

A definable E_0 class containing no definable elements

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Abstract

A generic extension $\mathbf{L}[x]$ of \mathbf{L} by a real x is defined, in which the E_0 -class of x is a Π_2^1 set containing no ordinal-definable reals.

1 Introduction

It is known that the existence of a non-empty OD (ordinal-definable) set of reals X with no OD element is consistent with **ZFC**; the set of all non-constructible reals gives an example in many generic models including *e.g.* the Solovay model or the extension of \mathbf{L} , the constructible universe, by a Cohen real.

Can such a set X be countable? That is, is it consistent with **ZFC** that there is a countable OD (or outright definable by a precise set-theoretic formula) set of reals X containing no OD element?

This question was initiated and discussed at the *Mathoverflow* website¹ and at FOM². In particular Ali Enayat (Footnote 2) conjectured that the problem can be solved by the finite-support countable product $\mathbb{P}^{<\omega}$ of the Jensen “minimal Π_2^1 real singleton forcing” \mathbb{P} defined in [4] (see also Section 28A of [3]). Enayat proved that a symmetric part of the $\mathbb{P}^{<\omega}$ -generic extension of \mathbf{L} definitely yields a model of **ZF** (not a model of **ZFC**!) in which there is a Dedekind-finite infinite OD set of reals with no OD elements — namely the set of all reals \mathbb{P} -generic over \mathbf{L} . In fact $\mathbb{P}^{<\omega}$ -generic extensions of \mathbf{L} and their symmetric submodels were considered in [1] (Theorem 3.3) with respect to some other questions.

Following the mentioned conjecture, we proved in [6] that indeed, in a $\mathbb{P}^{<\omega}$ -generic extension of \mathbf{L} , the set of all reals \mathbb{P} -generic over \mathbf{L} is a countable Π_2^1

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¹ A question about ordinal definable real numbers. *Mathoverflow*, March 09, 2010. <http://mathoverflow.net/questions/17608>.

² Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. <http://cs.nyu.edu/pipermail/fom/2010-July/014944.html>

set with no OD elements. The Π_2^1 definability is definitely the best one can get in this context since it easily follows from the Π_1^1 uniformisation theorem that any non-empty Σ_2^1 set of reals definitely contains a Δ_2^1 element.

Jindra Zapletal³ informed us that there is a totally different model of **ZFC** with an OD \mathbf{E}_0 -class⁴ X containing no OD elements. The construction of such a model, not yet published, but described to us in a brief communication, involves a combination of several forcing notions and some modern ideas in descriptive set theory, like models of the form $\mathbf{V}[x]_{\mathbf{E}}$ for $\mathbf{E} = \mathbf{E}_0$, recently presented in [7]; it also does not look to yield X being analytically definable, let alone Π_2^1 .

We prove the next theorem in this paper:

Theorem 1.1. *It is true in a suitable generic extension $\mathbf{L}[x]$ of \mathbf{L} , the constructible universe, by a real $x \in 2^\omega$ that the \mathbf{E}_0 -equivalence class $[x]_{\mathbf{E}_0}$ (hence a countable set) is Π_2^1 , but it has no OD elements.*

The forcing \mathbb{P} we use to prove the theorem is a clone of the abovementioned Jensen forcing, but defined on the base of the Silver forcing instead of the Sacks forcing. The crucial advantage of Silver's forcing here is that it leads to a Jensen-type forcing naturally closed under the 0-1 flip at any digit, so that the corresponding extension contains a Π_2^1 \mathbf{E}_0 -class of generic reals instead of a Π_2^1 generic singleton as in [4]. In fact a bigger family of \mathbf{E}_0 -large trees (perfect trees $T \subseteq 2^{<\omega}$ such that $\mathbf{E}_0 \upharpoonright [T]$ is not smooth, see [5, Section 10.9]) would also work similarly to Silver trees, an by similar reasons.

Remark 1.2. Theorem 1.1 also solves another question asked at the *Mathoverflow* website⁵ : namely,

is there an example of a set S definable in **ZFC** and provable in **ZFC** to be countably infinite, while at the same time, no set definable in **ZFC** can be proved in **ZFC** to be an element of S ?

To define such an example, let S be defined as (1) $[x]_{\mathbf{E}_0}$ provided the set universe is equal to the class $\mathbf{L}[x]$ as in Theorem 1.1, and (2) simply $S = \omega$ otherwise. Suppose towards the contrary that **ZFC** proves that the real x , uniquely defined by a certain fixed formula, outright belongs to S . Then in particular this must be true in case (1), contrary to the definition of S via Theorem 1.1. \square

It remains to note that a *finite* OD set of reals contains only OD reals by obvious reasons. On the other hand, by a result in [2] there can be two *sets* of reals X, Y such that the pair $\{X, Y\}$ is OD but neither X nor Y is OD.

³ Personal communication, Jul 31/Aug 01, 2014.

⁴ Recall that if $x, y \in \omega^\omega$ then $x \mathbf{E}_0 y$ iff $x(n) = y(n)$ for all but finite n .

⁵ A question about definable non-empty sets containing no definable elements. *Mathoverflow*, February 11, 2013, <http://mathoverflow.net/questions/121484>.

2 Trees and Silver-type forcing

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers $0, 1$. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\wedge k$ is the extension of t by k . If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s , while $s \subset t$ means proper extension. If $s \in 2^{<\omega}$ then $\text{lh } s$ is the length of s , and $2^n = \{s \in 2^{<\omega} : \text{lh } s = n\}$ (strings of length n).

Let any $s \in 2^{<\omega}$ **act** on 2^ω so that $(s \cdot x)(k) = x(k) + s(k) \pmod{2}$ whenever $k < \text{lh } s$ and simply $(s \cdot x)(k) = x(k)$ otherwise. If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$ then, as usual, let $s \cdot X = \{s \cdot x : x \in X\}$.

Similarly if $s \in 2^m$, $t \in 2^n$, $m \leq n$, then define $s \cdot t \in 2^n$ so that $(s \cdot t)(k) = t(k) + s(k) \pmod{2}$ whenever $k < \min\{m, n\}$ and $(s \cdot t)(k) = t(k)$ whenever $m \leq k < n$. Note that $\text{lh}(s \cdot t) = \text{lh } t$. Let $s \cdot T = \{s \cdot t : t \in T\}$ for $T \subseteq 2^{<\omega}$.

If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T \upharpoonright_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let **PT** be the set of all *perfect trees* $\emptyset \neq T \subseteq 2^{<\omega}$ (those with no endpoints and no isolated branches). If $T \in \mathbf{PT}$ then there is a largest string $s \in T$ such that $T = T \upharpoonright_s$; it is denoted by $s = \text{stem}(T)$ (the *stem* of T); we have $s^\wedge 1 \in T$ and $s^\wedge 0 \in T$ in this case. If $T \in \mathbf{PT}$ then

$$[T] = \{a \in 2^\omega : \forall n (a \upharpoonright n \in T)\} \subseteq 2^\omega$$

is the perfect set of all *paths through* T .

Let **ST** be the set of all *Silver trees*, that is, those $T \in \mathbf{PT}$ that is a partition $\omega = u_0 \cup u_1 \cup u_{0,1}$ such that $u_{0,1}$ is infinite and if $s \in T$ then

- if $\text{lh } s \in u_0$ then $s^\wedge 0 \in T$ but $s^\wedge 1 \notin T$;
- if $\text{lh } s \in u_1$ then $s^\wedge 1 \in T$ but $s^\wedge 0 \notin T$;
- if $\text{lh } s \in u_{0,1}$ then $s^\wedge 0 \in T$ and $s^\wedge 1 \in T$.

By a **Silver-type forcing (STF)** we understand any set $\mathbb{P} \subseteq \mathbf{ST}$ such that

- (1) \mathbb{P} contains the full tree $2^{<\omega}$;
- (2) if $u \in T \in \mathbb{P}$ then $T \upharpoonright_u \in \mathbb{P}$.
- (3) if $T \in \mathbb{P}$ and $s \in 2^{<\omega}$ then $s \cdot T \in \mathbb{P}$.

Such a set \mathbb{P} can be considered as a forcing notion (if $T \subseteq T'$ then T is a stronger condition), and then it adds a real in 2^ω .

3 Splitting construction over a Silver-type forcing

Assume that $\mathbb{P} \subseteq \mathbf{ST}$ is a **STF**. The set **SS**(\mathbb{P}) of *Silver splitting* constructions over \mathbb{P} consists of all finite systems of trees of the form $\varphi = \{T_s\}_{s \in 2^{<n}}$, where $n = \text{hgt}(\varphi) < \omega$ (the height of φ), satisfying the following conditions:

- (4) each tree $T_s = \varphi(s)$ belongs to \mathbb{P} , — we let $r_s = \text{stem}(T_s)$;

- (5) if $s^{\wedge i} \in 2^{<n}$ ($i = 0, 1$) then $T_{s^{\wedge i}} \subseteq T_s \upharpoonright_{r_{s^{\wedge i}}}$ — it easily follows that $[T_{s^{\wedge 0}}] \cap [T_{s^{\wedge 1}}] = \emptyset$;
- (6) there is an increasing sequence of numbers $h(0) < h(1) < \dots < h(n-1)$ such that $\text{lh } r_s = h(k)$ whenever $s \in 2^k$ and $k < n$;
- (7) if $k < m < n$, $u, v \in 2^m$, and $h(k) < j < h(k+1)$ then $r_u(j) = r_v(j)$.
- (8) if $m < n$, $u, v \in 2^m$, and $t \in 2^{<\omega}$ then $r_u^{\wedge} t \in T_u \iff r_v^{\wedge} t \in T_v$.

The tree $T = \bigcup_{s \in 2^{n-1}} T_s$ belongs to **ST** in this case.

Let φ, ψ be systems in **SS**(\mathbb{P}). Say that

- φ *extends* ψ , symbolically $\psi \preceq \varphi$, if $n = \text{hgt}(\psi) \leq \text{hgt}(\varphi)$ and $\psi(s) = \varphi(s)$ for all $s \in 2^{<n}$;
- ψ *properly extends* φ , symbolically $\psi \prec \varphi$, if in addition $\text{hgt}(\psi) < \text{hgt}(\varphi)$;
- φ *reduces* ψ , if $n = \text{hgt}(\psi) = \text{hgt}(\varphi)$, $\varphi(s) \subseteq \psi(s)$ for all $s \in 2^{n-1}$, and $\varphi(s) = \psi(s)$ for all $s \in 2^{<n-1}$.

In other words, the reduction allows to shrink trees in the top layer of the system, but keeps intact those in the lower layers.

Note that $\varphi = \Lambda$ (the empty system) is the only one with $\text{hgt}(\varphi) = 0$. To get a system φ with $\text{hgt}(\varphi) = 1$ (and then $\text{dom } \varphi = \{\Lambda\}$) put $\varphi(\Lambda) = T$, where $T \in \mathbf{ST}$. The following lemma leads to systems of bigger height.

Lemma 3.1. *Assume that $\mathbb{P} \subseteq \mathbf{ST}$ is a **STF** and $\varphi = \{T_s\}_{s \in 2^{<n}} \in \mathbf{SS}(\mathbb{P})$.*

- (i) *If $s_0 \in 2^{n-1}$, and $T \in \mathbf{ST}$, $T \subseteq T_{s_0}$, then there is a system $\varphi' = \{T'_s\}_{s \in 2^{<n}} \in \mathbf{SS}(\mathbb{P})$ which reduces φ and satisfies $T_{s_0} = T$.*
- (ii) *There is a system $\varphi' = \{T'_s\}_{s \in 2^{<n+1}} \in \mathbf{SS}(\mathbb{P})$ which properly extends φ .*
- (iii) *If a system ψ properly extends φ and a system ψ' reduces ψ then ψ' properly extends φ .*

Proof. By definition all strings $r_s = \text{stem}(T_s)$ with $s \in 2^{n-1}$ satisfy $\text{lh } r_s = h$ for one and the same $h = h(n-1)$.

(i) Put $T'_s = \{r_s^{\wedge} t : r_{s_0}^{\wedge} t \in T\}$ for all $s \in 2^{n-1}$, and still $T'_s = T_s$ for $s \in 2^{<n-1}$. The sets T'_s defined this way belong to \mathbb{P} by (3) of Section 2.

(ii) Put $T'_{s^{\wedge i}} = T_s \upharpoonright_{r_{s^{\wedge i}}}$ for all $s \in 2^{n-1}$ and $i = 0, 1$, and still $T'_s = T_s$ for $s \in 2^{<n}$. The sets $T'_{s^{\wedge i}}$ belong to \mathbb{P} by (2) of Section 2. \square

By the lemma, if $\mathbb{P} \subseteq \mathbf{ST}$ is a **STF** then there is a strictly \prec -increasing sequence $\{\varphi_n\}_{n < \omega}$ in **SS**(\mathbb{P}). The limit system $\varphi = \bigcup_n \varphi_n = \{T_s\}_{s \in 2^{<\omega}}$ then satisfies conditions (4) — (8) on the whole domain $2^{<\omega}$.

Proposition 3.2. *In this case, the tree $T = \bigcap_n \bigcup_{s \in 2^n} T_s$ is still a Silver tree in \mathbf{ST} (not necessarily in \mathbb{P}), and $[T] = \bigcap_n \bigcup_{s \in 2^n} [T_s]$. \square*

Say that a tree T occurs in $\varphi \in \mathbf{SS}(\mathbb{P})$ if $T = \varphi(s)$ for some $s \in 2^{\leq \text{hgt}(\varphi)}$.

We define $\mathbf{SS}^{<\omega}(\mathbb{P})$, **the finite-support product** of countably many copies of $\mathbf{SS}(\mathbb{P})$, to consist of all infinite sequences $\Phi = \{\varphi_k\}_{k \in \omega}$, where each $\varphi_k = \Phi(k)$ belongs to $\mathbf{SS}(\mathbb{P})$ and the set $|\Phi| = \{k : \varphi_k \neq \Lambda\}$ (the support of Φ) is finite. Sequences $\Phi \in \mathbf{SS}(\mathbb{P})$ will be called *multisystems*.

Say that a tree T occurs in $\Phi = \{\varphi_k\}$ if it occurs in some φ_k , $k \in |\Phi|$.

Let Φ, Ψ be multisystems in $\mathbf{SS}^{<\omega}(\mathbb{P})$. We define that

- Φ extends Ψ , symbolically $\Psi \preceq \Phi$, if $\Psi(k) \preceq \Phi(k)$ (in $\mathbf{SS}(\mathbb{P})$) for all k ;
- $\Psi \prec \Phi$, iff $|\Psi| \subseteq |\Phi|$ and $\Psi(k) \prec \Phi(k)$ for all $k \in |\Psi|$;
- Φ reduces Ψ iff $\Phi(k)$ reduces $\Psi(k)$ for all $k \in |\Psi|$.

Corollary 3.3 (of Lemma 3.1). *If $\mathbb{P} \subseteq \mathbf{ST}$ is a **STF** and $\Psi \in \mathbf{SS}^{<\omega}(\mathbb{P})$ then there is a multisystem $\Phi \in \mathbf{SS}^{<\omega}(\mathbb{P})$ such that $\Psi \prec \Phi$. \square*

4 Jensen's extension of a Silver-type forcing

Let \mathbf{ZFC}' be the subtheory of \mathbf{ZFC} including all axioms except for the power set axiom, plus the axiom saying that $\mathcal{P}(\omega)$ exists. (Then ω_1 and continual sets like \mathbf{PT} exist as well.) Let \mathfrak{M} be a countable transitive model of \mathbf{ZFC}' .

Suppose that $\mathbb{P} \in \mathfrak{M}$, $\mathbb{P} \subseteq \mathbf{ST}$ is a **STF**. Then the sets $\mathbf{SS}(\mathbb{P})$ and $\mathbf{SS}^{<\omega}(\mathbb{P})$ belong to \mathfrak{M} , too.

Definition 4.1. Consider any \preceq -increasing sequence $\Phi = \{\Phi^j\}_{j < \omega}$ of multisystems $\Phi^j = \{\varphi_k^j\}_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$, *generic over \mathfrak{M}* in the sense that it intersects every set $D \in \mathfrak{M}$, $D \subseteq \mathbf{SS}^{<\omega}(\mathbb{P})$, dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$ ⁶.

Then in particular it intersects every set of the form

$$D_k = \{\Phi \in \mathbf{SS}^{<\omega}(\mathbb{P}) : \forall k' \leq k (k \leq \text{hgt}(\Phi(k')))\}.$$

Hence if $k < \omega$ then the sequence $\{\varphi_k^j\}_{j < \omega}$ of systems $\varphi_k^j \in \mathbf{SS}(\mathbb{P})$ is *eventually strictly increasing*, so that $\varphi_k^j \prec \varphi_k^{j+1}$ for infinitely many indices j (and $\varphi_k^j = \varphi_k^{j+1}$ for other j). Therefore there is a system of trees $\{\mathbf{T}_k(s)\}_{k < \omega \wedge s \in 2^{<\omega}}$ in \mathbb{P} such that $\varphi_k^j = \{\mathbf{T}_k(s)\}_{s \in 2^{<h(j,k)}}$, where $h(j,k) = \text{hgt}(\varphi_k^j)$. Then

$$\mathbf{U}_k = \bigcap_n \bigcup_{s \in 2^n} \mathbf{T}_k(s) \quad \text{and} \quad \mathbf{U}_k(s) = \bigcap_{n \geq 1} \bigcup_{t \in 2^n, s \subseteq t} \mathbf{T}_k(t)$$

are trees in \mathbf{ST} (not necessarily in \mathbb{P}) by Proposition 3.2 for each k and $s \in 2^{<\omega}$; thus $\mathbf{U}_k = \mathbf{U}_k(\Lambda)$. In fact $\mathbf{U}_k(s) = \mathbf{U}_k \cap \mathbf{T}_k(s)$ by (5). \square

⁶ Meaning that for any $\Psi \in \mathbf{SS}^{<\omega}(\mathbb{P})$ there is $\Phi \in D$ with $\Psi \preceq \Phi$.

Lemma 4.2. *The set of trees $\mathbb{U} = \{t \cdot \mathbf{U}_k(s) : k < \omega \wedge s \in 2^{<\omega} \wedge t \in 2^{<\omega}\}$ satisfies (2) and (3) while the union $\mathbb{P} \cup \mathbb{U}$ is a **STF**. \square*

Lemma 4.3. *The set \mathbb{U} is dense in $\mathbb{U} \cup \mathbb{P}$.*

Proof. Suppose that $T \in \mathbb{P}$. The set $D(T)$ of all multisystems $\Phi = \{\varphi_k\}_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$, such that $\varphi_k(\Lambda) = T$ for some k , belongs to \mathfrak{M} and obviously is dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$. It follows that $\Phi^j \in D(T)$ for some j , by the choice of Φ . Then $\mathbf{T}_k(\Lambda) = T$ for some k . However $\mathbf{U}_k(\Lambda) \subseteq \mathbf{T}_k(\Lambda)$. \square

Lemma 4.4. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ is pre-dense in \mathbb{P} , and $U \in \mathbb{U}$, then $U \subseteq^{\text{fin}} \bigcup D$, that is, there is a finite $D' \subseteq D$ with $U \subseteq \bigcup D'$. Moreover D remains pre-dense in $\mathbb{U} \cup \mathbb{P}$.*

Proof. Suppose that $U = \mathbf{U}_K(s) \in \mathbb{U}$, $K < \omega$ and $s \in 2^{<\omega}$. (The general case, when $U = t \cdot \mathbf{U}_K(s)$ for some $t \in 2^{<\omega}$, is easily reducible to the particular case $U = \mathbf{U}_K(s)$ by substituting the set $t \cdot D$ for D .) Consider the set $\Delta \in \mathfrak{M}$ of all multisystems $\Phi = \{\varphi_k\}_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ such that $K \in |\Phi|$, $\mathbf{1}h s < h = \text{hgt}(\varphi_K)$, and for each $t \in 2^{h-1}$ there is a tree $S_t \in D$ with $\varphi_K(t) \subseteq S_t$. The set Δ is dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$ by Lemma 3.1 and the pre-density of D . Therefore there is an index j such that Φ^j belongs to Δ . Let this be witnessed by trees $S_t \in D$, $t \in 2^{h-1}$, where $\mathbf{1}h s < h = \text{hgt}(\varphi_K^j)$, so that $\varphi_K^j(t) \subseteq S_t$. Then

$$U = \mathbf{U}_K(s) \subseteq \mathbf{U}_K(\Lambda) \subseteq \bigcup_{t \in 2^{h-1}} \varphi_K^j(t) \subseteq \bigcup_{t \in 2^{h-1}} S_t \subseteq \bigcup D'$$

by construction, where $D' = \{S_t : t \in 2^{h-1}\} \subseteq D$ is finite.

To prove the pre-density, consider any string $t \in 2^{h-1}$ with $s \subset t$. Then $V = \mathbf{U}_K(t) \in \mathbb{U}$ and $V \subseteq U$. On the other hand, $V \subseteq S_t \in D$. Thus the tree V witnesses that U is compatible with $S_t \in D$ in $\mathbb{U} \cup \mathbb{P}$, as required. \square

5 Forcing a real away of a pre-dense set

Let \mathfrak{M} be still a countable transitive model of **ZFC'** and $\mathbb{P} \in \mathfrak{M}$, $\mathbb{P} \subseteq \mathbf{ST}$ be a **STF**. The goal of the following Theorem 5.5 is to prove that, in the conditions of Definition 4.1, for any \mathbb{P} -name c of a real in 2^ω , it is forced by the extended forcing $\mathbb{P} \cup \mathbb{U}$ that c does not belong to sets $[U]$ where u is a tree in \mathbb{U} — unless c is a name of one of reals in the \mathbf{E}_0 -class of the generic real x itself. We begin with a suitable notation.

Definition 5.1. A \mathbb{P} -real name is a system $\mathbf{c} = \{C_n^i\}_{n < \omega, i < 2}$ of sets $C_n^i \subseteq \mathbb{P}$ such that each set $C_n = C_n^0 \cup C_n^1$ is dense or at least pre-dense in \mathbb{P} and if $S \in C_n^0$ and $T \in C_n^1$ then S, T are incompatible in \mathbb{P} .

If in addition $\sigma \in 2^{<\omega}$ then define a \mathbb{P} -real name $\sigma \mathbf{c} = \{\sigma \cdot C_n^i\}_{n < \omega, i < 2}$, where $\sigma \cdot C_n^i = \{\sigma \cdot T : T \in C_n^i\}$.

If a set $G \subseteq \mathbb{P}$ is \mathbb{P} -generic at least over the collection of all sets C_n then we define $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_n^i \neq \emptyset$. \square

Thus any \mathbb{P} -real name $\mathbf{c} = \{C_n^i\}$ is a \mathbb{P} -name for a real in 2^ω .
Recall that \mathbb{P} adds a real $x \in 2^\omega$.

Example 5.2. Let $k < \omega$. Define a \mathbb{P} -real name $\dot{\mathbf{x}} = \{C_n^i\}_{n < \omega, i < 2}$ such that each set C_n^i contains a single tree $R_i^n = \{s \in 2^{<\omega} : \text{lh } s > n \implies s(n) = i\} \in \mathbf{ST}$. Then $\dot{\mathbf{x}}$ is a \mathbb{P} -name of the \mathbb{P} -generic real x , and accordingly each name $\sigma\dot{\mathbf{x}}$ ($\sigma \in 2^{<\omega}$) is a \mathbb{P} -name of $\sigma \cdot x$. \square

Let $\mathbf{c} = \{C_n^i\}$ and $\mathbf{d} = \{C_n^i\}$ be \mathbb{P} -real names. Say that $T \in \mathbf{ST}$:

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$ and $i = 0, 1$, iff $T \subseteq R_i^n$ (that is, the tree T satisfies $x(n) = i$ for all $x \in [T]$);
- *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \text{lh } s$, T directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{d} \neq \mathbf{c}$, iff there are strings $s, t \in 2^{<\omega}$, incomparable in $2^{<\omega}$ and such that T directly forces $s \subset \mathbf{c}$ and $t \subset \mathbf{d}$;
- *directly forces* $\mathbf{c} \notin [S]$, where $S \in \mathbf{PT}$, iff there is a string $s \in 2^{<\omega} \setminus S$ such that T directly forces $s \subset \mathbf{c}$;

Lemma 5.3. *If $S_1, \dots, S_n, T \in \mathbb{P}$ and \mathbf{c} is a \mathbb{P} -real name then there exist trees $S'_1, \dots, S'_n, T' \in \mathbb{P}$ such that $S'_i \subseteq S_i$ for all $i = 1, \dots, n$, $T' \subseteq T$, and T' directly forces $\mathbf{c} \notin [S']$, where $S' = \bigcup_{1 \leq i \leq n} S'_i$.*

Proof. Clearly there is a tree $T' \in \mathbb{P}$, $T' \subseteq T$, which directly forces $s \subset \mathbf{c}$ for some $s \in 2^{<\omega}$ satisfying $\text{lh } s > \text{lh}(\text{stem}(S_i))$ for every i . Then there is a collection of strings $u_i \in S_i$ incomparable with s . Put $S'_i = S \upharpoonright_{u_i}$; then obviously $s \notin S' = \bigcup_{1 \leq i \leq n} S'_i$. \square

Lemma 5.4. *If \mathbf{c} is a \mathbb{P} -real name, $\sigma \in 2^{<\omega}$, and $T \in \mathbb{P}$ directly forces $\sigma\mathbf{c} \neq \dot{\mathbf{x}}$, then there is a tree $S \in \mathbb{P}$, $S \subseteq T$, which directly forces $\mathbf{c} \notin [\sigma \cdot S]$.*

Proof. Taking $T' = \sigma \cdot T$ instead of T and $\mathbf{c}' = \sigma\mathbf{c}$ instead of \mathbf{c} , we reduce the problem to the case $\sigma = \Lambda$, that is, $\sigma\mathbf{c} = \mathbf{c}$ and $\sigma \cdot S = S$. Thus let's assume that T directly forces $\mathbf{c} \neq \dot{\mathbf{x}}$. There are incomparable strings $s, t \in 2^{<\omega}$ such that T directly forces $s \subset \mathbf{c}$ and $t \subset \dot{\mathbf{x}}$. Then by necessity $t \in T$, hence, $S = T \upharpoonright_t \in \mathbb{P}$ but $s \notin S$. By definition S directly forces $\mathbf{c} \notin [S]$, as required. \square

Theorem 5.5. *In the assumptions of Definition 4.1, suppose that $\mathbf{c} = \{C_m^i\}_{m < \omega, i < 2} \in \mathfrak{M}$ is a \mathbb{P} -real name, and for every $\sigma \in 2^{<\omega}$ the set*

$$D(\sigma) = \{T \in \mathbb{P} : T \text{ directly forces } \mathbf{c} \neq \sigma\dot{\mathbf{x}}\}$$

is dense in \mathbb{P} . Let $W \in \mathbb{P} \cup \mathbb{U}$ and $U \in \mathbb{U}$. Then there is a stronger condition $V \in \mathbb{U}$, $V \subseteq W$, which directly forces $\mathbf{c} \notin [U]$.

Proof. By construction, $U = \sigma \cdot \mathbf{U}_K(s_0)$, where $K < \omega$ and $\sigma, s_0 \in 2^{<\omega}$; we can assume that simply $s_0 = \Lambda$, so that $U = \sigma \cdot \mathbf{U}_K$. Further, by the same reasons as in the proof of Lemma 5.4, we can assume that $\sigma = \Lambda$, so that $U = \mathbf{U}_K$. Finally, by Lemma 4.3, we can assume that $W = \mathbf{U}_L(t_0) \in \mathbb{U}$, where $L < \omega$ and $t_0 \in 2^{<\omega}$. The indices K, L involved can be either equal or different.

There is an index J such that the multisystem $\Phi^J = \{\varphi_k^J\}_{k \in \omega}$ satisfies $K, L \in |\Phi^J|$ and $\text{hgt}(\varphi_L^J) > \text{lh } t_0$, so that the trees

$$S_0 = \varphi_K^J(\Lambda) = \mathbf{T}_K(\Lambda) \quad \text{and} \quad T_0 = \varphi_L^J(t_0) = \mathbf{T}_L(t_0)$$

in \mathbb{P} are defined. Note that $U \subseteq S_0$ and $W \subseteq T_0$.

Consider the set \mathcal{D} of all multisystems $\Phi = \{\varphi_k\}_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ such that $\Phi^J \preceq \Phi$ and there is a tree $T \in \mathbb{P}$, $T \subseteq T_0$ satisfying

- (9) T directly forces $\mathbf{c} \notin [S]$, where $S = \bigcup_{s \in 2^{h-1}} \varphi_K(s)$, $h = \text{hgt}(\varphi_K)$; and
- (10) the tree T occurs in Φ (see Section 3), and more specifically, $T = \varphi_L(t)$, where $t \in 2^{h'-1}$, $h' = \text{hgt}(\varphi_L)$, and $t_0 \subset t$.

Lemma 5.6. \mathcal{D} is dense in $\mathbf{SS}^{<\omega}(\mathbb{P})$ above Φ^J .

Proof. Consider any multisystem $\Phi^* = \{\varphi_k^*\}_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ with $\Phi^J \preceq \Phi^*$; the goal is to define a multisystem $\Phi' \in \mathcal{D}$ such that $\Phi^* \preceq \Phi'$. By Corollary 3.3 there is an intermediate multisystem $\Phi = \{\varphi_k\}_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ satisfying $\Phi^* \prec \Phi$; then any multisystem $\Phi' \in \mathbf{SS}^{<\omega}(\mathbb{P})$, which is a reduction of Φ , still satisfies $\Phi^* \prec \Phi'$ and $\Phi^* \preceq \Phi'$. Thus it suffices to find a multisystem $\Phi' \in \mathcal{D}$ which reduces Φ .

Let $h = \text{hgt}(\varphi_K)$ and $h' = \text{hgt}(\varphi_L)$. Then $\text{hgt}(\varphi_K^J) < h$ and $\text{hgt}(\varphi_L^J) < h'$ strictly. Pick a string $t \in 2^{h'-1}$ with $t_0 \subset t$; let $R = \varphi_L(t)$; $R \subseteq T_0$ is a tree in \mathbb{P} . Let $2^{h-1} = \{s_1, \dots, s_N\}$, where N is the integer 2^{h-1} , and $S_i = \varphi_K(s_i)$.

Case 1: $K \neq L$. By Lemma 5.3, there exist trees $S'_1 \subseteq S_1, \dots, S'_n \subseteq S_n$ and $T' \subseteq R$ in \mathbb{P} such that T' directly forces $\mathbf{c} \notin [S']$, where $S' = \bigcup_{1 \leq i \leq N} S'_i$. Define a multisystem $\Phi' = \{\varphi'_k\}_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ so that $\varphi'_L(t) = T'$, $\varphi'_K(s_i) = S'_i$ for all $i = 1, \dots, N$, and $\varphi'_k(s) = \varphi_k(s)$ for all other applicable values of k and s . Then Φ' belongs to \mathcal{D} and is a reduction of Φ , as required.

Case 2: $L = K$, and hence $h' = h$. Now t is one of s_i , say $t = s_{i(t)}$, and the construction as in Case 1 does not work. Nevertheless, following the same arguments, we find trees $S'_i \subseteq S_i$, $i = 1, \dots, N$, $i \neq i(t)$, and $T \subseteq R = \varphi_K(t)$ in \mathbb{P} such that T directly forces $\mathbf{c} \notin [S']$, where $S' = \bigcup_{1 \leq i \leq N, i \neq i(t)} S'_i$.

Further, as the set $D(\Lambda)$ is dense, there is a tree $T' \in \mathbb{P}$, $T' \subseteq T$, which directly forces $\mathbf{c} \neq \dot{\mathbf{x}}$. By Lemma 5.4, there is an even smaller tree $T'' \in \mathbb{P}$, $T'' \subseteq T'$, which directly forces $\mathbf{c} \notin [T'']$, that is, T'' directly forces $\mathbf{c} \notin [S' \cup T'']$. Define a multisystem $\Phi' = \{\varphi'_k\}_{k \in \omega} \in \mathbf{SS}^{<\omega}(\mathbb{P})$ so that $\varphi'_K(s_i) = S'_i$ for all $i = 1, \dots, N$, $i \neq i(t)$, $\varphi'_K(t) = T''$, and $\varphi'_k(s) = \varphi_k(s)$ for all other applicable values of k and s . Then $\Phi' \in \mathcal{D}$ and Φ' is a reduction of Φ . \square (Lemma)

Come back to the proof of the theorem. It follows from the lemma that there is an index $j \geq J$ such that the multisystem $\Phi^j = \{\varphi_k^j\}_{k \in \omega}$ belongs to \mathcal{D} , and let this be witnessed by a tree $T = \varphi_L^j(t) \subseteq T_0 = \varphi_L^j(t_0) = \mathbf{T}_L(t_0)$, where $t \in 2^{h'-1}$, $h' = \text{hgt}(\varphi_L^j)$, and $t_0 \subset t$, satisfying (9).

Consider the tree $V = \mathbf{U}_L(t) \in \mathcal{U}$. By construction we have both $V \subseteq W$ and $V \subseteq T \subseteq T_0$. Therefore V directly forces $\mathbf{c} \notin [S]$ by the choice of T (which satisfies (9)), where $S = \bigcup_{s \in 2^{h-1}} \varphi_K^j(s)$, $h = \text{hgt}(\varphi_K)$. And finally, we have $U \subseteq S$, so that V directly forces $\mathbf{c} \notin [S]$, as required. \square

6 Jensen's forcing

In this section, we argue in \mathbf{L} , the constructible universe. Let $\leq_{\mathbf{L}}$ be the canonical wellordering of \mathbf{L} .

Definition 6.1 (in \mathbf{L}). Following the construction in [4, Section 3] *mutatis mutandis*, we define, by induction on $\xi < \omega_1$, a countable set of trees $\mathcal{U}_\xi \subseteq \mathbf{ST}$ satisfying requirements (2) and (3) of Section 2, as follows.

Let \mathcal{U}_0 consist of all clopen trees $\emptyset \neq S \subseteq 2^{<\omega}$, including $2^{<\omega}$ itself.

Suppose that $0 < \lambda < \omega_1$, and countable sets $\mathcal{U}_\xi \subseteq \mathbf{ST}$ are already defined. Let \mathfrak{M}_ξ be the least model \mathfrak{M} of \mathbf{ZFC}' of the form \mathbf{L}_κ , $\kappa < \omega_1$, containing $\{\mathcal{U}_\xi\}_{\xi < \lambda}$ and such that $\alpha < \omega_1^{\mathfrak{M}}$ and all sets \mathcal{U}_ξ , $\xi < \lambda$, are countable in \mathfrak{M} .

Then $\mathbb{P}_\lambda = \bigcup_{\xi < \lambda} \mathcal{U}_\xi$ is countable in \mathfrak{M} , too. Let $\{\Phi^j\}_{j < \omega}$ be the $\leq_{\mathbf{L}}$ -least sequence of multisystems $\Phi^j \in \mathbf{SS}^{<\omega}(\mathbb{P}_\lambda)$, \preceq -increasing and generic over \mathfrak{M}_λ , and let $\mathcal{U}_\lambda = \mathcal{U}$ be defined, as in Definition 4.1 and Lemma 4.2.

Let $\mathbb{P} = \bigcup_{\xi < \omega_1} \mathcal{U}_\xi$. \square

Proposition 6.2 (in \mathbf{L}). *The sequence $\{\mathcal{U}_\xi\}_{\xi < \omega_1}$ belongs to Δ_1^{HC} .* \square

Lemma 6.3 (in \mathbf{L}). *If a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathbb{P}_\xi$ is pre-dense in \mathbb{P}_ξ then it remains pre-dense in \mathbb{P} . Hence if $\xi < \omega_1$ then \mathcal{U}_ξ is pre-dense in \mathbb{P} .*

Proof. By induction on $\lambda \geq \xi$, if D is pre-dense in \mathbb{P}_λ then it remains pre-dense in $\mathbb{P}_{\lambda+1} = \mathbb{P}_\lambda \cup \mathcal{U}_\lambda$ by Lemma 4.4. Limit steps are obvious. To prove the second part, note that \mathcal{U}_ξ is dense in $\mathbb{P}_{\xi+1}$ by Lemma 4.3, and $\mathcal{U}_\xi \in \mathfrak{M}_{\xi+1}$. \square

Lemma 6.4 (in \mathbf{L}). *If $X \subseteq \text{HC} = \mathbf{L}_{\omega_1}$ then the set W_X of all ordinals $\xi < \omega_1$ such that $\langle \mathbf{L}_\xi; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X \rangle$ and $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ is unbounded in ω_1 . More generally, if $X_n \subseteq \text{HC}$ for all n then the set W of all ordinals $\xi < \omega_1$, such that $\langle \mathbf{L}_\xi; \{X_n \cap \mathbf{L}_\xi\}_{n < \omega} \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; \{X_n\}_{n < \omega} \rangle$ and $\{X_n \cap \mathbf{L}_\xi\}_{n < \omega} \in \mathfrak{M}_\xi$, is unbounded in ω_1 .*

Proof. Let $\xi_0 < \omega_1$. By standard arguments, there are ordinals $\xi < \lambda < \omega_1$, $\xi > \xi_0$, such that $\langle \mathbf{L}_\lambda; \mathbf{L}_\xi, X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_2}; \mathbf{L}_{\omega_1}, X \rangle$. Then $\langle \mathbf{L}_\xi; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X \rangle$, of course. Moreover,

ξ is uncountable in \mathbf{L}_λ , hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\xi$. It follows that $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ since $X \cap \mathbf{L}_\xi \in \mathbf{L}_\lambda$ by construction. The second claim does not differ much. \square

Corollary 6.5 (compare to [4], Lemma 6). *The forcing \mathbb{P} satisfies CCC in \mathbf{L} .*

Proof. Suppose that $A \subseteq \mathbb{P}$ is a maximal antichain. By Lemma 6.4, there is an ordinal ξ such that $A' = A \cap \mathbb{P}_\xi$ is a maximal antichain in \mathbb{P}_ξ and $A' \in \mathfrak{M}_\xi$. But then A' remains pre-dense, therefore, still a maximal antichain, in the whole set \mathbb{P} by Lemma 6.3. It follows that $A = A'$ is countable. \square

7 The model

We consider the set $\mathbb{P} \in \mathbf{L}$ (Definition 6.1) as a forcing notion over \mathbf{L} .

Lemma 7.1 (compare to Lemma 7 in [4]). *A real $x \in 2^\omega$ is \mathbb{P} -generic over \mathbf{L} iff $x \in Z = \bigcap_{\xi < \omega_1^{\mathbf{L}}} \bigcup_{U \in \mathcal{U}_\xi} [U]$.*

Proof. If $\xi < \omega_1^{\mathbf{L}}$ then \mathcal{U}_ξ is pre-dense in \mathbb{P} by Lemma 6.3, therefore any real $x \in 2^\omega$ \mathbb{P} -generic over \mathbf{L} belongs to $\bigcup_{U \in \mathcal{U}_\xi} [U]$.

To prove the converse, suppose that $x \in Z$ and prove that x is \mathbb{P} -generic over \mathbf{L} . Consider a maximal antichain $A \subseteq \mathbb{P}$ in \mathbf{L} ; we have to prove that $x \in \bigcup_{T \in A} [T]$. Note that $A \subseteq \mathbb{P}_\xi$ for some $\xi < \omega_1^{\mathbf{L}}$ by Corollary 6.5. But then every tree $U \in \mathcal{U}_\xi$ satisfies $U \subseteq^{\text{fin}} \bigcup A$ by Lemma 4.4, so that $\bigcup_{U \in \mathcal{U}_\xi} [U] \subseteq \bigcup_{T \in A} [T]$, and hence $x \in \bigcup_{T \in A} [T]$, as required. \square

Corollary 7.2 (compare to Corollary 9 in [4]). *In any generic extension of \mathbf{L} , the set of all reals in 2^ω \mathbb{P} -generic over \mathbf{L} is Π_1^{HC} and Π_2^1 .*

Proof. Use Lemma 7.1 and Proposition 6.2. \square

Definition 7.3. From now on, let $G \subseteq \mathbb{P}$ be a set \mathbb{P} -generic over \mathbf{L} , so that $X = \bigcap_{T \in G} [T]$ is a singleton $X_G = \{x_G\}$. \square

Compare the next lemma to Lemma 10 in [4]. While Jensen's forcing notion in [4] guarantees that there is a single generic real in the extension, the forcing notion \mathbb{P} we use adds a whole \mathbf{E}_0 -class (a countable set) of generic reals!

Lemma 7.4 (in the assumptions of Definition 7.3). *If $y \in \mathbf{L}[G] \cap 2^\omega$ then y is a \mathbb{P} -generic real over \mathbf{L} iff $y \in [x_G]_{\mathbf{E}_0} = \{\sigma \cdot x_G : \sigma \in 2^{<\omega}\}$.*

Proof. The real x_G itself is \mathbb{P} -generic, of course. It follows that any real $y = \sigma \cdot x_G \in [x_G]_{\mathbf{E}_0}$ is \mathbb{P} -generic as well since the forcing \mathbb{P} is by definition invariant under the action of any $\sigma \in 2^{<\omega}$.

To prove the converse, suppose towards the contrary that there is a tree $T \in \mathbb{P}$ and a \mathbb{P} -real name $\mathbf{c} = \{C_n^i\}_{n < \omega, i=0,1} \in \mathbf{L}$ such that T \mathbb{P} -forces that \mathbf{c} is \mathbb{P} -generic while \mathbb{P} forces that $\mathbf{c} \neq \sigma \cdot \dot{x}$ for all $\sigma \in 2^{<\omega}$.

Let $C_n = C_n^0 \cup C_n^1$; this is a pre-dense set in \mathbb{P} . It follows from Lemma 6.4 that there is an ordinal $\lambda < \omega_1$ such that each set $C'_n = C_n \cap \mathbb{P}_\lambda$ is pre-dense in \mathbb{P}_λ , and the sequence $\{C'_{ni}\}_{n < \omega, i=0,1}$ belongs to \mathfrak{M}_λ , where $C'_{ni} = C'_n \cap C_n^i$ — then C'_n is pre-dense in \mathbb{P} too, by Lemma 6.3. Thus we can assume that in fact $C_n = C'_n$, that is, $\mathbf{c} \in \mathfrak{M}_\lambda$ and \mathbf{c} is a \mathbb{P} -real name.

Further, as $\mathbb{P}^{<\omega}$ forces that $\mathbf{c} \neq \sigma \cdot \dot{\mathbf{x}}$, the set $D(\sigma)$ of all conditions $S \in \mathbb{P}$ which directly force $\mathbf{c} \neq \sigma \cdot \dot{\mathbf{x}}$, is dense in \mathbb{P} — for every $\sigma \in 2^{<\omega}$. Therefore, still by Lemma 6.4, we may assume that the same ordinal λ as above satisfies the following: each set $D'(\sigma) = D(\sigma) \cap \mathbb{P}_\lambda$ is dense in \mathbb{P}_λ .

Applying Theorem 5.5 with $\mathbb{P} = \mathbb{P}_\lambda$, $\mathbb{U} = \mathbb{U}_\lambda$, and $\mathbb{P} \cup \mathbb{U} = \mathbb{P}_{\lambda+1}$, we conclude that for each $U \in \mathbb{U}_\lambda$ the set Q_U of all conditions $V \in \mathbb{P}_{\lambda+1}$ which directly force $\mathbf{c} \notin [U]$, is dense in $\mathbb{P}_{\lambda+1}$. As obviously $Q_U \in \mathfrak{M}_{\lambda+1}$, we further conclude that Q_U is pre-dense in the whole forcing \mathbb{P} by Lemma 6.3. This implies that \mathbb{P} forces $\mathbf{c} \notin \bigcup_{U \in \mathbb{U}_\lambda} [U]$, hence, forces that \mathbf{c} is not \mathbb{P} -generic, by Lemma 7.1. But this contradicts to the choice of T . \square

Lemma 7.5 (in the assumptions of Definition 7.3). *x_G is not OD in $\mathbf{L}[G]$.*

Proof. Suppose towards the contrary that there is a tree $T \in G$ and a formula $\vartheta(x)$ with ordinal parameters such that T \mathbb{P} -forces that x_G is the only $x \in 2^\omega$ satisfying $\vartheta(x)$. Let $s = \text{stem}(T)$, so that both $s^\wedge 0$ and $s^\wedge 1$ belong to T . Then either $s^\wedge 0 \subset x_G$ or $s^\wedge 1 \subset x_G$; let, say, $s^\wedge 0 \subset x_G$.

Let $n = \text{lh } s$ and $\sigma = 0^{n \wedge} 1$, so that all three strings $s^\wedge 0, s^\wedge 1, \sigma$ belong to 2^{n+1} , and $s^\wedge 1 = \sigma \cdot s^\wedge 0$. As the forcing \mathbb{P} is invariant under the action of σ , the set $G' = \sigma \cdot G$ is \mathbb{P} -generic over \mathbf{L} , and $T = \sigma \cdot T \in G'$. It follows that it is true in $\mathbf{L}[G'] = \mathbf{L}[G]$ that the real $x' = x_{G'} = \sigma \cdot x_G$ is still the only x satisfying $\vartheta(x)$. However obviously $x' \neq x$! \square

Now, arguing in the \mathbb{P} -generic model $\mathbf{L}[G] = \mathbf{L}[x_G]$, we observe that the countable set $X = [x_G]_{\mathbb{E}_0}$ is exactly the set of all \mathbb{P} -generic reals by Lemma 7.4, hence it belongs to Π_2^1 by Corollary 7.2, and finally it contains no OD elements by Lemma 7.5, as required.

\square (Theorem 1.1)

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