x(n)=W^e_n(x\restriction_{>n},\xi)\}$$

satisfies $\mathtt{dom}\,Z=P_0\times\mathbb{N}$. Therefore by Kreisel Selection there is a Δ^1_1 map $\varepsilon:P_0\times\mathbb{N}\to T$ such that $x(n)=W^e_n(x\restriction_{>n},\xi)$ holds for any $\langle x,\xi\rangle\in P_0$ and n, where $\langle e,k\rangle=\varepsilon(x,\xi,n)$ for some k.

The range $R=\mathtt{ran}\,\varepsilon$ of this function is a Σ^1_1 subset of the Π^1_1 set T. We conclude that there is a Δ^1_1 set B such that $R\subseteq B\subseteq T$. And since $T\subseteq\mathbb{N}\times\mathbb{N}$, it follows, by some known theorems of effective descriptive set theory, that the set $\widehat{E} = \mathtt{dom}\, B = \{e : \exists\, k\, (\langle e, k\rangle \in B)\}$ is $\varDelta^1_1$, and in addition there exists a $\varDelta^1_1$ map $K : \widehat{E} \to \mathbb{N}$ such that $\langle e, K(e)\rangle \in B$ (and $\in T$) for all $e \in \widehat{E}$.

And on the other hand it follows from the construction that

$$\forall\, \langle x, \xi\rangle \in P_0\, \forall\, n\, \exists\, e \in \widehat{E}\, (x(n) = W^e_n(x {\restriction}_{>n}, \xi))\,. \tag{3}$$

Let us fix any $\varDelta^1_1$ enumeration $\{e(i)\}_{i\in\mathbb{N}}$ of elements of $\widehat{E}$. Put $F^i_n = W^{e(i)}_n$. Then the last conclusion of the lemma follows from (3). Note that the functions F^i_n are uniformly $\varDelta^1_1$, $F^i_n \in {}^{\mathsf{T}}\!\mathscr{F}^k_n$ for some k, in particular, for $k = \kappa_i$, where $\kappa_i = K(e(i))$, and $\{\kappa_i\}_{i\in\mathbb{N}}$ is a $\varDelta^1_1$ sequence as well. □

Blanket Agreement 15. Below, we assume that the set P''_0 is chosen as above, that is, $\varDelta^1_1$ and $P''_0 \subseteq \mathbf{S}'$, while a system of functions F^i_n and a sequence $\{\kappa_i\}_{i\in\mathbb{N}}$ of natural numbers are chosen accordingly to Lemma 13. □

4 Case 1: countability of projections of equivalence classes

We prove here that in the assumption of Case 1 the equivalence $\mathsf{E}_{13} {\restriction} P''_0$ is Borel reducible to T_2, the equality of countable sets of reals. The main ingredient of this result will be the countability of the sets

$$C^\xi_x = \mathtt{dom}\,([\langle x, \xi\rangle]_{\mathsf{E}_{13}} \cap P''_0) = \{y \in \mathbb{R}^{\mathbb{N}} : y \mathrel{\mathsf{E}_1} x \wedge \exists\, \eta\, (\xi \mathrel{\mathsf{E}_3} \eta \wedge \langle y, \eta\rangle \in P''_0)\},$$

where $\langle x, \xi\rangle \in P''_0$ — projections of E_{13}-classes of elements of the set P''_0.

Lemma 16. *If $\langle x, \xi\rangle \in P''_0$ then $C^\xi_x \subseteq [x]_{\mathsf{E}_1}$ and C^ξ_x is at most countable.*

Proof. That $C^\xi_x \subseteq [x]_{\mathsf{E}_1}$ is obvious. The proof of countability begins with several definitions. In fact we are going to organize elements of any set of the form C^ξ_x in a countable sequence.

Recall that $\mathbb{R} = 2^{\mathbb{N}}$. If $u \subseteq \mathbb{N}$ and $b \in \mathbb{R}$ then define $u \cdot a \in \mathbb{R}$ so that $(u \cdot a)(j) = a(j)$ whenever $j \notin u$, and $(u \cdot a)(j) = 1 - a(j)$ otherwise.

If $f \subseteq \mathbb{N} \times \mathbb{N}$ and $a \in \mathbb{R}^k$ then define $f \cdot a \in \mathbb{R}^k$ so that $(f \cdot a)(j) = (f\,\text{"}j) \cdot a(j)$ for all $j < k$, where $f\,\text{"}j = \{m : \langle j, m\rangle \in f\}$. Note that $f \cdot a$ depends in this case only on the restricted set $f {\restriction} k = \{\langle j, m\rangle \in f : j < k\}$.

Put $\Phi = \mathscr{P}_{\mathtt{fin}}(\mathbb{N} \times \mathbb{N})$ and $D = \bigcup_n D_n$, where for every n:

$$D_n = \{\langle a, \varphi\rangle : a \in \mathbb{N}^n \wedge \varphi \in \Phi^n \wedge \forall\, j < n\, \big(\varphi(j) \subseteq \kappa_{a(j)} \times \mathbb{N}\big)\}.^{4}$$

(The inclusion $\varphi(j) \subseteq \kappa_{a(j)} \times \mathbb{N}$ here means that the set $\varphi(j) \subseteq \mathbb{N} \times \mathbb{N}$ satisfies $\varphi(j) = \varphi(j) {\restriction} \kappa_{a(j)}$, that is, every pair $\langle k, l\rangle \in \varphi(j)$ satisfies $k < \kappa_{a(j)}$.)

If $\langle a, \varphi\rangle \in D_n$ and $\langle x, \xi\rangle \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ then we define $y = \boldsymbol{\tau}^\xi_x(a, \varphi) \in \mathbb{R}^{\mathbb{N}}$ as follows: $y = \langle b_0, b_1, \dots, b_{n-1}\rangle^\wedge(x {\restriction}_{\geqslant n})$, where the reals $b_m \in \mathbb{R}$ $(m < n)$ are defined by inverse induction so that

$$b_m = F^{a(m)}_m\big(\langle b_{m+1}, b_{m+2}, \dots, b_{n-1}\rangle^\wedge(x {\restriction}_{\geqslant n})\,,\, \varphi(m) \cdot (\xi {\restriction}_{<\kappa_{a(m)}})\,,\, \xi {\restriction}_{\geqslant \kappa_{a(m)}}\big). \tag{4}$$

(See Remark 14 on notation. The element $\eta = \big(\varphi(m)\cdot(\xi\restriction_{<\kappa_{a(m)}})\big)^\wedge(\xi\restriction_{\geqslant\kappa_{a(m)}})$ belongs to $\mathbb{R}^{\mathbb{N}}$ and satisfies $\eta\,\mathsf{E}_3\,\xi$ because $\varphi(m)$ is a finite set.)

Put $\boldsymbol{\tau}_x^\xi(\Lambda,\Lambda) = x$ (Λ is the empty sequence).

Note that by definition the element $y = \boldsymbol{\tau}_x^\xi(a,\varphi) \in \mathbb{R}^{\mathbb{N}}$ satisfies $y\restriction_{\geqslant n} = x\restriction_{\geqslant n}$ provided $\langle a,\varphi\rangle \in D_n$, thus in any case $x\,\mathsf{E}_1\,\boldsymbol{\tau}_x^\xi(a,\varphi)$. Thus $\boldsymbol{\tau}_x^\xi$, the *trace* of $\langle x,\xi\rangle$, is a countable sequence, that is, a function defined on $D = \bigcup_n D_n$, a countable set, and the set $\mathtt{ran}\,\boldsymbol{\tau}_x^\xi = \{\boldsymbol{\tau}_x^\xi(a,\varphi) : \langle a,\varphi\rangle \in D\}$ of all terms of this sequence is at most countable and satisfies $x = \boldsymbol{\tau}_x^\xi(\Lambda,\Lambda) \in \mathtt{ran}\,\boldsymbol{\tau}_x^\xi \subseteq [x]_{\mathsf{E}_1}$.

Claim 17. *Suppose that $\langle x,\xi\rangle \in P_0''$. Then $C_x^\xi \subseteq \mathtt{ran}\,\boldsymbol{\tau}_x^\xi$ — and hence C_x^ξ is at most countable. More exactly if $y \in C_x^\xi$ and $y\restriction_{\geqslant n} = x\restriction_{\geqslant n}$ then there is a pair $\langle a,\varphi\rangle \in D_n$ such that $y = \boldsymbol{\tau}_x^\xi(a,\varphi)$.*

We prove the second, more exact part of the claim. By definition there is $\eta \in \mathbb{R}^{\mathbb{N}}$ such that $\langle y,\eta\rangle \in P_0''$ and $\xi\,\mathsf{E}_3\,\eta$. Put $b_m = y(m)$, $\forall m$. Note that for every $m < n$ there is a number $a(m)$ such that

$$\begin{aligned} b_m &= F_m^{a(m)}\big(\langle b_{m+1},\dots,b_{n-1}\rangle^\wedge(y\restriction_{\geqslant n})\,,\,\eta\big) && = \\ &= F_m^{a(m)}\big(\langle b_{m+1},\dots,b_{n-1}\rangle^\wedge(y\restriction_{\geqslant n})\,,\,\eta\restriction_{<\kappa_{a(m)}}\,,\,\eta\restriction_{\geqslant\kappa_{a(m)}}\big) \end{aligned}$$

for all $m < n$ (see Blanket Agreement 15), and hence

$$b_m = F_m^{a(m)}\big(\langle b_{m+1},\dots,b_{n-1}\rangle^\wedge(x\restriction_{\geqslant n})\,,\,\eta\restriction_{<\kappa_{a(m)}}\,,\,\xi\restriction_{\geqslant\kappa_{a(m)}}\big)$$

by the invariance of functions F_m^i and because $x\restriction_{\geqslant n} = y\restriction_{\geqslant n}$. On the other hand, it follows from the assumption $\xi\,\mathsf{E}_3\,\eta$ that for every $m < n$ there is a finite set $\varphi(m) \subseteq \kappa_{a(m)} \times \mathbb{N}$ such that $\eta\restriction_{<\kappa_{a(m)}} = \varphi(m)\cdot(\xi\restriction_{<\kappa_{a(m)}})$. Then

$$b_m = F_m^{a(m)}\big(\langle b_{m+1},\dots,b_{n-1}\rangle^\wedge(x\restriction_{\geqslant n})\,,\,\varphi(m)\cdot(\xi\restriction_{<\kappa_{a(m)}})\,,\,\xi\restriction_{\geqslant\kappa_{a(m)}}\big)$$

for every $m < n$, that is, $y = \boldsymbol{\tau}_x^\xi(a,\varphi)$, as required. □ (*Claim and Lemma 16*)

The next result reduces the equivalence relation $\mathsf{E}_{13}\restriction P_0''$ to the equality of sets of the form $\mathtt{ran}\,\boldsymbol{\tau}_x^\xi$, that is essentially to the equivalence relation T_2 of "equality of countable sets of reals".

Corollary 18. *Suppose that $\langle x,\xi\rangle$ and $\langle y,\eta\rangle$ belong to P_0''. Then $\langle x,\xi\rangle\,\mathsf{E}_{13}\,\langle y,\eta\rangle$ holds if and only if $\xi\,\mathsf{E}_3\,\eta$ and $\mathtt{ran}\,\boldsymbol{\tau}_x^\xi = \mathtt{ran}\,\boldsymbol{\tau}_y^\eta$.*

Proof. The "if" direction is rather easy. If $\xi\,\mathsf{E}_3\,\eta$ and $\mathtt{ran}\,\boldsymbol{\tau}_y^\eta = \mathtt{ran}\,\boldsymbol{\tau}_x^\xi$ then $x\,\mathsf{E}_1\,y$ because $\mathtt{ran}\,\boldsymbol{\tau}_y^\eta \subseteq [y]_{\mathsf{E}_1}$ and $\mathtt{ran}\,\boldsymbol{\tau}_x^\xi \subseteq [x]_{\mathsf{E}_1}$ by Lemma 16.

To prove the converse suppose that $\langle x,\xi\rangle\,\mathsf{E}_{13}\,\langle y,\eta\rangle$. Then $\xi\,\mathsf{E}_3\,\eta$, of course. Furthermore, $x\,\mathsf{E}_1\,y$, therefore $x\restriction_{\geqslant n} = y\restriction_{\geqslant n}$ for an appropriate n. Let us prove that $\mathtt{ran}\,\boldsymbol{\tau}^{\eta}_{y}=\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{x}$. First of all, by definition we have $y\in C^{\xi}_{x}$, and hence (see the proof of Claim 17) there exists a pair $\langle a,\varphi\rangle\in D_n$ such that $y=\boldsymbol{\tau}^{\xi}_{x}(a,\varphi)$.

Now, let us establish $\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{x}=\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{y}$ (with one and the same ξ). Suppose that $z\in\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{x}$, that is, $z=\boldsymbol{\tau}^{\xi}_{x}(b,\psi)$ for a pair $\langle b,\psi\rangle\in D_m$ for some m. If $m\ge n$ then obviously $z=\boldsymbol{\tau}^{\xi}_{x}(b,\psi)=\boldsymbol{\tau}^{\xi}_{y}(b,\psi)$, and hence (as $x\restriction_{\geqslant n}=y\restriction_{\geqslant n}$) $z\in\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{y}$. If $m<n$ then $z=\boldsymbol{\tau}^{\xi}_{x}(b,\psi)=\boldsymbol{\tau}^{\xi}_{y}(a',\varphi')$, where $a'=b^\wedge(a\restriction_{\geqslant m})$ and $\varphi'=\psi^\wedge(\varphi\restriction_{\geqslant m})$, and once again $z\in\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{y}$. Thus $\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{x}\subseteq\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{y}$. The proof of the inverse inclusion $\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{y}\subseteq\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{x}$ is similar.

Thus $\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{y}=\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{x}$. It remains to prove $\mathtt{ran}\,\boldsymbol{\tau}^{\eta}_{y}=\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{y}$ for all y,ξ,η such that $\xi\ \mathsf{E}_3\ \eta$. Here we need another block of definitions.

Let $\mathbb{H}$ be the set of all sets $\delta\subseteq\mathbb{N}\times\mathbb{N}$ such that $\delta\text{''}j=\{m:\langle j,m\rangle\in\delta\}$ is finite for all $j\in\mathbb{N}$. For instance if $\xi,\eta\in\mathbb{R}^{\mathbb{N}}$ satisfy $\xi\ \mathsf{E}_3\ \eta$ then the set

$$\delta_{\xi\eta}=\{\langle j,m\rangle:\xi(j)(m)\ne\eta(j)(m)\}$$

belongs to $\mathbb{H}$. The operation of symmetric difference Δ converts $\mathbb{H}$ into a Polish group equal to the product group $\langle\mathscr{P}_{\mathtt{fin}}(\mathbb{N});\Delta\rangle^{\mathbb{N}}$.

If $n\in\mathbb{N}$, $\langle a,\varphi\rangle\in D_n$, and $\delta\in\mathbb{H}$ then we define a sequence $\varphi'=H^a_\delta(\varphi)\in\Phi^n$ so that $\varphi'(m)=(\delta\restriction\kappa_{a(m)})\ \Delta\ \varphi(m)$ for every $m<n$.[5] Then the pair $\langle a,H^a_\delta(\varphi)\rangle$ obviously still belongs to D_n and $H^a_\delta(H^a_\delta(\varphi))=\varphi$.

Coming back to a triple of $y,\xi,\eta\in\mathbb{R}^{\mathbb{N}}$ such that $\xi\ \mathsf{E}_3\ \eta$, let $\delta=\delta_{\xi\eta}$. A routine verification shows that $\boldsymbol{\tau}^{\eta}_{y}(a,\varphi)=\boldsymbol{\tau}^{\xi}_{y}(a,H^a_\delta(\varphi))$ for all $\langle a,\varphi\rangle\in D$. It follows that $\mathtt{ran}\,\boldsymbol{\tau}^{\eta}_{y}=\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{y}$, as required. □

Corollary 19. *The restricted relation* $\mathsf{E}_{13}\restriction P''_0$ *is Borel reducible to* T_2*.*

Proof. Since all $\boldsymbol{\tau}^{\xi}_{x}$ are countable sequences of reals, the equality $\mathtt{ran}\,\boldsymbol{\tau}^{\eta}_{y}=\mathtt{ran}\,\boldsymbol{\tau}^{\xi}_{x}$ of Corollary 18 is Borel reducible to T_2. Thus $\mathsf{E}_{13}\restriction P''_0$ is Borel reducible to $\mathsf{E}_3\times\mathsf{T}_2$ by Corollary 18. However it is known that E_3 is Borel reducible to T_2, and so does $\mathsf{T}_2\times\mathsf{T}_2$. □

□ (*Case 1 of Theorem 2*)

5 Case 1: a more elementary (?) transformation group

Here we begin the proof of Theorem 3. Our plan is to define a countable group $\mathbb{G}$ of homeomorphisms of $\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$ such that the induced equivalence relation G satisfies Theorem 3. We continue to argue under the assumptions of Case 1.

First of all let us define the basic domain of transformations,

$$\boldsymbol{\Pi}=\{\langle x,\xi\rangle\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}:\forall n\,\exists\,\langle a,\varphi\rangle\in D_n\,(x=\boldsymbol{\tau}^{\xi}_{x}(a,\varphi))\}.$$

This is a closed subset of $\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$. Applying Claim 17 with $y=x$ we obtain

[5] Recall that $\delta\restriction k=\{\langle j,i\rangle\in\delta:j<k\}$.

Corollary 20. $P_0'' \subseteq \mathbf{\Pi}$. $\square$

Suppose that pairs $\langle a,\varphi\rangle$ and $\langle b,\psi\rangle$ belong to D_n for one and the same n, and $\langle x,\xi\rangle \in \mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$. We define $G^{b\psi}_{a\varphi}(x,\xi) = \langle y,\xi\rangle \in \mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$ so that

$$y = \begin{cases} \boldsymbol{\tau}^\xi_x(b,\psi) & \text{whenever} \quad x = \boldsymbol{\tau}^\xi_x(a,\varphi) \\ \boldsymbol{\tau}^\xi_x(a,\varphi) & \text{whenever} \quad x = \boldsymbol{\tau}^\xi_x(b,\psi) \\ x & \text{whenever} \quad \boldsymbol{\tau}^\xi_x(a,\varphi) \neq x \neq \boldsymbol{\tau}^\xi_x(b,\psi) \end{cases}$$

Note that if $\boldsymbol{\tau}^\xi_x(a,\varphi) = x = \boldsymbol{\tau}^\xi_x(b,\psi)$ then still $y = x$ by either of the two first cases of the definition. And in any case $y\restriction_{\geqslant n} = x\restriction_{\geqslant n}$ provided $\langle a,\varphi\rangle \in D_n$.

Lemma 21. *Suppose that* $n \in \mathbb{N}$ *and pairs* $\langle a,\varphi\rangle$, $\langle b,\psi\rangle$ *belong to* D_n. *Then* $G^{b\psi}_{a\varphi}$ *is a homeomorphism of* $\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$ *onto itself, and* $G^{b\psi}_{a\varphi} = G^{a\varphi}_{b\psi}$.

In addition, $G^{b\psi}_{a\varphi}$ *is a homeomorphism of* $\mathbf{\Pi}$ *onto itself.*

Proof. Suppose that $\langle x,\xi\rangle$ belongs to $\mathbf{\Pi}$ and prove that so does $\langle y,\xi\rangle = G^{b\psi}_{a\varphi}(x,\xi)$. By definition y coincides with one of $x, \boldsymbol{\tau}^\xi_x(a,\varphi), \boldsymbol{\tau}^\xi_x(b,\psi)$. So assume that $y = \boldsymbol{\tau}^\xi_x(b,\psi)$. Consider any m, we have to show that $y = \boldsymbol{\tau}^\xi_y(a',\varphi')$ for some $\langle a',\varphi'\rangle \in D_m$. If $m \le n$ then the pair of $a' = b\restriction m$ and $\varphi' = \psi\restriction m$ obviously works. If $m > n$ then take the pair of $a' = b^\wedge(b'\restriction_{\geqslant n})$ and $\varphi' = \psi^\wedge(\psi'\restriction_{\geqslant n})$ where $\langle b',\psi'\rangle \in D_m$ is an arbitrary pair satisfying $x = \boldsymbol{\tau}^\xi_x(b',\psi')$. $\square$

Lemma 22. *Suppose that* $\langle x,\xi\rangle \in \mathbf{\Pi}$. *Then*:

(i) *if* $\langle a,\varphi\rangle$, $\langle b,\psi\rangle \in D_n$ *and* $\langle y,\xi\rangle = G^{b\psi}_{a\varphi}(x,\xi)$ *then* $\texttt{ran}\,\boldsymbol{\tau}^\xi_x = \texttt{ran}\,\boldsymbol{\tau}^\xi_y$;

(ii) *if* $y \in \texttt{ran}\,\boldsymbol{\tau}^\xi_x$ *then there exist* n *and pairs* $\langle a,\varphi\rangle$, $\langle b,\psi\rangle \in D_n$ *such that* $\langle y,\xi\rangle = G^{b\psi}_{a\varphi}(x,\xi)$.

Proof. (i) Consider an arbitrary $z = \boldsymbol{\tau}^\xi_x(a',\varphi') \in \texttt{ran}\,\boldsymbol{\tau}^\xi_x$, where $\langle a',\varphi'\rangle \in D_m$. Once again y coincides with one of $x, \boldsymbol{\tau}^\xi_x(a,\varphi), \boldsymbol{\tau}^\xi_x(b,\psi)$, so assume that $y = \boldsymbol{\tau}^\xi_x(b,\psi)$. If $m \ge n$ then obviously $z = \boldsymbol{\tau}^\xi_y(a',\varphi') \in \texttt{ran}\,\boldsymbol{\tau}^\xi_y$. If $m < n$ then we have $z = \boldsymbol{\tau}^\xi_y(b',\psi')$, where $b' = a'^\wedge(b\restriction_{\geqslant m})$ and $\psi' = \varphi'^\wedge(\psi\restriction_{\geqslant m})$.

(ii) If $y \in \texttt{ran}\,\boldsymbol{\tau}^\xi_x$ then by definition there is a pair $\langle b,\psi\rangle$ in some D_n such that $y = \boldsymbol{\tau}^\xi_x(b,\psi)$. Then by the way $x\restriction_{\geqslant n} = y\restriction_{\geqslant n}$. As $\langle x,\xi\rangle \in \mathbf{\Pi}$, there is a pair $\langle a,\varphi\rangle \in D_n$ such that $x = \boldsymbol{\tau}^\xi_x(a,\varphi)$. Then $\langle y,\xi\rangle = G^{b\psi}_{a\varphi}(x,\xi)$. $\square$

Let $\mathbb{G}$ denote the group of all finite superpositions of maps of the form $G^{b\psi}_{a\varphi}$, where $\langle a,\varphi\rangle$, $\langle b,\psi\rangle$ belong to one and the same set D_n as in the lemma. Thus $\mathbb{G}$ is a countable group of homeomorphisms of $\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$. (We'll prove that $\mathbb{G}$ is even an increasing union of its finite subgroups!) Note that a superposition of the form $G^{a''\varphi''}_{a'\varphi'} \circ G^{a'\varphi'}_{a\varphi}$ does not necessarily coincide with $G^{a\varphi}_{a''\varphi''}$.

We are going to prove that the equivalence relation G induced by $\mathbb{G}$ on $\mathbf{\Pi}$ satisfies Theorem 3. To be more exact, G is defined on $\mathbf{\Pi}$ so that $\langle x,\xi\rangle \mathrel{\mathsf{G}} \langle y,\eta\rangle$ iff there exists a homeomorphism $g \in \mathbb{G}$ such that $g(x,\xi) = \langle y,\eta\rangle$. Note that then by definition $\eta = \xi$.

The hyperfiniteness G will be established in the next Section. Now let us study relations between $\mathbb{G}$ and $\mathbb{H}$, the other involved group introduced in the proof of Corollary 18. For any $\delta \in \mathbb{H}$ define a homeomorphism H_δ of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ so that $H_\delta(x,\xi) = \langle x,\eta\rangle$, where simply $\eta = \delta \,\Delta\, \xi$ in the sense that

$$\eta(m,j) = \begin{cases} \xi(m,j) & \text{whenever} \quad \langle m,j\rangle \notin \delta \\ 1-\xi(m,j) & \text{whenever} \quad \langle m,j\rangle \in \delta \end{cases}$$

(Then obviously $\delta = \delta_{\xi\eta}$.) If $\gamma,\delta \in \mathbb{H}$ then the superposition $H_\delta \circ H_\gamma$ coincides with $H_{\gamma\Delta\delta}$, where Δ is the symmetric difference, as usual.

Transformations of the form $G_{a\varphi}^{b\psi}$ do not commute with those of the form H_δ, yet there exists a convenient law of commutation:

Lemma 23. *Suppose that $n \in \mathbb{N}$ and pairs $\langle a,\varphi\rangle$ and $\langle b,\psi\rangle$ belong to D_n, and $\delta \in \mathbb{H}$. Then the superposition $G_{a\varphi}^{b\psi} \circ H_\delta$ coincides with $H_\delta \circ G_{a\varphi'}^{b\psi'}$, where $\varphi' = H_\delta^a(\varphi)$ and $\psi' = H_\delta^b(\psi)$.*

Proof. A routine argument is left for the reader. □

Let us consider the group $\mathbb{S}$ of all homeomorphisms $s : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ of the form

$$s = H_\delta \circ g_{\ell-1} \circ g_{\ell-2} \cdots \circ g_1 \circ g_0\,, \tag{5}$$

where $\ell \in \mathbb{N}$, $\delta \in \mathbb{H}$, and each g_i is a homeomorphism of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ of the form $G_{a_i\varphi_i}^{b_i\psi_i}$, where the pairs $\langle a_i,\varphi_i\rangle$, $\langle b_i,\psi_i\rangle$ belong to one and the same set D_n, $n = n_i$. (It follows that $g_{\ell-1} \circ g_{\ell-2} \cdots \circ g_1 \circ g_0 \in \mathbb{G}$.)

Lemma 23 implies that $\mathbb{S}$ is really a group under the operation of superposition. For instance if $g = G_{a\varphi}^{b\psi}$ and g_1 belong to $\mathbb{G}$ (and $\langle a,\varphi\rangle$, $\langle b,\psi\rangle$ belong to one and the same D_n) then the superposition $H_\delta \circ g \circ H_{\delta_1} \circ g_1$ coincides with $H_\delta \circ H_{\delta_1} \circ g' \circ g_1 = H_{\delta\Delta\delta_1} \circ (g' \circ g_1)$, where $g' = G_{a\varphi'}^{b\psi'}$ and $\varphi' = H_{\delta_1}^a(\varphi)$, $\psi' = H_{\delta_1}^b(\psi)$ as in Lemma 23.

Thus $\mathbb{S}$ seems to be a more complicated group than the direct cartesian product of $\mathbb{G}$ and $\mathbb{H}$, but on the other hand more elementary than the free product (of all formal superpositions of elements of both groups). A natural action of $\mathbb{S}$ on $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ is defined as follows: if s is as in (5) then $s\cdot\langle x,\xi\rangle = H_\delta(g_{\ell-1}(g_{\ell-2}(\dots g_1(g_0(x,\xi))\dots)))$. Let S denote the induced orbit equivalence relation. One can easily check that both the group $\mathbb{S}$ and the action are Polish. On the other hand, S is obviously the conjunction of G and the equivalence relation E_3 acting on the 2nd factor of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, in the sense of Theorem 3 and footnote 2 on page 3. Thus the next lemma, together with the result of Lemma 25 on the hyperfiniteness of G, accomplish the proof of Theorem 3.

Lemma 24. *Suppose that* $\langle x,\xi\rangle$, $\langle y,\eta\rangle \in P_0''$. *Then* $\langle x,\xi\rangle \,\mathsf{E}_{13}\, \langle y,\eta\rangle$ *if and only if* $\langle x,\xi\rangle \,\mathsf{S}\, \langle y,\eta\rangle$.

Proof. Suppose that $\langle x,\xi\rangle \,\mathsf{E}_{13}\, \langle y,\eta\rangle$. Then $y \in \mathtt{ran}\,\boldsymbol{\tau}_x^\xi$ by Corollary 18, and further $\langle x,\xi\rangle \,\mathsf{S}\, \langle y,\xi\rangle$ by Lemma 22(ii). It remains to note that $\langle y,\xi\rangle \,\mathsf{S}\, \langle y,\eta\rangle$ by obvious reasons.

Now suppose that $\langle x,\xi\rangle \,\mathsf{S}\, \langle y,\eta\rangle$. Then $\xi \,\mathsf{E}_3\, \eta$, and hence by Corollary 19 it suffices to prove that $\mathtt{ran}\,\boldsymbol{\tau}_x^\xi = \mathtt{ran}\,\boldsymbol{\tau}_y^\eta$. This follows from two observations saying that transformations in $\mathbb{H}$ and in $\mathbb{G}$ preserve $\mathtt{ran}\,\boldsymbol{\tau}_*^*$. First, if $\langle x,\xi\rangle \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $\delta \in \mathbb{H}$, and $\langle y,\xi\rangle = H_\delta(x,\xi)$ then $\boldsymbol{\tau}_x^\eta$ obviously is a permutation of $\boldsymbol{\tau}_y^\eta$, and hence $\mathtt{ran}\,\boldsymbol{\tau}_x^\xi = \mathtt{ran}\,\boldsymbol{\tau}_x^\eta$. Second, if $\langle x,\xi\rangle \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, pairs $\langle a,\varphi\rangle$, $\langle b,\psi\rangle$ belong to one and the same set D_n, and $\langle y,\xi\rangle = G_{a\varphi}^{b\psi}(x,\xi)$, then $\mathtt{ran}\,\boldsymbol{\tau}_x^\xi = \mathtt{ran}\,\boldsymbol{\tau}_y^\xi$ by Lemma 22. □

□ (*Theorem 3 modulo Lemma 25*)

6 Case 1: the "hyperfiniteness" of the countable group $\mathbb{G}$

Lemma 24 reduces further study of Case 1 of Theorem 2 to properties of the group $\mathbb{S}$ and its Polish actions. This is an open topic, and maybe the next result, the "hyperfiniteness" of $\mathbb{G}$, one of the two components of $\mathbb{S}$, can lead to a more comprehensive study. One might think that $\mathbb{G}$ is a rather complicated countable group, perhaps close to the free group on two (or countably many) generators. The reality is different:

Lemma 25. $\mathbb{G}$ *is the union of an increasing sequence of finite subgroups, therefore the induced equivalence relation* G *is hyperfinite.*

Proof. Let us show that a finite set of "generators" $G_{a\varphi}^{a'\varphi'}$ produces only finitely many superpositions — this obviously implies the lemma. Suppose that $m \in \mathbb{N}$, and $\langle a_i,\varphi_i\rangle \in D_{n(i)}$ for all $i < m$. Put $G_{ij} = G_{a_i\varphi_i}^{a_j\varphi_j}$ provided $n(i) = n(j)$, and let G_{ij} be the identity otherwise. Thus all G_{ij} are homeomorphisms of $\mathbf{\Pi}$. We are going to prove that the set of all superpositions of the form $f_0 \circ f_1 \circ \dots \circ f_\ell$, where ℓ is an arbitrary natural number and each of f_k is equal to one of G_{ij} (i,j depend on k) contains only finitely many really different functions.

Note that if $i,j < m$ and $n(i) < n(j)$ then the pair

$$\langle a_i{}^\wedge(a_j\restriction_{\geqslant n(i)})\,,\, \varphi_i{}^\wedge(\varphi_j\restriction_{\geqslant n(i)})\rangle$$

belongs to $D_{n(j)}$. We can *w.l.o.g.* assume that every such a pair occurs in the list of pairs $\langle a_i,\varphi_i\rangle$, $i < m$.

Let us associate a pair $q(x,\xi) = \langle u_{x\xi}, w_{x\xi}\rangle$ of finite sets

$$\begin{aligned} u_{x\xi} &= \{i < m : \boldsymbol{\tau}_x^\xi(a_i,\varphi_i) = x\}, \quad \text{and} \\ w_{x\xi} &= \{\langle i,j\rangle : i,j < m \wedge \boldsymbol{\tau}_x^\xi(a_i,\varphi_i) = \boldsymbol{\tau}_x^\xi(a_j,\varphi_j)\} \end{aligned}$$

with every point $\langle x,\xi\rangle\in\mathbf{\Pi}$. Put $Q=\mathscr{P}(m)\times\mathscr{P}(m\times m)$, a (finite) set including all possible values of $q(\pi)$.

Claim 26. *For every $q=\langle u,w\rangle\in Q$ and $i,j<m$ there exists $\tilde q=\langle\tilde u,\tilde w\rangle\in Q$ such that $q(G_{ij}(x,\xi))=\tilde q$ for all $\langle x,\xi\rangle\in\mathbf{\Pi}$ with $q(x,\xi)=q$.*

Proof (Claim)**.** We can assume that $i\ne j$ and $n(i)=n(j)$ since otherwise $G_{ij}(x,\xi)=\langle x,\xi\rangle$, and hence $\tilde q=q$ works. By the same reason we can *w. l. o. g.* assume that either $i\in u\wedge j\notin u$ or $i\notin u\wedge j\in u$. Let say $i\in u\wedge j\notin u$, that is, $\boldsymbol{\tau}^\xi_x(a_i,\varphi_i)=x\ne\boldsymbol{\tau}^\xi_x(a_j,\varphi_j)$. Then by definition the element $\langle y,\xi\rangle=G_{ij}(x,\xi)=G^{a_j\varphi_j}_{a_i\varphi_i}(x,\xi)$ coincides with $\langle\boldsymbol{\tau}^\xi_x(a_j,\varphi_j),\xi\rangle$. Let us compute $\tilde q=q(y,\xi)$.

Consider an arbitrary $k<m$. To figure out whether $k\in\tilde u=u_{y\xi}$ we have to determine whether $\boldsymbol{\tau}^\xi_y(a_k,\varphi_k)=y$ holds. If $n(k)\ge n(i)=n(j)$ then obviously $\boldsymbol{\tau}^\xi_y(a_k,\varphi_k)=\boldsymbol{\tau}^\xi_x(a_k,\varphi_k)$, and hence $\boldsymbol{\tau}^\xi_y(a_k,\varphi_k)=y$ iff $\langle j,k\rangle\in w$. Suppose that $n(k)<n(i)=n(j)$. Then

$$\boldsymbol{\tau}^\xi_y(a_k,\varphi_k)=\boldsymbol{\tau}^\xi_{\boldsymbol{\tau}^\xi_y(a_j,\varphi_j)}(a_k,\varphi_k)=\boldsymbol{\tau}^\xi_y(b,\psi)\,,$$

where the pair $\langle b,\psi\rangle=\langle a_k{}^\wedge(a_j\restriction_{\geqslant n(k)})\,,\,\varphi_k{}^\wedge(\varphi_j\restriction_{\geqslant n(k)})\rangle$ is equal to one of the pairs $\langle a_\nu,\varphi_\nu\rangle$, $\nu<m$ (and then $n(\nu)=n(i)=n(j)$). Thus $\boldsymbol{\tau}^\xi_y(a_k,\varphi_k)=y$ iff $\boldsymbol{\tau}^\xi_x(a_\nu,\varphi_\nu)=\boldsymbol{\tau}^\xi_x(a_j,\varphi_j)$ iff $\langle j,\nu\rangle\in w$.

Now consider arbitrary numbers $k,k'<m$. To figure out whether $\langle k,k'\rangle\in\tilde w=w_{y\xi}$ we have to determine whether $\boldsymbol{\tau}^\xi_y(a_k,\varphi_k)=\boldsymbol{\tau}^\xi_y(a_{k'},\varphi_{k'})$ holds. As above in the first part of the proof of the claim, there exist indices $\nu,\nu'<m$ (that depend on $q(\pi)=\langle u,v\rangle$ but not directly on $\langle x,\xi\rangle$) such that $\boldsymbol{\tau}^\xi_y(a_k,\varphi_k)=\boldsymbol{\tau}^\xi_x(a_\nu,\varphi_\nu)$ and $\boldsymbol{\tau}^\xi_y(a_{k'},\varphi_{k'})=\boldsymbol{\tau}^\xi_x(a_{\nu'},\varphi_{\nu'})$. And then the equality $\boldsymbol{\tau}^\xi_y(a_k,\varphi_k)=\boldsymbol{\tau}^\xi_y(a_{k'},\varphi_{k'})$ is equivalent to $\langle\nu,\nu'\rangle\in w$. □ (*Claim*)

Come back to the proof of Lemma 25.

Consider any $q=\langle u,w\rangle\in Q$. Then $\mathbf{\Pi}_q=\{\langle x,\xi\rangle\in\mathbf{\Pi}:q(x,\xi)=q\}$ is a Borel subset of $\mathbf{\Pi}$. It follows from the claim that for every superposition of the form $f=f_0\circ f_1\circ\dots\circ f_\ell$, where each of f_k is equal to one of G_{ij} (i,j depend on k) there exists a sequence $k_0,k_1,\dots,k_\ell$ of numbers $k_i<m$ such that

$$f(x,\xi)\;=\;\big(g_{a_{k_0}\varphi_{k_0}}\circ g_{a_{k_1}\varphi_{k_1}}\circ\dots\circ g_{a_{k_\ell}\varphi_{k_\ell}}\big)(x,\xi)$$

for all $\langle x,\xi\rangle\in\mathbf{\Pi}_q$, where $g_{a\varphi}$ is a map of $\mathbf{\Pi}\to\mathbf{\Pi}$ defined so that $g_{a\varphi}(x,\xi)=\langle\boldsymbol{\tau}^\xi_x(a,\varphi),\xi\rangle$ for all $\langle x,\xi\rangle\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$. In other words $f=f_0\circ\dots\circ f_\ell$ coincides with the superposition $g_{a_{k_0}\varphi_{k_0}}\circ\dots\circ g_{a_{k_\ell}\varphi_{k_\ell}}$ on $\mathbf{\Pi}_q$.

Note finally that if $\langle a,\varphi\rangle\in D_n$, $\langle b,\psi\rangle\in D_{n'}$, and $n'\le n$ then $g_{a\varphi}(g_{b\psi}(x,\xi))=g_{a\varphi}(x,\xi)$ for all $\langle x,\xi\rangle\in\mathbf{\Pi}$. It follows that the superposition $g_{a_{k_0}\varphi_{k_0}}\circ\dots\circ g_{a_{k_\ell}\varphi_{k_\ell}}$ will not change as a function if we remove all factors $g_{a_{k_i}\varphi_{k_i}}$ such that $n(k_i)\le n(k_j)$ for some $j<i$. The remaining superposition obviously contains at most

$n = \mathtt{max}_{i<m}\, n(i)$ terms, and hence there exist only finitely many superpositions of such a reduced form.

As Q itself is finite, this ends the proof of the lemma. □ (*Lemma 25*)

□ (*Theorem 3*)

7 Case 2

Then the Σ^1_1 set $R = P_0 \cap \mathbf{H}$, where $\mathbf{H} = 2^{\mathbb{N}} \smallsetminus \mathbf{S}$ is the chaotic domain, is non-empty. Our goal will be to prove that $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_{13} \restriction R$ in this case. The embedding $\vartheta : \mathbb{R}^{\mathbb{N}} \to R$ will have the property that any two elements $\langle x, \xi\rangle$ and $\langle x', \xi'\rangle$ in the range $\mathtt{ran}\,\vartheta \subseteq R$ satisfy $\xi \mathrel{\mathsf{E}_3} \xi'$, so that the ξ'-component in the range of ϑ is trivial. And as far as the x-component is concerned, the embedding will resemble the embedding defined in Case 1 of the proof of the 1st dichotomy theorem in [9] (see also [6, Ch. 8]).

Recall that sets S^k_n were defined in Corollary 10, and by definition

$$\left.\begin{aligned} \langle x,\xi\rangle \in \mathbf{H} &\implies \forall\, m\, \exists\, n \ge m\, \forall\, k\, (\langle x,\xi\rangle \notin S^k_n) \\ &\implies \forall\, m\, \exists\, n \ge m\, \forall\, k\, \forall\, \varphi \in \mathscr{F}^k_n \left(x(n) \ne \varphi(x\restriction_{>n}, \xi)\right) \end{aligned}\right\}. \tag{6}$$

in Case 2. Prove a couple of related technical lemmas.

Lemma 27. *Each set S^k_n is invariant in the following sense: if $\langle x, \xi\rangle \in S^k_n$, $\langle y, \eta\rangle \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $x\restriction_{\geqslant n} = y\restriction_{\geqslant n}$, and $\xi \mathrel{\mathsf{E}_3} \eta$ then $\langle y, \eta\rangle \in S^k_n$.*

Proof. Otherwise there is a Δ^1_1 function $\varphi \in {}^{\mathtt{T}}\mathscr{F}^k_n$ such that $y(n) = \varphi(y\restriction_{>n}, \eta)$. Then $x(n) = \varphi(x\restriction_{>n}, \eta)$ as well because $x\restriction_{\geqslant n} = y\restriction_{\geqslant n}$. We put

$$u_j = \xi(j) \,\Delta\, \eta(j) = \{m : \xi(j)(m) \ne \eta(j)(m)\}$$

for every $j < k$, these are finite subsets of $\mathbb{N}$. If $a \in 2^{\mathbb{N}}$ and $u \subseteq \mathbb{N}$ then define $u \cdot a \in 2^{\mathbb{N}}$ so that $(u \cdot a)(m) = a(m)$ for $m \notin u$, and $(u \cdot a)(m) = a(m)$ for $m \notin u$. If $\zeta \in \mathbb{R}^{\mathbb{N}}$ then define $f(\zeta) \in \mathbb{R}^{\mathbb{N}}$ so that $f(\zeta)(j) = u_j \cdot \zeta(j)$ for $j < k$, and $f(\zeta)(j) = \zeta(j)$ for $j \ge k$.

Finally, put $\psi(z, \zeta) = \varphi(z, f(\zeta))$ for every $\langle z, \zeta\rangle \in \mathbb{R}^{>n} \times \mathbb{R}^{\mathbb{N}}$. The map ψ obviously belongs to ${}^{\mathtt{T}}\mathscr{F}^k_n$ together with φ. Moreover

$$x(n) = \varphi(x\restriction_{>n}, \eta) = \psi(x\restriction_{>n}, f(\eta)) = \psi(x\restriction_{>n}, \xi)$$

because $f(\eta)\restriction_{<k} = \xi\restriction_{<k}$, and this contradicts to the choice of $\langle x, \xi\rangle$. □

The next simple lemma will allow us to split Σ^1_1 sets in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$.

Lemma 28. *If $P \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ is a Σ^1_1 set and $P \not\subseteq S^k_n$ then there exist points $\langle x, \xi\rangle$ and $\langle y, \eta\rangle$ in P with*

$$y\restriction_{>n} = x\restriction_{>n}, \quad \eta \mathrel{\mathsf{E}_3} \xi, \quad \eta\restriction_{<k} = \xi\restriction_{<k}, \quad \textit{but} \quad y(n) \ne x(n)\,.$$

Proof. Otherwise $\psi = \{\langle\langle y\restriction_{>n}, \eta\rangle, y(n)\rangle : \langle y, \eta\rangle \in P\}$ is a map in $\mathscr{F}^k_n$, and hence $P \subseteq S^k_n$, contradiction. □

8 Case 2: splitting system

We apply a splitting construction, developed in [5] for the study of "ill"founded Sacks iterations. Below, 2^n will typically denote the set of all dyadic sequences of length n, and $2^{<\omega} = \bigcup_n 2^n =$ all finite dyadic sequences.

The construction involves a map $\varphi : \mathbb{N} \to \mathbb{N}$ assuming **infinitely many** values and each its value infinitely many times (but $\mathtt{ran}\,\varphi$ may be a proper subset of $\mathbb{N}$), another map $\pi : \mathbb{N} \to \mathbb{N}$, and, for each $u \in 2^{<\omega}$, a non-empty Σ^1_1 subset $P_u \subseteq R = \mathbf{H} \cap P_0$ — which satisfy a quite long list of properties.

First of all, if φ is already defined at least on $[0, n)$ and $u \neq v \in 2^n$ then let $\nu_\varphi[u, v] = \mathtt{max}\{\varphi(\ell) : \ell < n \wedge u(\ell) \neq v(\ell)\}$. And put $\nu_\varphi[u, u] = -1$ for any u.

Now we present the list of requirements $1^\circ - 8^\circ$.

1°: if $\varphi(n) \notin \{\varphi(\ell) : \ell < n\}$ then $\varphi(n) > \varphi(\ell)$ for each $\ell < n$;

2°: if $u \in 2^n$ then $P_u \cap (\bigcup_k S^k_{\varphi(\ell)}) = \varnothing$ for each $\ell < n$;

3°: every P_u is a non-empty Σ^1_1 subset of $R \cap \mathbf{H}$;

4°: $P_{u^\wedge i} \subseteq P_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

Two further conditions are related rather to the sets $X_u = \mathtt{dom}\, P_u$.

5°: if $u, v \in 2^n$ then $X_u \restriction_{>\nu_\varphi[u,v]} = X_v \restriction_{>\nu_\varphi[u,v]}$;

6°: if $u, v \in 2^n$ then $X_u \restriction_{\geqslant\nu_\varphi[u,v]} \cap X_v \restriction_{\geqslant\nu_\varphi[u,v]} = \varnothing$.

The content of the next condition is some sort of genericity in the sense of the Gandy – Harrington forcing in the space $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, that is, the forcing notion

$$\mathbb{P} = \text{ all non-empty } \Sigma^1_1 \text{ subsets of } \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}.$$

Let us fix a countable transitive model $\mathfrak{M}$ of a sufficiently large fragment of **ZFC**. [6] For technical reasons, we assume that $\mathfrak{M}$ is an elementary submodel of the universe w. r. t. all analytic formulas. Then simple relations between sets in $\mathbb{P}$ in the universe, like $P = Q$ or $P \subseteq Q$, are adequately reflected as the same relations between their intersections $P \cap \mathfrak{M}$, $Q \cap \mathfrak{M}$ with the model $\mathfrak{M}$. In this sense $\mathbb{P}$ is a forcing notion in $\mathfrak{M}$.

A set $D \subseteq \mathbb{P}$ is *open dense* iff, first, for any $P \in \mathbb{P}$ there is $Q \in D$, $Q \subseteq P$, and given sets $P \subseteq Q \in \mathbb{R}$, if Q belongs to D then so does P. A set $D \subseteq \mathbb{P}$ is *coded in* $\mathfrak{M}$*,* iff the set $\{P \cap \mathfrak{M} : P \in D\}$ belongs to $\mathfrak{M}$. There exists at most countably many such sets because $\mathfrak{M}$ is countable. Let us fix an enumeration (**not** in $\mathfrak{M}$) $\{D_n : n \in \mathbb{N}\}$ of all open dense sets $D \subseteq \mathbb{P}$ coded in $\mathfrak{M}$.

The next condition essentially asserts the $\mathbb{P}$-genericity of each branch in the splitting construction over $\mathfrak{M}$.

[6] For instance remove the Power Set axiom but add the axiom saying that for any set X there exists the set of all countable subsets of X.

7°: for every n, if $u \in 2^{n+1}$ then $P_u \in D_n$.

Remark 29. It follows from 7° that for any $a \in 2^{\mathbb{N}}$ the sequence $\{P_{a \restriction n}\}_{n \in \mathbb{N}}$ is generic enough for the intersection $\bigcap_n P_{a \restriction n} \neq \varnothing$ to consist of a single point, say $\langle g(a), \gamma(a) \rangle$, and for the maps $g, \gamma : 2^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ to be continuous.

Note that g is $1-1$. Indeed if $a \neq b$ belong to $2^{\mathbb{N}}$ then $a(n) \neq b(n)$ for some n, and hence $\nu_\varphi[a \restriction m\,,\, b \restriction m] \geq \varphi(n)$ for all $m \geq n$. It follows by 6° that $X_{a \restriction m} \cap X_{b \restriction m} = \varnothing$ for $m > n$, therefore $g(a) \neq g(b)$. □

Our final requirement involves the ξ-parts of sets P_u. We'll need the following definition. Suppose that $\langle x, \xi \rangle$ and $\langle y, \eta \rangle$ belong to $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $p \in \mathbb{N}$, and $s \in \mathbb{N}^{<\omega}$, $\mathtt{lh}\, s = m$ (the length of s). Define $\langle x, \xi \rangle \cong^s_p \langle y, \eta \rangle$ iff

$$\xi \,\mathsf{E}_3\, \eta\,, \quad x \restriction_{>p} = y \restriction_{>p}\,, \quad \text{and} \quad \xi(k) \,\Delta\, \eta(k) \subseteq s(k) \text{ for all } k < m = \mathtt{lh}\, s\,,$$

where $\alpha \,\Delta\, \beta = \{j : \alpha(j) \neq \beta(j)\}$ for $\alpha, \beta \in 2^{\mathbb{N}}$. If $P, Q \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ are arbitrary sets then under the same circumstances $P \cong^s_p Q$ will mean that

$$\forall \langle x, \xi \rangle \in P\, \exists \langle y, \eta \rangle \in Q\, (\langle x, \xi \rangle \cong^s_p \langle y, \eta \rangle) \quad \text{and } \textit{vice versa}\,.$$

Obviously $\cong^s_p$ is an equivalence relation.

The following is the last condition:

8°: there exists a map $\pi : \mathbb{N} \to \mathbb{N}$, such that $P_u \cong^{\pi \restriction n}_{\nu_\varphi[u,v]} P_v$ holds for every n and all $u, v \in 2^n$ (and then $X_u \restriction_{>\nu_\varphi[u,v]} = X_v \restriction_{>\nu_\varphi[u,v]}$ as in 5°).

9 Case 2: splitting system implies the reducibility

Here we prove that any system of sets P_u and $X_u = \mathtt{dom}\, P_u$ and maps φ, π satisfying 1° – 8° implies Borel reducibility of E_1 to $\mathsf{E}_{13} \restriction R$. This completes Case 2. The construction of such a splitting system will follow in the remainder.

Let the maps g and γ be defined as in Remark 29. Put

$$W = \{\langle g(a), \gamma(a) \rangle : a \in 2^{\mathbb{N}}\}.$$

Lemma 30. *W is a closed set in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ and a function. Moreover if $\langle x, \xi \rangle$ and $\langle y, \eta \rangle$ belong to W then $\xi \,\mathsf{E}_3\, \eta$.*

Proof. W is closed as a continuous image of $2^{\mathbb{N}}$. That W is a function follows from the bijectivity of g, see Remark 29. Finally any two ξ, η as indikated satisfy $\xi(k) \,\Delta\, \eta(k) \subseteq \pi(k)$ for all k by 8°. □

Put $X = \mathtt{dom}\, W$. Thus W is a continuous map $X \to \mathbb{R}^{\mathbb{N}}$ by the lemma.

Corollary 31. *There exists a Borel reduction of $\mathsf{E}_1 \restriction X$ to $\mathsf{E}_{13} \restriction W$.*

Proof. As W is a function, we can use the notation $W(x)$ for $x \in X = \mathtt{dom}\, W$. Put $f(x) = \langle x, W(x)\rangle$. This is a Borel, even a continuous map $X \to W$. It remains to establish the equivalence

$$x \mathsf{E}_1 y \iff f(x) \mathsf{E}_{13} f(y) \qquad \text{for all} \quad x, y \in X. \tag{7}$$

If $x \mathsf{E}_1 y$ then $W(x) \mathsf{E}_3 W(y)$ by Lemma 30, and hence easily $f(x) \mathsf{E}_{13} f(y)$. If $x \mathsf{E}_1 y$ fails then obviously $f(x) \mathsf{E}_{13} f(y)$ fails, too. □

Thus to complete Case 2 it now suffices to define a Borel reduction of E_1 to $\mathsf{E}_1 \restriction X$. To get such a reduction consider the set $\Phi = \mathtt{ran}\, \varphi$, and let $\Phi = \{p_m : m \in \mathbb{N}\}$ in the increasing order; that the set $\Phi \subseteq \mathbb{N}$ is infinite follows from 1°.

Suppose that $n \in \mathbb{N}$. Then $\varphi(n) = p_m$ for some (unique) m : we put $\psi(n) = m$. Thus $\psi : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ and the preimage $\psi^{-1}(m) = \varphi^{-1}(p_m)$ is an infinite subset of $\mathbb{N}$ for any m. Define a parallel system of sets $Y_u \subseteq \mathbb{R}^{\mathbb{N}}$, $u \in 2^{<\omega}$, as follows. Put $Y_\Lambda = \mathbb{R}^{\mathbb{N}}$. Suppose that Y_u has been defined, $u \in 2^n$. Put $p = \varphi(n) = p_{\psi(n)}$. Let K be the number of all indices $\ell < n$ still satisfying $\varphi(\ell) = p$, perhaps $K = 0$. Put $Y_{u^\wedge i} = \{x \in Y_u : x(p)(K) = i\}$ for $i = 0, 1$.

Each of Y_u is clearly a basic clopen set in $\mathbb{R}^{\mathbb{N}}$, and one easily verifies that conditions 4°, 5°, 6° are satisfied for the sets Y_u and the map ψ (instead of φ in 5°, 6°), in particular

6*: if $u, v \in 2^n$ then $Y_u \restriction_{>\nu_\psi[u,v]} = Y_v \restriction_{>\nu_\psi[u,v]}$;

7*: if $u, v \in 2^n$ then $Y_u \restriction_{\geqslant\nu_\psi[u,v]} \cap Y_v \restriction_{\geqslant\nu_\psi[u,v]} = \varnothing$;

where $\nu_\psi[u, v] = \mathtt{max}\{\psi(\ell) : \ell < n \wedge u(\ell) \neq v(\ell)\}$ (compare with ν_φ above).

It is clear that for any $a \in 2^{\mathbb{N}}$ the intersection $\bigcap_n Y_{a\restriction n} = \{f(a)\}$ is a singleton, and the map f is continuous and $1-1$. (We can, of course, define f explicitly: $f(a)(p)(K) = a(n)$, where $n \in \mathbb{N}$ is chosen so that $\psi(n) = p$ and there is exactly K numbers $\ell < n$ with $\psi(\ell) = p$.) Note finally that $\{f(a) : a \in 2^{\mathbb{N}}\} = \mathbb{R}^{\mathbb{N}}$ since by definition $Y_{u^\wedge 1} \cup Y_{u^\wedge 0} = Y_u$ for all u.

We conclude that the map $\vartheta(x) = g(f^{-1}(x))$ is a continuous map (in fact a homeomorphism in this case by compactness) $\mathbb{R}^{\mathbb{N}} \xrightarrow{\text{onto}} X = \mathtt{dom}\, W$.

Lemma 32. *The map ϑ is a reduction of E_1 to $\mathsf{E}_1 \restriction X$, and hence ϑ witnesses $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_1 \restriction X$ and $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{E}_{13} \restriction W$ by Corollary 31.*

Proof. It suffices to check that the map ϑ satisfies the following requirement: for each $y, y' \in \mathbb{R}^{\mathbb{N}}$ and m,

$$y \restriction_{\geqslant m} = y' \restriction_{\geqslant m} \quad \text{iff} \quad \vartheta(y) \restriction_{\geqslant p_m} = \vartheta(y') \restriction_{\geqslant p_m}. \tag{8}$$

To prove (8) suppose that $y = f(a)$ and $x = g(a) = \vartheta(y)$, and similarly $y' = f(a')$ and $x' = g(a') = \vartheta(y')$, where $a, a' \in 2^{\mathbb{N}}$. Suppose that $y \restriction_{\geqslant m} = y' \restriction_{\geqslant m}$.

We then have $m > \nu_\psi[a \restriction n, a' \restriction n]$ for any n by 7*. It follows, by the definition of ψ, that $p_m > \nu_\varphi[a \restriction n, a' \restriction n]$ for any n, hence, $X_{a\restriction n}\restriction_{\geqslant p_m} = X_{a'\restriction n}\restriction_{\geqslant p_m}$ for any n by 5°. Therefore $x\restriction_{\geqslant p_m} = x'\restriction_{\geqslant p_m}$ by 7°, that is, the right-hand side of (8). The inverse implication in (8) is proved similarly. □ (*Lemma*)

It follows that we can now focus on the construction of a system satisfying 1° – 8°. The construction follows in Section 12, after several preliminary lemmas in Sections 10 and 11.

10 Case 2: how to shrink a splitting system

Let us prove some results related to preservation of condition 8° under certain transformations of shrinking type. They will be applied in the construction of a splitting system satisfying conditions 1° – 8° of Section 8.

Lemma 33. *Suppose that* $n \in \mathbb{N}$, $s \in \mathbb{N}^{<\omega}$, *and a system of* Σ^1_1 *sets* $\varnothing \neq P_u \subseteq \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$, $u \in 2^n$, *satisfies* $P_u \cong^s_{\nu_\varphi[u,v]} P_v$ *for all* $u, v \in 2^n$. *Assume also that* $w_0 \in 2^n$, *and* $\varnothing \neq Q \subseteq P_{w_0}$ *is a* Σ^1_1 *set. Then the system of* Σ^1_1 *sets*

$$P'_u = \{\langle x, \xi\rangle \in P_u : \exists\, \langle z, \zeta\rangle \in Q\, (\langle x, \xi\rangle \cong^s_{\nu_\varphi[u,w_0]} \langle z, \zeta\rangle)\}, \quad u \in 2^n,$$

still satisfies $P'_u \cong^s_{\nu_\varphi[u,v]} P'_v$ *for all* $u, v \in 2^n$, *and* $P'_{w_0} = Q$.

Proof. $P'_{w_0} = Q$ holds because $\nu_\varphi[w_0, w_0] = -1$. Let us verify 8°. Suppose that $u, v \in 2^n$. Each one of the three numbers $\nu_\varphi[u, w]$, $\nu_\varphi[v, w]$, $\nu_\varphi[u, v]$ is obviously not bigger than the largest of the two other numbers. This observation leads us to the following three cases.

Case a: $\nu_\varphi[u, w_0] = \nu_\varphi[u, v] \geq \nu_\varphi[v, w_0]$. Consider any $\langle x, \xi\rangle \in P'_u$. Then by definition there exists $\langle z, \zeta\rangle \in Q$ with $\langle x, \xi\rangle \cong^s_{\nu_\varphi[u,w_0]} \langle z, \zeta\rangle$. Then, as $P_{w_0} \cong^s_{\nu_\varphi[v,w_0]} P_v$ is assumed by the lemma, there is $\langle y, \eta\rangle \in P_v$ such that $\langle y, \eta\rangle \cong^s_{\nu_\varphi[v,w_0]} \langle z, \zeta\rangle$. Note that $\langle z, \zeta\rangle$ witnesses $\langle y, \eta\rangle \in P'_v$. On the other hand, $\langle x, \xi\rangle \cong^s_{\nu_\varphi[u,v]} \langle y, \eta\rangle$ because $\nu_\varphi[u, w_0] = \nu_\varphi[u, v] \geq \nu_\varphi[v, w_0]$. Conversely, suppose that $\langle y, \eta\rangle \in P'_v$. Then there is $\langle z, \zeta\rangle \in Q$ such that $\langle y, \eta\rangle \cong^s_{\nu_\varphi[v,w_0]} \langle z, \zeta\rangle$. Yet $P_{w_0} \cong^s_{\nu_\varphi[u,w_0]} P_u$, and hence there exists $\langle x, \xi\rangle \in P'_u$ with $\langle x, \xi\rangle \cong^s_{\nu_\varphi[u,w_0]} \langle z, \zeta\rangle$. Once again we conclude that $\langle x, \xi\rangle \cong^s_{\nu_\varphi[u,v]} \langle y, \eta\rangle$.

Case b: $\nu_\varphi[v, w] = \nu_\varphi[u, v] \geq \nu_\varphi[u, w]$. Absolutely similar to Case a.

Case c: $\nu_\varphi[u, w_0] = \nu_\varphi[v, w_0] \geq \nu_\varphi[u, v]$. This is a symmetric case, thus it is enough to carry out only the direction $P'_u \to P'_v$. Consider any $\langle x, \xi\rangle \in P'_u$. As above there is $\langle z, \zeta\rangle \in Q$ such that $\langle x, \xi\rangle \cong^s_{\nu_\varphi[u,w_0]} \langle z, \zeta\rangle$. On the other hand, as $P_u \cong^s_{\nu_\varphi[u,v]} P_v$, there exists a point $\langle y, \eta\rangle \in P_v$ such that $\langle y, \eta\rangle \cong^s_{\nu_\varphi[u,v]} \langle x, \xi\rangle$. Note that $\langle z, \zeta\rangle$ witnesses $\langle y, \eta\rangle \in P'_v$: indeed by definition we have $\langle y, \eta\rangle \cong^s_{\nu_\varphi[v,w_0]} \langle z, \zeta\rangle$. □

Corollary 34. *Assume that $n \in \mathbb{N}$, $s \in \mathbb{N}^{<\omega}$, and a system of Σ^1_1 sets $\varnothing \neq P_u \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $u \in 2^n$, satisfies $P_u \cong^s_{\nu_\varphi[u,v]} P_v$ for all $u, v \in 2^n$. Assume also that $\varnothing \neq W \subseteq 2^n$, and a Σ^1_1 set $\varnothing \neq Q_w \subseteq P_w$ is defined for every $w \in W$ so that still $Q_w \cong^s_{\nu_\varphi[w,w']} Q_{w'}$ for all $w, w' \in W$. Then the system of Σ^1_1 sets*

$$P'_u = \{\langle x, \xi\rangle \in P_u : \forall\, w \in W\ \exists\, \langle y, \eta\rangle \in Q_w\ (\langle x, \xi\rangle \cong^s_{\nu_\varphi[u,w]} \langle y, \eta\rangle)\}$$

still satisfies $P'_u \cong^s_{\nu_\varphi[u,v]} P'_v$ for all $u, v \in 2^n$, and $P'_w = Q_w$ for all $w \in W$.

Proof. Apply the transformation of Lemma 33 consecutively for all $w_0 \in W$ and the corresponding sets Q_{w_0}. Note that these transformations do not change the sets Q_w with $w \in W$ because $Q_w \cong^s_{\nu_\varphi[w,w']} Q_{w'}$ for all $w, w' \in W$. □

Remark 35. The sets P'_u in Corollary 34 can as well be defined by

$$P'_u = \{\langle x, \xi\rangle \in P_u : \exists\, \langle y, \eta\rangle \in Q_{w_u}\ (\langle x, \xi\rangle \cong^s_{\nu_\varphi[u,w_u]} \langle y, \eta\rangle)\}$$

where, for each $u \in 2^n$, w_u is an element of W such that the number $\nu_\varphi[u, w_u]$ is the least of all numbers of the form $\nu_\varphi[u, w]$, $w \in W$. (If there exist several $w \in W$ with the minimal $\nu_\varphi[u, w]$ then take the least of them.) □

11 Case 2: how to split a splitting system

Here we consider a different question related to the construction of systems satisfying conditions $1^\circ - 8^\circ$ of Section 8. Given a system of Σ^1_1 sets satisfying a 8°-like condition, how to shrink the sets so that 8° is preserved and in addition 6° holds. Let us begin with a basic technical question: given a pair of Σ^1_1 sets P, Q satisfying $P \cong^s_p Q$ for some p, s, how to define a pair of smaller Σ^1_1 sets $P' \subseteq P$, $Q' \subseteq Q$, still satisfying the same condition, but as disjoint as it is compatible with this condition.

Recall that $\mathtt{dom}\, P = \{x : \exists\, \xi\ (\langle x, \xi\rangle \in P\}$ for $P \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$.

Lemma 36. *If $P, Q \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ are non-empty Σ^1_1 sets, $p \in \mathbb{N}$, $s \in \mathbb{N}^{<\omega}$, $P \cong^s_p Q$, and $(P \cup Q) \cap S^k_p = \varnothing$, where $k = \mathtt{lh}\, s$, then there exist non-empty Σ^1_1 sets $P' \subseteq P$, $Q' \subseteq Q$ such that still $P' \cong^s_p Q'$ but in addition $(\mathtt{dom}\, P') {\restriction}_{\geqslant p} \cap (\mathtt{dom}\, Q') {\restriction}_{\geqslant p} = \varnothing$.*

Note that $P \cong^p_s Q$ implies $(\mathtt{dom}\, P) {\restriction}_{>p} = (\mathtt{dom}\, Q) {\restriction}_{>p}$.

Proof. It follows from Lemma 28 that there exist points $\langle x_0, \xi_0\rangle$ and $\langle x_1, \xi_1\rangle$ in P such that $\langle x_0, \xi_0\rangle \cong^s_p \langle x_1, \xi_1\rangle$ but $x_1(p) \neq x_0(p)$. Then there exists a number j such that, say, $x_1(p)(j) = 1 \neq 0 = x_0(p)(j)$. On the other hand, there exists $\langle y_0, \eta_0\rangle \in Q$ such that $\langle x_i, \xi_i\rangle \cong^s_p \langle y_0, \eta_0\rangle$ for $i = 0, 1$. Then $y_0(p)(j) \neq x_i(p)(j)$ for one of $i = 0, 1$. Let say $y_0(p)(j) = 0 \neq 1 = x_0(p)(j)$. Then the Σ^1_1 sets

$$P' = \{\langle x, \xi\rangle \in P : \exists\, \langle y, \eta\rangle \in Q\ \big(x(p)(j) = 1 \wedge y(p)(j) = 0 \wedge \langle x, \xi\rangle \cong^s_p \langle y, \eta\rangle\big)\};$$

$$Q' = \{\langle y, \eta\rangle \in Q : \exists\, \langle x, \xi\rangle \in P\ \big(x(p)(j) = 1 \wedge y(p)(j) = 0 \wedge \langle x, \xi\rangle \cong^s_p \langle y, \eta\rangle\big)\}$$

are Σ^1_1 and non-empty (contain resp. $\langle x_0, \xi_0\rangle$ and $\langle y_0, \eta_0\rangle$), and they satisfy $P' \cong^s_p Q'$, but $(\mathtt{dom}\, P')\restriction_{\geqslant p} \cap (\mathtt{dom}\, Q')\restriction_{\geqslant p} = \varnothing$ because $y(p)(j) = 0 \neq 1 = x(p)(j)$ whenever $\langle x, \xi\rangle \in P'$ and $\langle y, \eta\rangle \in Q'$. $\square$

Corollary 37. *Assume that* $n \in \mathbb{N}$, $s \in \mathbb{N}^{<\omega}$, *and a system of* Σ^1_1 *sets* $\varnothing \neq P_u \subseteq \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $u \in 2^n$, *satisfies* $P_u \cong^s_{\nu_\varphi[u,v]} P_v$ *for all* $u, v \in 2^n$. *Then there exists a system of* Σ^1_1 *sets* $\varnothing \neq P'_u \subseteq P_u$, $u \in 2^n$, *such that still* $P'_u \cong^s_{\nu_\varphi[u,v]} P_v$, *and in addition* $(\mathtt{dom}\, P'_u)\restriction_{\geqslant \nu_\varphi[u,v]} \cap (\mathtt{dom}\, P'_v)\restriction_{\geqslant \nu_\varphi[u,v]} = \varnothing$, *for all* $u \neq v \in 2^n$.

Proof. Consider any pair of $u_0 \neq v_0$ in 2^n. Apply Lemma 36 for the sets $P = P_{u_0}$ and $Q = P_{v_0}$ and $p = \nu_\varphi[u_0, v_0]$. Let P' and Q' be the Σ^1_1 sets obtained, in particular $P' \cong^s_{\nu_\varphi[u_0,v_0]} Q'$ and $(\mathtt{dom}\, P')\restriction_{\geqslant \nu_\varphi[u_0,v_0]} \cap (\mathtt{dom}\, Q')\restriction_{\geqslant \nu_\varphi[u_0,v_0]} = \varnothing$. Then by Corollary 34 there is a system of Σ^1_1 sets $\varnothing \neq P'_u \subseteq P_u$ such that still $P'_u \cong^s_{\nu_\varphi[u,v]} P'_v$ for all $u, v \in 2^n$, and $P_{u_0} = P'$, $P_{v_0} = Q'$ — and hence

$$(\mathtt{dom}\, P'_{u_0})\restriction_{\geqslant \nu_\varphi[u_0,v_0]} \cap (\mathtt{dom}\, P'_{v_0})\restriction_{\geqslant \nu_\varphi[u_0,v_0]} = \varnothing.$$

Take any other pair of $u_1 \neq v_1$ in 2^n and transform the system of sets P'_u the same way. Iterate this construction sufficient (finite) number of steps. $\square$

12 Case 2: the construction of a splitting system

We continue the proof of Theorem 2 – Case 2. Recall that $R = P_0 \cap \mathbf{H}$ is a Σ^1_1 set. By Lemma 32, it suffices to define functions φ and π and a system of Σ^1_1 sets $P_u \subseteq R$ together satisfying conditions $1^\circ - 8^\circ$. The construction of such a system will go on by induction on n. That is, at any step n the sets P_u with $u \in 2^n$, as well as the values of $\varphi(k)$ and $\pi(k)$ with $k < n$, will be defined.

For $n = 0$, we put $P_\Lambda = R$. ($\Lambda \in 2^0$ is the only sequence of length 0.)

Suppose that sets $P_u \subseteq R$ with $u \in 2^n$, and also all values $\varphi(\ell)$, $\ell < n$, and $\pi(k)$, $k < n$, have been defined and satisfy the applicable part of $1^\circ - 8^\circ$. The content of the inductive step $n \mapsto n+1$ will consist in definition of $\varphi(n)$, $\pi(n)$, and sets $P_{u^\wedge i}$ with $u^\wedge i \in 2^{n+1}$, that is, $u \in 2^n$ (a dyadic sequence of length n) and $i = 0, 1$. This goes on in four steps A,B,C,D.

12.1 Step A: definition of $\varphi(n)$

Suppose that, in the order of increase,

$$\{\varphi(\ell) : \ell < n\} = \{p_0 < \dots < p_m\}\,.$$

For $j \leq m$, let K_j be the number of all $\ell < n$ with $\varphi(\ell) = p_j$.

Case A: $K_j \geq m$ for all $j \leq m$. Then consider any $u_0 \in 2^n$ and an arbitrary point $\langle x_0, \xi_0\rangle \in P_{u_0}$. Note that by (6) of Section 7 there is a number $p > \mathtt{max}_{\ell<n}\, \varphi(\ell)$ such that $\langle x_0, \xi_0\rangle \notin \bigcup_k S^k_p$. Put $\varphi(n) = p$.

We claim that the sets $P'_u = P_u \smallsetminus \bigcup_k S^k_{\varphi(n)}$ still satisfy condition 8° (and then 5° for $X'_u = \mathtt{dom}\, P'_u$). Indeed suppose that $u, v \in 2^n$ and $\langle x, \xi\rangle \in P'_u$. Then $\langle x, \xi\rangle \in P_u$, and hence there is a point $\langle y, \eta\rangle \in P_v$ such that $\langle x, \xi\rangle \cong^{\pi\restriction n}_{\nu_\varphi[u,v]} \langle y, \eta\rangle$. It remains to show that $\langle y, \eta\rangle \notin \bigcup_k S^k_{\varphi(n)}$. Suppose towards the contrary that $\langle y, \eta\rangle \in S^k_{\varphi(n)}$ for some k. By definition $\varphi(n) > \nu_\varphi[u, v]$, therefore $x\restriction_{\geqslant\varphi(n)} = y\restriction_{\geqslant\varphi(n)}$. It follows that $\langle x, \xi\rangle \in S^k_{\varphi(n)}$ by Lemma 27, contradiction.

Case B. If some numbers K_j are $< m$ then choose $\varphi(n)$ among p_j with the least K_j, and among them take the least one. Thus $\varphi(n) = \varphi(\ell)$ for some $\ell < n$. It follows that in this case $P_u \cap (\bigcup_k S^k_{\varphi(n)}) = \varnothing$ for all $u \in 2^n$ by the inductive assumption of 2°. Put $P'_u = P_u$.

Note that this manner of choice of $\varphi(n)$ implies 1°, 2° and also implies that φ takes infinitely many values and takes each its value infinitely many times. In addition, the construction given above proves:

Lemma 38. *There exists a system of Σ^1_1 sets $\varnothing \ne P'_u \subseteq P_u$ satisfying 8° and $P'_u \cap (\bigcup_k S^k_{\varphi(n)}) = \varnothing$ for all $u \in 2^n$.* □

12.2 Step B: definition of $\pi(n)$

We work with the sets P'_u such as in Lemma 38. The next goal is to prove the following result:

Lemma 39. *There exist a number $r \in \mathbb{N}$ and a system of Σ^1_1 sets $\varnothing \ne P''_u \subseteq P'_u$ satisfying $P''_u \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u,v]} P''_v$ for all $u, v \in 2^n$.*

Proof. Let $2^n = \{u_j : j < K\}$ be an arbitrary enumeration of all dyadic sequences of length n; $K = 2^n$, of course. The method of proof will be to define, for any $k \le K$, a number $r_k \in \mathbb{N}$ and a system of Σ^1_1 sets $\varnothing \ne Q^k_{u_j} \subseteq P'_{u_j}$, $j < k$, by induction on k so that

(∗) $Q^k_{u_i} \cong^{(\pi\restriction n)^\wedge r_k}_{\nu_\varphi[u_i,u_j]} Q^k_{u_j}$ for all $i < j < k$. (Where $(\pi \restriction n)^\wedge r$ is the extension of the finite sequence $\pi \restriction n$ by r as the new rightmost term.)

After this is done, $r = r_K$ and the sets $P''_u = Q^K_u$ prove the lemma.

We begin with $k = 2$. Then $P'_{u_0} \cong^{\pi\restriction n}_{\nu_\varphi[u_0,u_1]} P'_{u_1}$ by 8°, and hence there exist points $\langle x_0, \xi_0\rangle \in P'_{u_0}$, $\langle x_1, \xi_1\rangle \in P'_{u_1}$ such that $\langle x_0, \xi_0\rangle \cong^{\pi\restriction n}_{\nu_\varphi[u_0,u_1]} \langle x_1, \xi_1\rangle$. Then $\xi_0 \mathsf{E}_3 \xi_1$, so that there is a number $r \in \mathbb{N}$ with $\xi_0(n) \,\Delta\, \xi_1(n) \subseteq r_2$. Note that for any $p \in \mathbb{N}$ and any points $\langle x, \xi\rangle, \langle y, \eta\rangle \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, $\langle x, \xi\rangle \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_0,u_1]} \langle y, \eta\rangle$ is equivalent to the conjunction

$$\langle x, \xi\rangle \cong^{\pi\restriction n}_{\nu_\varphi[u_0,u_1]} \langle y, \eta\rangle \quad\wedge\quad \xi(n) \,\Delta\, \eta(n) \subseteq r\,.$$

It follows that the sets

$$S_0 \ = \ \{\langle x,\xi\rangle \in P'_{u_0} : \exists\,\langle y,\eta\rangle \in P'_{u_1}\,\big(\langle x,\xi\rangle \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_0,u_1]} \langle y,\eta\rangle\big)\}\,, \quad \text{and}$$
$$S_1 \ = \ \{\langle y,\eta\rangle \in P'_{u_1} : \exists\,\langle x,\xi\rangle \in P'_{u_0}\,\big(\langle x,\xi\rangle \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_0,u_1]} \langle y,\eta\rangle\big)\}$$

are $\varSigma^1_1$ and non-empty (contain resp. $\langle x_0,\xi_0\rangle$ and $\langle x_1,\xi_1\rangle$), and they obviously satisfy $S_0 \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_0,u_1]} S_1$. Therefore by Corollary 34 there exists a system of $\varSigma^1_1$ sets $\varnothing \ne Q^2_u \subseteq P'_u$, $u\in 2^n$, such that $Q^2_{u_0} = S_0$, $Q^2_{u_1} = S_1$, 8° still holds, and in addition $Q^2_{u_0} \cong^{(\pi\restriction n)^\wedge r_2}_{\nu_\varphi[u_0,u_1]} Q^2_{u_1}$. Put $r_2 = r$.

Now let us carry out the step $k \mapsto k+1$. Suppose that r_k and sets $Q^k_{u_j}$, $j<k$, satisfy $(*)$. Of all numbers $\nu_\varphi[u_j,u_k]$, $j<k$, consider the least one. Let this be, say, $\nu_\varphi[u_\ell,u_k]$, so that $\ell<k$ and $\nu_\varphi[u_\ell,u_k] \le \nu_\varphi[u_j,u_k]$ for all $j<k$. As above there exists a number r and a pair of non-empty $\varSigma^1_1$ sets $S_\ell \subseteq Q^k_{u_\ell}$ and $S_k \subseteq Q^k_{u_k}$ such that $S_\ell \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_\ell,u_k]} S_k$. We can assume that $r \ge r_k$. Put

$$Q'_{u_j} = \{\langle y,\eta\rangle \in S_{u_j} : \exists\,\langle x,\xi\rangle \in S_\ell\,\big(\langle x,\xi\rangle \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_\ell,u_j]} \langle y,\eta\rangle\big)\}$$

for all $j<k$. The proof of Lemma 33 shows that Q'_{u_j} are non-empty $\varSigma^1_1$ sets still satisfying $(*)$ in the form of $Q'_{u_i} \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_i,u_j]} Q'_{u_j}$ for $i<j<k$ — since $r\ge r_k$, and obviously $Q'_{u_\ell} = S_\ell$. In addition, put $Q'_{u_k} = S_k$. Then still $Q'_{u_\ell} \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_\ell,u_k]} Q'_{u_k}$ by the choice of S_ℓ and S_k. We claim that also

$$Q'_{u_j} \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_j,u_k]} Q'_{u_k} \quad \text{for all } j<k\,. \tag{9}$$

Indeed we have $Q'_{u_j} \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_j,u_\ell]} Q'_{u_\ell}$ and $Q'_{u_\ell} \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_\ell,u_k]} Q'_{u_k}$ by the above. It follows that $Q'_{u_j} \cong^{(\pi\restriction n)^\wedge r}_{p} Q'_{u_k}$, where $p = \mathtt{max}\{\nu_\varphi[u_j,u_\ell],\nu_\varphi[u_\ell,u_k]\}$. Thus it remains to show that $p \le \nu_\varphi[u_j,u_k]$. That $\nu_\varphi[u_\ell,u_k] \le \nu_\varphi[u_j,u_k]$ holds by the choice of ℓ. Prove that $\nu_\varphi[u_j,u_\ell] \le \nu_\varphi[u_j,u_k]$. Indeed in any case

$$\nu_\varphi[u_j,u_\ell] \le \mathtt{max}\{\nu_\varphi[u_j,u_k],\nu_\varphi[u_\ell,u_k]\}.$$

But once again $\nu_\varphi[u_\ell,u_k] \le \nu_\varphi[u_j,u_k]$, so $\nu_\varphi[u_j,u_\ell] \le \nu_\varphi[u_j,u_k]$ as required.

Thus (9) is established. It follows that $Q'_{u_i} \cong^{(\pi\restriction n)^\wedge r}_{\nu_\varphi[u_i,u_j]} Q'_{u_j}$ for all $i<j\le k$. We end the inductive step of the lemma by putting $r_{k+1} = r$. □ (*Lemma*)

12.3 Step C: splitting to the next level

We work with the number r and sets P''_u such as in Lemma 39. Put $\pi(n) = r$. (Recall that $\varphi(n)$ was defined at Step A.) The next step is to split each one of the sets P''_u in order to define sets $P_{u^\wedge i}$, $u^\wedge i \in 2^{n+1}$, of the next splitting level.

To begin with, put $Q_{u^\wedge i} = P''_u$ for all $u \in 2^n$ and $i = 0, 1$. It is easy to verify that the system of sets $Q_{u^\wedge i}$, $u^\wedge i \in 2^{n+1}$, satisfies conditions 1° – 8° for the level $n+1$, except for 7° and 6°. In particular, 2° was fixed at Step A, and 8° in the form that $Q_{u^\wedge i} \cong^{\pi \restriction (n+1)}_{\nu_\varphi[u^\wedge i, v^\wedge j]} Q_{v^\wedge j}$ for all $u^\wedge i$ and $v^\wedge j$ in 2^{n+1} (and then 5° as well) at Step B — because $(\pi \restriction n)^\wedge r = \pi \restriction (n+1)$.

Recall that by definition all sets involved have no common point with $\bigcup_k S^k_{\varphi(n)}$ by 2°. Therefore Corollary 37 is applicable. We conclude that there exists a system of non-empty Σ^1_1 sets $W_{u^\wedge i} \subseteq Q_{u^\wedge i}$, $u^\wedge i \in 2^{n+1}$, still satisfying 8°, and also satisfying 6°.

12.4 Step D: genericity

We have to further shrink the sets $W_{u^\wedge i}$, $u^\wedge i \in 2^{n+1}$, obtained at Step C, in order to satisfy 7°, the last condition not yet fulfilled in the course of the construction. The goal is to define a new system of Σ^1_1 sets $\varnothing \neq P_{u^\wedge i} \subseteq W_{u^\wedge i}$, $u^\wedge i \in 2^{n+1}$, such that still 8° holds, and in addition $P_{u^\wedge i} \in D_n$ for all $u^\wedge i \in 2^{n+1}$, where D_n is the n-th open dense subset of $\mathbb{P}$ coded in $\mathfrak{M}$.

Take any $u_0{}^\wedge i_0 \in 2^{n+1}$. As D_n is a dense subset of $\mathbb{P}$, there exists a set $W_0 \in D_n$, therefore, a non-empty Σ^1_1 set, such that $W_0 \subseteq W_{u_0{}^\wedge i_0}$. It follows from Lemma 33 that there exists a system of non-empty Σ^1_1 sets $W'_{u^\wedge i} \subseteq W_{u^\wedge i}$, $u^\wedge i \in 2^{n+1}$, still satisfying 8°, and such that $W'_{u_0{}^\wedge i_0} = Q_0$.

Now take any other $u_1{}^\wedge i_1 \neq u_0{}^\wedge i_0$ in 2^{n+1}. The same construction yields a system of non-empty Σ^1_1 sets $W''_{u^\wedge i} \subseteq W'_{u^\wedge i}$, $u^\wedge i \in 2^{n+1}$, still satisfying 8°, and such that $W''_{u_1{}^\wedge i_1} = W_1 \subseteq W'_{u_1{}^\wedge i_1}$ is a set in D_n.

Iterating this construction 2^{n+1} times, we obtain a system of sets $P_{u^\wedge i}$ satisfying 7° as well as all other conditions in the list 1° – 8°, as required.

□ (*Construction and Case 2 of Theorem 2*)

□ (*Theorems 2 and 1*)

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