Linearization of partial quasi-orderings in the Solovay model revisited. *

Vladimir Kanovei† Vassily Lyubetsky‡

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Abstract

We modify arguments in [5] to reprove a linearization theorem on real-ordinal definable partial quasi-orderings in the Solovay model.

1 Introduction

The following theorem is the main content of this note.

Theorem 1.1 (in the Solovay model). *Let* $\preccurlyeq$ *be a* **ROD** *(real-ordinal definable) partial quasi-ordering on* ω^ω *and* $\approx$ *be the associated equivalence relation. Then exactly one of the following two conditions is satisfied*:

(I) *there is an antichain* $A \subseteq 2^{<\omega_1}$ *and a* **ROD** *map* $F : \omega^\omega \to A$ *such that*

1) *if* $a, b \in \omega^\omega$ *then*: $x \preccurlyeq y \Longrightarrow F(x) \leqslant_{\texttt{lex}} F(y)$, *and*

2) *if* $a, b \in \omega^\omega$ *then*: $x \not\approx y \implies F(x) \neq F(y)$;

(II) *there exists a continuous* $1-1$ *map* $F : 2^\omega \to \omega^\omega$ *such that*

3) *if* $a, b \in 2^\omega$ *then*: $a \leq_0 b \Longrightarrow F(a) \preccurlyeq F(b)$, *and*

4) *if* $a, b \in 2^\omega$ *then*: $a \not\mathsf{E}_0 b \implies F(a) \not\preccurlyeq F(b)$.

Here $\leqslant_{\texttt{lex}}$ is the lexicographical order on sets of the form 2^α, $\alpha \in \texttt{Ord}$ — it linearly orders any antichain $A \subseteq 2^{<\omega_1}$, while $\leq_0$ is the partial quasi-ordering on 2^ω defined so that $x \leq_0 y$ iff $x \mathsf{E}_0 y$ and either $x = y$ or $x(k) < y(k)$, where k is the largest number with $x(k) \neq y(k)$. [1]

The proof of this theorem (Theorem 6) in [5, Section 6]) contains a reference to Theorem 5 on page 91 (top), which is in fact not immediately applicable in

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†IITP RAS and MIIT, Moscow, Russia, kanovei@googlemail.com — contact author

‡IITP RAS, Moscow, Russia, lyubetsk@iitp.ru

[1] Clearly $\leq_0$ orders each E_0-class similarly to the (positive and negative) integers, except for the class $[\omega \times \{0\}]_{\mathsf{E}_0}$ ordered as ω and the class $[\omega \times \{1\}]_{\mathsf{E}_0}$ ordered the inverse of ω.

the Solovay model. The goal of this note is to present a direct and self-contained proof of Theorem 1.1.

The combinatorial side of the proof follows the proof of a theorem on Borel linearization in [4], in turn based on earlier results in [2, 1]. This will lead us to (I) in a weaker form, with a function F mapping ω^ω into 2^{ω_2}. To reduce this to an antichain in $2^{<\omega_1}$, a compression lemma (Lemma 5.1 below) is applied, which has no counterpart in the Borel case.

Our general notation follows [6, 8], but for the convenience of the reader, we add a review of notation.

PQO, *partial quasi-order*: reflexive ($x \le x$) and transitive in the domain;

LQO, *linear quasi-order*: PQO and $x \le y \vee y \le x$ in the domain;

LO, *linear order*: LQO and $x \le y \wedge y \le x \Longrightarrow x = y$;

associated equivalence relation: $x \approx y$ iff $x \le y \wedge y \le x$.

associated strict ordering: $x < y$ iff $x \le y \wedge y \not\le x$;

LR (left–right) order preserving map: any map $f : \langle X\,; \le\rangle \to \langle X'\,; \le'\rangle$ such that we have $x \le y \Longrightarrow f(x) \le' f(y)$ for all $x, y \in \mathtt{dom}\, f$;

$<_{\mathtt{lex}}$, $\leqslant_{\mathtt{lex}}$: the lexicographical LOs on sets of the form 2^α, $\alpha \in \mathtt{Ord}$, resp. strict and non-strict;

$[x]_{\mathsf{E}} = \{y \in \mathtt{dom}\,\mathsf{E} : x \;\mathsf{E}\; y\}$ (the E-*class* of x) and $[X]_{\mathsf{E}} = \bigcup_{x\in X}[x]_{\mathsf{E}}$ — whenever E is an equivalence relation and $x \in \mathtt{dom}\,\mathsf{E}$, $X \subseteq \mathtt{dom}\,\mathsf{E}$.

Remark 1.2. We shall consider only the case of a parameterfree OD ordering $\preccurlyeq$ in Theorem 1.1; the case of OD(p) with a fixed real parameter p does not differ much.

2 The Solovay model and OD forcing

We start with a brief review of the Solovay model. Let Ω be an ordinal. Let Ω-SM be the following hypothesis:

Ω-SM: $\Omega = \omega_1$, Ω is strongly inaccessible in $\mathbf{L}$, the constructible universe, and the whole universe $\mathbf{V}$ is a generic extension of $\mathbf{L}$ via the Levy collapse forcing $\mathbf{Coll}(\omega, {<}\Omega)$, as in [9].

Assuming Ω-SM, let $\mathbf{P}$ be the set of all **non-empty** OD sets $Y \subseteq \omega^\omega$. We consider $\mathbf{P}$ as a forcing notion (smaller sets are stronger). A set $D \subseteq \mathbf{P}$ is:

- *dense*, iff for every $Y \in \mathbf{P}$ there exists $Z \in D$, $Z \subseteq Y$;
- *open dense*, iff in addition we have $Y \in D \Longrightarrow X \in D$ whenever sets $Y \subseteq X$ belong to $\mathbf{P}$;

A set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic, iff 1) if $X, Y \in G$ then there is a set $Z \in G$, $Z \subseteq X \cap Y$, and 2) if $D \subseteq \mathbf{P}$ is OD and dense then $G \cap D \neq \varnothing$.

Given an OD equivalence relation E on ω^ω, a *reduced product* forcing notion $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ consists of all sets of the form $X \times Y$, where $X, Y \in \mathbf{P}$ and $[X]_{\mathsf{E}} \cap [Y]_{\mathsf{E}} \neq \varnothing$. For instance $X \times X$ belongs to $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ whenever $X \in \mathbf{P}$. The notions of sets dense and open dense in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$, and $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic sets are similar to the case of $\mathbf{P}$

A condition $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ is *saturated* iff $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$.

Lemma 2.1. *If $X \times Y$ is a condition in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ then there is a stronger saturated subcondition $X' \times Y'$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$.*

Proof. Let $X' = X \cap [Y]_{\mathsf{E}}$ and $Y' = Y \cap [X]_{\mathsf{E}}$. □

Proposition 2.2 (lemmas 14, 16 in [3])**.** *Assume Ω-SM.*

If a set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic then the intersection $\bigcap G = \{x[G]\}$ consists of a single real $x[G]$, called $\mathbf{P}$-generic — its name will be $\dot{\boldsymbol{x}}$.

Given an OD equivalence relation E on ω^ω, if a set $G \subseteq \mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ is $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic then the intersection $\bigcap G = \{\langle x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]\rangle\}$ consists of a single pair of reals $x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]$, called an $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic pair — their names will be $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$; either of $x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]$ is separately $\mathbf{P}$-generic. □

As the set $\mathbf{P}$ is definitely uncountable, the existence of $\mathbf{P}$-generic sets does not immediately follow from Ω-SM by a cardinality argument. Yet fortunately $\mathbf{P}$ is *locally countable*, in a sense.

Definition 2.3 (assuming Ω-SM)**.** A set $X \in \mathrm{OD}$ is *OD-1st-countable* if the set $\mathscr{P}_{\mathtt{OD}}(X) = \mathscr{P}(X) \cap \mathrm{OD}$ of all OD subsets of X is at most countable.

For instance, assuming Ω-SM, the set $X = \omega^\omega \cap \mathrm{OD} = \omega^\omega \cap \mathbf{L}$ of all OD reals is OD-1st-countable. Indeed $\mathscr{P}_{\mathtt{OD}}(X) = \mathscr{P}(X) \cap \mathbf{L}$, and hence $\mathscr{P}_{\mathtt{OD}}(X)$ admits an OD bijection onto the ordinal $\omega_2^{\mathbf{L}} < \omega_1 = \Omega$.

Lemma 2.4 (assuming Ω-SM)**.** *If a set $X \in$ OD is OD-1st-countable then the set $\mathscr{P}_{\mathtt{OD}}(X)$ is OD-1st-countable either.*

Proof. There is an ordinal $\lambda < \omega_1 = \Omega$ and an OD bijection $b : \lambda \xrightarrow{\text{onto}} \mathscr{P}_{\mathtt{OD}}(X)$. Any OD set $Y \subseteq \lambda$ belongs to $\mathbf{L}$, hence, the OD power set $\mathscr{P}_{\mathtt{OD}}(\lambda) = \mathscr{P}(\lambda) \cap \mathbf{L}$ belongs to $\mathbf{L}$ and $\mathtt{card}(\mathscr{P}_{\mathtt{OD}}(\lambda)) \leq \lambda^+ < \Omega$ in $\mathbf{L}$. We conclude that $\mathscr{P}_{\mathtt{OD}}(\lambda)$ is countable. It follows that $\mathscr{P}_{\mathtt{OD}}(\mathscr{P}_{\mathtt{OD}}(X))$ is countable, as required. □

Lemma 2.5 (assuming Ω-SM)**.** *If $\lambda < \Omega$ then the set $\mathrm{C{\small OH}}_\lambda$ of all elements $f \in \lambda^\omega$, $\mathbf{Coll}(\omega, \lambda)$-generic over $\mathbf{L}$, is OD-1st-countable.*

Proof. If $Y \subseteq \mathrm{C{\small OH}}_\lambda$ is OD and $x \in Y$ then "$\check{x} \in \check{Y}$" is $\mathbf{Coll}(\omega, \lambda)$-forced over $\mathbf{L}$. It follows that there is a set $S \subseteq \lambda^{<\omega} = \mathbf{Coll}(\omega, \lambda)$, $S \in \mathbf{L}$, such that

$Y = \textsc{Coh}_\lambda \cap \bigcup_{t\in S} \mathscr{N}_t$, where $\mathscr{N}_t = \{x \in \lambda^{<\omega} : t \subset x\}$, a Baire interval in $\lambda^{<\omega}$. But the collection of all such sets S belongs to $\mathbf{L}$ and has cardinality λ^+ in $\mathbf{L}$, hence, is countable under Ω-SM. □

Let $\mathbf{P}^*$ be the set of all OD-1st-countable sets $X \in \mathbf{P}$. We also define

$$\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^* = \{X \times Y \in \mathbf{P} \times_{\mathsf{E}} \mathbf{P} : X, Y \in \mathbf{P}^*\}.$$

Lemma 2.6 (assuming Ω-SM)**.** *The set $\mathbf{P}^*$ is dense in $\mathbf{P}$, that is, if $X \in \mathbf{P}$ then there is a condition $Y \in \mathbf{P}^*$ such that $Y \subseteq X$.*

If E is an OD equivalence relation on ω^ω then the set $\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$ is dense in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ and any $X \times Y$ in $\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$ is OD-1st-countable.

Proof. Let $X \in \mathbf{P}$. Then $X \neq \varnothing$, hence, there is a real $x \in X$. It follows from Ω-SM that there is an ordinal $\lambda < \omega_1 = \Omega$, an element $f \in \textsc{Coh}_\lambda$, and an OD map $H : \lambda^\omega \to \omega^\omega$, such that $x = H(f)$. The set $P = \{f' \in \textsc{Coh}_\lambda : H(f') \in X\}$ is then OD and non-empty (contains f), and hence so is its image $Y = \{H(f') : f' \in P\} \subseteq X$ (contains x). Finally, $Y \in \mathbf{P}^*$ by Lemma 2.5.

To prove the second claim, let $X \times Y$ be a condition in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$. By Lemma 2.1 there is a stronger saturated subcondition $X' \times Y' \subseteq X \times Y$. By the first part of the lemma, let $X'' \subseteq X'$ be a condition in $\mathbf{P}^*$, and $Y'' = Y' \cap [X'']_{\mathsf{E}}$. Similarly, let $Y''' \subseteq Y''$ be a condition in $\mathbf{P}^*$, and $X''' = X'' \cap [Y''']_{\mathsf{E}}$. Then $X''' \times Y'''$ belongs to $\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$. □

Corollary 2.7 (assuming Ω-SM)**.** *If $X \in \mathbf{P}$ then there exists a $\mathbf{P}$-generic set $G \subseteq \mathbf{P}$ containing X. If $X \times Y$ is a condition in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ then there exists a $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic set $G \subseteq \mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ containing $X \times Y$.*

Proof. By Lemma 2.6, assume that $X \in \mathbf{P}^*$. Then the set $\mathbf{P}_{\subseteq X}$ of stronger conditions contains only countably many OD subsets by Lemma 2.4. □

3 The OD forcing relation

The forcing notion $\mathbf{P}$ will play the same role below as the Gandy – Harrington forcing in [2, 7]. There is a notable technical difference: under Ω-SM, OD-generic sets exist in the ground Solovay-model universe by Corollary 2.7. Another notable difference is connected with the forcing relation.

Definition 3.1 (assuming Ω-SM)**.** Let $\varphi(x)$ be an `Ord`-*formula*, that is, a formula with ordinals as parameters.

A condition $X \in \mathbf{P}$ is said to $\mathbf{P}$-*force* $\varphi(\dot{\boldsymbol{x}})$ iff $\varphi(x)$ is true (in the Solovay-model set universe considered) for any $\mathbf{P}$-generic real x.

If E is an OD equivalence relation on ω^ω then a condition $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ is said to $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-*force* $\varphi(\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}})$ iff $\varphi(x, y)$ is true for any $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic pair $\langle x, y\rangle$. □

Lemma 3.2 (assuming Ω-SM)**.** *Given an* $\mathtt{Ord}$*-formula* $\varphi(x)$ *and a* $\mathbf{P}$*-generic real* x*, if* $\varphi(x)$ *is true* (in the Solovay-model set universe considered) *then there is a condition* $X \in \mathbf{P}$ *containing* x*, which* $\mathbf{P}$*-forces* $\varphi(\dot{\boldsymbol{x}})$*.*

Let E *be an OD equivalence relation on* ω^ω*. Given an* $\mathtt{Ord}$*-formula* $\varphi(x,y)$ *and a* $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$*-generic pair* $\langle x,y\rangle$*, if* $\varphi(x,y)$ *is true then there is a condition in* $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ *containing* $\langle x,y\rangle$*, which* $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$*-forces* $\varphi(\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}})$*.*

Proof. To prove the first claim, put $X = \{x' \in \omega^\omega : \varphi(x')\}$. But this argument does not work for $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$. To fix the problem, we propose a longer argument which equally works in both cases — but we present it in the case of $\mathbf{P}$ which is slightly simpler.

Formally the forcing notion $\mathbf{P}$ does not belong to $\mathbf{L}$. But it is order-isomorphic to a certain forcing notion $P \in \mathbf{L}$, namely, the set P of *codes* [2] of OD sets in $\mathbf{P}$. The order between the codes in P, which reflects the relation $\subseteq$ between the OD sets themselves, is expressible in $\mathbf{L}$, too. Furthermore dense OD sets in $\mathbf{P}$ correspond to dense sets in the coded forcing P in $\mathbf{L}$.

Now, let x be $\mathbf{P}$-generic and $\varphi(x)$ be true. It is a known property of the Solovay model that there is another $\mathtt{Ord}$-formula $\psi(x)$ such that $\varphi(x)$ iff $\mathbf{L}[x] \models \psi(x)$. Let $g \subseteq P$ be the set of all codes of conditions $X \in \mathbf{P}$ such that $x \in X$. Then g is a P-generic set over $\mathbf{L}$ by the choice of x, and x is the corresponding generic object. Therefore there is a condition $p \in g$ which P-forces $\psi(\dot{\boldsymbol{x}})$ over $\mathbf{L}$. Let $X \in \mathbf{P}$ be the OD set coded by p, so that $x \in X$. To prove that X OD-forces $\varphi(\dot{\boldsymbol{x}})$, let $x' \in X$ be a $\mathbf{P}$-generic real. Let $g' \subseteq P$ be the P-generic set of all codes of conditions $Y \in \mathbf{P}$ such that $x' \in Y$. Then $p \in g'$, hence $\psi(x')$ holds in $\mathbf{L}[x']$, by the choice of p. Then $\varphi(x')$ holds (in the Solovay-model set universe) by the choice of ψ, as required. □

Corollary 3.3 (assuming Ω-SM)**.** *Given an* $\mathtt{Ord}$*-formula* $\varphi(x)$*, if* $X \in \mathbf{P}$ *does not* $\mathbf{P}$*-force* $\varphi(\dot{\boldsymbol{x}})$ *then there is a condition* $Y \in \mathbf{P}$, $Y \subseteq X$*, which* $\mathbf{P}$*-forces* $\neg\,\varphi(\dot{\boldsymbol{x}})$*. The same for* $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$*.* □

4 Some similar and derived forcing notions

Some forcing notions similar to $\mathbf{P}$ and $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ will be considered:

1°. $\mathbf{P}_{\subseteq W} = \{Q \subseteq W : \varnothing \neq Q \in \mathrm{OD}\}$, where $W \subseteq \omega^\omega$ or $W \subseteq \omega^\omega \times \omega^\omega$ is an OD set. Especially, in the case when $W \subseteq \mathsf{E}$, where E is an OD equivalence relation on ω^ω (that is, $\langle x,y\rangle \in W \Longrightarrow x \mathrel{\mathsf{E}} y$) — note that $[\mathtt{dom}\, W]_{\mathsf{E}} = [\mathtt{ran}\, W]_{\mathsf{E}}$ in this case.

2°. $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})_{\subseteq X \times Y} = \{X' \times Y' \in \mathbf{P} \times_{\mathsf{E}} \mathbf{P} : X' \subseteq X \wedge Y' \subseteq Y\}$, where E is an OD equivalence relation on ω^ω and $X \times Y \in \mathbf{P} \times_{\mathsf{E}} \mathbf{P}$.

[2] A code of an OD set X is a finite sequence of logical symbols and ordinals which correspond to a definition in the form $X = \{x \in \mathbf{V}_\alpha : \mathbf{V}_\alpha \models \varphi(x)\}$.

3°. $\mathbf{P}_{\subseteq W} \times_{\mathsf{E}} \mathbf{P}_{\subseteq X} = \{P \times Y : P \in \mathbf{P}_{\subseteq W} \wedge Y \in \mathbf{P}_{\subseteq X} \wedge [Y]_{\mathsf{E}} \cap [\mathtt{dom}\, P]_{\mathsf{E}} \neq \varnothing\}$, where E is an OD equivalence relation on ω^ω, $W \subseteq \mathsf{E}$ is OD, $X \in \mathbf{P}$, and $[X]_{\mathsf{E}} \cap [\mathtt{dom}\, W]_{\mathsf{E}} \neq \varnothing$ (equivalently, $[X]_{\mathsf{E}} \cap [\mathtt{ran}\, W]_{\mathsf{E}} \neq \varnothing$).

4°. $\mathbf{P}_{\subseteq W} \times_{\mathsf{E}} \mathbf{P}_{\subseteq W} = \{P \times Q : P, Q \in \mathbf{P}_{\subseteq W} \wedge [\mathtt{dom}\, P]_{\mathsf{E}} \cap [\mathtt{dom}\, Q]_{\mathsf{E}} \neq \varnothing\}$, where E is an OD equivalence relation on ω^ω and $W \subseteq \mathsf{E}$ is OD.

They have the same basic properties as $\mathbf{P}$ — the forcing notions of the form 1°, or as $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ — 2°, 3°, 4°. This includes such results and concepts as 2.2, 2.6, 2.7, the associated forcing relation as in 3.1, and 3.2, 3.3, with suitable and rather transparent corrections, of course.

5 Compression lemma

A set $A \subseteq 2^{<\Omega}$ is an antichain if its elements are pairwise $\subset$-incomparable, that is, no sequence in A properly extends another sequence in A. Clearly any antichain is linearly ordered by $\leqslant_{\mathtt{lex}}$.

Let $\Theta = \Omega^+$; the cardinal successor of Ω in both $\mathbf{L}$, the ground model, and its $\mathbf{Coll}(\omega, {<}\Omega)$-generic extension postulated by Ω-SM to be the set universe; in the latter, $\Omega = \omega_1$ and $\Theta = \omega_2$.

Lemma 5.1 (compression lemma)**.** *Assume that $\Omega \le \vartheta \le \Theta$ and $X \subseteq 2^\Theta$ is the image of ω^ω via an OD map. Then there is an OD antichain $A(X) \subseteq 2^{<\Omega}$ and an OD isomorphism $f : \langle X\,; \leqslant_{\mathtt{lex}}\rangle \xrightarrow{\text{onto}} \langle A(X)\,; \leqslant_{\mathtt{lex}}\rangle$.*

Proof. If $\vartheta = \Theta$ then, as $\mathtt{card}\, X \le \mathtt{card}\, \omega^\omega = \Omega$, there is an ordinal $\vartheta < \Theta$ such that $x \restriction \vartheta \neq y \restriction \vartheta$ whenever $x \neq y$ belong to X — this reduces the case $\vartheta = \Theta$ to the case $\Omega \le \vartheta < \Theta$. We prove the latter by induction on ϑ.

The nontrivial step is the step $\mathtt{cof}\, \lambda = \Omega$, so that let $\vartheta = \bigcup_{\alpha<\Omega} \vartheta_\alpha$, for an increasing OD sequence of ordinals ϑ_α. Let $I_\alpha = [\vartheta_\alpha, \vartheta_{\alpha+1})$. Then, by the induction hypothesis, for any $\alpha < \Omega$ the set $X_\alpha = \{S \restriction I_\alpha : S \in X\} \subseteq 2^{I_\alpha}$ is $<_{\text{lex}}$-order-isomorphic to an antichain $A_\alpha \subseteq 2^{<\Omega}$ via an OD isomorphism i_α, and the map, which sends α to A_α and i_α, is OD. It follows that the map, which sends each $S \in X$ to the concatenation of all sequences $i_\alpha(x \restriction I_\alpha)$, is an OD $<_{\text{lex}}$-order-isomorphism X onto an antichain in 2^Ω. Therefore, in fact it suffices to prove the lemma in the case $\vartheta = \Omega$. Thus let $X \subseteq 2^\Omega$.

First of all, note that each sequence $S \in X$ is ROD. Lemma 7 in [3] shows that, in this case, we have $S \in \mathbf{L}[S \restriction \eta]$ for an ordinal $\eta < \Omega$. Let $\eta(S)$ be the least such an ordinal, and $h(S) = S \restriction \eta(S)$, so that $h(S)$ is a countable initial segment of S and $S \in \mathbf{L}[h(S)]$. Note that h is still OD.

Consider the set $U = \mathtt{ran}\, h = \{h(S) : S \in X\} \subseteq 2^{<\Omega}$. We can assume that every sequence $u \in U$ has a limit length. Then $U = \bigcup_{\gamma<\Omega} U_\gamma$, where $U_\gamma = U \cap 2^{\omega\gamma}$ ($\omega\gamma$ is the the γ-th limit ordinal). For $u \in U_\gamma$, let $\gamma_u = \gamma$.

If $u \in U$ then by construction the set $X_u = \{S \in X : h(S) = u\}$ is $\mathrm{OD}(u)$ and satisfies $X_u \subseteq \mathbf{L}[u]$. Therefore, it follows from the known properties of the

Solovay model that X_u belongs to $\mathbf{L}[u]$ and is of cardinality $\le \Omega$ in $\mathbf{L}[u]$. Fix an enumeration $X_u = \{S_u(\alpha) : \gamma_u \le \alpha < \Omega\}$ for all $u \in U$. We can assume that the map $\alpha, u \longmapsto S_u(\alpha)$ is OD.

If $u \in U$ and $\gamma_u \le \alpha < \Omega$, then we define a shorter sequence, $s_u(\alpha) \in 3^{\omega\alpha+1}$, as follows.

(i) $s_u(\alpha)(\xi+1) = S_u(\alpha)(\xi)$ for any $\xi < \omega\alpha$.

(ii) $s_u(\alpha)(\omega\alpha) = 1$.

(iii) Let $\delta < \alpha$. If $S_u(\alpha) \restriction \omega\delta = S_v(\delta) \restriction \omega\delta$ for some $v \in U$ (equal to or different from u) then $s_u(\alpha)(\omega\delta) = 0$ whenever $S_u(\alpha) <_{\mathtt{lex}} S_v(\delta)$, and $s_u(\alpha)(\omega\delta) = 2$ whenever $S_v(\delta) <_{\rm lex} S_u(\alpha)$.

(iv) Otherwise (*i.e.*, if there is no such v), $s_u(\alpha)(\omega\delta) = 1$.

To demonstrate that (iii) is consistent, we show that $S_{u'}(\delta) \restriction \omega\delta = S_{u''}(\delta) \restriction \omega\delta$ implies $u' = u''$. Indeed, as by definition $u' \subset S_{u'}(\delta)$ and $u'' \subset S_{u''}(\delta)$, u' and u'' must be $\subseteq$-compatible: let, say, $u' \subseteq u''$. Now, by definition, $S_{u''}(\delta) \in \mathbf{L}[u'']$, therefore $\in \mathbf{L}[S_{u'}(\delta)]$ because $u'' \subseteq S_{u''}(\delta) \restriction \omega\delta = S_{u'}(\delta) \restriction \omega\delta$, finally $\in \mathbf{L}[u']$, which shows that $u' = u''$ as $S_{u''}(\delta) \in X_{u''}$.

We are going to prove that the map $S_u(\alpha) \longmapsto s_u(\alpha)$ is a $<_{\mathtt{lex}}$-order isomorphism, so that $S_v(\beta) <_{\mathtt{lex}} S_u(\alpha)$ implies $s_v(\beta) <_{\mathtt{lex}} s_u(\alpha)$.

We first observe that $s_v(\beta)$ and $s_u(\alpha)$ are $\subseteq$-incomparable. Indeed assume that $\beta < \alpha$. If $S_u(\alpha) \restriction \omega\beta \ne S_v(\beta) \restriction \omega\beta$ then clearly $s_v(\beta) \not\subseteq s_u(\alpha)$ by (i). If $S_u(\alpha) \restriction \omega\beta = S_v(\beta) \restriction \omega\beta$ then $s_u(\alpha)(\omega\beta) = 0$ or 2 by (iii) while $s_v(\beta)(\omega\beta) = 1$ by (ii). Thus all $s_u(\alpha)$ are mutually $\subseteq$-incomparable, so that it suffices to show that conversely $s_v(\beta) <_{\rm lex} s_u(\alpha)$ implies $S_v(\beta) <_{\rm lex} S_u(\alpha)$. Let ζ be the least ordinal such that $s_v(\beta)(\zeta) < s_u(\alpha)(\zeta)$; then $s_u(\alpha) \restriction \zeta = s_v(\beta) \restriction \zeta$ and $\zeta \le \mathtt{min}\{\omega\alpha, \omega\beta\}$.

The case when $\zeta = \xi + 1$ is clear: then by definition $S_u(\alpha) \restriction \xi = S_v(\beta) \restriction \xi$ while $S_v(\beta)(\xi) < S_u(\alpha)(\xi)$, so let us suppose that $\zeta = \omega\delta$, where $\delta \le \mathtt{min}\{\alpha, \beta\}$. Then obviously $S_u(\alpha) \restriction \omega\delta = S_v(\beta) \restriction \omega\delta$. Assume that one of the ordinals α, β is equal to δ, say, $\beta = \delta$. Then $s_v(\beta)(\omega\delta) = 1$ while $s_u(\alpha)(\omega\delta)$ is computed by (iii). Now, as $s_v(\beta)(\omega\delta) < s_u(\alpha)(\omega\delta)$, we conclude that $s_u(\alpha)(\omega\delta) = 2$, hence $S_v(\beta) <_{\rm lex} S_u(\alpha)$, as required. Assume now that $\delta < \mathtt{min}\{\alpha, \beta\}$. Then easily α and β appear in one and the same class (iii) or (iv) with respect to the δ. However this cannot be (iv) because $s_v(\beta)(\omega\delta) \ne s_u(\alpha)(\omega\delta)$. Hence we are in (iii), so that, for some (unique) $w \in U$. $0 = S_v(\beta) <_{\rm lex} S_w(\delta) <_{\mathtt{lex}} S_u(\alpha) = 2$, as required.

This ends the proof of the lemma, except for the fact that the sequences $s_u(\alpha)$ belong to $3^{<\Omega}$, but improvement to $2^{<\Omega}$ is easy. □

6 The dichotomy

Here we begin the proof of Theorem 1.1. **We assume Ω-SM in the course of the proof**. And we assume that the ordering $\preccurlyeq$ of the theorem is just OD — then so is the associated equivalence relation $\approx$ and strict order $\prec$.

Let $\mathscr{F}$ be the set of all OD LR order preserving maps $F : \langle \omega^\omega ; \preccurlyeq \rangle \to \langle A ; \leqslant_{\texttt{lex}} \rangle$, where $A \subseteq 2^{<\Omega}$ is an OD antichain. Let

$$x \mathrel{\mathsf{E}} y \ \text{ iff } \ \forall\, F \in \mathscr{F}\ (F(x) = F(y))$$

for $x, y \in \omega^\omega$. Then E is an OD equivalence relation, OD-smooth in the sense that it admits an obvious OD reduction to the equality on the set $2^{\mathscr{F}}$.

Lemma 6.1. *If $R(x,y)$ is an OD relation and $\forall\, x, y\, (x \mathrel{\mathsf{E}} y \Longrightarrow R(x,y))$ then there is a function $F \in \mathscr{F}$ such that $\forall\, x, y\, (F(x) = F(y) \Longrightarrow R(x,y))$.*

Proof. Clearly $\texttt{card}\, \mathscr{F} = \Theta = \Omega^+$ and $\mathscr{F}$ admits an OD enumeration $\mathscr{F} = \{F_\xi : \xi < \Theta\}$. If $x \in \omega^\omega$ then let $f(x) = F_0(x)^\wedge F_1(x)^\wedge \dots {}^\wedge F_\xi(x)^\wedge\ \dots$ — the concatenation of all sequences $F_\xi(x)$. Then $f : \langle \omega^\omega ; \preccurlyeq \rangle \to \langle X ; \leqslant_{\texttt{lex}} \rangle$ is an OD LR order preserving map, where $X = \texttt{ran}\, f = \{f(r) : r \in \omega^\omega\} \subseteq 2^\Theta$, and $f(x) = f(y) \Longrightarrow R(x,y)$ by the construction. By Lemma 5.1 there is an OD isomorphism $g : \langle X ; \leqslant_{\texttt{lex}} \rangle \xrightarrow{\text{onto}} \langle A ; \leqslant_{\texttt{lex}} \rangle$ onto an antichain $A \subseteq 2^{<\Omega}$. The superposition $F(x) = g(f(x))$ proves the lemma. □

Lemma 6.2. *Let OD sets $\varnothing \neq X, Y \subseteq \omega^\omega$ satisfy $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$. Then the set $B = \{\langle x, y \rangle \in X \times Y : x \mathrel{\mathsf{E}} y \wedge x \preccurlyeq y\}$ is non-empty, $\texttt{dom}\, B = X$, $\texttt{ran}\, B = Y$.*

Proof. It suffices to establish $B \neq \varnothing$. The OD set

$$X' = \{x' \in \omega^\omega : \exists\, x \in X\, (x' \mathrel{\mathsf{E}} x \wedge x' \preccurlyeq x)\}$$

is downwards $\preccurlyeq$-closed in each E-class, and if $B = \varnothing$ then $X' \cap Y = \varnothing$. By Lemma 6.1, there is a function $F \in \mathscr{F}$ such that $x \in X' \Longrightarrow x' \in X'$ holds whenever $F(x) = F(x')$ and $x' \preccurlyeq x$. It follows that the derived function

$$G(x) = \begin{cases} F(x)^\wedge 0\,, & \text{whenewer} \quad x \in X' \\ F(x)^\wedge 1\,, & \text{whenewer} \quad x \in \omega^\omega \smallsetminus X' \end{cases}$$

belongs to $\mathscr{F}$. Thus if $x \in X \subseteq X'$ and $y \in Y \subseteq \omega^\omega \smallsetminus X'$ then $G(x) \neq G(y)$ and hence $x \mathrel{\not\mathsf{E}} y$. In other words, $[X]_{\mathsf{E}} \cap [Y]_{\mathsf{E}} = \varnothing$, a contradiction. □

We'll make use of the OD-forcing notions $\mathbf{P}$ and $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$.

Lemma 6.3. *Condition $\omega^\omega \times \omega^\omega$ $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\texttt{le}} \mathrel{\mathsf{E}} \dot{\boldsymbol{x}}_{\texttt{ri}}$.*

Proof. Otherwise, by Lemma 3.2, there is a function $F \in \mathscr{F}$ and a condition $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ which $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-forces $F(\dot{\boldsymbol{x}}_{\texttt{le}})(\xi) = 0 \neq 1 = F(\dot{\boldsymbol{x}}_{\texttt{ri}})(\xi)$ for a certain ordinal $\xi < \Omega$. We may assume that $X \times Y$ is a saturated condition. Then easily $F(x)(\xi) = 0 \neq 1 = F(y)(\xi)$ holds for any pair $\langle x, y \rangle \in X \times Y$, so that we have $F(x) \neq F(y)$ and $x \mathrel{\not\mathsf{E}} y$ whenever $\langle x, y \rangle \in X \times Y$, which contradicts the choice of $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$. □

Case 1: $\approx$ and E coincide on ω^ω, so that $x \mathsf{E} y \iff x \approx y$ for $x, y \in \omega^\omega$. By Lemma 6.1 there is a single function $F \in \mathscr{F}$ such that $F(x) = F(y)$ implies $x \approx y$ for all $x, y \in U^*$, as required for (I) of Theorem 1.1.

Case 2: $\approx$ is a *proper* subrelation of E, hence, the OD set

$$U_0 = \{x \in \omega^\omega : \exists y \in \omega^\omega \, (x \not\approx y \wedge x \mathsf{E} y)\}$$

(the domain of singularity) is non-empty. It follows that $U_0 \in \mathbf{P}$ and $U_0 \times U_0$ is a condition in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$. We'll work towards (II) of Theorem 1.1.

7 The domain of singularity

Since the set U_0 belongs to $\mathbf{P}$, there is a set $U^* \in \mathbf{P}^*$, $U^* \subseteq U_0$. Then obviously $U^* \times U^*$ belongs to $\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$.

Lemma 7.1. *Condition $U^* \times U^*$ $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-forces that the reals $\dot{\boldsymbol{x}}_{\mathtt{le}}$ and $\dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-incomparable.*

Proof. Suppose to the contrary that, by Corollary 3.3, a subcondition $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ either $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ or $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. We will get a contradiction in both cases. Note that $X, Y \subseteq U^*$ are non-empty OD sets and $[X]_{\mathsf{E}} \cap [Y]_{\mathsf{E}} \neq \varnothing$.

Claim 7.2. *The set $W = \{\langle x, x' \rangle \in X \times X : x \mathsf{E} x' \wedge x' \not\approx x\}$ is non-empty.*

Proof. Suppose to the contrary that $W = \varnothing$, so E coincides with $\approx$ on X. As $X \subseteq U^*$, at least one of the OD sets

$$Z = \{z : \exists x \in X \, (z \mathsf{E} x \wedge z \not\preccurlyeq x)\}, \; Z' = \{z : \exists x \in X \, (z \mathsf{E} x \wedge x \not\preccurlyeq z)\}$$

is non-empty; assume that, say, $Z \neq \varnothing$. Consider the OD set

$$U = \{z : \exists x \in X \, (z \mathsf{E} x \wedge z \preccurlyeq x)\}.$$

Then $X \subseteq U$ and $U \cap Z = \varnothing$, U is downwards $\preccurlyeq$-closed while Z is upwards $\preccurlyeq$-closed in each E-class, therefore $y \not\preccurlyeq x$ whenever $x \in U \wedge y \in Z \wedge x \mathsf{E} y$, and hence we have $[U]_{\mathsf{E}} \cap [Z]_{\mathsf{E}} = \varnothing$ be Lemma 6.2. Yet by definition $[X]_{\mathsf{E}} \cap [Z]_{\mathsf{E}} \neq \varnothing$ and $X \subseteq U$, which is a contradiction. □ (Claim)

Suppose that condition $X \times Y$ $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$. As $W \neq \varnothing$ by Claim 7.2, the forcing $\mathbf{P}_{\subseteq W}$ of all non-empty OD sets $P \subseteq W$ adds pairs $\langle x, x' \rangle \in W$ of $\mathbf{P}$-generic (separately) reals $x, x' \in X$ which satisfy $x' \mathsf{E} x$ and $x' \not\approx x$. If $P \in \mathbf{P}_{\subseteq W}$ then obviously $[\mathtt{dom}\, P]_{\mathsf{E}} = [\mathtt{ran}\, P]_{\mathsf{E}}$. Consider a more complex forcing $\mathscr{P} = \mathbf{P}_{\subseteq W} \times_{\mathsf{E}} \mathbf{P}$ of all pairs $P \times Y'$, where $P \in \mathbf{P}_{\subseteq W}$, $Y' \in \mathbf{P}$, $Y' \subseteq Y$, and $[\mathtt{dom}\, P]_{\mathsf{E}} \cap [Y']_{\mathsf{E}} \neq \varnothing$. For instance, $W \times Y \in \mathbf{P}_{\subseteq W} \times_{\mathsf{E}} \mathbf{P}$. Then $\mathscr{P}$ adds a pair $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}} \rangle \in W$ and another real $\dot{\boldsymbol{x}} \in Y$ such that both pairs

$\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}} \rangle$ and $\langle \dot{\boldsymbol{x}}_{\mathtt{ri}}, \dot{\boldsymbol{x}} \rangle$ belong to $X \times Y$ and are $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic, hence, we have $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ by the choice of $X \times Y$. On the other hand, $\dot{\boldsymbol{x}}_{\mathtt{le}} \not\approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ since the pair belongs to W, which is a contradiction.

Now suppose that condition $X \times Y$ $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. The set

$$B = \{\langle x, y \rangle \in X \times Y : y \mathrel{\mathsf{E}} x \wedge y \preccurlyeq x\}$$

is non-empty by Lemma 6.2. Consider the forcing $\mathbf{P}_{\subseteq B}$ of all non-empty OD sets $P \subseteq B$; if $P \in \mathbf{P}_{\subseteq B}$ then obviously $[\mathtt{dom}\, P]_{\mathsf{E}} = [\mathtt{ran}\, P]_{\mathsf{E}}$. Consider a more complex forcing $\mathbf{P}_{\subseteq B} \times_{\mathsf{E}} \mathbf{P}_{\subseteq B}$ of all products $P \times Q$, where $P, Q \in \mathbf{P}_{\subseteq B}$ and $[\mathtt{dom}\, P]_{\mathsf{E}} \cap [\mathtt{dom}\, Q]_{\mathsf{E}} \neq \varnothing$. In particular $B \times B \in \mathbf{P}_{\subseteq B} \times_{\mathsf{E}} \mathbf{P}_{\subseteq B}$.

Let $\langle x, y; x', y' \rangle$ be a $\mathbf{P}_{\subseteq B} \times_{\mathsf{E}} \mathbf{P}_{\subseteq B}$-generic quadruple in $B \times B$, so that both $\langle x, y \rangle \in B$ and $\langle x', y' \rangle \in B$ are $\mathbf{P}_{\subseteq B}$-generic pairs in B, and both $y \preccurlyeq x$ and $y' \preccurlyeq x'$ hold by the definition of B. On the other hand, an easy argument shows that both criss-cross pairs $\langle x, y' \rangle \in X \times Y$ and $\langle x', y \rangle \in X \times Y$ are $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$-generic, hence $x \prec y'$ and $x' \prec y$ by the choice of $X \times Y$. Altogether $y \preccurlyeq x \prec y' \preccurlyeq x' \prec y$, which is a contradiction. □

8 The splitting construction

Our aim is to define, in the universe of Ω-SM, a splitting system of sets which leads to a function F satisfying (II) of Theorem 1.1. Let

$$\boldsymbol{B} = \{\langle x, y \rangle \in U^* \times U^* : x \mathrel{\mathsf{E}} y \wedge x \preccurlyeq y\}; \quad \boldsymbol{B} \neq \varnothing \text{ by Lemma 6.2.}$$

The construction will involve three forcing notions: $\mathbf{P}$, $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$, and

$$\mathbf{P}_{\subseteq \boldsymbol{B}}, \text{ the collection of all non-empty OD sets } P \subseteq \boldsymbol{B}.$$

We also consider the dense (by Lemma 2.6) subforcings $\mathbf{P}^* \subseteq \mathbf{P}$, $\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^* \subseteq \mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ (see Section 2), and

$$\mathbf{P}^*_{\subseteq \boldsymbol{B}} = \{Q \in \mathbf{P}_{\subseteq \boldsymbol{B}} : Q \text{ is OD-1st-countable}\} \subseteq \mathbf{P}_{\subseteq \boldsymbol{B}}.$$

Now note the following.

1. As $U^* \in \mathbf{P}^*$, the set $\mathscr{D}$ of all sets open dense in the restricted forcing $\mathbf{P}_{\subseteq U^*}$, is countable by Lemma 2.6; hence we can fix an enumeration $\mathscr{D} = \{D_n : n \in \omega\}$ such that $D_n \subseteq D_m$ whenever $m < n$.
2. As $U^* \times U^* \in \mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$, the set $\mathscr{D}'$ of all sets, open dense in the restricted forcing $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})_{\subseteq U^* \times U^*}$, is countable as above; fix an enumeration $\mathscr{D}' = \{D'_n : n \in \omega\}$ s.t. $D'_n \subseteq D'_m$ for $m < n$.
3. If $Q \in \mathbf{P}^*_{\subseteq \boldsymbol{B}}$ then the set $\mathscr{D}(Q)$ of all sets open dense in the restricted forcing $\mathbf{P}_{\subseteq Q}$, is countable by Lemma 2.6; hence we can fix an enumeration $\mathscr{D}(Q) = \{D_n(Q) : n \in \omega\}$ such that $D_n(Q) \subseteq D_m(Q)$ whenever $m < n$.

The chosen enumerations are not necessarily OD, of course.

A pair $\langle u, v\rangle$ of strings $u, v \in 2^n$ is called *crucial* iff $u = 1^k{}^\wedge 0^\wedge w$ and $v = 0^k{}^\wedge 1^\wedge w$ for some $k < n$ and $w \in 2^{n-k-1}$. Note that each pair of the form $\langle 1^k{}^\wedge 0, 0^k{}^\wedge 1\rangle$ is a minimal crucial pair, and if $\langle u, v\rangle$ is a crucial pair then so is $\langle u^\wedge i, v^\wedge i\rangle$, but not $\langle u^\wedge i, v^\wedge j\rangle$ whenever $i \neq j$. The graph of all crucial pairs in 2^n is actually a chain connecting all members of 2^n.

We are going to define, **in the assumption of Ω-SM**, a system of sets $X_u \in \mathbf{P}^*$, where $u \in 2^{<\omega}$, and sets $Q_{uv} \in \mathbf{P}^*_{\subseteq \boldsymbol{B}}$, $\langle u, v\rangle$ being a crucial pair in some 2^n, satisfying the following conditions:

(1) $X_u \in \mathbf{P}^*$ and $Q_{uv} \in \mathbf{P}^*_{\subseteq \boldsymbol{B}}$;

(2) $X_{u^\wedge i} \subseteq X_u$;

(3) $Q_{u^\wedge i\,,\,v^\wedge i} \subseteq Q_{uv}$;

(4) if $\langle u, v\rangle$ is a crucial pair in 2^n then $\texttt{dom}\, Q_{uv} = X_u$ and $\texttt{ran}\, Q_{uv} = X_v$;

(5) $X_u \in D_n$ whenever $u \in 2^{n+1}$;

(6) if $u, v \in 2^{n+1}$ and $u(n) \neq v(n)$ then $X_u \times X_v \in D'_n$ and $X_u \cap X_v = \varnothing$.

(7) if $\langle u, v\rangle = \langle 1^k{}^\wedge 0^\wedge w, 0^k{}^\wedge 1^\wedge w\rangle$ is a crucial pair in 2^{n+1} and $k < n$ (so that w in not the empty string) then $Q_{uv} \in D_n(Q_{1^k{}^\wedge 0, 0^k{}^\wedge 1})$;

Remark 8.1. It follows from (4) that $[X_u]_{\mathsf{E}} = [X_v]_{\mathsf{E}}$ for all $u, v \in 2^n$, because $Q_{uv} \subseteq \boldsymbol{B} \subseteq \mathsf{E}$ and u, v are connected in 2^n by a chain of crucial pairs. □

Why this implies the existence of a function as in (II) **of Theorem 1.1?**

First of all, if $a \in 2^\omega$ then the sequence of sets $X_{a \restriction n}$ is $\mathbf{P}$-generic by (5), therefore the intersection $\bigcap_{n\in\omega} X_{a\restriction n}$ is a singleton by Proposition 2.2. Let $F(a) \in \omega^\omega$ be its only element.

It does not take much effort to prove that F is continuous and $1-1$.

Consider any $a, b \in 2^\omega$ satisfying $a \not\mathsf{E}_0\, b$. Then $a(n) \neq b(n)$ for infinitely many n, hence the pair $\langle F(a), F(b)\rangle$ is $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$-generic by (7), thus $F(a)$ and $F(b)$ are $\preccurlyeq$-incomparable by Lemma 7.1.

Consider $a, b \in 2^\omega$ satisfying $a <_0 b$. We may assume that a and b are $<_0$-neighbours, *i.e.*, $a = 1^k{}^\wedge 0^\wedge w$ while $b = 0^k{}^\wedge 1^\wedge w$ for some $k \in \omega$ and $w \in 2^\omega$. The sequence of sets $Q_{a\restriction n\,,\,b\restriction n}$, $n > k$, is $\mathbf{P}_{\subseteq \boldsymbol{B}}$-generic by (6), hence it results in a pair of reals satisfying $x \preccurlyeq y$. However $x = F(a)$ and $y = F(b)$ by (4).

9 The construction of a splitting system

Now the goal is to define, **in the assumption of Ω-SM**, a system of sets X_u and Q_{uv} satisfying (1) – (7) above. Suppose that the construction has been completed up to a level n, and expand it to the next level. From now on s, t will denote strings in 2^n while u, v will denote strings in 2^{n+1}.

Step 0. To start with, we set $X_{s^\wedge i} = X_s$ for all $s \in 2^n$ and $i = 0, 1$, and $Q_{s^\wedge i\,,\,t^\wedge i} = Q_{st}$ whenever $i = 0, 1$ and $\langle s, t\rangle$ is a crucial pair in 2^n. For the initial crucial pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$ at this level, let $Q_{1^n{}^\wedge 0\,,\,0^n{}^\wedge 1} = X_{1^n} \times X_{0^n}$. The newly defined sets satisfy (1) – (4) except for the requirement $Q_{uv} \in \mathbf{P}^*_{\subseteq \boldsymbol{B}}$ in (1) for the pair $\langle u, v\rangle = \langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$.

This ends the definition of "initial values" of X_u and Q_{uv} at the $(n+1)$-th level. The plan is to gradually shrink the sets in order to fulfill (5) – (7).

Step 1. We take care of item (5). Consider an arbitrary $u_0 = s_0{}^\wedge i \in 2^{n+1}$. As D_n is dense there is a set $X' \in D_n$, $X' \subseteq X_{u_0}$. The intention is to take X' as the "new" X_{u_0}. But this change has to be propagated through the chain of crucial pairs, in order to preserve (4).

Thus put $X'_{u_0} = X'$. Suppose that $u \in 2^{n+1}$, a set $X'_u \subseteq X_u$ has been defined, and $\langle u, v\rangle$ is a crucial pair, $v \in 2^{n+1}$ being not yet encountered. Define $Q'_{uv} = (X'_u \times \omega^\omega) \cap Q_{uv}$ and $X'_v = \mathtt{ran}\, Q'_{uv}$. Clearly (4) holds for the "new" sets X'_u, X'_v, Q'_{uv}. Similarly if $\langle v, u\rangle$ is a crucial pair, then define $Q'_{vu} = (\omega^\omega \times X'_u) \cap Q_{vu}$ and $X'_v = \mathtt{dom}\, Q'_{uv}$. Note that still $Q'_{1^n{}^\wedge 0\,,\,0^n{}^\wedge 1} = X'_{1^n} \times X'_{0^n}$.

The construction describes how the original change from X_{u_0} to X'_{u_0} spreads through the chain of crucial pairs in 2^{n+1}, resulting in a system of new sets, X'_u and Q'_{uv}, which satisfy (5) for the particular $u_0 \in 2^{n+1}$. We iterate this construction consecutively for all $u_0 \in 2^{n+1}$, getting finally a system of sets satisfying (5) (fully) and (4), which we denote by X_u and Q_{uv} from now on.

Step 2. We take care of item (6). Consider a pair of u_0 and v_0 in 2^{n+1}, such that $u_0(n) = 0$ and $v_0(n) = 1$. By the density of D'_n, there is a set $X'_{u_0} \times X'_{v_0} \in D'_n$ included in $X_{u_0} \times X_{v_0}$. We may assume that $X'_{u_0} \cap X'_{v_0} = \varnothing$. (Indeed it easily follows from Claim 7.2 that there exist reals $x_0 \in X_{u_0}$ and $y_0 \in X_{v_0}$ satisfying $x_0 \mathsf{E}\, y_0$ but $x_0 \neq y_0$, say $x_0(k) = 0$ while $y_0(k) = 1$. Define

$$X = \{x \in X_0 : x(k) = 0 \wedge \exists\, y \in Y_0\, (y(k) = 1 \wedge x \mathsf{E}\, y)\}\,,$$

and Y correspondingly; then $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$ and $X \cap Y = \varnothing$.)

Spread the change from X_{u_0} to X'_{u_0} and from X_{v_0} to X'_{v_0} through the chain of crucial pairs in 2^{n+1}, by the method of Step 1, until the wave of spreading from u_0 meets the wave of spreading from u_0 at the crucial pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$. This leads to a system of sets X'_u and Q'_{uv} which satisfy (7) for the particular pair $\langle u_0, v_0\rangle$ and still satisfy (6) possibly except for the crucial pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$ (for which basically the set $Q'_{1^n{}^\wedge 0\,,\,0^n{}^\wedge 1}$ is not yet defined for this step).

By construction the previous steps leave $Q_{1^n{}^\wedge 0\,,\,0^n{}^\wedge 1}$ in the form $X_{1^n{}^\wedge 0} \times X_{0^n{}^\wedge 1}$, where $X_{1^n{}^\wedge 0}$ and $X_{0^n{}^\wedge 1}$ are the "versions" at the end of Step 1). We now have the new sets, $X'_{1^n{}^\wedge 0}$ and $X'_{0^n{}^\wedge 1}$, included in resp. $X_{1^n{}^\wedge 0}$ and $X_{0^n{}^\wedge 1}$ and satisfying $[X'_{0^n{}^\wedge 0}]_{\mathsf{E}} = [X'_{0^n{}^\wedge 1}]_{\mathsf{E}}$. (Indeed $[X'_{u_0}]_{\mathsf{E}} = [X'_{v_0}]_{\mathsf{E}}$ held at the beginning of the change.) Now we put $Q'_{1^n{}^\wedge 0\,,\,0^n{}^\wedge 1} = (X'_{1^n{}^\wedge 0} \times X'_{0^n{}^\wedge 1}) \cap \boldsymbol{B}$. Then $Q'_{1^n{}^\wedge 0\,,\,0^n{}^\wedge 1} \in \mathbf{P}_{\subseteq \boldsymbol{B}}$, and we have $\mathtt{dom}\, Q'_{1^n{}^\wedge 0\,,\,0^n{}^\wedge 1} = X'_{1^n{}^\wedge 0}$, $\mathtt{ran}\, Q'_{1^n{}^\wedge 0\,,\,0^n{}^\wedge 1} = X'_{0^n{}^\wedge 1}$ by Remark 8.1 and Lemma 6.2.

This ends the consideration of the pair $\langle u_0, v_0\rangle$.

Applying this construction consecutively for all pairs of u_0 and v_0 with $u_0(n) = 0$, $v_0(n) = 1$ (including the pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$) we finally get a system of sets satisfying (1) – (6), except for the requirement $Q_{uv} \in \mathbf{P}^*_{\subseteq \boldsymbol{B}}$ in (1) for the pair $\langle u, v\rangle = \langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$, — and these sets will be denoted still by X_u and Q_{uv} from now on.

Step 3. Now we take care of (7). Consider a crucial pair in 2^{n+1},

$$\langle u_0, v_0\rangle = \langle 1^k{}^\wedge 0^\wedge w, 0^k{}^\wedge 1^\wedge w\rangle \in 2^{n+1}.$$

If $k < n$ then $\langle u_0, v_0\rangle \neq \langle 1^k{}^\wedge 0, 0^k{}^\wedge 1\rangle$, the set $Q_{1^k{}^\wedge 0, 0^k{}^\wedge 1} \in \mathbf{P}^*_{\subseteq \boldsymbol{B}}$ is defined at a previous level, and $Q_{u_0,v_0} \subseteq Q_{1^k{}^\wedge 0, 0^k{}^\wedge 1}$. By the density, there exists a set $Q'_{u_0,v_0} \in D_n(Q_{1^k{}^\wedge 0, 0^k{}^\wedge 1})$, $Q'_{u_0,v_0} \subseteq Q_{u_0,v_0}$. If $k = n$ then $\langle u_0, v_0\rangle = \langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$, and by Lemma 2.6 there is a set $Q'_{u_0,v_0} \in \mathbf{P}^*_{\subseteq \boldsymbol{B}}$, $Q'_{u_0,v_0} \subseteq Q_{u_0,v_0}$.

In both cases define $X'_{u_0} = \mathtt{dom}\, Q'_{u_0,v_0}$ and $X'_{v_0} = \mathtt{ran}\, Q'_{u_0,v_0}$ and spread this change through the chain of crucial pairs in 2^{n+1}, exactly as above. Note that $[X'_{u_0}]_{\mathsf{E}} = [X'_{v_0}]_{\mathsf{E}}$ as sets in $\mathbf{P}_{\subseteq \boldsymbol{B}}$ are included in E. This keeps $[X'_u]_{\mathsf{E}} = [X'_v]_{\mathsf{E}}$ for all $u, v \in 2^{n+1}$ through the spreading.

Executing this step for all crucial pairs in 2^{n+1}, we finally accomplish the construction of a system of sets satisfying (1) through (7).

□ (Theorem 1.1)

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