Bounding and decomposing thin analytic partial orderings *

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Abstract

We modify arguments in [2] to reprove extensions of two key results there in the context of bounding and decomposing of analytic subsets of Borel partial quasi-orderings.

1 Introduction

The following theorem is the main content of this note.

Theorem 1.1. *Let* $\preccurlyeq$ *be a* $\varDelta^1_1$ *PQO on* ω^ω, $\approx$ *be the associated equivalence relation, and* $X^* \subseteq \omega^\omega$ *be a* $\varSigma^1_1$ *set such that* $\preccurlyeq \restriction X^*$ *is thin*[1]. *Then*

(i) *there is an ordinal* $\alpha < \omega_1^{\mathrm{CK}}$ *and a* $\varDelta^1_1$ *LR order preserving map* $F : \langle \omega^\omega ; \preccurlyeq \rangle \to \langle 2^\alpha ; \leqslant_{\mathtt{lex}} \rangle$ *satisfying the following additional requirement: if* $x, y \in X^*$ *then* $x \not\approx y \implies F(x) \neq F(y)$;

(ii) X^* *is covered by the countable union of all* $\varDelta^1_1$ $\preccurlyeq$*-chains* $C \subseteq \omega^\omega$.

The theorem is essentially proved in [2, 3.1 and 5.1]. Literally, only the case of $\varDelta^1_1$ subsets X^* is considered in [2], but the case of $\varSigma^1_1$ sets X^* can be obtained by a rather transparent rearrangement of the arguments in [2]. See also [3] in matters of the additional requirement in claim (i) of the theorem, which also is presented in [2] implicitly. Our proofs will largely follow the arguments in [2], but by necessity we modify those here and there in order to streamline some key arguments. On the other hand, we substitute reflection arguments in [2] with more transparent constructions.

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[1] Meaning that there is no perfect set of pairwise $\preccurlyeq$-incomparable elements.

2 Notation and an important lemma

Non-strict relations

PQO, *partial quasi-order*: reflexive ($x \le x$) and transitive in the domain;

LQO, *linear quasi-order*: PQO and $x \le y \vee y \le x$ in the domain;

LO, *linear order*: LQO and $x \le y \wedge y \le x \Longrightarrow x = y$;

associated equivalence relation: $x \approx y$ iff $x \le y \wedge y \le x$.

Strict relations

strict PQO : irreflexive ($x \not< x$) and transitive;

strict LQO : strict PQO and $x < y \Longrightarrow \forall z\,(z < y \vee x < z)$;

strict LO : strict PQO and the trichotomy $\forall x, y\,(x < y \vee y < x \vee x = y)$.

By default we consider only *non-strict* orderings. All cases of consideration of *strict* PQOs will be explicitly specified.

Any non-strict PQO $\le$ defines an associated strict one so that $x < y$ iff $x \le y \wedge y \not\le x$. In the opposite direction, given a strict PQO $<$, we define an equivalence relation $x \approx y$ iff $x < z \Longleftrightarrow y < z$ and $z < x \Longleftrightarrow z < y$ for all z in the domain, and then define $x \le y$ iff $x < y$ or $x \approx y$.

Order preserving maps

LR (left–right) order preserving map: any map $f : \langle X\,; \le\rangle \to \langle X'\,; \le'\rangle$ such that we have $x \le y \Longrightarrow f(x) \le' f(y)$ for all $x, y \in \mathtt{dom}\, f$;

RL (right–left) order preserving map: any map $f : \langle X\,; \le\rangle \to \langle X'\,; \le'\rangle$ such that we have $x \le y \Longleftarrow f(x) \le' f(y)$ for all $x, y \in \mathtt{dom}\, f$;

2-ways order preserving map: any map $f : \langle X\,; \le\rangle \to \langle X'\,; \le'\rangle$ such that we have $x \le y \Longleftrightarrow f(x) \le' f(y)$ for all $x, y \in \mathtt{dom}\, f$.

Varia

sub-order: restriction of the given PQO to a subset of its domain.

$<_{\mathtt{lex}}$, $\leqslant_{\mathtt{lex}}$: the lexicographical LOs on sets of the form 2^α, $\alpha \in \mathtt{Ord}$, resp. strict and non-strict;

a $\le$-chain in a PQO : any set of 2wise $\le$-comparable elements, *i.e.*, LQO;

a $\le$-thin set in a PQO: any set in the domain of $\le$ containing no perfect subsets of 2wise $\le$-incomparable elements;

$[x]_{\mathsf{E}} = \{y \in \mathtt{dom}\,\mathsf{E} : x \mathrel{\mathsf{E}} y\}$ (the E-*class* of x) and $[X]_{\mathsf{E}} = \bigcup_{x\in X}[x]_{\mathsf{E}}$ — whenever E is an equivalence relation and $x \in \mathtt{dom}\,\mathsf{E}$, $X \subseteq \mathtt{dom}\,\mathsf{E}$.

Lemma 2.1 (Kreisel selection). *Let D be the set of all $\varDelta^1_1$ points in ω^ω. If $P \subseteq \omega^\omega \times D$ is a $\varPi^1_1$ set, and $X \subseteq \mathtt{dom}\,P$ is $\varSigma^1_1$ then there is a $\varDelta^1_1$ set $Y \subseteq \mathtt{dom}\,P$ and a $\varDelta^1_1$ function $F : Y \to D$ such that $X \subseteq Y$ and $F \subseteq P$.*

Proof. The set $X_0 = \mathtt{dom}\,P$ is $\varPi^1_1$ since $\varPi^1_1$ is closed under $\exists\, y \in \varDelta^1_1$. Therefore by Separation there is a $\varDelta^1_1$ set Y, $X \subseteq Y \subseteq X_0$. By $\varPi^1_1$ Uniformization, there is a $\varPi^1_1$ set $F \subseteq P$ such that $\mathtt{dom}\,F = Y$ and Y is a function. To show that F is in fact $\varDelta^1_1$ note that $F(x) = y$ iff $x \in Y$ and $\forall\, y' \in D\,(y \ne y' \Longrightarrow \langle x, n'\rangle \notin F)$, which leads to a $\varSigma^1_1$ definition. □

3 Ingredient 1: coding $\varDelta^1_1$ functions

The proof of Theorem 1.1 involves several technical methods of rather general nature, which we present in the three following sections.

Recall that ω_1^{CK} is the least non-recursive (= the least non-$\varDelta^1_1$) ordinal. If $\alpha < \omega_1^{\mathrm{CK}}$ then let $\mathscr{F}_\alpha$ be the set of all $\varDelta^1_1$ LR order preserving maps $F : \langle \omega^\omega\,; \preccurlyeq\rangle \to \langle 2^\alpha\,; \leqslant_{\mathtt{lex}}\rangle$, so that

$$x \preccurlyeq y \implies F(x) \leqslant_{\mathtt{lex}} F(y) \quad \text{for all} \quad x, y \in \omega^\omega\,. \tag{1}$$

Such a function F has to be $\approx$-invariant on ω^ω. Let $\mathscr{F} = \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} \mathscr{F}_\alpha$.

If, in addition, $X \subseteq \omega^\omega$ is a $\varSigma^1_1$ set then let $\mathscr{F}_X$ consist of all $\varDelta^1_1$ functions $F \in \mathscr{F}$ such that

$$x, y \in X \text{ are } \preccurlyeq\text{-incomparable} \implies F(x) = F(y)\,. \tag{2}$$

or equivalently, $F(x) <_{\mathtt{lex}} F(y) \Longrightarrow x \prec y$ for all $x, y \in X$.

Note that a function $F \in \mathscr{F}_X$ has to be not just $\approx$-invariant, but also invariant w.r.t. the common equivalence hull of the relation $\approx$ and the (non-equivalence) relation of being $\preccurlyeq$-incomparable. In particular, if for any $x, y \in X$ there is $z \in X$ $\preccurlyeq$-incomparable with both x and y, then the only maps in $\mathscr{F}_X$ are those constant on X.

Definition 3.1. Let, for $x, y \in \omega^\omega$: $x \mathrel{\mathsf{E}_{\mathscr{F}}} y$ iff $\forall\, F \in \mathscr{F}\ (F(x) = F(y))$,

$$x \mathrel{\mathsf{E}_{\mathscr{F}_X}} y \quad \text{iff} \quad \forall\, F \in \mathscr{F}_X\ (F(x) = F(y)) \quad (\text{here } X \subseteq \omega^\omega \text{ is } \varSigma^1_1). \qquad \square$$

Lemma 3.2. *$\mathsf{E}_{\mathscr{F}}$ is a smooth Σ^1_1 equivalence relation, and if $R(x,y)$ is a Π^1_1 relation and $\forall\,x,y\,(x\mathrel{\mathsf{E}_{\mathscr{F}}}y \Longrightarrow R(x,y))$ then there is a single function $F\in\mathscr{F}$ such that $\forall\,x,y\,(F(x)=F(y)\Longrightarrow R(x,y))$.*

Similarly, if $X\subseteq\omega^\omega$ is a Σ^1_1 set then $\mathsf{E}_{\mathscr{F}_X}$ is a smooth Σ^1_1 equivalence relation, and if $R(x,y)$ is a Π^1_1 relation and $\forall\,x,y\,(x\mathrel{\mathsf{E}_{\mathscr{F}_X}}y \Longrightarrow R(x,y))$ then there is a function $F\in\mathscr{F}_X$ such that $\forall\,x,y\,(F(x)=F(y)\Longrightarrow R(x,y))$.

Proof. We concentrate on the first part; the result for the second part is pretty similar. We'll make use of an appropriate coding of functions in $\mathscr{F}$, based on a standard coding system of Δ^1_1 sets. A *code* will be a such-and-such pair $f=\langle\varepsilon,k\rangle\in\omega^\omega\times\omega$. We require that:

(I) the relation $\le_\varepsilon=\{\langle i,j\rangle:\varepsilon(2^i\cdot 3^j)=0\}$ is a (non-strict) wellordering of the set $\mathtt{dom}\,(\le_\varepsilon)$ — in this case, we let:

- $|\varepsilon|=\operatorname{otp}(\varepsilon)<\omega_1$ be the order type of $\le_\varepsilon$,
- $\beta_\varepsilon:\mathtt{dom}\,(\le_\varepsilon)\xrightarrow{\text{onto}}|\varepsilon|$ be the order-preserving bijection,
- $H_\varepsilon:\omega^\omega\xrightarrow{\text{onto}}(\omega^\omega)^{|\varepsilon|}$ be the induced homeomorphism;

(II) k belongs to the set $\mathbb{B}\subseteq\omega$ of *codes of Δ^1_1 sets* $B\subseteq\omega^\omega\times\omega^\omega$, so that it is assumed that $\mathbb{B}$ is a Π^1_1 set, and for any $k\in\mathbb{B}$ a Δ^1_1 set $B_k\subseteq\omega^\omega\times\omega^\omega$ is defined, and conversely, for any Δ^1_1 set $B\subseteq\omega^\omega\times\omega^\omega$ there is a code $k\in\mathbb{B}$ with $B=B_k$, and finally there exist two Π^1_1 sets $W,W'\subseteq\omega\times\omega^\omega\times\omega^\omega$ such that if $k\in\mathbb{B}$ and $x,y\in\omega^\omega$ then

$$\langle x,y\rangle\in B_k \iff W(k,x,y) \iff \neg\,W'(k,x,y)\,;$$

(III) we define $\mathbb{F}\mathrm{un}=\{k\in\mathbb{B}:B_k\text{ is a total map }\omega^\omega\to\omega^\omega\}$, the set of codes of all Δ^1_1 functions $F:\omega^\omega\to\omega^\omega$ — this is still a Π^1_1 set because the key condition $\mathtt{dom}\,B_f=\omega^\omega$ can be expressed by

$$\forall\,x\,\exists\,y\in\Delta^1_1(x)\,W(k,x,y)\,,$$

where the quantifier $\exists\,y\in\Delta^1_1(x)$ is known to preserve the type Π^1_1;

(IV) if ε satisfies (I) and $k\in\mathbb{F}\mathrm{un}$ then let F^ε_k be the Δ^1_1 map $\omega^\omega\to(\omega^\omega)^{|\varepsilon|}$ defined by $F^\varepsilon_k(x)=H_\varepsilon(B_k(x))$ for all $x\in\omega^\omega$;

Definition 3.3. Let $\mathbb{F}$ be the set of all pairs $f=\langle\varepsilon,k\rangle$ such that $\varepsilon\in\omega^\omega$ satisfies (I), $\varepsilon\in\Delta^1_1$, $k\in\mathbb{F}\mathrm{un}$, and $F^\varepsilon_k\in\mathscr{F}_{|\varepsilon|}$.

If $X\subseteq\omega^\omega$ is a Σ^1_1 set then let $\mathbb{F}(X)=\{\langle\varepsilon,k\rangle\in\mathbb{F}:F^\varepsilon_k\in\mathscr{F}_{|\varepsilon|}(X)\}$. □

The following is a routine fact.

Claim 3.4. *$\mathbb{F} \subseteq \omega^\omega \times \omega$ is a countable Π^1_1 set of Δ^1_1 elements, and $\mathscr{F} = \{F^\varepsilon_k : \langle \varepsilon, k \rangle \in \mathbb{F}\}$. If $X \subseteq \omega^\omega$ is a Σ^1_1 set then $\mathbb{F}(X) \subseteq \omega^\omega \times \omega$ is a countable Π^1_1 set of Δ^1_1 elements, and $\mathscr{F}_X = \{F^\varepsilon_k : \langle \varepsilon, k \rangle \in \mathbb{F}(X)\}$.* □

In continuation of the proof of Lemma 3.2, note that

$$x \mathsf{E}_{\mathscr{F}} y \iff \forall \langle \varepsilon, k \rangle \in \mathbb{F} \, (F^\varepsilon_k(x) = F^\varepsilon_k(y)) \iff \forall \langle \varepsilon, k \rangle \in \mathbb{F} \, (B_k(x) = B_k(y)),$$

and this easily implies that $\mathsf{E}_{\mathscr{F}}$ is Σ^1_1 by Claim 3.4. Now prove the claim of Lemma 3.2 related to $R(x, y)$. We re-write the assumption as follows:

$$\forall x, y \, (\neg R(x, y) \implies \neg (x \mathsf{E}_{\mathscr{F}} y)),$$

or, equivalently by Claim 3.4, as

$$\forall x, y \left(\neg R(x, y) \implies \exists \langle \varepsilon, k \rangle \in \Delta^1_1 \underbrace{(\langle \varepsilon, k \rangle \in \mathbb{F} \wedge F^\varepsilon_k(x) \neq F^\varepsilon_k(y))}_{P(x, y \, ; \, \varepsilon, k)} \right).$$

The relation P is expressible by a Π^1_1 formula by means of (II) and Claim 3.4. It follows by Lemma 2.1 that there is a Δ^1_1 set $W \subseteq \omega^\omega \times \omega^\omega$ satisfying $\neg R(x, y) \implies W(x, y)$, and a Δ^1_1 map $\Phi(x, y) = \langle \varepsilon(x, y), k(x, y) \rangle : W \to \mathbb{F}$ such that we have $F^{\varepsilon(x,y)}_{k(x,y)}(x) \neq F^{\varepsilon(x,y)}_{k(x,y)}(y)$ for all $\langle x, y \rangle \in W$ — then, in particular, for all x, y with $\neg R(x, y)$.

The range $H = \{\Phi(x, y) : \langle x, y \rangle \in W\}$ is then a Σ^1_1 subset of the (countable) Π^1_1 set $\mathbb{F}$. By Separation, there is a Δ^1_1 set D with $H \subseteq D \subseteq \mathbb{F}$. As a countable Δ^1_1 set, it admits a Δ^1_1 enumeration $D = \{\langle \varepsilon_n, k_n \rangle : n \in \mathbb{N}\}$, and by construction $\forall n \, (F^{\varepsilon_n}_{k_n}(x) = F^{\varepsilon_n}_{k_n}(y))$ implies $R(x, y)$. Let

$$F(x) = F^{\varepsilon_0}_{k_0}(x)^\wedge F^{\varepsilon_1}_{k_1}(x)^\wedge F^{\varepsilon_2}_{k_2}(x)^\wedge \dots$$

for $x \in \omega^\omega$. Then $F \in \mathscr{F}$ and $F(x) = F(y) \implies R(x, y)$. □ (Lemma 3.2)

4 Ingredient 2: invariant separation

In the assumptions of Theorem 1.1, let E be a Σ^1_1 equivalence relation containing $\approx$ (so that $x \approx y$ implies $x \mathsf{E} y$). A set $X \subseteq \omega^\omega$ is *downwards $\preccurlyeq$-closed in each* E*-class* iff we have $x \in X \implies y \in X$ whenever $x \mathsf{E} y$ and $y \preccurlyeq x$. The notion of a set *upwards $\preccurlyeq$-closed in each* E*-class* is similar.

Lemma 4.1. *Let* E *be a* Σ^1_1 *equivalence relation containing* $\approx$, X, Y *be disjoint* Σ^1_1 *sets, satisfying* $y \not\preccurlyeq x$ *whenever* $x \in X \wedge y \in Y \wedge x \mathrel{\mathsf{E}} y$. *Then there is a* Δ^1_1 *set* Z, *downwards* $\preccurlyeq$*-closed in each* E*-class and satisfying* $X \subseteq Z$ *and* $Y \cap Z = \varnothing$.

Proof. Let $Y' = \{y' : \exists\, y \in Y\, (y \preccurlyeq y')\}$; still $Y' \cap X = \varnothing$ and Y' is Σ^1_1. Using Separation, define an increasing sequence of sets

$$X = X_0 \subseteq A_0 \subseteq X_1 \subseteq A_1 \subseteq \ldots \subseteq X_n \subseteq A_n \subseteq \ldots \subseteq \omega^\omega \smallsetminus Y'$$

so that $A_n \in \Delta^1_1$ and $X_{n+1} = \{x' \in \omega^\omega : \exists\, x \in A_n\, (x' \mathrel{\mathsf{E}} x \wedge x' \preccurlyeq x)\}$ for all n. If $A_n \cap Y' = \varnothing$ then $X_{n+1} \cap Y' = \varnothing$ as well since Y' is upwards closed, which justifies the inductive construction. Furthermore, a proper execution of the construction yields the final set $Z = \bigcup_n A_n = \bigcup_n X_n$ in Δ^1_1. (We refer to the proof of an "invariant" effective separation theorem in [1] or a similar construction in [4, Lemma 10.4.2].) Note that by construction $X \subseteq Z$, but $Z \cap Y = \varnothing$, and Z is downwards $\preccurlyeq$-closed in each E-class. □

Corollary 4.2. *Let* E *be* $\mathsf{E}_{\mathscr{F}}$. *If* $X, Y \subseteq \omega^\omega$ *are disjoint* Σ^1_1 *sets and* $[X]_{\mathsf{E}} \cap [Y]_{\mathsf{E}} \neq \varnothing$ *then there are points* $x \in X$, $y \in Y$ *with* $x \mathrel{\mathsf{E}} y$ *and* $y \preccurlyeq x$.

Proof. Otherwise by Lemma 4.1 there is a Δ^1_1 set Z such that $X \subseteq Z$ and $Y \cap Z = \varnothing$, and downwards $\preccurlyeq$-closed in each E-class. Then, by Lemma 3.2, there is a function $F \in \mathscr{F}$ such that $x \in Z \Longrightarrow y \in Z$ holds whenever $F(x) = F(y)$ and $x \preccurlyeq y$. It follows that the derived function

$$G(x) = \begin{cases} F(x)^\wedge 0\,, & \text{whenewer} \quad x \in Z \\ F(x)^\wedge 1\,, & \text{whenewer} \quad x \in \omega^\omega \smallsetminus Z \end{cases}$$

belongs to $\mathscr{F}$. Thus if $x \in Z$ and $y \notin Z$, say $x \in X$ and $y \in Y$, then $G(x) \neq G(y)$ and hence $x \not\mathrel{\mathsf{E}} y$, a contradiction. □

5 Ingredient 3: the Gandy – Harrington forcing

The Gandy – Harrington forcing notion $\mathbb{P}$ is the set of all Σ^1_1 sets $\varnothing \neq X \subseteq \omega^\omega$, ordered so that smaller sets are stronger conditions. We also define $\mathbb{P}_n$ $(n \geq 2)$ to be the set of all Σ^1_1 sets $\varnothing \neq X \subseteq (\omega^\omega)^n$.

It is known that $\mathbb{P}$ adds a point of ω^ω, whose name will be $\dot{\boldsymbol{x}}$.

Together with $\mathbb{P}$, some other related forcing notions will be considered below, for instance, the product $\mathbb{P}^2 = \mathbb{P} \times \mathbb{P}$ which consists of all cartesian products of the form $X \times Y$, where $X, Y \in \mathbb{P}$. It follows from the above that $\mathbb{P}^2$ forces a pair of points of 2^ω, whose name will be $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}} \rangle$.

There is another important subforcing introduced in [2]. If E is a Σ^1_1 equivalence relation on ω^ω then let $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ consist of all sets of the form $X \times Y$, where $X, Y \in \mathbb{P}$ and $(X \times Y) \cap \mathsf{E} \neq \varnothing$.

A condition $X \times Y$ in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ is *saturated* iff $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$.

Lemma 5.1. *If $X \times Y$ is a condition in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ then there is a stronger saturated subcondition $X' \times Y'$ in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$.*

Proof. $X' = \{x \in X : \exists\, y \in Y\, (x \mathrel{\mathsf{E}} y)\}$, $Y' = \{y \in Y : \exists\, x \in X\, (x \mathrel{\mathsf{E}} y)\}$. □

Remark 5.2. If $X \times Y$ is a saturated condition in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$, and $\varnothing \neq X' \subseteq X$ is a Σ^1_1 set, then $Y' = Y \cap [X']_{\mathsf{E}}$ is Σ^1_1 and $X' \times Y'$ is still a saturated condition in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$. It follows that $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ forces a pair of $\mathbb{P}$-generic reals, whose names will be $\dot{\boldsymbol{x}}_{\mathtt{le}}$ and $\dot{\boldsymbol{x}}_{\mathtt{ri}}$ as above.

Lemma 5.3 (2.9 in [2])**.** *Suppose that E is a smooth Σ^1_1 equivalence relation. Then $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \mathrel{\mathsf{E}} \dot{\boldsymbol{x}}_{\mathtt{ri}}$.* □

Note that Lemma 5.3, generally speaking, fails in the non-smooth case.

The next result will be pretty important.

Lemma 5.4 (2.9 in [2])**.** *Suppose that $\preccurlyeq$ is a Δ^1_1 PQO on ω^ω, and for any $A \in \mathbb{P}$ there is a Σ^1_1 equivalence relation E_A on ω^ω such that if $A \subseteq B$ then $x \mathrel{\mathsf{E}_A} y$ implies $x \mathrel{\mathsf{E}_B} y$. Assume that $X^* \in \mathbb{P}$, and if $B \in \mathbb{P}$, $B \subseteq X^*$ then $B \times B$ does* **not** *$(\mathbb{P} \times_{\mathsf{E}_B} \mathbb{P})$-force that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable.*

Then X^* is not $\preccurlyeq$-thin, in other words, there is a perfect set $Y \subseteq X^*$ of pairwise $\preccurlyeq$-incomparable elements. □

The forcing $\mathbb{P}$, as well as some of its derivates like $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$, will be used below as forcing notions over the ground set universe $\mathbf{V}$.

Lemma 5.5 (see [1, 2])**.** *If $X \in \mathbb{P}$ then X $\mathbb{P}$-forces that $\dot{\boldsymbol{x}} \in X$. Moreover if $\Phi(x)$ is a Π^1_2 formula and $\Phi(x)$ holds for all $x \in X$ then X $\mathbb{P}$-forces that $\dot{\boldsymbol{x}}$ satisfies $\Phi(X)$.*

The same is true for other similar forcing notions like $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$. □

Here (and below in some cases), given a Σ^1_1 (or Π^1_1) set X in the ground universe $\mathbf{V}$, **we denote by the same letter X the extended set** (*i.e.*, defined by the same formula) **in any generic extension of $\mathbf{V}$**. By the Shoenfield absoluteness theorem, there is no ambiguity here. See [5, 2.4] in more detail.

6 Bounding thin partial orderings

Here **we prove claim** (i) **of Theorem 1.1**. We'll make use of the family $\mathscr{F}$ of $\varDelta^1_1$ functions, introduced in Section 3, and the corresponding smooth $\varSigma^1_1$ equivalence relation $\mathsf{E} = \mathsf{E}_{\mathscr{F}}$. Then $\approx$ is a subrelation of E by Lemma 3.2.

The following partition on cases is quite common in this sort of proofs.

Case 1: $\approx$ and E coincide on X^*, so that $x \mathrel{\mathsf{E}} y \Longrightarrow x \approx y$ for $x, y \in X^*$. Then, by Lemma 3.2, there is a single function $F \in \mathscr{F}$ such that $F(x) = F(y)$ implies $x \approx y$ for all $x, y \in X^*$, as required.

Case 2: $\approx$ is a *proper* subrelation of E on X^*, hence, the $\varSigma^1_1$ set

$$V^* = \{x \in X^* : \exists\, y \in X^* \ (x \not\approx y \wedge x \mathrel{\mathsf{E}} y)\}$$

is non-empty. Our final goal will be to infer a contradiction; then the result for Case 1 proves Claim (i) of the theorem.

Note that $V^* \times V^*$ is a saturated condition in $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$.

Lemma 6.1. *Condition $V^* \times V^*$ $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}}$ and $\dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-incomparable.*

Proof. Suppose to the contrary that a subcondition $Y \times Z$ either forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ or forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. We will get a contradiction in both cases. Note that $Y, Z \subseteq V^*$ are non-empty $\varSigma^1_1$ sets and $[Y]_{\mathsf{E}} \cap [Z]_{\mathsf{E}} \neq \varnothing$.

Case A: $Y \times Z$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$.

Subcase A1: the $\varSigma^1_1$ set $W = \{\langle y, y' \rangle \in Y \times Y : y \mathrel{\mathsf{E}} y' \wedge y' \not\approx y\}$ is empty, or in other words E coincides with $\approx$ on Y. By the non-emptiness of V^* at least one of the $\varSigma^1_1$ sets

$$B = \{x : \exists\, y \in Y \, (x \mathrel{\mathsf{E}} y \wedge x \not\preccurlyeq y)\}\,, \ B' = \{x : \exists\, y \in Y \, (x \mathrel{\mathsf{E}} y \wedge y \not\preccurlyeq x)\}$$

is non-empty; assume that, say, $B \neq \varnothing$. Consider the $\varSigma^1_1$ set

$$A = \{x : \exists\, y \in Y \, (x \mathrel{\mathsf{E}} y \wedge x \preccurlyeq y)\}\,; \quad Y \subseteq A\,.$$

Then $A \cap B = \varnothing$, A is downwards closed while B is upwards closed in each E-class, therefore $y \not\preccurlyeq x$ whenever $x \in A$, $y \in B$, and $x \mathrel{\mathsf{E}} y$. Then $[A]_{\mathsf{E}} \cap [B]_{\mathsf{E}} = \varnothing$ by Corollary 4.2. Yet by definition $[Y]_{\mathsf{E}} \cap [B]_{\mathsf{E}} \neq \varnothing$ and $Y \subseteq A$, which is a contradiction.

Subcase A2: $W \neq \varnothing$. Then the forcing notion $\mathbb{P}(W)$ of all non-empty $\varSigma^1_1$ sets $P \subseteq W$ adds pairs of $\mathbb{P}$-generic (separately) reals $y, y' \in Y$ which belong to W and satisfy $y' \mathrel{\mathsf{E}} y$ and $y' \not\approx y$, by Lemma 5.5.

If $P \in \mathbb{P}(W)$ then obviously $[\mathtt{dom}\, P]_{\mathsf{E}} = [\mathtt{ran}\, P]_{\mathsf{E}}$.

Consider a more complex forcing notion $\mathscr{P} = \mathbb{P}(W) \times_{\mathsf{E}} \mathbb{P}$ of all pairs $P \times Z'$, where $P \in \mathbb{P}(W)$, $Z' \in \mathbb{P}$, $Z' \subseteq Z$, and $[\mathtt{dom}\, P]_{\mathsf{E}} \cap [Z']_{\mathsf{E}} \neq \varnothing$. For instance, $W \times Z \in \mathbb{P}(W) \times_{\mathsf{E}} \mathbb{P}$. Then $\mathscr{P}$ adds a pair $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}} \rangle \in W$ and a separate real $\dot{\boldsymbol{x}} \in B$ such that both pairs $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}} \rangle$ and $\langle \dot{\boldsymbol{x}}_{\mathtt{ri}}, \dot{\boldsymbol{x}} \rangle$ are $(\mathbb{P} \times_{\mathsf{E}} \mathbb{P})$-generic, hence, we have $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}} \approx \dot{\boldsymbol{x}}_{\mathtt{le}}$ (in the extended universe $\mathbf{V}[\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}, \dot{\boldsymbol{x}}]$) by the choice of $Y \times Z$. On the other hand, $\dot{\boldsymbol{x}}_{\mathtt{le}} \not\approx \dot{\boldsymbol{x}}_{\mathtt{le}}$ by Lemma 5.5, since the pair belongs to W, which is a contradiction.

Case B: $Y \times Z$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$.

Subcase B1: the Σ^1_1 set $W = \{\langle y, z \rangle \in Y \times Z : z \mathrel{\mathsf{E}} y \wedge z \preccurlyeq y\}$ is empty. Then the Σ^1_1 sets

$$Y_0 = \{y' : \exists\, y \in Y\, (y \mathrel{\mathsf{E}} y' \wedge y' \preccurlyeq y)\}, \quad Z_0 = \{z' : \exists\, z \in Z\, (z \mathrel{\mathsf{E}} z' \wedge z \preccurlyeq z')\}$$

are disjoint and $\preccurlyeq$-closed resp. downwards and upwards, hence we have $[Z_0]_{\mathsf{E}} \cap [Y_0]_{\mathsf{E}} = \varnothing$ by Corollary 4.2. However $[Z]_{\mathsf{E}} \cap [Y]_{\mathsf{E}} \neq \varnothing$, which is a contradiction as $Z \subseteq Z_0$, $Y \subseteq Y_0$.

Subcase B2: $W \neq \varnothing$. Consider the forcing $\mathbb{P}(W)$ of all non-empty Σ^1_1 sets $P \subseteq W$; if $P \in \mathbb{P}(W)$ then obviously $[\mathtt{dom}\, P]_{\mathsf{E}} = [\mathtt{ran}\, P]_{\mathsf{E}}$. Consider a more complicated forcing $\mathbb{P}(W) \times_{\mathsf{E}} \mathbb{P}(W)$ of all products $P \times Q$, where $P, Q \in \mathbb{P}(W)$ and $[\mathtt{dom}\, P]_{\mathsf{E}} \cap [\mathtt{dom}\, Q]_{\mathsf{E}} \neq \varnothing$. In particular $W \times W \in \mathbb{P}(W) \times_{\mathsf{E}} \mathbb{P}(W)$.

Let $\langle x, y; x', y' \rangle$ be a $\mathbb{P}(W) \times_{\mathsf{E}} \mathbb{P}(W)$-generic quadruple in $W \times W$, so that both $\langle x, y \rangle \in W$ and $\langle x', y' \rangle \in W$ are $\mathbb{P}(W)$-generic pairs in W, and both $y \preccurlyeq x$ and $y' \preccurlyeq x'$ hold by the definition of W. On the other hand, an easy argument shows that both criss-cross pairs $\langle x, y' \rangle \in X \times Y$ and $\langle x', y \rangle \in X \times Y$ are $\mathbb{P} \times_{\mathsf{E}} \mathbb{P}$-generic, hence $x \prec y'$ and $x' \prec y$ by the choice of $X \times Y$. Altogether $y \preccurlyeq x \prec y' \preccurlyeq x' \prec y$, which is a contradiction. □

To accomplish the proof of (i) of Theorem 1.1, note that by Lemma 6.1 and Lemma 5.4 (with $\mathsf{E}_A = \mathsf{E}$ for all A) there is a perfect 2wise $\approx$-inequivalent set, so $\preccurlyeq$ is not thin, contrary to our assumptions.

7 Decomposing thin partial orderings

We prove **claim** (ii) **of Theorem 1.1** in this Section. Let U^* be the Σ^1_1 set of all reals $x \in X^*$ such that there is no Δ^1_1 $\preccurlyeq$-chain C containing x.

We assume to the contrary that $U^* \neq \varnothing$.

The proof will make heavy use of the functions in families of the form $\mathscr{F}_X$, introduced in Section 3. If $X \subseteq \omega^\omega$ is a Σ^1_1 set then $\mathsf{E}_X = \mathsf{E}_{\mathscr{F}_X}$ is a smooth Σ^1_1 equivalence relation by Lemma 3.2.

If $X \subseteq X'$ then $\mathscr{F}'_X \subseteq \mathscr{F}_X$, and hence $x \mathsf{E}_X y$ implies $x \mathsf{E}_{X'} y$.

Corollary 7.1 (of Lemma 5.3)**.** *If $X \subseteq U^*$ is a non-empty Σ^1_1 set then the condition $X \times X$ $(\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}} \mathsf{E}_X \dot{\boldsymbol{x}}_{\mathtt{ri}}$.* $\square$

Lemma 7.2. *Let $X \subseteq U^*$ be a non-empty Σ^1_1 set. Then $X \times X$ does* ***not*** *$(\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P})$-force that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable.*

Proof. Suppose to the contrary that $X \times X$ forces the comparability. Then there is a subcondition $Y \times Z$ which either forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ or forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$; $Y, Z \subseteq X$ are non-empty Σ^1_1 sets and $[Y]_{\mathsf{E}_X} \cap [Z]_{\mathsf{E}_X} \neq \varnothing$.

Case A: $Y \times Z$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$.

Subcase A1: the Σ^1_1 set $W = \{\langle y, y' \rangle \in Y \times Y : y \mathsf{E}_X y' \wedge y' \not\approx y\}$ is empty. Then Y is a $\preccurlyeq$-chain: indeed if $x, y \in Y$ are $\preccurlyeq$-incomparable then by definition we have $x \mathsf{E}_X y$, hence $x \approx y$, contradiction. Let C be the Π^1_1 set of all reals $\preccurlyeq$-comparable with each $y \in Y$; then $Y \subseteq C$. By Separation there is a Δ^1_1 set D, $Y \subseteq D \subseteq C$. Let C' be the Π^1_1 set of all reals in D, $\preccurlyeq$-comparable with each $d \in D$; then $Y \subseteq C' \subseteq D$. Take any Δ^1_1 set B with $Y \subseteq B \subseteq C'$. By construction B is a Δ^1_1 $\preccurlyeq$-chain with $\varnothing \neq Y \subseteq B$, contrary to the definition of U^*.

Subcase A2: $W \neq \varnothing$: yields a contradiction similarly to Subcase A2 in the proof of Lemma 6.1.

Case B: $Y \times Z$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$.

Subcase B1: the Σ^1_1 set $W = \{\langle y, z \rangle \in Y \times Z : y \mathsf{E}_X z \wedge y \not\prec z\}$ is non-empty. Let $Y' = \mathtt{dom}\, W$. As $Y' \subseteq X$, the condition $Y' \times Y'$ $(\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable. Therefore by the result in Case A there is a condition $A \times B$ in $\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P}$, with $A \cup B \subseteq Y'$, which forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$; for if it forces $\dot{\boldsymbol{x}}_{\mathtt{ri}} \prec \dot{\boldsymbol{x}}_{\mathtt{le}}$ then just consider $B \times A$ instead of $A \times B$. Consider the forcing notion $\mathscr{P}$ of all non-$\varnothing$ Σ^1_1 sets of the form $P \times B'$, where

$$P \subseteq W, \ \mathtt{dom}\, P \subseteq A, \ B' \subseteq B, \quad \text{and} \quad [B']_{\mathsf{E}_X} = [\mathtt{dom}\, P]_{\mathsf{E}_X} = [\mathtt{ran}\, P]_{\mathsf{E}_X}.$$

For instance if $B' = B$ and $P = \{\langle x, y \rangle \in W : x \in A\}$ then $P \times B' \in \mathscr{P}$. Then $\mathscr{P}$ forces a pair $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}} \rangle \in W$ and a separate real $\dot{\boldsymbol{x}} \in B$ such that both pairs $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}} \rangle$ and $\langle \dot{\boldsymbol{x}}_{\mathtt{ri}}, \dot{\boldsymbol{x}} \rangle$ are $(\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P})$-generic. It follows that $\mathscr{P}$ forces both $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}$ (as this pair belongs to $A \times B$) and $\dot{\boldsymbol{x}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$ (as this pair belongs to $Y \times Z$), hence, forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. On the other hand $\mathscr{P}$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \not\prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$ (as this pair belongs to W), which is a contradiction.

Subcase B2: $W = \varnothing$, in other words, if $y \in Y$, $z \in Z$, and $y \mathsf{E}_X z$ then $y \prec z$ strictly. Then by Lemma 4.1 there is a Δ^1_1 set $C \subseteq \omega^\omega$, downwards

$\preccurlyeq$-closed in each E_X-class, such that $Y \subseteq C$ and still $Z \cap C = \varnothing$. We claim that, moreover,

$$\textit{if } y \in C \cap X\,,\ z \in X \smallsetminus C\,,\ \textit{and } y \mathrel{\mathsf{E}_X} z\,,\ \textit{then } y \prec z\,.$$

Indeed otherwise, the following Σ^1_1 set

$$H_0 = \{z \in X \smallsetminus C : \exists\, y \in C \cap X\, (y \mathrel{\mathsf{E}_X} z \wedge y \not\prec z)\} \subseteq X$$

is non-$\varnothing$. As above, there is a saturated condition $H \times H'$ in $\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P}$, with $H \cup H' \subseteq H_0$, which forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$, and then $z \prec z'$ holds whenever $\langle z, z' \rangle \in H \times H'$ and $z \mathrel{\mathsf{E}_X} z'$. By construction the Σ^1_1 set

$$C_1 = \{y \in C \cap X : \exists\, z' \in H'\, (y \mathrel{\mathsf{E}_X} z' \wedge y \not\prec z')\}$$

satisfies $[C_1]_{\mathsf{E}_X} = [H]_{\mathsf{E}_X} = [H']_{\mathsf{E}_X}$, hence $C_1 \times H$ is a condition in $\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P}$. Let $\langle y_1, z \rangle \in C_1 \times H$ be any $(\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P})$-generic pair. Then $y_1 \mathrel{\mathsf{E}_X} z$ by Corollary 7.1, and, by the choice of X and the result in Case A, we have $y_1 \prec z$ or $z \prec y_1$. However by construction $y_1 \in C$, $z \notin C$, and C is downwards closed in each E_X-class. Thus in fact $y_1 \prec z$. Therefore, for all $z' \in H'$, if $y_1 \mathrel{\mathsf{E}_X} z'$ then $y_1 \prec z \prec z'$, which contradicts to $y_1 \in C_1$.

Thus indeed $y \prec z$ holds whenever $y \in C \cap X$, $z \in X \smallsetminus C$, and $y \mathrel{\mathsf{E}_X} z$. By Lemma 3.2 *there is a single function* $F \in \mathscr{F}_X$ *such that if* $y \in C \cap X$, $z \in X \smallsetminus C$, *and* $F(y) = F(z)$, *then* $y \prec z$.

We claim that *the derived function*

$$G(x) = \begin{cases} F(x)^\wedge 0\,, & \text{whenewer} \quad x \in C \\ F(x)^\wedge 1\,, & \text{whenewer} \quad x \in \omega^\omega \smallsetminus C \end{cases}$$

belongs to $\mathscr{F}_X$. First of all, still $G \in \mathscr{F}$ since C is downwards $\preccurlyeq$-closed in each $\mathbb{P} \times_{\mathsf{E}_X} \mathbb{P}$-class. Now suppose that $z, y \in X$ and $G(y) <_{\mathtt{lex}} G(z)$. Then either $F(y) <_{\mathtt{lex}} F(z)$, or $F(z) = F(y)$ and $y \in C$ but $z \notin C$. In the "either" case immediately $y \prec z$ since $F \in \mathscr{F}_X$ [2]. In the "or" case we have $y \prec z$ by the choice of F and the definition of G. Thus $G \in \mathscr{F}_X$.

Now pick any pair of reals $y \in Y$ and $z \in Z$ with $y \mathrel{\mathsf{E}_X} z$. Then we have $G(x) = G(y)$ since $G \in \mathscr{F}_X$. But $y \in C$ and $z \notin C$ hold since $Y \subseteq C$ and $Z \cap C = \varnothing$ by construction, and in this case surely $G(y) \neq G(z)$ by the definition of G. This contradiction completes the proof of Lemma 7.2. $\square$

Lemma 7.2 plus Lemma 5.4 imply claim (ii) of Theorem 1.1.

$\square$ (Theorem 1.1)

[2] The family $\mathscr{F}$ would not work in the passage; here we have to use $\mathscr{F}_X$ instead.

References

[1] L. A. Harrington, A. S. Kechris, and A. Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. *J. Amer. Math. Soc.*, 3(4):903–928, 1990.

[2] L. A. Harrington, D. Marker, and S. Shelah. Borel orderings. *Trans. Am. Math. Soc.*, 310(1):293–302, 1988.

[3] Vladimir Kanovei. When a partial Borel order is linearizable. *Fund. Math.*, 155(3):301–309, 1998.

[4] Vladimir Kanovei. *Borel equivalence relations. Structure and classification.* Providence, RI: American Mathematical Society (AMS), 2008.

[5] Vladimir Kanovei, Martin Sabok, and Jindřich Zapletal. *Canonical Ramsey theory on Polish spaces.* Cambridge: Cambridge University Press, 2013.