Reasonable non–Radon–Nikodym idealss

Vladimir Kanovei[*] Vassily Lyubetsky[†]

June 9, 2018

Abstract

For any abelian Polish σ-compact group $\mathbb{H}$ there exist an $\mathbf{F}_\sigma$ ideal $\mathscr{Z} \subseteq \mathscr{P}(\mathbb{N})$ and a Borel $\mathscr{Z}$-approximate homomorphism $f : \mathbb{H} \to \mathbb{H}^{\mathbb{N}}$ which is not $\mathscr{Z}$-approximable by a continuous true homomorphism $g : \mathbb{H} \to \mathbb{H}^{\mathbb{N}}$.

Introduction

Let G, H be abelian Polish groups, and $\mathscr{Z}$ be an ideal over a countable set A. We consider H^A as a product group. For $s, t \in H^A$ put

$$\Delta_{s,t} = \{a \in A : s(a) \neq t(a)\}.$$

Suppose that $\mathscr{Z}$ is an ideal over A. A map $f : G \to H^A$ is a *$\mathscr{Z}$-approximate homomorphism* iff $\Delta_{f(x)+f(y),\, f(x+y)} \in \mathscr{Z}$ for all $x, y \in G$. Thus it is required that the set of all $a \in A$ such that $f_a(x) + f_a(y) \neq f_a(x+y)$ belongs to $\mathscr{Z}$. Here $f_a : G \to H$ is the a-th co-ordinate map of the map $f : G \to H^A$.

And $\mathscr{Z}$ is a *Radon–Nikodym* ideal (for this pair of groups) iff for any measurable $\mathscr{Z}$-approximate homomorphism $f : G \to H^{\mathbb{N}}$ there is a continuous exact homomorphism $g : G \to H^{\mathbb{N}}$ which $\mathscr{Z}$-approximates f in the sense that $\Delta_{f(x),\, g(x)} \in \mathscr{Z}$ for all $x \in G$. Here the measurability condition can be understood as Baire measurability, or, if G is equipped with a σ-additive Borel measure, as measurability with respect to that measure.

The idea of this (somewhat loose) concept is quite clear: the Radon–Nikodym ideals are those which allow us to approximate non-exact homomorphisms by true ones. This type of problems appears in different domains of mathematics. Closer to the context of this note, Velickovic [7] proved that any Baire-measurable FIN-approximate Boolean-algebra automorphism f of $\mathscr{P}(\mathbb{N})$ (so that the symmetric differences between $f(x) \cup f(y)$ and $f(x \cup y)$ and between $f(\mathbb{N} \setminus x)$ and $\mathbb{N} \setminus f(x)$ are finite for all $x, y \subseteq \mathbb{N}$) is FIN-approximable by

[*]IITP, Moscow. Partially supported by RFFI grants 06-01-00608 and 07-01-00445.
[†]IITP, Moscow. Partially supported by RFFI grant 07-01-00445.

a true automorphism g induced by a bijection betveen two cofinite subsets of $\mathbb{N}$. Kanovei and Reeken proved that any Baire measurable $\mathbb{Q}$-approximate homomorphism $f:\mathbb{R}\to\mathbb{R}$ is $\mathbb{Q}$-approximable by a homomorphism of the form $f(x)=cx$, c being a real constant. See also some results in [1, 4, 5].

The term "Radon–Nikodym ideal" was introduced by Farah [1, 2] in the context of Baire measurable Boolean algebra homomorphisms of $\mathscr{P}(\mathbb{N})$. Many known Borel ideals were demonstrated to be Radon–Nikodym, see [1, 2, 4, 5]. Suitable counterexamples, again in the context of Boolean algebra homomorphisms, were defined by Farah on the base of so called pathological submeasures. A different and, perhaps, more transparent counterexample, related to homomorphisms $\mathbb{T}\to\mathbb{T}^{\mathbb{N}}$ (where $\mathbb{T}=\mathbb{R}/\mathbb{N}$), is defined in [5] as a modification of an ideal introduced in [6]. The next theorem generalizes this result.

Theorem 1. *Suppose that $\mathbb{H}$ is an uncountable abelian Polish group. Then there is an analytic ideal $\mathscr{Z}$ over $\mathbb{N}$ that is not a Radon – Nikodym ideal for maps $\mathbb{H}\to\mathbb{H}^{\mathbb{N}}$ in the sense that there is a Borel and $\mathscr{Z}$-approximate homomorphism $f:\mathbb{H}\to\mathbb{H}^{\mathbb{N}}$ not $\mathscr{Z}$-approximable by a continuous homomorphism $g:\mathbb{H}\to\mathbb{H}^{\mathbb{N}}$. If moreover $\mathbb{H}$ is σ-compact then $\mathscr{Z}$ can be chosen to be $\mathbf{F}_\sigma$.*

Note that the theorem will not become stronger if we require g to be only Baire-measurable, or just measurable with respect to a certain Borel measure on $\mathbb{H}$ — because by the Pettis theorem any such a measurable group homomorphism must be continuous.

The remainder of the note contains the proof of Theorem 1. It would be interesting to prove the theorem for non-abelian Polish groups. (The assumption that $\mathbb{H}$ is abelian is used in the proof of Lemma 7.) And it will be interesting to find non–Radon–Nikodym ideals for homomorphisms $G\to H^{\mathbb{N}}$ in the case when the Polish groups G and H are not necessarily equal.

1 Countable subgroup

Let us fix a group $\mathbb{H}$ as in the theorem, that is, an uncountable abelian Polish group. By $\mathbb{0}$ we denote the neutral element, by $\oplus$ the group operation, by d a compatible complete separable distance (and we do not assume it to be invariant). The first step is to choose a certain countable subgroup $D\subseteq\mathbb{H}$ of "rational elements".

It is quite clear that there exists a countable dense subgroup $D\subseteq\mathbb{H}$ satisfying the following requirement of elementary equivalence type.

(∗) Suppose that $n\geq 1$, $c_1,\dots,c_n\in D$, ε is a positive rational, $U_i=\{x\in\mathbb{H}: d(x,c_i)\leq\varepsilon\}$, and $P(x_1,\dots,x_n)$ is a finite system of linear equations with integer coefficients, unknowns $x_1,\dots,x_n$, and constants in D, of the form:

$$b_1x_1\oplus\ldots\oplus b_nx_n=r\,,\quad\text{where}\quad b_i\in\mathbb{Z}\ \text{ and }\ r\in D\,.$$

Suppose also that this system P has a solution $\langle x_1, \dots, x_n\rangle$ in $\mathbb{H}$ such that $x_i \in U_i$ for all i. Then P has a solution in D as well. (That is, all x_i belong to $D \cap U_i$.)

Let us fix such a subgroup D.

2 The index set

Let *rational ball* mean any subset of $\mathbb{H}$ of the form $\{x \in \mathbb{H} : d(c, x) < \varepsilon\}$, where $c \in D$ (the center), and ε is a positive rational number.

Definition 2. Let A, the index set, consist of all objects a of the following kind. Each $a \in A$ consists of:

– an open non-empty set $U^a \subsetneqq \mathbb{H}$,

– a partition $U^a = U^a_1 \cup \dots \cup U^a_n$ of U^a onto a finite number $n = n^a$ of pairwise disjoint non-empty rational balls $U^a_i \subseteq \mathbb{H}$, and

– a set of points $r^a_i \in U^a_i \cap D$ such that, for all i, $j = 1, 2, \dots, n$:

(1) *either* $r^a_i \oplus r^a_j$ is r^a_k for some k, and $(U^a_i \oplus U^a_j) \cap U^a \subseteq U^a_k$,

(2) *or* $(U^a_i \oplus U^a_j) \cap U^a = \varnothing$. □

Under the conditions of Definition 2, if $\mathbb{0} \in U^a_i$ then $s_i = \mathbb{0}$: for take $j = i$.

Lemma 3. *A is an infinite (countable) set.*

Proof. For any $\varepsilon > 0$ there is $a \in A$ such that U^a a set of diameter $\le \varepsilon$: just take $n^a = 1$, $r^a_1 = \mathbb{0}$, and let $U^a = U^a_1$ be the $\frac{\varepsilon}{2}$-nbhd of $\mathbb{0}$ in $\mathbb{H}$. □

The next lemma will be used below.

Lemma 4. *If $y_1, \dots, y_n \in \mathbb{H}$ are pairwise distinct then there exists $a \in A$ such that $n^a = n$ and $y_i \in U^a_i$ for all $i = 1, \dots, n$.*

Proof. As the operation is continuous, we can pick pairwise disjoint rational balls $B_1, \dots, B_n$ such that $y_i \in B_i$ for all i and the following holds: If $1 \le i, j \le n$ then either there exists k such that $(B_i \oplus B_j) \cap B \subseteq B_k$, where $B = B_1 \cup \dots \cup B_n$, or just $(B_i \oplus B_j) \cap B = \varnothing$. Put $U^a_i = B_i$.

To obtain a system of points r^a_i required, let $P(x_1, \dots, x_n)$ be the system of all equations of the form $x_i + x_j = x_k$ with unknowns x_i, x_j, x_k, where $1 \le i, j, k \le n$ and in reality $y_i + y_j = y_k$. It follows from the choice of D that this system has a solution $\langle r_1, \dots, r_n\rangle$ such that $r_i \in U^a_i \cap D$ for all i. In other words we have: $r_i + r_j = r_k$ whenever $y_i + y_j = y_k$. Let $r^a_i = r_i$. This ends the definition of $a \in A$ as required. (An extra care to guarantee that $U^a = \bigcup_{1 \le i \le n} U^a_i$ is a proper subset of $\mathbb{H}$ is left to the reader.) □

3 The ideal

Let $\mathscr{X}$ be the set of all sets $X \subseteq A$ such that there is a finite set $u \subseteq \mathbb{H}$ satisfying the following: for any $a \in X$ we have $u \not\subseteq U^a$.

The idea of this ideal goes back to Solecki [6], where a certain ideal over the set Ω of all clopen sets $U \subseteq 2^{\mathbb{N}}$ of measure $\frac{1}{2}$ (also a countable set) is considered. In our case the index set A is somewhat more complicated.

Lemma 5. *$\mathscr{X}$ is an ideal containing all finite sets $X \subseteq A$, but $A \notin \mathscr{X}$.*

Proof. If $a \in A$ then the singleton $\{a\}$ belongs to $\mathscr{X}$. Indeed by definition U^a is a non-empty subset of $\mathbb{H}$. Therefore there is a point $x \in \mathbb{H} \smallsetminus U^a$. Then $u = \{x\}$ witnesses $A \in \mathscr{X}$. To see that $\mathscr{X}$ is closed under finite unions, suppose that finite sets $u, v \subseteq \mathbb{H}$ witness that resp. X, Y belong to $\mathscr{X}$. Then $w = u \cup v$ obviously witnesses that $Z = X \cup Y \in \mathscr{X}$. Finally by Lemma 4 for any finite $u = \{x_1, ..., x_n\} \subseteq \mathbb{H}$ there is an element $a \in A$ such that $u \subseteq U^a$. This implies that A itself does not belong to $\mathscr{X}$. □

Proposition 6. *$\mathscr{X}$ is an analytic ideal. If $\mathbb{H}$ is σ-compact then $\mathscr{X}$ is $\mathbf{F}_\sigma$.*

Proof. We claim that $X \in \mathscr{X}$ iff there are a natural n and a partition $X = \bigcup_{1 \le k \le n} X_k$ such that for any k the set $X_k \subseteq A$ satisfies $\bigcup_{a \in X_k} U^a \ne \mathbb{H}$. Indeed suppose that $X \in \mathscr{X}$ and this is witnessed by a finite set $u = \{x_1, \dots, x_n\} \subseteq \mathbb{H}$, that is, $u \not\subseteq U^a$ for all $a \in X$. It follows that $X = \bigcup_{1 \le k \le n} X_k$, where $X_k = \{a \in X : x_k \notin U^a\}$. Clearly $x_k \notin \bigcup_{a \in X_k} U^a$. To prove the converse suppose that $X = \bigcup_{1 \le k \le n} X_k \subseteq A$ and $\bigcup_{a \in X_k} U^a \ne \mathbb{H}$ for all k. Let us pick arbitrary points $x_k \in \mathbb{H} \smallsetminus \bigcup_{a \in X_k} U^a$ for all k. Then $u = \{x_1, \dots, x_n\}$ witnesses $X \in \mathscr{X}$, as required.

It easily follows that $\mathscr{X}$ is analytic.

Now suppose that $\mathbb{H} = \bigcup_{\ell \in \mathbb{N}} H_\ell$, where all sets H_ℓ are compact. Then the inequality $\bigcup_{a \in X_k} U^a \ne \mathbb{H}$ is equivalent to $\exists \ell \, (H_\ell \not\subseteq \bigcup_{a \in X_k} U^a)$. And by the compactness, the non-inclusion $H_\ell \not\subseteq \bigcup_{a \in X_k} U^a$ is equivalent to the following statement: $H_\ell \not\subseteq \bigcup_{a \in X'} U^a$ for every finite $X' \subseteq X_k$. Fix an enumeration $A = \{a_n\}_{n \in \mathbb{N}}$. Put $A \restriction m = \{a_j : j < m\}$. Using König's lemma, we conclude that $X \in \mathscr{X}$ iff there exist natural ℓ, n such that for any m there exists a partition $X \cap (A \restriction m) = \bigcup_{k < n} X_k$, where for every k we have $H_\ell \not\subseteq \bigcup_{a \in X_k} U^a$. And this is a $\mathbf{F}_\sigma$ definition for $\mathscr{X}$. □

4 The main result

Here we prove Theorem 1. Define a Borel map $f : \mathbb{H} \to \mathbb{H}^A$ as follows. Suppose that $x \in \mathbb{H}$ and $a \in A$, $n^a = n$. If $x \in U^a_i$, $1 \le i \le n$, then put $f_a(x) = x \ominus r^a_i$. ($\ominus$ in the sense of the group $\mathbb{H}$.) If $x \notin U^a$ then put simply $f_a(x) = \mathbb{0}$.

Finally define $f(x) = \{f_a(x)\}_{a \in A}$. Clearly f is a Borel map.

The maps f_a do not look like homomorphisms $\mathbb{H} \to \mathbb{H}$. Nevertheless their combination surprisingly turns out to be an approximate homomorphism!

Lemma 7. *$f : \mathbb{H} \to \mathbb{H}^A$ is a Borel and $\mathscr{Z}$-approximate homomorphism.*

Proof. Let $x, y \in \mathbb{H}$ and $z = x \oplus y$. Prove that the set

$$C_{xy} = \{a : f_a(x) \oplus f_a(y) \neq f_a(z)\}$$

belongs to $\mathscr{Z}$. We assert that this is witnessed by the set $u = \{x, y, z\}$, that is, if $a \in C_{xy}$ then at least one of the points x, y, z is not a point in U^a. Or, equivalently, if $a \in A$ and x, y, z belong to U^a then $f_a(x) \oplus f_a(y) = f_a(z)$.

To prove this fact suppose that $a \in A$ and $x, y, z \in U^a$. By definition, $U^a = U^a_1 \cup \dots \cup U^a_n$, where $n = n^a$ and U^a_i are disjoint rational balls in $\mathbb{H}$. We have $x \in U^a_i$, $y \in U^a_j$, $z \in U^a_k$, where $1 \le i, j, k \le n$. Then by definition

$$f_a(x) = x \ominus r^a_i\,, \quad f_a(y) = y \ominus r^a_j\,, \quad f_a(z) = z \ominus r^a_k\,.$$

Therefore $f_a(x) \oplus f_a(y) = x \oplus y \ominus (s_i \oplus s_j)$. (Here we clearly use the assumption that the group is abelian.) We assert that $r^a_i \oplus r^a_j = r^a_k$ — then obviously $f_a(x) \oplus f_a(y) = f_a(z)$ by the above, and we are done.

Note that $z = x \oplus y \in U^a$, hence $(U^a_i \oplus U^a_j) \cap U^a \neq \varnothing$. We conclude that (2) of Definition 2 fails. Therefore (1) holds, $r^a_i \oplus r^a_j = r^a_{k'}$ for some k' and $(U^a_i \oplus U^a_j) \cap U^a \subseteq U^a_{k'}$. But the set $(U^a_i \oplus U^a_j) \cap U^a$ obviously contains z, and $z \in U^a_k$. It follows that $k' = k$, $r^a_{k'} = r^a_k$, $r^a_i \oplus r^a_j = r^a_k$, as required. □ (Lemma)

Lemma 8. *The approximate homomorphism f is not $\mathscr{Z}$-approximable by a continuous homomorphism $g : \mathbb{H} \to \mathbb{H}^A$.*

Proof. Assume towards the contrary that $g : \mathbb{H} \to \mathbb{H}^A$ is a continuous homomorphism which $\mathscr{Z}$-approximates f. Thus if $x \in \mathbb{H}$ then the set $\Delta_x = \{a : f_a(x) \neq g_a(x)\}$ belongs to $\mathscr{Z}$, where, as usual, $g_a(x) = g(x)(a)$. Note that all of these projection maps $g_a : \mathbb{H} \to \mathbb{H}$ are continuous group homomorphisms since such is g itself.

Thus if $x \in \mathbb{H}$ then $\Delta_x \in \mathbb{Z}$, and hence there is a finite set $u_x \subseteq D$ satisfying the following: if $a \in A$ and $u_x \subseteq U^a$ then $a \notin \Delta_x$, that is, $f_a(x) = g_a(x)$. Put

$$X_u = \{x \in \mathbb{H} : \forall\, a \in A\,(u \subseteq U^a \Longrightarrow f_a(x) = g_a(x))\}$$

for every finite $u \subseteq D$. These sets are Borel since so are maps f, g (and g even continuous). Moreover $\mathbb{H} = \bigcup_{u \subseteq D \text{ finite}} X_u$ since every $x \in \mathbb{H}$ belongs to X_{u_x}. Thus at least one of the sets X_u is not meager, therefore, is comeager on a certain rational ball $B \subseteq \mathbb{H}$. Fix u and B. By definition for comeager-many $x \in B$ and all $a \in A$ satisfying $u \subseteq U^a$ we have $f_a(x) = g_a(x)$.

Arguing as in the proof of Lemma 4, we obtain an element $a \in A$ satisfying the following properties: $u \subseteq U^a$, $U^a \cap B \neq \varnothing$, but the set $B \smallsetminus U^a$ is non-empty and moreover is not dense in B. Fix such a. Thus there exists a non-empty rational ball $B' \subseteq B$ that does not intersect U^a. By definition $f_a(x) = \mathbb{0}$ for all $x \in B'$, and hence $g_a(x) = \mathbb{0}$ for comeager-many $x \in B'$ by the choice of B. We conclude that $g_a(x) = \mathbb{0}$ for all $x \in B$ in general, because g is continuous.

Now, let $n^a = n$. Then $U^a = U_1^a \cup \dots \cup U_n^a$. Recall that the intersection $B \cap U^a$ of two open sets is non-empty by the choice of a. It follows that there exists an index i, $1 \le i \le n$, and a non-empty rational ball $B'' \subseteq B \cap U_i^a$. Then by definition $f_a(x) = x \ominus r$ for all $x \in B''$, where $r = r_i^a$. Therefore $g_a(x) = x \ominus r$ for comeager-many $x \in B''$, and then $g_a(x) = x \ominus r$ for all $x \in B''$ since g is continuous.

To conclude, g_a, a continuous group homomorphism, is constant $\mathbb{0}$ on a non-empty open set B', and is bijective on another non-empty open set B''. But this cannot be the case. □ (Lemma)

Lemmas 7 and 8 complete the proof of Theorem 1.

References

[1] I. Farah, *Approximate homomorphisms. II: Group homomorphisms*, *Combinatorica*, 2000, 20, No.1, pp. 47-60.

[2] I. Farah, Analytic quotients. Theory of liftings for quotients over analytic ideals on the integers, *Mem. Am. Math. Soc.*, 2000, 702, 171 p.

[3] V. Kanovei, M. Reeken, On Baire measurable homomorphisms of quotients of the additive group of the reals, *Math. Log. Q.*, 2000, 46, No.3, pp. 377–384.

[4] V. Kanovei, M. Reeken, On Ulam stability of the real line, Abe, Jair Minoro (ed.) et al., *Unsolved problems on mathematics for the 21st century. A tribute to Kiyoshi Iséki's 80th birthday*. Amsterdam: IOS Press, 2001, pp. 169-181.

[5] V. Kanovei, M. Reeken, On Ulam's problem of stability of non-exact homomorphisms, Grigorchuk, R. I. (ed.), *Dynamical systems, automata, and infinite groups. Proc. Steklov Inst. Math.*, 2000, 231, pp. 238-270.

[6] S. Solecki, Filters and sequences, *Fundam. Math.*, 2000, 163, No.3, pp. 215–228.

[7] B. Veličković, Definable automorphisms of $\mathscr{P}(\omega)/\mathtt{fin}$, *Proc. Amer. Math. Soc.* 1986, 96, pp. 130 – 135.