Abstract

We modify arguments in [5] to reprove a linearization theorem on real-ordinal definable partial quasi-orderings in the Solovay model.

1 Introduction

The following theorem is the main content of this note.

Theorem 1.1 (in the Solovay model). Let $\preceq$ be a ROD (real-ordinal definable) partial quasi-ordering on $\omega^\omega$ and $\approx$ be the associated equivalence relation. Then exactly one of the following two conditions is satisfied:

(I) there is an antichain $A \subseteq 2^{<\omega_1}$ and a ROD map $F : \omega^\omega \to A$ such that

1) if $a,b \in \omega^\omega$ then: $x \preceq y \implies F(x) \leq_{\text{lex}} F(y)$, and  
2) if $a,b \in \omega^\omega$ then: $x \not\approx y \implies F(x) \neq F(y)$;

(II) there exists a continuous 1–1 map $F : 2^\omega \to \omega^\omega$ such that

3) if $a,b \in 2^\omega$ then: $a \leq_0 b \implies F(a) \prec F(b)$, and  
4) if $a,b \in 2^\omega$ then: $a \not\preceq_0 b \implies F(a) \not\prec F(b)$.

Here $\leq_{\text{lex}}$ is the lexicographical order on sets of the form $2^\alpha$, $\alpha \in \text{Ord}$ — it linearly orders any antichain $A \subseteq 2^{<\omega_1}$, while $\leq_0$ is the partial quasi-ordering on $2^\omega$ defined so that $x \leq_0 y$ iff $x \not\approx_0 y$ and either $x = y$ or $x(k) < y(k)$, where $k$ is the largest number with $x(k) \neq y(k)$.

The proof of this theorem (Theorem 6) in [5, Section 6]) contains a reference to Theorem 5 on page 91 (top), which is in fact not immediately applicable in

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1 Clearly $\leq_0$ orders each $\not\approx_0$-class similarly to the (positive and negative) integers, except for the class $[\omega \times \{0\}]_{\not\approx_0}$ ordered as $\omega$ and the class $[\omega \times \{1\}]_{\not\approx_0}$ ordered the inverse of $\omega$. 

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the Solovay model. The goal of this note is to present a direct and self-contained
proof of Theorem 1.1.

The combinatorial side of the proof follows the proof of a theorem on Borel
linearization in [4], in turn based on earlier results in [2, 1]. This will lead us to
(U) in a weaker form, with a function \( F \) mapping \( \omega^\omega \) into \( 2^{\omega^2} \). To reduce this
to an antichain in \( 2^{<\omega_1} \), a compression lemma (Lemma 5.1 below) is applied,
which has no counterpart in the Borel case.

Our general notation follows [6, 8], but for the convenience of the reader, we
add a review of notation.

PQO, partial quasi-order: reflexive \((x \leq x)\) and transitive in the domain;
LQO, linear quasi-order: PQO and \( x \leq y \lor y \leq x \) in the domain;
LO, linear order: LQO and \( x \leq y \land y \leq x \implies x = y \);
associated equivalence relation: \( x \equiv y \) iff \( x \leq y \land y \leq x \).
associated strict ordering: \( x < y \) iff \( x \leq y \land y \nleq x \);
LR (left–right) order preserving map: any map \( f: \langle X; \leq \rangle \to \langle X'; \leq' \rangle \) such
that we have \( x \leq y \implies f(x) \leq' f(y) \) for all \( x, y \in \text{dom } f \);
\(<_{\text{lex}}, \leq_{\text{lex}}\): the lexicographical LOs on sets of the form \( 2^\alpha, \alpha \in \text{Ord} \), resp.
strict and non-strict;
\([x]_E = \{ y \in \text{dom } E : x \in E y \} \) (the \( E \)-class of \( x \)) and
\([X]_E = \bigcup_{x \in X} [x]_E \) —
whenever \( E \) is an equivalence relation and \( x \in \text{dom } E \), \( X \subseteq \text{dom } E \).

Remark 1.2. We shall consider only the case of a parameterfree OD ordering
\( \preceq \) in Theorem 1.1 the case of OD(\( p \)) with a fixed real parameter \( p \) does not
differ much.

2 The Solovay model and OD forcing

We start with a brief review of the Solovay model. Let \( \Omega \) be an ordinal. Let \( \Omega\-SM \) be the following hypothesis:

\( \Omega\-SM:\ \Omega = \omega_1, \ \Omega \text{ is strongly inaccessible in } L, \ \text{the constructible universe, and}
\text{the whole universe } V \text{ is a generic extension of } L \text{ via the Levy collapse}
\text{forcing } \text{Coll}(\omega, <\Omega), \text{ as in [9].}

Assuming \( \Omega\-SM \), let \( P \) be the set of all non-empty OD sets \( Y \subseteq \omega^\omega \). We
consider \( P \) as a forcing notion (smaller sets are stronger). A set \( D \subseteq P \) is:

- dense, iff for every \( Y \in P \) there exists \( Z \in D, \ Z \subseteq Y \);
- open dense, iff in addition we have \( Y \in D \implies X \in D \) whenever sets
\( Y \subseteq X \) belong to \( P \);
A set $G \subseteq P$ is $P$-generic, iff 1) if $X, Y \in G$ then there is a set $Z \in G$, $Z \subseteq X \cap Y$, and 2) if $D \subseteq P$ is OD and dense then $G \cap D \neq \emptyset$.

Given an OD equivalence relation $E$ on $\omega^\omega$, a reduced product forcing notion $P \times_E P$ consists of all sets of the form $X \times Y$, where $X, Y \in P$ and $[X]_E \cap [Y]_E \neq \emptyset$. For instance $X \times X$ belongs to $P \times_E P$ whenever $X \in P$. The notions of sets dense and open dense in $P \times_E P$, and $(P \times_E P)$-generic sets are similar to the case of $P$.

A condition $X \times Y$ in $P \times_E P$ is saturated iff $[X]_E = [Y]_E$.

**Lemma 2.1.** If $X \times Y$ is a condition in $P \times_E P$ then there is a stronger saturated subcondition $X' \times Y'$ in $P \times_E P$.

**Proof.** Let $X' = X \cap [Y]_E$ and $Y' = Y \cap [X]_E$. □

**Proposition 2.2** (lemmas 14, 16 in [3]). Assume $\Omega$-SM.

If a set $G \subseteq P$ is $P$-generic then the intersection $\bigcap G = \{x[G]\}$ consists of a single real $x[G]$, called $P$-generic — its name will be $\mathfrak{x}$.

Given an OD equivalence relation $E$ on $\omega^\omega$, if a set $G \subseteq P \times_E P$ is $(P \times_E P)$-generic then the intersection $\bigcap G = \langle x_{1E}[G], x_{r1}[G] \rangle$ consists of a single pair of reals $x_{1E}[G], x_{r1}[G]$, called an $(P \times_E P)$-generic pair — their names will be $\mathfrak{x}_{1E}, \mathfrak{x}_{r1}$; either of $x_{1E}[G], x_{r1}[G]$ is separately $P$-generic. □

As the set $P$ is definitely uncountable, the existence of $P$-generic sets does not immediately follow from $\Omega$-SM by a cardinality argument. Yet fortunately $P$ is locally countable, in a sense.

**Definition 2.3** (assuming $\Omega$-SM). A set $X \in OD$ is OD-1st-countable if the set $P_{OD}(X) = P(X) \cap OD$ of all OD subsets of $X$ is at most countable.

For instance, assuming $\Omega$-SM, the set $X = \omega^\omega \cap OD = \omega^\omega \cap L$ of all OD reals is OD-1st-countable. Indeed $P_{OD}(X) = P(X) \cap L$, and hence $P_{OD}(X)$ admits an OD bijection onto the ordinal $\omega_2^L < \omega_1 = \Omega$.

**Lemma 2.4** (assuming $\Omega$-SM). If a set $X \subseteq OD$ is OD-1st-countable then the set $P_{OD}(X)$ is OD-1st-countable.

**Proof.** There is an ordinal $\lambda < \omega_1 = \Omega$ and an OD bijection $b : \lambda \onto P_{OD}(X)$. Any OD set $Y \subseteq L$ belongs to $L$, hence, the OD power set $P_{OD}(\lambda) = P(\lambda) \cap L$ belongs to $L$ and $\text{card}(P_{OD}(\lambda)) \leq \omega^+ < \Omega$ in $L$. We conclude that $P_{OD}(\lambda)$ is countable. It follows that $P_{OD}(P_{OD}(X))$ is countable, as required. □

**Lemma 2.5** (assuming $\Omega$-SM). If $\lambda < \Omega$ then the set $\text{Coll}(\lambda)$ of all elements $f \in \lambda^\omega$, $\text{Coll}(\omega, \lambda)$-generic over $L$, is OD-1st-countable.

**Proof.** If $Y \subseteq \text{Coll}(\lambda)$ is OD and $x \in Y$ then “$\mathfrak{x} \in \mathfrak{Y}$” is $\text{Coll}(\omega, \lambda)$-forced over $L$. It follows that there is a set $S \subseteq \lambda^{<\omega} = \text{Coll}(\omega, \lambda)$, $S \in L$, such that
Let $P^*$ be the set of all OD-1st-countable sets $X \in P$. We also define

$$P^* \times_\mathcal{E} P^* = \{X \times Y \in P \times_\mathcal{E} P : X, Y \in P^*\}.$$

**Lemma 2.6** (assuming $\Omega$-SM). *The set $P^*$ is dense in $P$, that is, if $X \in P$ then there is a condition $Y \in P^*$ such that $Y \subseteq X$. If $\mathcal{E}$ is an OD equivalence relation on $\omega^\omega$ then the set $P^* \times_\mathcal{E} P^*$ is dense in $P \times_\mathcal{E} P$ and any $X \times Y$ in $P^* \times_\mathcal{E} P^*$ is OD-1st-countable.*

**Proof.** Let $X \in P$. Then $X \neq \emptyset$, hence, there is a real $x \in X$. It follows from $\Omega$-SM that there is an ordinal $\lambda < \omega_1 = \Omega$, an element $f \in \text{COH}_\lambda$, and an OD map $H : \lambda^\omega \to \omega^\omega$, such that $x = H(f)$. The set $P = \{f' \in \text{COH}_\lambda : H(f') \subseteq X\}$ is then OD and non-empty (contains $f$), and hence so is its image $Y = \{H(f') : f' \in P\} \subseteq X$ (contains $x$). Finally, $Y \in P^*$ by Lemma 2.5.

To prove the second claim, let $X \times Y$ be a condition in $P \times_\mathcal{E} P$. By Lemma 2.4 there is a stronger saturated subcondition $X' \times Y' \subseteq X \times Y$. By the first part of the lemma, let $X'' \subseteq X'$ be a condition in $P^*$, and $Y'' = Y' \cap [X'']_\mathcal{E}$. Similarly, let $Y''' \subseteq Y''$ be a condition in $P^*$, and $X''' = X'' \cap [Y''']_\mathcal{E}$. Then $X''' \times Y'''$ belongs to $P^* \times_\mathcal{E} P^*$.

**Corollary 2.7** (assuming $\Omega$-SM). *If $X \in P$ then there exists a $P$-generic set $G \subseteq P$ containing $X$. If $X \times Y$ is a condition in $P \times_\mathcal{E} P$ then there exists a $(P \times_\mathcal{E} P)$-generic set $G \subseteq P \times_\mathcal{E} P$ containing $X \times Y$.*

**Proof.** By Lemma 2.6 assume that $X \in P^*$. Then the set $P_{\subseteq X}$ of stronger conditions contains only countably many OD subsets by Lemma 2.4.

### 3 The OD forcing relation

The forcing notion $P$ will play the same role below as the Gandy – Harrington forcing in [2, 7]. There is a notable technical difference: under $\Omega$-SM, OD-generic sets exist in the ground Solovay-model universe by Corollary 2.7. Another notable difference is connected with the forcing relation.

**Definition 3.1** (assuming $\Omega$-SM). Let $\varphi(x)$ be an 0rd-formula, that is, a formula with ordinals as parameters.

A condition $X \in P$ is said to $P$-force $\varphi(\dot{x})$ iff $\varphi(x)$ is true (in the Solovay-model set universe considered) for any $P$-generic real $x$.

If $\mathcal{E}$ is an OD equivalence relation on $\omega^\omega$ then a condition $X \times Y$ in $P \times_\mathcal{E} P$ is said to $(P \times_\mathcal{E} P)$-force $\varphi(\dot{x}_1, \dot{x}_2)$ iff $\varphi(x, y)$ is true for any $(P \times_\mathcal{E} P)$-generic pair $\langle x, y \rangle$. 

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Lemma 3.2 (assuming Ω-SM). Given an $\text{ord}$-formula $\varphi(x)$ and a $P$-generic real $x$, if $\varphi(x)$ is true (in the Solovay-model set universe considered) then there is a condition $X \in P$ containing $x$, which $P$-forces $\varphi(\hat{x})$.

Let $E$ be an $\text{OD}$ equivalence relation on $\omega^\omega$. Given an $\text{ord}$-formula $\varphi(x,y)$ and a $(P \times E P)$-generic pair $\langle x,y \rangle$, if $\varphi(x,y)$ is true then there is a condition in $P \times E P$ containing $\langle x,y \rangle$, which $(P \times E P)$-forces $\varphi(\hat{x}_1, \hat{x}_2)$.

Proof. To prove the first claim, put $X = \{ x' \in \omega^\omega : \varphi(x') \}$. But this argument does not work for $P \times E P$. To fix the problem, we propose a longer argument which equally works in both cases — but we present it in the case of $P$ which is slightly simpler.

Formally the forcing notion $P$ does not belong to $L$. But it is order-isomorphic to a certain forcing notion $P \in L$, namely, the set $P$ of codes of OD sets in $P$. The order between the codes in $P$, which reflects the relation $\subseteq$ between the OD sets themselves, is expressible in $L$, too. Furthermore dense OD sets in $P$ correspond to dense sets in the coded forcing $P$ in $L$.

Now, let $x$ be $P$-generic and $\varphi(x)$ be true. It is a known property of the Solovay model that there is another $\text{ord}$-formula $\psi(x)$ such that $\varphi(x) \iff L[x] \models \psi(x)$. Let $g \subseteq P$ be the set of all codes of conditions $X \in P$ such that $x \in X$. Then $g$ is a $P$-generic set over $L$ by the choice of $x$, and $x$ is the corresponding generic object. Therefore there is a condition $p \in g$ which $P$-forces $\psi(\hat{x})$ over $L$. Let $X \in P$ be the OD set coded by $p$, so that $x \in X$.

To prove that $X$ OD-forces $\varphi(\hat{x})$, let $x' \in X$ be a $P$-generic real. Let $g' \subseteq P$ be the $P$-generic set of all codes of conditions $Y \in P$ such that $x' \in Y$. Then $p \in g'$, hence $\psi(x')$ holds in $L[x']$, by the choice of $p$. Then $\varphi(x')$ holds (in the Solovay-model set universe) by the choice of $\psi$, as required. \hfill \square

Corollary 3.3 (assuming Ω-SM). Given an $\text{ord}$-formula $\varphi(x)$, if $X \in P$ does not $P$-force $\varphi(\hat{x})$ then there is a condition $Y \in P$, $\subseteq X$, which $P$-forces $\neg \varphi(\hat{x})$. The same for $P \times E P$. \hfill \square

4 Some similar and derived forcing notions

Some forcing notions similar to $P$ and $P \times E P$ will be considered:

1°. $P_{\subseteq W} = \{ Q \subseteq W : \emptyset \neq Q \in \text{OD} \}$, where $W \subseteq \omega^\omega$ or $W \subseteq \omega^\omega \times \omega^\omega$ is an OD set. Especially, in the case when $W \subseteq E$, where $E$ is an OD equivalence relation on $\omega^\omega$ (that is, $\langle x,y \rangle \in W \implies x \in y$) — note that $[\text{dom} W]_E = [\text{ran} W]_E$ in this case.

2°. $\langle P \times E P \rangle_{\subseteq X \times Y} = \{ X' \times Y' \in P \times E P : X' \subseteq X \land Y' \subseteq Y \}$, where $E$ is an OD equivalence relation on $\omega^\omega$ and $X \times Y \in P \times E P$.

A code of an OD set $X$ is a finite sequence of logical symbols and ordinals which correspond to a definition in the form $X = \{ x \in V_\alpha : V_\alpha \models \varphi(x) \}$. 


3. \( P_{\mathbb{C}W} \times_E P_{\mathbb{C}X} = \{ P \times Y : P \in P_{\mathbb{C}W} \land Y \in P_{\mathbb{C}X} \land [Y]_E \cap [\text{dom} P]_E \neq \emptyset \} \), where \( E \) is an OD equivalence relation on \( \omega^\omega \), \( W \subseteq E \) is OD, \( X \in P \), and \( [X]_E \cap [\text{dom} W]_E \neq \emptyset \) (equivalently, \( [X]_E \cap [\text{ran} W]_E \neq \emptyset \)).

4. \( P_{\mathbb{C}W} \times_E P_{\mathbb{C}W} = \{ P \times Q : P, Q \in P_{\mathbb{C}W} \land [\text{dom} P]_E \cap [\text{dom} Q]_E \neq \emptyset \} \), where \( E \) is an OD equivalence relation on \( \omega^\omega \) and \( W \subseteq E \) is OD.

They have the same basic properties as \( P \) — the forcing notions of the form \( 1 \) or as \( P \times_E P \rightarrow 2 \). This includes such results and concepts as 2.2, 2.6, 2.7, the associated forcing relation as in 3.1 and 3.2, 3.3, with suitable and rather transparent corrections, of course.

5. **Compression lemma**

A set \( A \subseteq 2^{<\Omega} \) is an antichain if its elements are pairwise \( \preceq \)-incomparable, that is, no sequence in \( A \) properly extends another sequence in \( A \). Clearly any antichain is linearly ordered by \( \leq_{\text{lex}} \).

Let \( \Theta = \Omega^+ \); the cardinal successor of \( \Omega \) in both \( L \), the ground model, and its \( \text{Coll}(\omega, < \Omega) \)-generic extension postulated by \( \Omega \)-SM to be the set universe; in the latter, \( \Omega = \omega_1 \) and \( \Theta = \omega_2 \).

**Lemma 5.1** (compression lemma). Assume that \( \Omega \leq \vartheta \leq \Theta \) and \( X \subseteq 2^\Theta \) is the image of \( \omega^\omega \) via an OD map. Then there is an OD antichain \( A(X) \subseteq 2^{<\Omega} \) and an OD isomorphism \( f : \langle X; \leq_{\text{lex}} \rangle \overset{\text{onto}}{\longrightarrow} \langle A(X); \leq_{\text{lex}} \rangle \).

**Proof.** If \( \vartheta = \Theta \) then, as \( \text{card} X \leq \text{card} \omega^\omega = \Omega \), there is an ordinal \( \vartheta < \Theta \) such that \( x \upharpoonright \vartheta \neq y \upharpoonright \vartheta \) whenever \( x \neq y \) belong to \( X \) — this reduces the case \( \vartheta = \Theta \) to the case \( \Omega \leq \vartheta < \Theta \). We prove the latter by induction on \( \vartheta \).

The nontrivial step is the cof \( \lambda = \Omega \), so that let \( \vartheta = \bigcup_{\alpha < \Omega} \vartheta_\alpha \), for an increasing OD sequence of ordinals \( \vartheta_\alpha \). Let \( I_\alpha = \langle \vartheta_\alpha, \vartheta_{\alpha+1} \rangle \). Then, by the induction hypothesis, for any \( \alpha < \Omega \) the set \( X_\alpha = \{ S \upharpoonright I_\alpha : S \in X \} \subseteq 2^{I_\alpha} \) is \( <_{\text{lex}} \)-order-isomorphic to an antichain \( A_\alpha \subseteq 2^{<\Omega} \) via an OD isomorphism \( i_\alpha \), and the map, which sends \( \alpha \) to \( A_\alpha \) and \( i_\alpha \), is OD. It follows that the map, which sends each \( S \in X \) to the concatenation of all sequences \( i_\alpha(x \upharpoonright I_\alpha) \), is an OD \( <_{\text{lex}} \)-order-isomorphic \( X \) onto an antichain in \( 2^\Omega \). Therefore, in fact it suffices to prove the lemma in the case \( \vartheta = \Omega \). Thus let \( X \subseteq 2^\Omega \).

First of all, note that each sequence \( S \in X \) is ROD. Lemma 7 in [3] shows that, in this case, we have \( S \in L[S \upharpoonright \eta] \) for an ordinal \( \eta < \Omega \). Let \( \eta(S) \) be the least such an ordinal, and \( h(S) = S \upharpoonright \eta(S) \), so that \( h(S) \) is a countable initial segment of \( S \) and \( S \in L[h(S)] \). Note that \( h \) is still OD.

Consider the set \( U = \text{ran} h = \{ h(S) : S \in X \} \subseteq 2^{<\Omega} \). We can assume that every sequence \( u \in U \) has a limit length. Then \( U = \bigcup_{\gamma < \Omega} U_\gamma \), where \( U_\gamma = U \cap 2^{<\gamma} \) (\( \omega^\gamma \) is the \( \gamma \)-th limit ordinal). For \( u \in U_\gamma \), let \( \gamma_u = \gamma \).

If \( u \in U \) then by construction the set \( X_u = \{ S \in X : h(S) = u \} \) is OD(\( u \)) and satisfies \( X_u \subseteq L[u] \). Therefore, it follows from the known properties of the
Solovay model that \( X_u \) belongs to \( L[u] \) and is of cardinality \( \leq \Omega \) in \( L[u] \). Fix an enumeration \( X_u = \{ S_u(\alpha) : \gamma_u \leq \alpha < \Omega \} \) for all \( u \in U \). We can assume that the map \( \alpha, u \mapsto S_u(\alpha) \) is OD.

If \( u \in U \) and \( \gamma_u \leq \alpha < \Omega \), then we define a shorter sequence, \( s_u(\alpha) \in 3^{\omega_1+1} \), as follows.

(i) \( s_u(\alpha)(\xi + 1) = S_u(\alpha)(\xi) \) for any \( \xi < \omega_1 \).

(ii) \( s_u(\alpha)(\omega_1) = 1 \).

(iii) Let \( \delta < \alpha \). If \( S_u(\alpha) \upharpoonright \omega_1 = S_v(\delta) \upharpoonright \omega_1 \) for some \( v \in U \) (equal to or different from \( u \)) then \( s_u(\alpha)(\omega_1) = 0 \) whenever \( S_u(\alpha) <_{\text{lex}} S_v(\delta) \), and \( s_u(\alpha)(\omega_1) = 2 \) whenever \( S_v(\delta) <_{\text{lex}} S_u(\alpha) \).

(iv) Otherwise (i.e., if there is no such \( v \) ), \( s_u(\alpha)(\omega_1) = 1 \).

To demonstrate that (iii) is consistent, we show that \( S_v(\delta) \upharpoonright \omega_1 = S_v(\delta) \upharpoonright \omega_1 \) implies \( u' = u'' \). Indeed, as by definition \( u' \subseteq S_u(\delta) \) and \( u'' \subseteq S_u(\delta) \), \( u' \) and \( u'' \) must be \( \sqsubseteq \)-compatible: let, say, \( u' \subseteq u'' \). Now, by definition, \( S_u(\delta) \in L[u''] \), therefore \( \in L[S_u(\delta)] \) because \( u'' \subseteq S_u(\delta) \upharpoonright \omega_1 = S_v(\delta) \upharpoonright \omega_1 \), finally in \( L[u'] \), which shows that \( u' = u'' \) as \( S_u(\delta) \in X_{u''} \).

We are going to prove that the map \( S_u(\alpha) \mapsto s_u(\alpha) \) is a \( <_{\text{lex}} \)-order isomorphism, so that \( S_v(\beta) <_{\text{lex}} S_u(\alpha) \) implies \( s_v(\beta) <_{\text{lex}} s_u(\alpha) \).

We first observe that \( s_v(\beta) \) and \( s_u(\alpha) \) are \( \sqsubseteq \)-incomparable. Indeed assume that \( \beta < \alpha \). If \( S_u(\alpha) \upharpoonright \omega_1 \neq S_v(\beta) \upharpoonright \omega_1 \) then clearly \( s_u(\alpha) \upharpoonright \omega_1 \neq s_u(\alpha) \) by (i). If \( S_u(\alpha) \upharpoonright \omega_1 = S_v(\beta) \upharpoonright \omega_1 \) then \( s_u(\alpha)(\omega_1) = 0 \) or 2 by (iii) while \( s_v(\beta)(\omega_1) = 1 \) by (ii). Thus all \( s_u(\alpha) \) are mutually \( \sqsubseteq \)-incomparable, so that it suffices to show that conversely \( s_v(\beta) <_{\text{lex}} s_u(\alpha) \) implies \( S_v(\beta) <_{\text{lex}} S_u(\alpha) \). Let \( \zeta \) be the least ordinal such that \( s_v(\beta)(\zeta) < s_u(\alpha)(\zeta) \); then \( s_v(\beta)(\zeta) = s_u(\alpha)(\zeta) \) and \( \zeta \leq \min\{\omega_1, \alpha, \beta\} \).

The case when \( \zeta = \xi + 1 \) is clear: then by definition \( S_u(\alpha) \upharpoonright \xi = S_v(\beta) \upharpoonright \xi \) while \( S_v(\beta)(\xi) < S_u(\alpha)(\xi) \), so let us suppose that \( \zeta = \omega_1 \), where \( \delta \leq \min\{\alpha, \beta\} \). Then obviously \( S_u(\alpha) \upharpoonright \omega_1 = S_v(\beta) \upharpoonright \omega_1 \). Assume that one of the ordinals \( \alpha, \beta \) is equal to \( \delta \), say, \( \beta = \delta \). Then \( s_v(\beta)(\omega_1) = 1 \) while \( s_u(\alpha)(\omega_1) \) is computed by (iii). Now, as \( s_v(\beta)(\omega_1) = s_u(\alpha)(\omega_1) \), we conclude that \( s_u(\alpha)(\omega_1) = 2 \), hence \( S_v(\beta) <_{\text{lex}} S_u(\alpha) \), as required. Assume now that \( \delta < \min\{\alpha, \beta\} \). Then easily \( \alpha \) and \( \beta \) appear in one and the same class \( \{\text{iii}\} \) or \( \{\text{iv}\} \) with respect to \( \delta \). However this cannot be \( \{\text{iv}\} \) because \( s_v(\beta)(\omega_1) \neq s_u(\alpha)(\omega_1) \). Hence we are in (iii), so that, for some (unique) \( w \in U \), \( 0 = S_v(\beta) <_{\text{lex}} S_w(\delta) <_{\text{lex}} S_u(\alpha) = 2 \), as required.

This ends the proof of the lemma, except for the fact that the sequences \( s_u(\alpha) \) belong to \( 3^{\omega_1} \), but improvement to \( 2^{\omega_1} \) is easy.

6 The dichotomy

Here we begin the proof of Theorem 1.1. We assume \( \Omega\text{-SM} \) in the course of the proof. And we assume that the ordering \( \preceq \) of the theorem is just OD — then
so is the associated equivalence relation $\approx$ and strict order ${\prec}$. Let $\mathcal{F}$ be the set of all OD LR order preserving maps $F : \langle \omega^\omega ; \preceq \rangle \to \langle A ; \leq_{\text{lex}} \rangle$, where $A \subseteq 2^{<\Omega}$ is an OD antichain. Let

$$x \in y \text{ iff } \forall F \in \mathcal{F} \ (F(x) = F(y))$$

for $x, y \in \omega^\omega$. Then $E$ is an OD equivalence relation, OD-smooth in the sense that it admits an obvious OD reduction to the equality on the set $2^\mathcal{F}$.

**Lemma 6.1.** If $R(x, y)$ is an OD relation and $\forall x, y \ (x \in E \implies R(x, y))$ then there is a function $F \in \mathcal{F}$ such that $\forall x, y \ (F(x) = F(y) \implies R(x, y))$.

**Proof.** Clearly $\text{card } \mathcal{F} = \Theta = \Omega^+$ and $\mathcal{F}$ admits an OD enumeration $\mathcal{F} = \{F_\xi : \xi < \Theta\}$. If $x \in \omega^\omega$ then let $f(x) = F_0(x) \upharpoonright F_1(x) \upharpoonright \ldots \upharpoonright F_\xi(x) \upharpoonright \ldots$ — the concatenation of all sequences $F_\xi(x)$. Then $f : \langle \omega^\omega ; \preceq \rangle \to \langle X ; \leq_{\text{lex}} \rangle$ is an OD LR order preserving map, where $X = \text{ran } f = \{f(r) : r \in \omega^\omega\} \subseteq 2^\Theta$, and $f(x) = f(y) \implies R(x, y)$ by the construction. By Lemma 5.1 there is an OD isomorphism $g : \langle X ; \leq_{\text{lex}} \rangle \overset{\text{onto}}{\longrightarrow} \langle A ; \leq_{\text{lex}} \rangle$ onto an antichain $A \subseteq 2^{<\Omega}$. The superposition $F(x) = g(f(x))$ proves the lemma. □

**Lemma 6.2.** Let OD sets $\emptyset \neq X, Y \subseteq \omega^\omega$ satisfy $[X]_E = [Y]_E$. Then the set $B = \{\langle x, y \rangle \in X \times Y : x \in E \wedge x \preceq y\}$ is non-empty, $\text{dom } B = X$, $\text{ran } B = Y$.

**Proof.** It suffices to establish $B \neq \emptyset$. The OD set

$$X' = \{x' \in \omega^\omega : \exists x \in X \ (x \in E \wedge x' \preceq x)\}$$

is downwards $\preceq$-closed in each $E$-class, and if $B = \emptyset$ then $X' \cap Y = \emptyset$. By Lemma 6.1 there is a function $F \in \mathcal{F}$ such that $x \in X' \implies x' \in X'$ holds whenever $F(x) = F(x')$ and $x' \preceq x$. It follows that the derived function

$$G(x) = \begin{cases} F(x) \upharpoonright 0, \text{ whenever } x \in X' \\ F(x) \upharpoonright 1, \text{ whenever } x \in \omega^\omega \setminus X' \end{cases}$$

belongs to $\mathcal{F}$. Thus if $x \in X \subseteq X'$ and $y \in Y \subseteq \omega^\omega \setminus X'$ then $G(x) \neq G(y)$ and hence $x \in E y$. In other words, $[X]_E \cap [Y]_E = \emptyset$, a contradiction. □

We’ll make use of the OD-forcing notions $P$ and $P \times \mathcal{P}$.

**Lemma 6.3.** Condition $\omega^\omega \times \omega^\omega \ (P \times E P)$-forces $\dot{x}_{\text{le}} E \dot{x}_{\text{rl}}$.

**Proof.** Otherwise, by Lemma 3.2 there is a function $F \in \mathcal{F}$ and a condition $X \times Y$ in $P \times E P$ which $(P \times E P)$-forces $F(\dot{x}_{\text{le}})(\xi) = 0 \neq 1 = F(\dot{x}_{\text{rl}})(\xi)$ for a certain ordinal $\xi < \Omega$. We may assume that $X \times Y$ is a saturated condition. Then easily $F(x)(\xi) = 0 \neq 1 = F(y)(\xi)$ holds for any pair $(x, y) \in X \times Y$, so that we have $F(x) \neq F(y)$ and $x \in E y$ whenever $(x, y) \in X \times Y$, which contradicts the choice of $X \times Y$ in $P \times E P$. □
Case 1: \( \approx \) and \( E \) coincide on \( \omega^\omega \), so that \( x \in E y \iff x \approx y \) for \( x, y \in \omega^\omega \).

By Lemma 6.1 there is a single function \( F \in \mathcal{F} \) such that \( F(x) = F(y) \) implies \( x \approx y \) for all \( x, y \in U^* \), as required for [I] of Theorem 1.1.

Case 2: \( \approx \) is a proper subrelation of \( E \), hence, the OD set

\[
U_0 = \{ x \in \omega^\omega : \exists y \in \omega^\omega (x \not\approx y \land x \in E y) \}
\]

(the domain of singularity) is non-empty. It follows that \( U_0 \in P \) and \( U_0 \times U_0 \) is a condition in \( P \times_E P \). We’ll work towards [III] of Theorem 1.1.

7 The domain of singularity

Since the set \( U_0 \) belongs to \( P \), there is a set \( U^* \in P^* \), \( U^* \subseteq U_0 \). Then obviously \( U^* \times U^* \) belongs to \( P^* \times_E P^* \).

Lemma 7.1. Condition \( U^* \times U^* (P \times_E P) \)-forces that the reals \( \hat{x}_{1e} \) and \( \hat{x}_{r1} \) are \( \preccurlyeq \)-incomparable.

Proof. Suppose to the contrary that, by Corollary 3.3, a subcondition \( X \times Y \) in \( P \times_E P \) either \( (P \times_E P) \)-forces \( \hat{x}_{1e} \approx \hat{x}_{r1} \) or \( (P \times_E P) \)-forces \( \hat{x}_{1e} < \hat{x}_{r1} \). We will get a contradiction in both cases. Note that \( X, Y \subseteq U^* \) are non-empty OD sets and \([X]_E \cap [Y]_E \neq \emptyset \).

Claim 7.2. The set \( W = \{ \langle x, x' \rangle \in X \times X : x \in E x' \land x' \not\approx x \} \) is non-empty.

Proof. Suppose to the contrary that \( W = \emptyset \), so \( E \) coincides with \( \approx \) on \( X \). As \( X \subseteq U^* \), at least one of the OD sets

\[
Z = \{ z : \exists x \in X (z \in E x \land z \not\approx x) \}, \quad Z' = \{ z : \exists x \in X (z \in E x \land z \approx x) \}
\]

is non-empty; assume that, say, \( Z \neq \emptyset \). Consider the OD set

\[
U = \{ z : \exists x \in X (z \in E x \land z \not\approx x) \}.
\]

Then \( X \subseteq U \) and \( U \cap Z = \emptyset \), \( U \) is downwards \( \preccurlyeq \)-closed while \( Z \) is upwards \( \preccurlyeq \)-closed in each \( E \)-class, therefore \( y \not\approx x \) whenever \( x \in U \land y \in Z \land x \in E y \), and hence we have \( [U]_E \cap [Z]_E = \emptyset \) be Lemma 6.2. Yet by definition \([X]_E \cap [Z]_E \neq \emptyset \) and \( X \subseteq U \), which is a contradiction. \( \square \) (Claim)

Suppose that condition \( X \times Y (P \times_E P) \)-forces \( \hat{x}_{1e} \approx \hat{x}_{r1} \). As \( W \neq \emptyset \) by Claim 7.2, the forcing \( P \subseteq W \) of all non-empty OD sets \( P \subseteq W \) adds pairs \( \langle x, x' \rangle \in W \) of \( P \)-generic (separately) reals \( x, x' \in X \) which satisfy \( x' \in E x \) and \( x' \not\approx x \). If \( P \in P \subseteq W \) then obviously \( [\text{dom } P]_E = [\text{ran } P]_E \). Consider a more complex forcing \( \mathcal{P} = P \subseteq W \times_E P \) of all pairs \( P \times Y' \), where \( P \in P \subseteq W \), \( Y' \in P \), \( Y' \subseteq Y \), and \( [\text{dom } P]_E \cap [Y']_E \neq \emptyset \). For instance, \( W \times Y \in P \subseteq W \times_E P \). Then \( \mathcal{P} \) adds a pair \( \langle \hat{x}_{1e}, \hat{x}_{r1} \rangle \in W \) and another real \( \hat{x} \in Y \) such that both pairs
\[ \langle \dot{x}_{1e}, \dot{x} \rangle \text{ and } \langle \dot{x}_{r1}, \dot{x} \rangle \text{ belong to } X \times Y \text{ and are } (P \times E P)\text{-generic, hence, we have } \dot{x}_{1e} \approx \dot{x} \approx \dot{x}_{r1} \text{ by the choice of } X \times Y. \text{ On the other hand, } \dot{x}_{1e} \not\approx \dot{x}_{r1} \text{ since the pair belongs to } W, \text{ which is a contradiction.}

Now suppose that condition \( X \times Y \) \((P \times E P)\text{-forces } \dot{x}_{1e} < \dot{x}_{r1}. \) The set
\[ B = \{ (x, y) \in X \times Y : y \leq x \} \]
is non-empty by Lemma \([6,2]\) Consider the forcing \( P_{\subseteq B} \) of all non-empty OD sets \( P \subseteq B; \) if \( P \in P_{\subseteq B} \) then obviously \( \text{dom } P|_E = \text{ran } P|_E. \) Consider a more complex forcing \( P_{\subseteq B} \times_E P_{\subseteq B} \) of all products \( P \times Q, \) where \( P, Q \in P_{\subseteq B} \) and \( \text{dom } P|_E \cap \text{dom } Q|_E \neq \emptyset. \) In particular \( B \times B \in P_{\subseteq B} \times_E P_{\subseteq B}. \)

Let \( \langle x, y; x', y' \rangle \) be a \( P_{\subseteq B} \times E P_{\subseteq B} \)-generic quadruple in \( B \times B, \) so that both \( \langle x, y \rangle \in B \) and \( \langle x', y' \rangle \in B \) are \( P_{\subseteq B} \)-generic pairs in \( B, \) and both \( y \leq x \) and \( y' \leq x' \) hold by the definition of \( B. \) On the other hand, an easy argument shows that both criss-cross pairs \( \langle x, y' \rangle \in X \times Y \) and \( \langle x', y \rangle \in X \times Y \) are \( P \times E P \)-generic, hence \( x < y' \) and \( x' < y \) by the choice of \( X \times Y. \) Altogether \( y \leq x < y' \leq x' < y, \) which is a contradiction. \( \square \)

8 The splitting construction

Our aim is to define, in the universe of \( \Omega\text{-SM,} \) a splitting system of sets which leads to a function \( F \) satisfying \([II]\) of Theorem \([II.1]\) Let
\[ B = \{ (x, y) \in U^* \times U^* : x \leq y \}; \quad B \neq \emptyset \text{ by Lemma } [6,2] \]
The construction will involve three forcing notions: \( P, P \times_E P, \) and \( P_{\subseteq B}, \) the collection of all non-empty OD sets \( P \subseteq B. \)

We also consider the dense (by Lemma \([2,6]\) subforcings \( P^* \subseteq P, P^* \times_E P^* \subseteq P \times E P \) (see Section \([2]\), and
\[ P_{\subseteq B}^* = \{ Q \in P_{\subseteq B} : Q \text{ is OD-1st-countable} \} \subseteq P_{\subseteq B}. \]

Now note the following.

1. As \( U^* \in P^* \), the set \( \mathcal{D} \) of all sets open dense in the restricted forcing \( P_{\subseteq U^*}, \) is countable by Lemma \([2,6]\) hence we can fix an enumeration \( \mathcal{D} = \{ D_n : n \in \omega \} \) such that \( D_n \subseteq D_m \) whenever \( m < n. \)

2. As \( U^* \times U^* \in P^* \times E P^* \), the set \( \mathcal{D}' \) of all sets, open dense in the restricted forcing \( (P \times E P)_{\subseteq U^* \times U^*}, \) is countable as above; fix an enumeration \( \mathcal{D}' = \{ D'_n : n \in \omega \} \) s.t. \( D'_n \subseteq D'_m \) for \( m < n. \)

3. If \( Q \in P_{\subseteq B}^* \) then the set \( \mathcal{D}(Q) \) of all sets open dense in the restricted forcing \( P_{\subseteq Q}, \) is countable by Lemma \([2,6]\) hence we can fix an enumeration \( \mathcal{D}(Q) = \{ D_n(Q) : n \in \omega \} \) such that \( D_n(Q) \subseteq D_m(Q) \) whenever \( m < n. \)
The chosen enumerations are not necessarily OD, of course.

A pair \( \langle u, v \rangle \) of strings \( u, v \in 2^n \) is called crucial iff \( u = 1^k \wedge 0^k \wedge w \) and \( v = 0^k \wedge 1^k \wedge w \) for some \( k < n \) and \( w \in 2^{n-k-1} \). Note that each pair of the form \( 1^k \wedge 0^k \wedge 1 \) is a minimal crucial pair, and if \( \langle u, v \rangle \) is a crucial pair then so is \( \langle u \wedge i, v \wedge i \rangle \), but not \( \langle u \wedge i, v \wedge j \rangle \) whenever \( i \neq j \). The graph of all crucial pairs in \( 2^n \) is actually a chain connecting all members of \( 2^n \).

We are going to define, in the assumption of \( \Omega\)-SM, a system of sets \( X_u \in \mathbb{P}^\ast \), where \( u \in 2^{< \omega} \), and sets \( Q_{uv} \in \mathbb{P}_{\leq B}^\ast \), \( \langle u, v \rangle \) being a crucial pair in some \( 2^n \), satisfying the following conditions:

1. \( X_u \in \mathbb{P}^\ast \) and \( Q_{uv} \in \mathbb{P}_{\leq B}^\ast \);
2. \( X_u \wedge i \subseteq X_u \);
3. \( Q_{u \wedge i, v \wedge i} \subseteq Q_{uv} \);
4. If \( \langle u, v \rangle \) is a crucial pair in \( 2^n \) then \( \text{dom} Q_{uv} = X_u \) and \( \text{ran} Q_{uv} = X_v \);
5. \( X_u \in D_n \) whenever \( u \in 2^{n+1} \);
6. If \( u, v \in 2^{n+1} \) and \( u(n) \neq v(n) \) then \( X_u \times X_v \in D'_n \) and \( X_u \cap X_v = \emptyset \).
7. If \( \langle u, v \rangle = \langle 1^k \wedge 0^k \wedge w, 0^k \wedge 1^k \wedge w \rangle \) is a crucial pair in \( 2^{n+1} \) and \( k < n \) (so that \( w \) in not the empty string) then \( Q_{uv} \in D_n(Q_{1^k \wedge 0^k \wedge 1}) \).

**Remark 8.1.** It follows from [4] that \( [X_u]_E = [X_v]_E \) for all \( u, v \in 2^n \), because \( Q_{uv} \subseteq B \subseteq E \) and \( u, v \) are connected in \( 2^n \) by a chain of crucial pairs. \( \square \)

**Why this implies the existence of a function as in [11] of Theorem 1.1**

First of all, if \( a \in 2^\omega \) then the sequence of sets \( X_a \upharpoonright n \) is \( \mathbb{P} \)-generic by [5] therefore the intersection \( \bigcap_{n \in \omega} X_a \upharpoonright n \) is a singleton by Proposition 2.2. Let \( F(a) \in \omega^\omega \) be its only element.

It does not take much effort to prove that \( F \) is continuous and \( 1 - 1 \).

Consider any \( a, b \in 2^\omega \) satisfying \( a \leq_0 b \). Then \( a(n) \neq b(n) \) for infinitely many \( n \), hence the pair \( \langle F(a), F(b) \rangle \) is \( \mathbb{P} \times E \)-generic by [7] thus \( F(a) \) and \( F(b) \) are \( \leq \)-incomparable by Lemma 7.1.

Consider \( a, b \in 2^\omega \) satisfying \( a <_0 b \). We may assume that \( a \) and \( b \) are \( <_0 \)-neighbours, i.e., \( a = 1^k \wedge 0^k \wedge w \) while \( b = 0^k \wedge 1^k \wedge w \) for some \( k \in \omega \) and \( w \in 2^\omega \). The sequence of sets \( Q_a \upharpoonright n, b \upharpoonright n, n > k \), is \( \mathbb{P}_{\leq B} \)-generic by [6] hence it results in a pair of reals satisfying \( x \preceq y \). However \( x = F(a) \) and \( y = F(b) \) by [4].

**9 The construction of a splitting system**

Now the goal is to define, in the assumption of \( \Omega\)-SM, a system of sets \( X_u \) and \( Q_{uv} \) satisfying [1] – [7] above. Suppose that the construction has been completed up to a level \( n \), and expand it to the next level. From now on \( s, t \) will denote strings in \( 2^n \) while \( u, v \) will denote strings in \( 2^{n+1} \).
Step 0. To start with, we set $X_{s^i} = X_s$ for all $s \in 2^n$ and $i = 0, 1$, and $Q_{s^i} = Q_{st}$ whenever $i = 0, 1$ and $(s, t)$ is a crucial pair in $2^n$. For the initial crucial pair $(1^n \wedge 0, 0^n \wedge 1)$ at this level, let $Q_{1^n \wedge 0, 0^n \wedge 1} = X_{1^n} \times X_{0^n}$ The newly defined sets satisfy $[1] - [4]$ except for the requirement $Q_{uv} \in P_{\subseteq \mathcal{B}}$ in [1] for the pair $(u, v) = (1^n \wedge 0, 0^n \wedge 1)$.

This ends the definition of "initial values" of $X_u$ and $Q_{uv}$ at the $(n + 1)$-th level. The plan is to gradually shrink the sets in order to fulfill (5) – (7).

Step 1. We take care of item (5). Consider an arbitrary $u_0 = s_0^i, \ i \in 2^{n+1}$. As $D_n$ is dense, there is a set $X'_u \subseteq X_u$ has been defined, and $(u, v)$ is a crucial pair, $v \in 2^{n+1}$ being not yet encountered. Define $Q_{u_0} = (X'_u \times \omega^n) \cap Q_{uv}$ and $X'_u = \text{ran} Q_{u_0}$. Clearly [4] holds for the "new" sets $X'_u, X'_v, Q'_{uv}$. Similarly if $(v, u)$ is a crucial pair, then define $Q'_{v_0} = (\omega^n \times X'_v) \cap Q_{uv}$ and $X'_v = \text{dom} Q'_{v_0}$. Note that still $Q'_{1^n \wedge 0, 0^n \wedge 1} = X'_{1^n} \times X'_{0^n}$.

The construction describes how the original change from $X_{u_0}$ to $X'_{u_0}$ spreads through the chain of crucial pairs in $2^{n+1}$, resulting in a system of new sets, $X'_u$ and $Q'_{uv}$, which satisfy (5) for the particular $u_0 \in 2^{n+1}$. We iterate this construction consecutively for all $u_0 \in 2^{n+1}$, getting finally a system of sets satisfying (5) fully and (4) which we denote by $X_u$ and $Q_{uv}$ from now on.

Step 2. We take care of item (6). Consider a pair of $u_0$ and $v_0$ in $2^{n+1}$, such that $u_0(n) = 0$ and $v_0(n) = 1$. By the density of $D'_n$, there is a set $X'_{u_0} \times X'_{v_0} \subseteq D'_n$ included in $X_{u_0} \times X_{v_0}$. We may assume that $X'_{u_0} \cap X'_{v_0} = \emptyset$.

(Indeed it easily follows from Claim 7.2 that there exist reals $x_0 \in X_{u_0}$ and $y_0 \in X_{v_0}$ satisfying $x_0 \in y_0$ and $x_0 \neq y_0$, say $x_0(k) = 0$ while $y_0(k) = 1$. Define $X = \{x \in X_0 : x(k) = 0 \land \exists y \in Y_0 \ (y(k) = 1 \land x \in y)\}$, and $Y$ correspondingly; then $[X]_E = [Y]_E$ and $X \cap Y = \emptyset$.)

Spread the change from $X_{u_0}$ to $X'_{u_0}$ and from $X_{v_0}$ to $X'_{v_0}$ through the chain of crucial pairs in $2^{n+1}$, by the method of Step 1, until the wave of spreading from $u_0$ meets the wave of spreading from $u_0$ at the crucial pair $(1^n \wedge 0, 0^n \wedge 1)$. This leads to a system of sets $X'_u$ and $Q'_{uv}$ which satisfy (7) for the particular pair $(u_0, v_0)$ and still satisfy (6) possibly except for the crucial pair $(1^n \wedge 0, 0^n \wedge 1)$ (for which basically the set $Q'_{1^n \wedge 0, 0^n \wedge 1}$ is not yet defined for this step).

By construction the previous steps leave $Q_{1^n \wedge 0, 0^n \wedge 1}$ in the form $X_{1^n \wedge 0} \times X_{0^n \wedge 1}$, where $X_{1^n \wedge 0}$ and $X_{0^n \wedge 1}$ are the “versions” at the end of Step 1). We now have the new sets, $X'_{1^n \wedge 0}$ and $X'_{0^n \wedge 1}$, included in resp. $X_{1^n \wedge 0}$ and $X_{0^n \wedge 1}$ and satisfying $[X'_{1^n \wedge 0}]_E = [X'_{0^n \wedge 1}]_E$. (Indeed $[X'_{u_0}]_E = [X'_{u_0}]_E$ held at the beginning of the change.) Now we put $Q'_{1^n \wedge 0, 0^n \wedge 1} = (X'_{1^n \wedge 0} \times X'_{0^n \wedge 1}) \cap B$. Then $Q'_{1^n \wedge 0, 0^n \wedge 1} \subseteq B$, and we have $\text{dom} Q'_{1^n \wedge 0, 0^n \wedge 1} = X'_{1^n \wedge 0}$, $\text{ran} Q'_{1^n \wedge 0, 0^n \wedge 1} = X'_{0^n \wedge 1}$ by Remark 8.1 and Lemma 6.2.
This ends the consideration of the pair $\langle u_0, v_0 \rangle$.

Applying this construction consecutively for all pairs of $u_0$ and $v_0$ with $u_0(n) = 0$, $v_0(n) = 1$ (including the pair $\langle 1^n \wedge 0, 0^n \wedge 1 \rangle$) we finally get a system of sets satisfying (1) – (6), except for the requirement $Q_{uv} \in \mathcal{P} \subseteq \mathcal{B}$ in (1) for the pair $\langle u, v \rangle = \langle 1^n \wedge 0, 0^n \wedge 1 \rangle$, — and these sets will be denoted still by $X_u$ and $Q_{uv}$ from now on.

**Step 3.** Now we take care of (7). Consider a crucial pair in $2^{n+1}$, $\langle u_0, v_0 \rangle = \langle 1^k \wedge 0^k \wedge w, 0^k \wedge 1^k \wedge w \rangle \in 2^{n+1}$.

If $k < n$ then $\langle u_0, v_0 \rangle \neq \langle 1^k \wedge 0^k \wedge 1 \rangle$, the set $Q_{u_0,v_0} \in \mathcal{P} \subseteq \mathcal{B}$ is defined at a previous level, and $Q_{u_0,v_0} \subseteq Q_{1^k \wedge 0^k \wedge 1}$. By the density, there exists a set $Q'_{u_0,v_0} \in D_n(Q_{1^k \wedge 0^k \wedge 1})$, $Q'_{u_0,v_0} \subseteq Q_{u_0,v_0}$. If $k = n$ then $\langle u_0, v_0 \rangle = \langle 1^n \wedge 0^n \wedge 1 \rangle$, and by Lemma 2.6 there is a set $Q'_{u_0,v_0} \in \mathcal{P} \subseteq \mathcal{B}$, $Q'_{u_0,v_0} \subseteq Q_{u_0,v_0}$.

In both cases define $X'_{u_0} = \text{dom} Q'_{u_0,v_0}$ and $X'_{v_0} = \text{ran} Q'_{u_0,v_0}$ and spread this change through the chain of crucial pairs in $2^{n+1}$, exactly as above. Note that $[X'_{u_0}]_E = [X'_{v_0}]_E$ as sets in $\mathcal{P} \subseteq \mathcal{B}$ are included in $E$. This keeps $[X'_u]_E = [X'_v]_E$ for all $u, v \in 2^{n+1}$ through the spreading.

Executing this step for all crucial pairs in $2^{n+1}$, we finally accomplish the construction of a system of sets satisfying (1) through (7).

\[\square\] (Theorem 1.1)

**References**


