Counterexamples to countable-section $\mathit{\Pi}^1_2$ uniformization and $\mathit{\Pi}^1_3$ separation

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Abstract

We make use of a finite support product of the Jensen minimal $\mathit{\Pi}^1_2$ singleton forcing to define a model in which $\mathit{\Pi}^1_2$ Uniformization fails for a set with countable cross-sections. We also define appropriate submodels of the same model in which Separation fails for $\mathit{\Pi}^1_3$.

1 Introduction

The uniformization problem, introduced by Luzin [13], is well known in modern set theory. (See Moschovakis [14] and Kechris [12] for both older and more recent studies.) In particular, it is known that every $\mathbf{\Sigma}^1_2$ set can be uniformized by a set of the same class $\mathbf{\Sigma}^1_2$, but on the other hand, there is a $\mathbf{\Pi}^1_2$ set (in fact, a lightface $\mathit{\Pi}^1_2$ set), not uniformizable by any set in $\mathbf{\Pi}^1_2$.

The negative part of this result cannot be strengthened much further in the direction of more complicated uniformizing sets since any $\mathbf{\Pi}^1_2$ set admits a $\mathbf{\Delta}^1_3$-uniformization assuming $\mathbb{V} = \mathbf{L}$ and admits a $\mathbf{\Pi}^1_3$-uniformization assuming the existence of sharps (the Martin – Solovay – Mansfield theorem, [14, 8H.10]).

However, the mentioned $\mathbf{\Pi}^1_2$-non-uniformization theorem can be strengthened in the context of consistency. For instance, the $\mathit{\Pi}^1_2$ set

$$P = \{\langle x, y\rangle : x, y \in 2^\omega \wedge y \notin \mathbf{L}[x]\}$$

is not uniformizable by any ROD (real-ordinal definable) set in the Solovay model and many other models of **ZFC** in which it is not true that $\mathbb{V} = \mathbf{L}[x]$ for a real x, and then the cross-sections of P can be considered as "large", in particular, they are definitely uncountable. Therefore one may ask:

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Question 1.1. Can such a ROD-non-uniformizable Π^1_2 set P have the property that all his cross-sections are at most countable?

This question is obviously connected with another question, initiated and briefly discussed at the *Mathoverflow* exchange desk [1] and at FOM [2] :

Question 1.2. Is it consistent with **ZFC** that there is a *countable* definable set of reals $X \neq \varnothing$ which has no OD (ordinal definable) elements.

Ali Enayat (Footnote 2) conjectured that Question 1.2 can be solved in the positive by the finite-support product $\mathbb{P}^{<\omega}$ of countably many copies of the Jensen "minimal Π^1_2 real singleton forcing" $\mathbb{P}$ defined in [7] (see also Section 28A of [5]). Enayat demonstrated that a symmetric part of the $\mathbb{P}^{<\omega}$-generic extension of **L** definitely yields a model of **ZF** (not a model of **ZFC**!) in which there is a Dedekind-finite infinite OD set of reals with no OD elements.

Following the mentioned conjecture, we proved in [8] that indeed it is true in a $\mathbb{P}^{<\omega}$-generic extension of **L** that the set of $\mathbb{P}$-generic reals is a countable non-empty Π^1_2 set with no OD elements. [3] Using a finite-support product $\prod_{\xi<\omega_1} {\mathbb{P}_\xi}^{<\omega}$, where all $\mathbb{P}_\xi$ are forcings similar to, but different from, Jensen's forcing $\mathbb{P}$ (and from each other), we answer Question 1.1 in the positive.

Theorem 1.3. *In a suitable generic extension of* **L**, *it is true that there is a lightface* Π^1_2 *set* $P \subseteq 2^\omega \times 2^\omega$ *whose all cross-sections* $P_x = \{y : \langle x, y\rangle \in P\}$ *are at most countable, but* P *is not uniformizable by a ROD set.*

Using an appropriate generic extension of a submodel of the same model, similar to models considered in Harrington's unpublished notes [3], we also prove

Theorem 1.4. *In a suitable generic extension of* **L**, *it is true that there is a pair of disjoint lightface* Π^1_3 *sets* $X, Y \subseteq 2^\omega$, *not separable by disjoint* $\mathbf{\Sigma}^1_3$ *sets, and hence* $\mathbf{\Pi}^1_3$ *Separation and* Π^1_3 *Separation fail.*

This result was first proved by Harrington in [3] on the base of almost disjoint forcing of Jensen – Solovay [6], and in this form has never been published, but was mentioned, *e.g.*, in [14, 5B.3] and [4, page 230]. A complicated alternative proof of Theorem 1.4 can be obtained with the help of *countable-support* products and iterations of Jensen's forcing studied earlier in [1, 10, 11]. The

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[1] A question about ordinal definable real numbers. Mathoverflow, March 09, 2010. `http://mathoverflow.net/questions/17608`.

[2] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. `http://cs.nyu.edu/pipermail/fom/2010-July/014944.html`

[3] We also proved in [9] that the existence of a Π^1_2 E_0-class with no OD elements is consistent with **ZFC**, using a E_0-invariant version of the Jensen forcing.

finite-support approach which we pursue here yields a significantly more compact proof. As far as Theorem 1.3 is concerned, countable-support products and iterations hardly can lead to the countable-section non-uniformization results.

We recall that the $\mathbf{\Pi}^1_3$ Separation *hold* in $\mathbf{L}$, the constructible universe. Thus Theorem 1.4 in fact shows that the $\mathbf{\Pi}^1_3$ Separation principle is "killed" in an appropriate generic extension of $\mathbf{L}$. It would be interesting to find a generic extension in which, the other way around, the $\mathbf{\Sigma}^1_3$ Separation (false in $\mathbf{L}$) holds.

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2 Trees, perfect-tree forcing notions, splitting

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers $0,1$. If $t \in 2^{<\omega}$ and $i = 0,1$ then $t^\wedge k$ is the extension of t by k. If $s,t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s, while $s \subset t$ means proper extension. If $s \in 2^{<\omega}$ then $\mathtt{lh}\,s$ is the length of s, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}\,s = n\}$ (strings of length n).

A set $T \subseteq 2^{<\omega}$ is a *tree* iff for any strings $s \subset t$ in $2^{<\omega}$, if $t \in T$ then $s \in T$. Thus every non-empty tree $T \subseteq 2^{<\omega}$ contains the empty string Λ. If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T{\restriction}_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let $\mathbf{PT}$ be the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$. Thus a non-empty tree $T \subseteq 2^{<\omega}$ belongs to $\mathbf{PT}$ iff it has no endpoints and no isolated branches. Then there is a largest string $s \in T$ such that $T = T{\restriction}_s$; it is denoted by $s = \mathtt{stem}(T)$ (the *stem* of T); we have $s^\wedge 1 \in T$ and $s^\wedge 0 \in T$ in this case.

Each perfect tree $T \in \mathbf{PT}$ defines $[T] = \{a \in 2^\omega : \forall n\,(a{\restriction}n \in T)\} \subseteq 2^\omega$, the perfect set of all *paths through* T.

Definition 2.1. A **perfect-tree forcing notion** is any set $\mathbb{P} \subseteq \mathbf{PT}$ such that if $u \in T \in \mathbb{P}$ then $T{\restriction}_u \in \mathbb{P}$. Let $\mathbb{PTF}$ be the set of all such $\mathbb{P} \subseteq \mathbf{PT}$. □

Such a set $\mathbb{P}$ can be considered as a forcing notion (if $T \subseteq T'$ then T is a stronger condition); such a forcing $\mathbb{P}$ adds a real in 2^ω.

Example 2.2. If $s \in 2^{<\omega}$ then the tree $I_s = \{t \in 2^{<\omega} : s \subseteq t \vee t \subseteq s\}$ belongs to $\mathbf{PT}$ and the set $\mathbb{P}_0 = \{I_s : s \in 2^{<\omega}\}$ is a perfect-tree forcing. □

Lemma 2.3. *If* $\mathbb{P}, \mathbb{P}' \in \mathbf{PT}$, $T \in \mathbb{P}$, $T' \in \mathbb{P}'$, *then there are trees* $S \in \mathbb{P}$, $S' \in \mathbb{P}'$ *such that* $S \subseteq T$, $S' \subseteq T'$, *and* $[S] \cap [S'] = \varnothing$.

Proof. If $T = T'$ then let $s = \mathtt{stem}(T)$ and $S = T{\restriction}_{s^\wedge 0}$, $S' = T'{\restriction}_{s^\wedge 1}$. If say $T \not\subseteq T'$ then let $s \in T \smallsetminus T'$, $S = T{\restriction}_s$, and simply $S' = T'$. □

If $\mathbb{P} \in \mathbb{PTF}$ then let $\mathbf{FSS}(\mathbb{P})$ be the set of all *finite splitting systems* over $\mathbb{P}$, that is, systems of the form $\varphi = \langle T_s \rangle_{s \in 2^{<n}}$, where $n = \mathtt{hgt}(\varphi) < \omega$ (the height of φ), each value $T_s = \varphi(s)$ is a tree in $\mathbb{P}$, and

(∗) if $s^\wedge i \in 2^{<n}$ $(i = 0, 1)$ then $T_{s^\wedge i} \subseteq T_s$ and $\mathtt{stem}(T_s)^\wedge i \subseteq \mathtt{stem}(T_{s^\wedge i})$ — it easily follows that $[T_{s^\wedge 0}] \cap [T_{s^\wedge 0}] = \varnothing$.

Let φ, ψ be systems in $\mathbf{FSS}(\mathbb{P})$. Say that

- φ *extends* ψ, symbolically $\psi \preccurlyeq \varphi$, if $n = \mathtt{hgt}(\psi) \le \mathtt{hgt}(\varphi)$ and $\psi(s) = \varphi(s)$ for all $s \in 2^{<n}$;
- *properly extends* ψ, symbolically $\psi \prec \varphi$, if in addition $\mathtt{hgt}(\psi) < \mathtt{hgt}(\varphi)$;
- *reduces* ψ, if $n = \mathtt{hgt}(\psi) = \mathtt{hgt}(\varphi)$, $\varphi(s) \subseteq \psi(s)$ for all $s \in 2^{\mathtt{hgt}(\varphi)-1}$, and $\varphi(s) = \psi(s)$ for all $s \in 2^{<\mathtt{hgt}(\varphi)-1}$.

In other words, reduction allows to shrink trees in the top layer of the system, but keeps intact those in the lower layers.

The empty system Λ is the only one in $\mathbf{FSS}(\mathbb{P})$ satisfying $\mathtt{hgt}(\Lambda) = 0$. To get a system $\varphi \in \mathbf{FSS}(\mathbb{P})$ with $\mathtt{hgt}(\varphi) = 1$ take any $T \in \mathbb{P}$ and put $\varphi(\Lambda) = T$. The next lemma provides systems of bigger height.

Lemma 2.4. *Assume that* $\mathbb{P} \in \mathbb{PTF}$. *If* $n \ge 1$ *and* $\psi = \langle T_s \rangle_{s \in 2^{<n}} \in \mathbf{FSS}(\mathbb{P})$ *then there is a system* $\varphi = \langle T_s \rangle_{s \in 2^{<n+1}} \in \mathbf{FSS}(\mathbb{P})$ *which properly extends* ψ.

Proof. If $s \in 2^{n-1}$ and $i = 0, 1$ then let $T_{s^\wedge i} = T_s \restriction_{\mathtt{stem}(T_s)^\wedge i}$. □

Corollary 2.5. *Let* $\mathbb{P} \in \mathbb{PTF}$. *Then there is an* $\prec$*-increasing sequence* $\langle \varphi_n \rangle_{n<\omega}$ *of systems in* $\mathbf{FSS}(\mathbb{P})$. *In this case the limit system* $\varphi = \bigcup_n \varphi_n = \langle T_s \rangle_{s \in 2^{<\omega}}$ *satisfies* (∗) *of Section 2 on the whole domain* $2^{<\omega}$, $T = \bigcap_n \bigcup_{s \in 2^n} T_s$ *is a perfect tree in* $\mathbf{PT}$ (yet not necessarily in $\mathbb{P}$), *and* $[T] = \bigcap_n \bigcup_{s \in 2^n} [T_s]$. □

Say that a tree T *occurs in* $\varphi \in \mathbf{FSS}(\mathbb{P})$ if $T = \varphi(s)$ for some $s \in 2^{<\mathtt{hgt}(\varphi)}$.

3 Multitrees and splitting multisystems

Suppose that $\vartheta \in \mathbf{Ord}$ and $\mathbb{p} = \langle \mathbb{P}_\xi \rangle_{\xi<\vartheta}$ is a sequence of sets $\mathbb{P}_\xi \in \mathbb{PTF}$. We'll systematically consider such sequences below, and if $\mathbb{q} = \langle \mathbb{Q}_\xi \rangle_{\xi<\vartheta}$ is another such a sequence of the same length then let $\mathbb{p} \vee \mathbb{q} = \langle \mathbb{P}_\xi \cup \mathbb{Q}_\xi \rangle_{\xi<\vartheta}$.

Definition 3.1. A $\mathbb{p}$-*multitree* is a "matrix" of the form $\boldsymbol{\tau} = \langle T_{\xi k} \rangle^{\xi<\vartheta}_{k<\omega}$, where each $\boldsymbol{\tau}(\xi, k) = T_{\xi k}$ belongs to $\mathbb{P}_\xi$, and the *support* $|\boldsymbol{\tau}| = \{\langle \xi, k \rangle : T_{\xi k} \ne 2^{<\omega}\}$ is finite. Let $\mathbf{MT}(\mathbb{p})$ be the set of all $\mathbb{p}$-multitrees. If $\boldsymbol{\tau} \in \mathbf{MT}(\mathbb{p})$ then let

$$[\boldsymbol{\tau}] = \{x \in 2^{\vartheta \times \omega} : \forall \langle \xi, k \rangle \in |\boldsymbol{\tau}| \, (x(\xi, k) \in [\boldsymbol{\tau}(\xi, k)])\} ;$$

this is a cofinite-dimensional perfect cube in $2^{\vartheta\times\omega}$.

A $\mathbb{p}$-*multisystem* is a "matrix" of the form $\Phi = \langle\varphi_{\xi m}\rangle^{\xi<\vartheta}_{m<\omega}$, where each $\Phi(\xi,m) = \varphi_{\xi m}$ belongs to $\mathbf{FSS}(\mathbb{P}_\xi)$, and the *support* $|\Phi| = \{\langle\xi,m\rangle : \varphi_{\xi m} \neq 2^{<\omega}\}$ is finite. Let $\mathbf{MS}(\mathbb{p})$ be the set of all $\mathbb{p}$-multisystems.

Say that a multitree $\boldsymbol{\tau} = \langle T_{\xi k}\rangle^{\xi<\vartheta}_{k<\omega}$ *occurs* in a multisystem $\Phi = \langle\varphi_{\xi m}\rangle^{\xi<\vartheta}_{m<\omega}$ if $|\boldsymbol{\tau}| \subseteq |\Phi|$ and for each $\langle\xi,k\rangle \in |\boldsymbol{\tau}|$ there is a number $m < \omega$ and a string $s \in 2^{<\omega}$ with $\mathtt{lh}\, s < \mathtt{hgt}(\varphi_{\xi m})$ such that $T_{\xi k} = \varphi_{\xi m}(s)$. $\square$

The set $\mathbf{MT}(\mathbb{p})$ is equal to the finite support product $\prod_{\xi<\vartheta}(\mathbb{P}_\xi)^\omega$ of $\vartheta\times\omega$-many factors, with each factor $\mathbb{P}_\xi$ in ω-many copies. Accordingly, the set $\mathbf{MS}(\mathbb{p})$ is equal to the finite support product $\prod_{\xi<\vartheta}(\mathbf{FSS}(\mathbb{P}_\xi))^\omega$ of $(\vartheta\times\omega)$-many factors, with each factor $\mathbf{FSS}(\mathbb{P}_\xi)$ in ω-many copies. We order $\mathbf{MT}(\mathbb{p})$ componentwise: $\boldsymbol{\sigma} \leqslant \boldsymbol{\tau}$ iff $\boldsymbol{\sigma}(\xi,k) \subseteq \boldsymbol{\tau}(\xi,k)$ in $\mathbb{P}_\xi$ for all ξ,k. The forcing $\mathbf{MT}(\mathbb{p})$ adds a "matrix" $\langle x_{\xi k}\rangle^{\xi<\vartheta}_{k<\omega}$, where each $x_{\xi k} \in 2^\omega$ is a $\mathbb{P}_\xi$-generic real.

If $\Phi,\Psi \in \mathbf{MS}(\mathbb{p})$ then we define

- $\Psi \preccurlyeq \Phi$ iff $\Psi(\xi,m) \preccurlyeq \Phi(\xi,m)$ (in $\mathbf{FSS}(\mathbb{P}_\xi)$) for all ξ,m;
- Φ reduces Ψ iff $|\Psi| \subseteq |\Phi|$ and $\Phi(\xi,m)$ reduces $\Psi(\xi,m)$ for all pairs $\langle\xi,m\rangle \in |\Psi|$;
- $\Phi \prec\!\!\prec \Psi$ iff $|\Phi| \subseteq |\Psi|$ and $\Phi(\xi,m) \prec \Psi(\xi,m)$ for all $\langle\xi,m\rangle \in |\Phi|$.

Lemma 3.2. *If $\Phi \prec\!\!\prec \Psi$ and Φ' reduces Ψ then still $\Phi \prec\!\!\prec \Phi'$ and $\Phi \preccurlyeq \Phi'$.* $\square$

4 Jensen's extension of a perfect tree forcing

Let $\mathbf{ZFC}'$ be the subtheory of $\mathbf{ZFC}$ including all axioms except for the power set axiom, plus the axiom saying that $\mathscr{P}(\omega)$ exists. (Then ω_1 and continual sets like $\mathbf{PT}$ exist as well.) Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$.

Suppose that $\mathbb{p} = \langle\mathbb{P}_\xi\rangle_{\xi<} \in \mathfrak{M}$ is a sequence of (countable) sets $\mathbb{P}_\xi \in \mathbb{PTF}$, of length $\mathbb{\theta} < \omega_1^{\mathfrak{M}}$. Then the sets $\mathbb{P}_\xi$ and $\mathbf{FSS}(\mathbb{P}_\xi)$ for all $\xi < \mathbb{\theta}$, as well as the sets $\mathbf{MT}(\mathbb{p})$ and $\mathbf{MS}(\mathbb{p})$, belong to $\mathfrak{M}$, too.

Definition 4.1. (i) Let us fix any $\preccurlyeq$-increasing sequence $\mathbb{\Phi} = \langle\Phi^j\rangle_{j<\omega}$ of multisystems $\Phi^j = \langle\varphi^j_{\xi m}\rangle^{\xi<}_{m<\omega} \in \mathbf{MS}(\mathbb{p})$, *generic over* $\mathfrak{M}$ in the sense that it intersects every set $D \in \mathfrak{M}$, $D \subseteq \mathbf{MS}(\mathbb{p})$, dense in $\mathbf{MS}(\mathbb{p})$ [4].

(ii) Suppose that $\xi < \mathbb{\theta}$ and $m < \omega$. In particular, the sequence $\mathbb{\Phi}$ intersects every set of the form

$$D_{\xi m h} = \{\Phi \in \mathbf{MS}(\mathbb{p}) : \mathtt{hgt}(\Phi(\xi,m)) \geq h\}\,, \quad \text{where } h < \omega\,.$$

[4] Meaning that for any $\Psi \in \mathbf{MS}(\mathbb{p})$ there is $\Phi \in D$ with $\Psi \preccurlyeq \Phi$.

It follows that the sequence $\langle\varphi^j_{\xi m}\rangle_{j<\omega}$ of systems $\varphi^j_{\xi m} \in \mathbf{FSS}(\mathbb{P}_\xi)$ satisfies $\varphi^j_{\xi m} \prec \varphi^{j+1}_{\xi m}$ for infinitely many indices j (and $\varphi^j_{\xi m} = \varphi^{j+1}_{\xi m}$ for other j).

(iii) We conclude that the limit system $\varphi^\infty_{\xi m} = \bigcup_{j<\omega}\varphi^j_{\xi m}$ has the form $\langle\boldsymbol{T}_{\xi m}(s)\rangle_{s\in 2^{<\omega}}$ such that each $\boldsymbol{T}_{\xi m}(s)$ is a tree in $\mathbb{P}_\xi$, and if $j<\omega$ then we have $\varphi^j_{\xi m} = \langle\boldsymbol{T}_{\xi m}(s)\rangle_{s\in 2^{<h(j,\xi,m)}}$, where $h(j,\xi,m) = \mathtt{hgt}(\varphi^j_{\xi m})$.

(iv) Moreover, by Corollary 2.5, the trees

$$\boldsymbol{U}_{\xi m} = \bigcap_n \bigcup_{s\in 2^n} \boldsymbol{T}_{\xi m}(s)\,, \quad \boldsymbol{U}_{\xi m}(s) = \bigcap_{n\geq \mathtt{lh}\, s} \bigcup_{t\in 2^n,\, s\subseteq t} \boldsymbol{T}_{\xi m}(t)$$

belong to **PT** (not necessarily to $\mathbb{P}_\xi$) for each $s\in 2^{<\omega}$; thus $\boldsymbol{U}_{\xi m} = \boldsymbol{U}_{\xi m}(\Lambda)$.

(v) If $\xi<\vartheta$ then let $\mathbb{U}_\xi = \{\boldsymbol{U}_{\xi m}(s) : m<\omega \wedge s\in 2^{<\omega}\}$.

Let $\mathbb{u} = \langle\mathbb{U}_\xi\rangle_{\xi< }$.

Finally let $\mathbb{p}\vee\mathbb{u} = \langle\mathbb{P}_\xi\cup\mathbb{U}_\xi\rangle_{\xi< }$. □

Lemma 4.2. (i) *if* $\langle\xi,m\rangle \neq \langle\eta,n\rangle$ *then* $[\boldsymbol{U}_{\xi m}]\cap[\boldsymbol{U}_{\eta n}] = \varnothing$;

(ii) *if* $\xi<\vartheta$, $m<\omega$, $s\in 2^{<\omega}$, *then* $\boldsymbol{U}_{\xi m}(s) = \boldsymbol{U}_{\xi m}\cap\boldsymbol{T}_{\xi m}(s)$;

(iii) *if* $\xi<\vartheta$, $m<\omega$, *and strings* $s\subseteq t$ *belong to* $2^{<\omega}$ *then* $[\boldsymbol{T}_{\xi m}(s)]\subseteq[\boldsymbol{T}_{\xi m}(t)]$ *and* $[\boldsymbol{U}_{\xi m}(s)]\subseteq[\boldsymbol{U}_{\xi m}(t)]$;

(iv) *If* $\xi<\vartheta$, $m<\omega$, *and strings* $t'\neq t$ *in* $2^{<\omega}$ *are* $\subseteq$*-incomparable then* $[\boldsymbol{U}_{\xi m}(t')]\cap[\boldsymbol{U}_{\xi m}(t)] = [\boldsymbol{T}_{\xi m}(t')]\cap[\boldsymbol{T}_{\xi m}(t)] = \varnothing$.

Proof. (i) By Lemma 2.3, the set D of all multisystems Φ such that the pairs $\langle\xi,m\rangle$, $\langle\eta,n\rangle$ belong to $|\Phi|$ and, for some $h<\min\{\mathtt{hgt}(\Phi(\xi,m)),\mathtt{hgt}(\Phi(\eta,n))\}$, we have $[\Phi(\xi,m)(s)]\cap[\Phi(\eta,n)(t)] = \varnothing$ for all $s,t\in 2^h$, is dense.

(ii) easily follows from $(*)$ of Section 2. (iii) is obvious.

(iv) Note that $[\varphi(s^\wedge 0)]\cap[\varphi(s^\wedge 1)] = \varnothing$ for any system φ with $\mathtt{hgt}(\varphi) > 1+\mathtt{lh}\, s$ by $(*)$ of Section 2. Therefore $[\boldsymbol{T}_{\xi m}(s^\wedge 0)]\cap[\boldsymbol{T}_{\xi m}(s^\wedge 1)] = \varnothing$. □

It follows that if $U\in\bigcup_{\xi< }\mathbb{U}_\xi$ then there is a unique triple of $\xi<\vartheta$, $m<\omega$, and $s\in 2^{<\omega}$ such that $U = \boldsymbol{U}_{\xi m}(s)$!

Lemma 4.3. *If* $\xi<\vartheta$ *then the sets* $\mathbb{U}_\xi$ *and* $\mathbb{P}_\xi\cup\mathbb{U}_\xi$ *belong to* $\mathbb{PTF}$. □

Lemma 4.4. *Let* $\xi<\vartheta$. *The set* $\mathbb{U}_\xi$ *is dense in* $\mathbb{U}_\xi\cup\mathbb{P}_\xi$.

Proof. If $T\in\mathbb{P}_\xi$ then the set $D(T)$ of all multisystems $\Phi = \langle\varphi_{\xi m}\rangle^{\xi<}_{m<\omega}$ in $\mathbf{MS}(\mathbb{p})$, such that $\varphi_{\xi m}(\Lambda) = T$ for some k, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{MS}(\mathbb{p})$. It follows that $\Phi^J\in D(T)$ for some $J<\omega$, by the choice of Φ. Then $\boldsymbol{T}_{\xi m}(\Lambda) = T$ for some $m<\omega$. However $\boldsymbol{U}_{\xi m}(\Lambda)\subseteq\boldsymbol{T}_{\xi m}(\Lambda)$. □

Lemma 4.5. *If $\xi < \vartheta$ and a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}_\xi$ is pre-dense in $\mathbb{P}_\xi$, and $U \in \mathbb{U}_\xi$, then $U \subseteq^{\mathtt{fin}} \bigcup D$, that is, there is a finite set $D' \subseteq D$ with $U \subseteq \bigcup D'$.*

Proof. Suppose that $U = \boldsymbol{U}_{\xi M}(s)$, $M < \omega$ and $s \in 2^{<\omega}$. Consider the set $\Delta \in \mathfrak{M}$ of all multisystems $\Phi = \langle \varphi_{\xi m} \rangle \in \mathbf{MS}(\mathbb{p})$ such that $\langle \xi, M \rangle \in |\Phi|$, $\mathtt{lh}\, s < h = \mathtt{hgt}(\varphi_{\xi M})$, and for each $t \in 2^{h-1}$ there is a tree $S_t \in D$ with $\varphi_{\xi M}(t) \subseteq S$. The set Δ is dense in $\mathbf{SC}^{<\omega}(\mathbb{P})$ by the pre-density of D. Therefore there is an index J such that Φ^J belongs to Δ. Let this be witnessed by trees $S_t \in D$, $t \in 2^{h-1}$, where $\mathtt{lh}\, s < h = \mathtt{hgt}(\varphi^J_{\xi M})$, so that $\varphi^J_{\xi M}(t) \subseteq S_t$. Then

$$U = \boldsymbol{U}_{\xi M}(s) \subseteq \boldsymbol{U}_{\xi M}(\Lambda) \subseteq \textstyle\bigcup_{t \in 2^{h-1}} \varphi^J_{\xi M}(t) \subseteq \bigcup_{t \in 2^{h-1}} S_t \subseteq \bigcup D'$$

by construction, where $D' = \{S_t : t \in 2^{h-1}\} \subseteq D$ is finite. □

Lemma 4.6. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbf{MT}(\mathbb{p})$ is pre-dense in $\mathbf{MT}(\mathbb{p})$ then it remains pre-dense in $\mathbf{MT}(\mathbb{p} \vee \mathbb{u})$.*

Proof. Given a multitree $\boldsymbol{\tau} \in \mathbf{MT}(\mathbb{p} \vee \mathbb{u})$, prove that $\boldsymbol{\tau}$ is compatible in $\mathbf{MT}(\mathbb{p} \vee \mathbb{u})$ with a multitree $\boldsymbol{\sigma} \in D$. For the sake of brevity, assume that $\boldsymbol{\tau} \in \mathbf{MT}(\mathbb{u})$ and $|\boldsymbol{\tau}| = \{\langle \eta, K \rangle, \langle \zeta, L \rangle\}$, where $\zeta < \eta < \vartheta$ and $K, L < \omega$. Then by construction $\boldsymbol{\tau}(\eta, K) = \boldsymbol{U}_{\eta M}(s)$ and $\boldsymbol{\tau}(\zeta, L) = \boldsymbol{U}_{\zeta N}(t)$ for some $M, N < \omega$ and $s, t \in 2^{<\omega}$.

Consider the set $\Delta \in \mathfrak{M}$ of all multisystems $\Phi = \langle \varphi_{\xi m} \rangle^{\xi <}_{m < \omega} \in \mathbf{MS}(\mathbb{p})$ such that there are strings $s', t' \in 2^{<\omega}$ with $s \subset s'$, $t \subset t'$, $\mathtt{lh}\, s' < \mathtt{hgt}(\varphi_{\eta M})$, $\mathtt{lh}\, t' < \mathtt{hgt}(\varphi_{\zeta N})$, and multitrees $\boldsymbol{\sigma} \in D$ and $\boldsymbol{\sigma}' \in \mathbf{MT}(\mathbb{p})$, such that $\boldsymbol{\sigma}' \leqslant \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ occurs in Φ in such a way that $\boldsymbol{\sigma}'(\eta, K) = \varphi_{\eta M}(s')$ and $\boldsymbol{\sigma}'(\zeta, L) = \varphi_{\zeta N}(t')$.

The set Δ is dense in $\mathbf{MS}(\mathbb{p})$ by the pre-density of D. Therefore there is an index j such that Φ^j belongs to Δ. Let this be witnessed by strings $s', t' \in 2^{<\omega}$, and multitrees $\boldsymbol{\sigma} \in D$, and $\boldsymbol{\sigma}' \in \mathbf{MT}(\mathbb{p})$, $\boldsymbol{\sigma}' \leqslant \boldsymbol{\sigma}$, as above. In other words, $s \subset s'$, $t \subset t'$, $\mathtt{lh}\, s' < \mathtt{hgt}(\varphi^j_{\eta M})$, $\mathtt{lh}\, t' < \mathtt{hgt}(\varphi^j_{\zeta N})$, and $\boldsymbol{\sigma}'$ occurs in Φ in such a way that $\boldsymbol{\sigma}'(\eta, K) = \varphi^j_{\eta M}(s')$ and $\boldsymbol{\sigma}'(\zeta, L) = \varphi^j_{\zeta N}(t')$. The set $|\boldsymbol{\sigma}'| = \{\langle \xi_1, k_1 \rangle, \langle \xi_2, k_2 \rangle, \dots, \langle \xi_n, k_n \rangle\} \subseteq \vartheta \times \omega$ is finite and contains the pairs $\langle \eta, K \rangle, \langle \zeta, L \rangle$; let, say, $\langle \xi_1, k_1 \rangle = \langle \eta, K \rangle$, $\langle \xi_2, k_2 \rangle = \langle \zeta, L \rangle$.

And if $i = 1, 2, \dots, n$ then by definition $\boldsymbol{\sigma}'(\xi_i, k_i) = \varphi^j_{\xi_i m_i}(s_i) = \boldsymbol{T}_{\xi_i m_i}(s_i)$ holds for some $m_i < \omega$ and $s_i \in 2^{<\omega}$. In particular $\boldsymbol{\sigma}'(\eta, K) = \varphi^j_{\eta M}(s') = \boldsymbol{T}_{\eta M}(s')$ and $\boldsymbol{\sigma}'(\zeta, L) = \varphi^j_{\zeta N}(t') = \boldsymbol{T}_{\zeta N}(t')$, for $i = 1, 2$.

Consider the multitree $\boldsymbol{\tau}' \in \mathbf{MT}(\mathbb{u})$ defined so that $|\boldsymbol{\tau}'| = |\boldsymbol{\sigma}'|$ and $\boldsymbol{\tau}'(\xi_i, k_i) = \boldsymbol{U}_{\xi_i m_i}(s_i)$ for all $i = 1, \dots, n$. In particular $\boldsymbol{\tau}'(\eta, K) = \boldsymbol{U}_{\eta M}(s')$ and $\boldsymbol{\tau}'(\zeta, L) = \boldsymbol{U}_{\zeta N}(t')$. Then $\boldsymbol{\tau}' \leqslant \boldsymbol{\sigma}'$ (since $\boldsymbol{U}_{\xi_i m_i}(s_i) \subseteq \boldsymbol{T}_{\xi_i m_i}(s_i)$), therefore $\boldsymbol{\tau}' \leqslant \boldsymbol{\sigma} \in D$.

It remains to prove that $\boldsymbol{\tau}' \leqslant \boldsymbol{\tau}$, which amounts to $\boldsymbol{\tau}'(\eta, K) \subseteq \boldsymbol{\tau}(\eta, K)$ and $\boldsymbol{\tau}'(\zeta, L) \subseteq \boldsymbol{\tau}(\zeta, L)$. However $\boldsymbol{\tau}(\eta, K) = \boldsymbol{U}_{\eta M}(s) \subseteq \boldsymbol{U}_{\eta M}(s') = \boldsymbol{\tau}'(\eta, K)$ since $s \subset s'$, and the same for the pair $\langle \zeta, L \rangle$. □

5 Forcing a real away of a pre-dense set

Let $\mathfrak{M}$ be still a countable transitive model of $\mathbf{ZFC}'$ and $\mathbb{p} = \langle \mathbb{P}_\xi \rangle_{\xi<\omega_1^{\mathfrak{M}}} \in \mathfrak{M}$ be as in Section 4. The goal of the following Theorem 5.3 is to prove that, under the conditions and notation of Definition 4.1, if $\xi < \vartheta$ and c is a $\mathbf{MT}(\mathbb{p})$-name of a real in 2^ω then it is forced by the extended forcing $\mathbf{MT}(\mathbb{p}\vee\mathbb{u})$ that c does not belong to sets $[U]$ where U is a tree in $\mathbb{U}_\xi$ — unless c is a name of one of generic reals $x_{\xi k}$ themselves. We begin with a suitable notation.

Definition 5.1. A $\mathbf{MT}(\mathbb{p})$-*real name* is a system $\mathbf{c} = \langle C_{ni} \rangle_{n<\omega,\, i<2}$ of sets $C_{ni} \subseteq \mathbf{MT}(\mathbb{p})$ such that each set $C_n = C_{n0} \cup C_{n1}$ is dense or at least pre-dense in $\mathbf{MT}(\mathbb{p})$ and if $\boldsymbol{\sigma} \in C_{n0}$ and $\boldsymbol{\tau} \in C_{n1}$ then $\boldsymbol{\sigma}, \boldsymbol{\tau}$ are incompatible in $\mathbf{MT}(\mathbb{p})$.

If a set $G \subseteq \mathbf{MT}(\mathbb{p})$ is $\mathbf{MT}(\mathbb{p})$-generic at least over the collection of all sets C_n then we define $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_{ni} \neq \varnothing$. □

Thus any $\mathbf{MT}(\mathbb{p})$-real name $\mathbf{c} = \langle C_{ni} \rangle$ is a $\mathbf{MT}(\mathbb{p})$-name for a real in 2^ω.

Recall that $\mathbf{MT}(\mathbb{p})$ adds a generic sequence $\langle x_{\xi k} \rangle_{\xi<\ ,k<\omega}$ of reals $x_{\xi k} \in 2^\omega$.

Example 5.2. If $\xi < \vartheta$ and $k < \omega$ then define a $\mathbf{MT}(\mathbb{p})$-real name $\dot{\boldsymbol{x}}_{\xi k} = \langle C^{\xi k}_{ni} \rangle_{n<\omega,\, i<2}$ such that each set $C^{\xi k}_{ni}$ contains a single multitree $\boldsymbol{\rho}^{\xi k}_{ni} \in \mathbf{MT}(\mathbb{p})$, such that $|\boldsymbol{\rho}^{\xi k}_{ni}| = \{\langle \xi, k \rangle\}$ and finally $\boldsymbol{\rho}^{\xi k}_{ni}(\xi, k) = R_{ni}$, where

$$R_{ni} = \{ s \in 2^{<\omega} : \mathtt{lh}\, s > n \Longrightarrow s(n) = i \}\,.$$

Then $\dot{\boldsymbol{x}}_{\xi k}$ is a $\mathbf{MT}(\mathbb{p})$-real name of the real $x_{\xi k}$, the (ξ, k)th term of a $\mathbf{MT}(\mathbb{p})$-generic sequence $\langle x_{\xi k} \rangle_{\xi<\ ,k<\omega}$. □

Let $\mathbf{c} = \langle C_{ni} \rangle$ and $\mathbf{d} = \langle D_{ni} \rangle$ be $\mathbf{MT}(\mathbb{p})$-real names. Say that $\boldsymbol{\tau} \in \mathbf{MT}(\mathbb{p})$:

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$ and $i = 0, 1$, iff there is a finite set $\Sigma \subseteq C_{ni}$ such that $[\boldsymbol{\tau}] \subseteq \bigcup_{\boldsymbol{\sigma} \in \Sigma} [\boldsymbol{\sigma}]$;
- *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \mathtt{lh}\, s$, $\boldsymbol{\tau}$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{d} \neq \mathbf{c}$, iff there are strings $s, t \in 2^{<\omega}$, incomparable in $2^{<\omega}$ and such that $\boldsymbol{\tau}$ directly forces $s \subset \mathbf{c}$ and $t \subset \mathbf{d}$;
- *directly forces* $\mathbf{c} \notin [T]$, where $T \in \mathbf{PT}$, iff there is a string $s \in 2^{<\omega} \smallsetminus T$ such that $\boldsymbol{\tau}$ directly forces $s \subset \mathbf{c}$;

Theorem 5.3. *In the assumptions of Definition 4.1, suppose that* $\eta < \vartheta$, $\mathbf{c} = \langle C^i_m \rangle_{m<\omega,\, i<2} \in \mathfrak{M}$ *is a* $\mathbf{MT}(\mathbb{p})$*-real name, and for all* k *the set*

$$D(k) = \{ \boldsymbol{\tau} \in \mathbf{MT}(\mathbb{p}) : \boldsymbol{\tau} \text{ directly forces } \mathbf{c} \neq \dot{\boldsymbol{x}}_{\eta k} \}$$

is dense in $\mathbf{MT}(\mathbb{p})$. *Let* $\boldsymbol{u} \in \mathbf{MT}(\mathbb{p}\vee\mathbb{u})$, $\eta < \vartheta$, *and* $U \in \mathbb{U}_\eta$. *Then there is a stronger multitree* $\boldsymbol{v} \in \mathbf{MT}(\mathbb{u})$, $\boldsymbol{v} \leqslant \boldsymbol{u}$, *which directly forces* $\mathbf{c} \notin [U]$.

Proof. By construction $U \subseteq \boldsymbol{U}_{\eta M}$ for some $M < \omega$; thus we can assume that simply $U = \boldsymbol{U}_{\eta M}$. The indices η, M are fixed in the proof. We can assume by Lemma 4.4 that $\boldsymbol{u} \in \mathbf{MT}(\mathbb{u})$. The support $|\boldsymbol{u}| = \{\langle\xi_1, k_1\rangle, \dots, \langle\xi_\nu, k_\nu\rangle\} \subseteq \vartheta \times \omega$ is a finite set ($\nu < \omega$), and if $i = 1, \dots, \nu$ then, as $\boldsymbol{u} \in \mathbf{MT}(\mathbb{u})$, there is a string s_i and a number m_i such that $\boldsymbol{u}(\xi_i, k_i) = \boldsymbol{U}_{\xi_i m_i}(s_i)$. We can assume that

(a) if $i \neq i'$ and $\xi_i = \xi_{i'}$ then $k_i \neq k_i'$;

(b) $s_i \neq s_{i'}$ whenever $i \neq i'$, and there is $h < \omega$ such that $\mathtt{lh}\, s_i = h$, $\forall\, i$; [5]

(c) there is a number $\mu \leq \nu$ such that $\xi_1 = \dots = \xi_\mu = \eta$ and $m_1 = \dots = m_\mu = M$ (then $\mu \leq 2^h$), but if $\mu < i \leq \nu$ then $\langle\xi_i, m_i\rangle \neq \langle\eta, M\rangle$.

In these assumptions, define a multitree $\boldsymbol{\tau} \in \mathbf{MT}(\mathbb{p})$ so that $|\boldsymbol{\tau}| = |\boldsymbol{u}| = \{\langle\xi_1, k_1\rangle, \dots, \langle\xi_\nu, k_\nu\rangle\}$ and $\boldsymbol{\tau}(\xi_i, k_i) = \boldsymbol{T}_{\xi_i m_i}(s_i)$ for $i = 1, \dots, \nu$, so that $\boldsymbol{u} \leqslant \boldsymbol{\tau}$.

Consider the set $\mathscr{D}$ of all multisystems $\Phi = \langle\varphi_{\xi m}\rangle^{\xi<}_{m<\omega} \in \mathbf{MS}(\mathbb{p})$ such that

(1) there is a number $H > h$ and strings $\underline{s}_i \in 2^H$ satisfying $s_i \subset \underline{s}_i$ and $\mathtt{hgt}(\varphi_{\xi_i m_i}) = H + 1$ for $i = 1, \dots, \nu$;

(2) there is a multitree $\boldsymbol{\sigma} \in \mathbf{MT}(\mathbb{p})$ which occurs in Φ (Definition 3.1) and satisfies conditions (3), (4) below;

(3) $\boldsymbol{\sigma}(\xi_i, k_i) = \varphi_{\xi_i m_i}(\underline{s}_i)$ for $i = 1, \dots, \nu$;

(4) $\boldsymbol{\sigma}$ directly forces $\mathbf{c} \notin [T]$, where $T = \bigcup_{s \in 2^H} \varphi_{\eta M}(s)$.

Lemma 5.4. *$\mathscr{D}$ is dense in* $\mathbf{MS}(\mathbb{p})$.

Proof. By Lemma 3.2, it suffices to prove that for any multisystem $\Phi = \langle\varphi_{\xi m}\rangle^{\xi<}_{m<\omega} \in \mathbf{MS}(\mathbb{p})$ which already satisfies (1) by means of a number H and strings $\underline{s}_i \in 2^H$, $1 \leq i \leq \nu$, there is a multisystem $\Phi' \in \mathscr{D}$ which reduces Φ.

Let $p = 2^H$ (a number) and let $\{t_1, \dots, t_p\} = 2^H = \{t \in 2^{<\omega} : \mathtt{lh}\, t = H\}$. We suppose that the enumeration is chosen so that $t_i = \underline{s}_i$ for $i = 1, \dots, \mu$. Let $\ell_i = k_i$ whenewer $1 \leq i \leq \mu$. If $\mu + 1 \leq n \leq p$ then let

$$\ell_n = n + 1 + \max_{1 \leq i \leq \nu} \{k_i : \xi_i = \eta\},$$

so that pairs of the form $\langle\eta, \ell_n\rangle$, $n \geq \mu + 1$, do not belong to $|\boldsymbol{\tau}|$.

Consider a multitree $\boldsymbol{\rho} \in \mathbf{MT}(\mathbb{p})$, defined so that

- $|\boldsymbol{\rho}| = |\boldsymbol{\tau}| \cup \{\langle\eta, \ell_n\rangle : \mu + 1 \leq n \leq p\}$;

[5] If $s_i \subset s_i' \in 2^{<\omega}$ for all i, and $\boldsymbol{u}' \in \mathbf{MT}(\mathbb{u})$, $|\boldsymbol{u}'| = |\boldsymbol{u}|$ and $\boldsymbol{u}'(\xi_i, k_i) = \boldsymbol{U}^{\Phi}_{\xi_i m_i}(s_i')$ for all i, then $\boldsymbol{u}' \leqslant \boldsymbol{u}$. Thus if we prove the theorem for $\boldsymbol{u}'$ then it implies the result for $\boldsymbol{u}$ as well.

- $\boldsymbol{\rho}(\xi_i,k_i)=\varphi_{\xi_i m_i}(\underline{s}_i)$ for all $i=1,\dots,\nu$;
- $\boldsymbol{\rho}(\eta,\ell_n)=\varphi_{\eta M}(t_n)$ for all n, $\mu+1\le n\le p$ — note that by construction the equality $\boldsymbol{\rho}(\eta,\ell_i)=\varphi_{\eta M}(t_i)$ also holds for $i=1,\dots,\mu$, being just a reformulation of $\boldsymbol{\rho}(\xi_i,k_i)=\varphi_{\xi_i m_i}(\underline{s}_i)$.

By the density of sets $D(k)$, there exists a multitree $\boldsymbol{\sigma}\in\mathbf{MT}(\mathbb{p})$, $\boldsymbol{\sigma}\leqslant\boldsymbol{\rho}$, which directly forces $\mathbf{c}\ne\dot{\boldsymbol{x}}_{\eta\ell_n}$ for all $n=1,\dots,p$ — including $\mathbf{c}\ne\dot{\boldsymbol{x}}_{\eta k_i}$ for $i=1,\dots,\mu$. Then there are strings $u,v_1,\dots,v_p\in 2^{<\omega}$ such that u is incompatible in $2^{<\omega}$ with each v_n and $\boldsymbol{\sigma}$ directly forces each of the formulas

$$u\subset\mathbf{c},\quad\text{and}\quad v_n\subseteq\dot{\boldsymbol{x}}_{\eta\ell_n}\ \text{for all}\ n,\ 1\le n\le p.$$

However $\boldsymbol{\sigma}$ directly forces $v_n\subseteq\dot{\boldsymbol{x}}_{\eta\ell_n}$ iff $v_n\subseteq\mathtt{stem}(\boldsymbol{\sigma}(\eta,\ell_n))$. We conclude that $\boldsymbol{\sigma}$ directly forces $\mathbf{c}\notin[T]$, where $T=\bigcup_{1\le n\le p}\boldsymbol{\sigma}(\eta,\ell_n)$.

Now let $\Phi'=\langle\varphi'_{\xi m}\rangle^{\xi<}_{m<\omega}\in\mathbf{MS}(\mathbb{p})$ be defined as follows.

(I) we let $\varphi'_{\xi_i m_i}(\underline{s}_i)=\boldsymbol{\sigma}(\xi_i,k_i)$ for $i=1,\dots,\nu$;

(II) if $\mu+1\le n\le p$ then let $\varphi'_{\eta M}(t_n)=\boldsymbol{\sigma}(\eta,\ell_n)$ — the equality is also true for $n\le\mu$ by (I);

(III) if $\langle\xi,m\rangle\in|\Phi|$, $s\in 2^{<\omega}$, and $\mathtt{lh}\,s<\mathtt{hgt}(\varphi_{\xi m})$ (that is, $\varphi_{\xi m}(s)$ is defined), but $\varphi'_{\xi m}(s)$ is **not** defined by (I) and (II) [6], then we keep $\varphi'_{\xi m}(s)=\varphi_{\xi m}(s)$;

(IV) for any $\langle\xi,k\rangle\in|\boldsymbol{\sigma}|\smallsetminus|\boldsymbol{\rho}|$ add to $|\Phi'|$ a pair $\langle\xi,m\rangle\notin|\Phi|$ and define $\mathtt{hgt}(\varphi'_{\xi m})=1$, $\varphi'_{\xi m}(\Lambda)=\boldsymbol{\sigma}(\xi,k)$ — to make sure that $\boldsymbol{\sigma}$ occurs in Φ'.

By construction, the multisystem $\Phi'\in\mathbf{MS}(\mathbb{p})$ reduces Φ, the multitree $\boldsymbol{\sigma}$ occurs in Φ' by (IV) and satisfies $\boldsymbol{\sigma}\leqslant\boldsymbol{\rho}$. Finally to check (4) note that by construction $\bigcup_{1\le n\le p}\boldsymbol{\sigma}(\eta,\ell_n)=\bigcup_{s\in 2^H}\varphi'_{\eta M}(s)$. Thus $\Phi'\in\mathscr{D}$, as required. □ (Lemma)

Come back to the proof of the theorem. It follows from the lemma that there is an index j such that the system $\Phi^j=\langle\varphi^j_{\xi m}\rangle^{\xi<}_{m<\omega}$ belongs to $\mathscr{D}$. Let this be witnessed by a number $H>h$, a collection of strings $\underline{s}_i\in 2^H$ $(1\le i\le\nu)$, and a multitree $\boldsymbol{\sigma}\in\mathbf{MT}(\mathbb{p})$, so that conditions (1), (2), (3), (4) are satisfied for $\Phi=\Phi^j$ and $\boldsymbol{\sigma}$. Then, for instance, $\varphi^j_{\xi_i m_i}(\underline{s}_i)=\boldsymbol{T}_{\xi_i m_i}(\underline{s}_i)$ (see Definition 4.1(iii)). However $\boldsymbol{\sigma}(\xi_i,k_i)=\varphi^j_{\xi_i m_i}(\underline{s}_i)$ by (3) while $\boldsymbol{\tau}(\xi_i,k_i)=\boldsymbol{T}_{\xi_i m_i}(s_i)$ by the construction, and $s_i\subset\underline{s}_i$. It follows that $\boldsymbol{\sigma}\leqslant\boldsymbol{\tau}$.

Finally consider a multitree $\boldsymbol{v}\in\mathbf{MT}(\mathbb{u})$, defined so that $|\boldsymbol{u}|=|\boldsymbol{\sigma}|$, $\boldsymbol{u}(\xi_i,k_i)=\boldsymbol{U}_{\xi_i m_i}(\underline{s}_i)$ for $i=1,\dots,\nu$, and if $\langle\xi,k\rangle\in|\boldsymbol{\sigma}|\smallsetminus\{\langle\xi_i,k_i\rangle:1\le i\le\nu\}$ then let $\boldsymbol{v}(\xi,k)$ be any tree in $\mathbb{U}_{\xi k}$ satisfying $\boldsymbol{v}(\xi,k)\subseteq\boldsymbol{\sigma}(\xi,k)$ (we refer to Lemma 4.4).

[6] That is, except for the triples $\langle\xi,m,s\rangle=\langle\xi_i,m_i,\underline{s}_i\rangle$ and $\langle\eta,M,t_n\rangle$.

Recall that by construction $s_i \subset \underline{s}_i$ for all i. It follows that $\boldsymbol{v} \leqslant \boldsymbol{u}$. On the other hand, $\boldsymbol{v} \leqslant \boldsymbol{\sigma}$, therefore $\boldsymbol{v}$ directly forces $\mathbf{c} \notin [T]$ by (4), where $T = \bigcup_{s \in 2^H} \varphi^j_{\eta M}(s) = \bigcup_{s \in 2^H} \boldsymbol{T}_{\eta M}(s)$. And finally by definition $U = \boldsymbol{U}_{\eta M} \subseteq \bigcup_{s \in 2^H} \boldsymbol{T}_{\eta M}(s)$, so $\boldsymbol{v}$ directly forces $\mathbf{c} \notin [U]$, as required. □

6 The product forcing

In this section, **we argue in L, the constructible universe.** Let $\leqslant_{\mathbf{L}}$ be the canonical wellordering of $\mathbf{L}$.

Definition 6.1 (in $\mathbf{L}$)**.** We define, by induction on $\alpha < \omega_1$, sequences $\mathbb{u}^\alpha = \langle \mathbb{U}^\alpha_\xi \rangle_{\xi < \alpha}$, $\mathbb{p}^\alpha = \langle \mathbb{P}^\alpha_\xi \rangle_{\xi < \alpha}$ of countable sets of trees $\mathbb{U}^\alpha_\xi$, $\mathbb{P}^\alpha_\xi$ in $\mathbb{PTF}$, as follows.

First of all, we let $\mathbb{P}^\alpha_\alpha = 0$ and $\mathbb{U}^\alpha_\alpha = \mathbb{P}_0$ (see Example 2.2) for all α; note that the terms $\mathbb{P}^\alpha_\alpha, \mathbb{U}^\alpha_\alpha$ do not participate in the sequences $\mathbb{p}^\alpha$ and $\mathbb{u}^\alpha$.

The case $\boldsymbol{\alpha = 0}$. Let $\mathbb{p}^0 = \mathbb{u}^0 = \Lambda$ (the empty sequence).

The step. Suppose that $0 < \lambda < \omega_1$, and $\mathbb{u}^\alpha$, $\mathbb{p}^\alpha$ as above are already defined for every $\alpha < \lambda$. Let $\mathfrak{M}_\lambda$ be the least model $\mathfrak{M}$ of $\mathbf{ZFC}'$ of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, containing $\langle \mathbb{u}^\alpha \rangle_{\alpha < \lambda}$ and $\langle \mathbb{p}^\alpha \rangle_{\alpha < \lambda}$, and such that $\lambda < \omega_1^{\mathfrak{M}}$ and $\mathbb{U}^\alpha_\xi$, $\mathbb{P}^\alpha_\xi$ are countable in $\mathfrak{M}$ for all $\xi < \alpha < \lambda$.

We first define a sequence $\mathbb{p}^\lambda = \langle \mathbb{P}^\lambda_\xi \rangle_{\xi < \lambda}$ so that $\mathbb{P}^\lambda_\xi = \bigcup_{\xi \le \alpha < \lambda} \mathbb{U}^\alpha_\xi$ for all $\xi < \lambda$. In particular if $\lambda = \alpha + 1$ then $\mathbb{P}^{\alpha+1}_\xi = \mathbb{P}^\alpha_\xi \cup \mathbb{U}^\alpha_\xi$ for all $\xi < \alpha + 1$ (because $\mathbb{P}^\alpha_\xi = \bigcup_{\xi \le \alpha' < \alpha} \mathbb{U}^{\alpha'}_\xi$ at the previous step), and, for $\xi = \alpha$, $\mathbb{P}^{\alpha+1}_\alpha = \mathbb{P}^\alpha_\alpha \cup \mathbb{U}^\alpha_\alpha = \mathbb{P}_0$ (see above). Thus $\mathbb{p}^{\alpha+1}$ is the extension of $\mathbb{p}^\alpha \vee \mathbb{u}^\alpha$ (see Section 3) by the default assignment $\mathbb{P}^{\alpha+1}_\alpha = \mathbb{P}_0$. For instance, $\mathbb{p}^1 = \langle \mathbb{P}_0 \rangle$.

Thus a sequence $\mathbb{p}^\lambda = \langle \mathbb{P}^\lambda_\xi \rangle_{\xi < \lambda}$ is defined.

To define $\mathbb{u}^\lambda$ and accomplish the step, let $\Phi = \langle \Phi^j \rangle_{j < \omega}$ be the $\leqslant_{\mathbf{L}}$-least sequence of multisystems $\Phi^j \in \mathbf{MS}(\mathbb{p}^\lambda)$, $\preccurlyeq$-increasing and generic over $\mathfrak{M}_\lambda$, and let $\mathbb{u}^\lambda = \langle \mathbb{U}^\lambda_\xi \rangle_{\xi < \lambda}$ be defined, on the base of this sequence, as in Definition 4.1.

After the sequences $\mathbb{u}^\alpha = \langle \mathbb{U}^\alpha_\xi \rangle_{\xi < \alpha}$ and $\mathbb{p}^\alpha = \langle \mathbb{P}^\alpha_\xi \rangle_{\xi < \alpha}$, and the model $\mathfrak{M}_\alpha$, have been defined for all $\alpha < \omega_1$, we let $\mathbb{P}_\xi = \bigcup_{\xi \le \alpha < \omega_1} \mathbb{U}^\alpha_\xi$ for all $\xi < \omega_1$, and let $\mathbb{p} = \mathbb{p}^{\omega_1} = \langle \mathbb{P}_\xi \rangle_{\xi < \omega_1}$. The set $\mathbf{MT}(\mathbb{p})$ of all $\mathbb{p}$-multitrees (Definition 3.1) will be our principal forcing notion. □

Proposition 6.2. *The sequences $\langle \mathbb{u}^\alpha \rangle_{\alpha < \omega_1}$, $\langle \mathbb{p}^\alpha \rangle_{\alpha < \omega_1}$ belong to $\varDelta^{\mathrm{HC}}_1$.* □

Remark 6.3. If $\alpha < \gamma \le \omega_1$ then the sets $\mathbf{MT}(\mathbb{p}^\alpha)$ and $\mathbf{MT}(\mathbb{p}^\gamma)$ of multitrees are formally disjoint. However we can naturally embed the former in the latter. Indeed each multitree $\boldsymbol{\tau} = \langle T_{\xi k} \rangle^{\xi < \alpha}_{k < \omega} \in \mathbf{MT}(\mathbb{p}^\alpha)$ can be identified as an element of $\mathbf{MT}(\mathbb{p}^\gamma)$ by the default extension $T_{\xi k} = 2^{<\omega}$ whenever $\alpha \le \xi < \gamma$. With such an identification, we can assume that $\mathbf{MT}(\mathbb{p}^\alpha) \subseteq \mathbf{MT}(\mathbb{p}^\gamma) \subseteq \mathbf{MT}(\mathbb{p})$, and similarly $\mathbf{MT}(\mathbb{p}^\lambda) = \bigcup_{\alpha < \lambda} \mathbf{MT}(\mathbb{u}^\alpha)$ for all limit λ, and the like. □

Lemma 6.4. *If* $\alpha < \omega_1$ *and a set* $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbf{MT}(\mathbb{p}^\alpha)$ *is pre-dense in* $\mathbf{MT}(\mathbb{p}^\alpha)$ *then it remains pre-dense in* $\mathbf{MT}(\mathbb{p})$.

Therefore $\mathbf{MT}(\mathbb{u}^\alpha)$ *is pre-dense in* $\mathbf{MT}(\mathbb{p})$.

Proof. By induction on γ, $\xi \le \gamma < \omega_1$, if D is pre-dense in $\mathbf{MT}(\mathbb{p}^\gamma)$ then it remains pre-dense in $\mathbf{MT}(\mathbb{p}^\gamma \vee \mathbb{u}^\gamma)$ by Lemma 4.6, hence in $\mathbf{MT}(\mathbb{p}^{\gamma+1})$ too by constructions. Limit steps including the step ω_1 are obvious.

To prove the second part, note that $\mathbf{MT}(\mathbb{u}^\alpha)$ is dense in $\mathbf{MT}(\mathbb{p}^\alpha \vee \mathbb{u}^\alpha)$ by Lemma 4.4, therefore, pre-dense in $\mathbf{MT}(\mathbb{p}^{\alpha+1})$, and $\mathbf{MT}(\mathbb{u}^\alpha) \in \mathfrak{M}_{\alpha+1}$. □

Corollary 6.5. *If* $\xi < \alpha < \omega_1$ *then the set* $\mathbb{U}^\alpha_\xi$ *is pre-dense in* $\mathbb{P}_\xi$.

Proof. Let $T \in \mathbb{P}_\xi$. Consider a multitree $\boldsymbol{\tau} \in \mathbf{MT}(\mathbb{p})$ defined so that $\boldsymbol{\tau}(\xi,0) = T$ and $\boldsymbol{\tau}(\eta,k) = 2^{<\omega}$ whenever $\langle \eta,k\rangle \ne \langle \xi,0\rangle$. By Lemma 6.4 $\boldsymbol{\tau}$ is compatible in $\mathbf{MT}(\mathbb{p})$ with some $\boldsymbol{u} \in \mathbf{MT}(\mathbb{u}^\alpha)$. We conclude that T is compatible in $\mathbb{P}_\xi$ with $U = \boldsymbol{u}(\xi,0) \in \mathbb{U}^\alpha_\xi$. □

Lemma 6.6. *If* $X \subseteq \mathbb{HC} = \mathbf{L}_{\omega_1}$ *then the set* W_X *of all ordinals* $\alpha < \omega_1$ *such that* $\langle \mathbf{L}_\alpha\,; X \cap \mathbf{L}_\alpha\rangle$ *is an elementary submodel of* $\langle \mathbf{L}_{\omega_1}\,; X\rangle$ *and* $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$ *is unbounded in* ω_1. *More generally, if* $X_n \subseteq \mathbb{HC}$ *for all* n *then the set* W *of all ordinals* $\alpha < \omega_1$, *such that* $\langle \mathbf{L}_\alpha\,; \langle X_n \cap \mathbf{L}_\alpha\rangle_{n<\omega}\rangle$ *is an elementary submodel of* $\langle \mathbf{L}_{\omega_1}\,; \langle X_n\rangle_{n<\omega}\rangle$ *and* $\langle X_n \cap \mathbf{L}_\alpha\rangle_{n<\omega} \in \mathfrak{M}_\alpha$, *is unbounded in* ω_1.

Proof. Let $\alpha_0 < \omega_1$. Let M be a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing α_0, ω_1, X, and such that $M \cap \mathbb{HC}$ is transitive. Let $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and let $\alpha = \phi(\omega_1)$. Then $\alpha_0 < \alpha < \lambda < \omega_1$ and $\phi(X) = X \cap \mathbf{L}_\alpha$ by the choice of M. It follows that $\langle \mathbf{L}_\alpha\,; X \cap \mathbf{L}_\alpha\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}\,; X\rangle$. Moreover, α is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\alpha$. We conclude that $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$ since $X \cap \mathbf{L}_\alpha \in \mathbf{L}_\lambda$ by construction.

The second claim does not differ much. □

Corollary 6.7. *The forcing* $\mathbf{MT}(\mathbb{p})$ *satisfies CCC.*

Proof. Suppose that $A \subseteq \mathbf{MT}(\mathbb{p})$ is a maximal antichain. By Lemma 6.6, there is an ordinal α such that $A' = A \cap \mathbf{MT}(\mathbb{p}^\alpha)$ is a maximal antichain in $\mathbf{MT}(\mathbb{p}^\alpha)$ and $A' \in \mathfrak{M}_\alpha$. But then A' remains pre-dense, therefore, maximal, in the whole set $\mathbf{MT}(\mathbb{p})$ by Lemma 6.4. It follows that $A = A'$ is countable. □

7 The extension: non-uniformizable set and Theorem 1.3

Working in terms of Definition 6.1, we consider the set $\mathbf{MT}(\mathbb{p}) \in \mathbf{L}$ as a forcing notion over $\mathbf{L}$. It is equal to the finite-support product $\prod_{\xi<\omega_1} {\mathbb{P}_\xi}^{<\omega}$, which also can be understood as the finite-support product $\prod_{\xi<\omega_1,\, k<\omega} \mathbb{P}_{\xi k}$, where each $\mathbb{P}_{\xi k}$ is equal to one and the same $\mathbb{P}_\xi = \bigcup_{\xi\le\alpha<\omega_1} \mathbb{U}^\alpha_\xi$ of Definition 6.1.

We make use of this forcing to **prove Theorem 1.3**.

Lemma 7.1 (= Lemma 7 in [7]). *Let* $\xi < \omega_1^{\mathbf{L}}$. *A real* $x \in 2^\omega$ *is* $\mathbb{P}_\xi$*-generic over* $\mathbf{L}$ *iff* $x \in Z_\xi = \bigcap_{\xi<\alpha<\omega_1^{\mathbf{L}}} \bigcup_{U\in\mathbb{U}_\xi^\alpha} [U]$.

Proof. All sets $\mathbb{U}_\xi^\alpha$ are pre-dense in $\mathbb{P}_\xi$ by Corollary 6.5. On the other hand, if $A \subseteq \mathbb{P}_\xi$, $A \in \mathbf{L}$ is a maximal antichain in $\mathbb{P}_\xi$, then easily $A \subseteq \mathbb{P}_\xi^\alpha$ for some α, $\xi < \alpha < \omega_1^{\mathbf{L}}$, by Corollary 6.7. But then every tree $U \in \mathbb{U}_\xi^\alpha$ satisfies $U \subseteq^{\mathtt{fin}} \bigcup A$ by Lemma 4.5, so that $\bigcup_{U\in\mathbb{U}_\xi^\alpha}[U] \subseteq \bigcup_{T\in A}[T]$. □

Corollary 7.2. *In any generic extension of* $\mathbf{L}$ *with the same* ω_1, *the set*

$$P = \{\langle\xi,x\rangle : \xi < \omega_1^{\mathbf{L}} \wedge x \in 2^\omega \text{ is } \mathbb{P}_\xi\text{-generic over } \mathbf{L}\} \subseteq \omega_1^{\mathbf{L}} \times 2^\omega$$

is Π_1^{HC}, *and* Π^1_2 *in terms of a usual coding system of ordinals* $< \omega_1$ *by reals.*

Proof. Use Lemma 7.1 and Proposition 6.2. □

Definition 7.3. From now on, let $G \subseteq \mathbb{P}^{<\omega}$ be a set $\mathbf{MT}(\mathbb{p})$-generic over $\mathbf{L}$. Note that $\omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}}$ by Corollary 6.7.

If $\xi < \omega_1^{\mathbf{L}}$ and $k < \omega$ then let $G_{\xi k} = \{\boldsymbol{\tau}(\xi,k) : \boldsymbol{\tau} \in G\}$, so that each $G_{\xi k}$ is $\mathbb{P}_\xi$-generic over $\mathbf{L}$ and $X_{\xi k} = \bigcap_{T\in G_{\xi k}}[T]$ is a singleton $X_{\xi k} = \{x_{\xi k}\}$ whose only element $x_{\xi k} \in 2^\omega$ is a real $\mathbb{P}_\xi$-generic over $\mathbf{L}$. □

The whole extension $\mathbf{L}[G]$ is then equal to $\mathbf{L}[\langle x_{\xi k}\rangle_{\xi<\omega_1^{\mathbf{L}},\,k<\omega}]$, and our goal is now to prove that it contains no $\mathbb{P}_\xi$-generic reals except for the reals $x_{\xi k}$.

Lemma 7.4 (in the assumptions of Definition 7.3). *If* $\xi < \omega_1^{\mathbf{L}}$ *and* $x \in \mathbf{L}[G] \cap 2^\omega$ *then* $x \in \{x_{\xi k} : k < \omega\}$ *iff* x *is a* $\mathbb{P}_\xi$*-generic real over* $\mathbf{L}$.

Proof. Otherwise there is a multitree $\boldsymbol{\tau} \in \mathbf{MT}(\mathbb{p})$ and a $\mathbf{MT}(\mathbb{p})$-real name $\mathbf{c} = \langle C_{ni}\rangle_{n<\omega,\,i=0,1} \in \mathbf{L}$ such that $\boldsymbol{\tau}$ $\mathbf{MT}(\mathbb{p})$-forces that $\mathbf{c}$ is $\mathbb{P}_\xi$-generic over $\mathbf{L}$ while $\mathbf{MT}(\mathbb{p})$ forces $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\xi k}$, $\forall k$. (Recall that $\dot{\boldsymbol{x}}_{\xi k}$ is a $\mathbf{MT}(\mathbb{p})$-name for $x_{\xi k}$.)

The sets $C_n = C_{n0} \cup C_{n1}$ are pre-dense in $\mathbf{MT}(\mathbb{p})$. It follows from Lemma 6.6 that there is an ordinal λ, $\xi < \lambda < \omega_1$, such that each set $C'_n = C_n \cap \mathbf{MT}(\mathbb{p}^\lambda)$ is pre-dense in $\mathbf{MT}(\mathbb{p}^\lambda)$, and the sequence $\langle C'_{ni}\rangle_{n<\omega,\,i=0,1}$ belongs to $\mathfrak{M}_\lambda$, where $C'_{ni} = C'_n \cap C_{ni}$ — then C'_n is pre-dense in $\mathbf{MT}(\mathbb{p})$, too, by Lemma 6.4. Thus we can assume that in fact $C_n = C'_n$, that is, $\mathbf{c} \in \mathfrak{M}_\lambda$ and $\mathbf{c}$ is a $\mathbf{MT}(\mathbb{p}^\lambda)$-name.

Further, as $\mathbf{MT}(\mathbb{p})$ forces that $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\xi k}$, the set D_k of all multitrees $\boldsymbol{\sigma} \in \mathbf{MT}(\mathbb{p})$ which directly force $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\xi k}$, is dense in $\mathbf{MT}(\mathbb{p})$ — for every k. Therefore, still by Lemma 6.6, we may assume that the same ordinal λ as above satisfies the following: each set $D'_k = D_k \cap \mathbf{MT}(\mathbb{p}^\lambda)$ is dense in $\mathbf{MT}(\mathbb{p}^\lambda)$.

Applying Theorem 5.3 with $\mathbb{p} = \mathbb{p}^\lambda$, $\mathbb{u} = \mathbb{u}^\lambda$, $\theta = \lambda$, $\eta = \xi$, we conclude that for each $U \in \mathbb{U}_\xi^\lambda$ the set Q_U of all multitrees $\boldsymbol{v} \in \mathbf{MT}(\mathbb{u}^\lambda)$ which directly force $\mathbf{c} \notin [U]$, is dense in $\mathbf{MT}(\mathbb{u}^\lambda \vee \mathbb{p}^\lambda)$, therefore, pre-dense in $\mathbf{MT}(\mathbb{p}^{\lambda+1})$. As obviously $Q_U \in \mathfrak{M}_{\lambda+1}$, we further conclude that Q_U is pre-dense in $\mathbf{MT}(\mathbb{p})$ by Lemma 6.4. Therefore $\mathbf{MT}(\mathbb{p})$ forces $\mathbf{c} \notin \bigcup_{U \in \mathbb{U}^\lambda_\xi}[U]$, hence, forces that $\mathbf{c}$ is not $\mathbb{P}_\xi$-generic, by Lemma 7.1. But this contradicts to the choice of $\boldsymbol{\tau}$. $\square$

Lemma 7.5 (in the assumptions of Definition 7.3)**.** *If* $\xi < \omega_1^{\mathbf{L}}$ *and* $k < \omega$ *then*

(i) $x_{\xi k} \notin \mathbf{L}[\langle x_{\eta\ell}\rangle_{\langle \eta,\ell\rangle \neq \langle \xi,k\rangle}]$,

(ii) $x_{\xi k}$ *is not* $OD(\langle x_{\eta\ell}\rangle_{\eta\neq\xi,\, k<\omega})$ *in* $\mathbf{L}[G]$. $\square$

Proof. (i) is a usual property of product forcing, while to prove (ii) we need to make use of the fact that by construction the ξ-part of the forcing is itself a finite-support product of countably many copies of $\mathbb{P}_\xi$. $\square$

Example 7.6 (non-uniformizable Π_1^{HC} set)**.** Arguing in the assumptions of Definition 7.3, we consider, in $\mathbf{L}[G] = \mathbf{L}[\langle x_{\xi k}\rangle_{\xi<\omega_1^{\mathbf{L}},\, k<\omega}]$, the set P of Corollary 7.2. First of all P is Π_1^{HC} in $\mathbf{L}[G]$ by Corollary 7.2. Further, it follows from Lemma 7.4 that

$$P = \{\langle \xi, x_{\xi k}\rangle : \xi < \omega_1^{\mathbf{L}} \wedge k < \omega\},$$

and hence all vertical cross-sections of P are countable. And by Lemma 7.5 it is not ROD uniformizable since any real in $\mathbf{L}[G]$ belongs to a submodel of the form $\mathbf{L}[\langle x_{\xi k}\rangle_{\xi<\zeta,\, k<\omega}]$, where $\zeta < \omega_1^{\mathbf{L}}$. $\square$

Example 7.7 (non-uniformizable Π_2^1 set)**.** To get a non-uniformizable Π_2^1 set in $2^\omega \times 2^\omega$ on the base of the abovedefined set $P \subseteq \omega_1^{\mathbf{L}} \times 2^\omega$, we make use of a usual coding of countable ordinals by reals. Let $\mathbf{WO} \subseteq 2^\omega$ be the Π_1^1 set of codes, and for $w \in \mathbf{WO}$ let $|w| < \omega_1$ be the ordinal coded by w. We consider

$$P' = \{\langle w, x\rangle \in \mathbf{WO} \times 2^\omega : \langle |w|, x\rangle \in P\},$$

a Π_2^1 set in $\mathbf{L}[G]$. Suppose towards the contrary that, in $\mathbf{L}[G]$, P' is uniformizable by a ROD set $Q' \subseteq P'$. As $\omega_1^{\mathbf{L}} = \omega_1$ by Corollary 6.7, for any $\xi < \omega_1$ there is a code $w \in \mathbf{WO} \cap \mathbf{L}$ with $|w| = \xi$. Let w_ξ be the $\leqslant_{\mathbf{L}}$-least of those. Then

$$Q = \{\langle \xi, x\rangle \in P : \langle w_\xi, x\rangle \in Q'\}$$

is a ROD subset of P which uniformizes P, contrary to Example 7.6. $\square$

$\square$ (Theorem 1.3)

8 Non-separation model

Here we prove Theorem 1.4. The model we use will be defined on the base of the model $\mathbf{L}[G] = \mathbf{L}[\langle x_{\xi k}\rangle_{\xi<\omega_1^{\mathbf{L}},\,k<\omega}]$ of Definition 7.3, of the form $\mathfrak{N}_\Xi = \mathbf{L}[\langle x_{\xi 0}\rangle_{\xi\in\Xi}]$, where $\Xi \subseteq \omega_1^{\mathbf{L}}$ will be a generic subset of $\omega_1^{\mathbf{L}}$, so that, strictly speaking, $\mathfrak{N}_\Xi$ is not going to be a submodel of $\mathbf{L}[G]$.

To define Ξ, we recall first of all that the ordinal product 2ν is considered as the ordered sum of ν copies of $2 = \{0,1\}$. Thus if $\nu = \lambda + m$, where λ is a limit ordinal or 0 and $m < \omega$, then $2\nu = \lambda + 2m$ and $2\nu + 1 = \lambda + 2m + 1$.

Now let $\mathbb{Q} = 3^{\omega_1^{\mathbf{L}}}$ with finite support, so that a typical element of $\mathbb{Q}$ is a partial map $q : \omega_1^{\mathbf{L}} \to 3 = 0,1,2$ with a finite domain $\mathtt{dom}\,q \subseteq \omega_1^{\mathbf{L}}$; this is a version of the Cohen forcing, of course.

Definition 8.1 (in the assumptions of Definition 7.3)**.** Let $H \subseteq \mathbb{Q}$ be a set generic over $\mathbf{L}[G]$. It naturally yiels a Cohen-generic map $F_H : \omega_1^{\mathbf{L}} \to 3$. Let

$$\begin{aligned} A_H &= \{\nu < \omega_1^{\mathbf{L}} : F_H(\nu) = 0\}, \qquad & B_H &= \{\nu < \omega_1^{\mathbf{L}} : F_H(\nu) = 1\}, \\ D_H &= \{\nu < \omega_1^{\mathbf{L}} : F_H(\nu) = 2\}, \qquad & &\text{and} \end{aligned}$$

$$\Xi_H = \{2\nu : \nu \in A_H \cup D_H\} \cup \{2\nu + 1 : \nu \in B_H \cup D_H\}.$$

We consider the model $\mathfrak{N}_H = \mathbf{L}[\langle x_{\xi 0}\rangle_{\xi\in\Xi_H}]$. Let $\mathbb{HC}(H) = (\mathbb{HC})^{\mathfrak{N}_H}$. □

Note that $\mathfrak{N}_H$ is not a submodel of $\mathbf{L}[G]$ since the set Ξ_H does not belong to $\mathbf{L}[G]$; but $\mathfrak{N}_H \subseteq \mathbf{L}[G][H]$, of course.

Theorem 8.2 (in the assumptions of Definition 8.1)**.** *It is true in $\mathfrak{N}_H$ that A_H and B_H are disjoint $\Pi_2^{\mathbb{HC}(H)}$ sets not separable by disjoint $\mathbf{\Sigma}_2^{\mathbb{HC}}$ sets.*

Example 8.3 (non-separable Π^1_3 sets)**.** In the notation of Example 7.7, let

$$X = \{w_\xi : \xi \in A_H\} \quad \text{and} \quad Y = \{w_\xi : \xi \in B_H\}.$$

The sets $X, Y \subseteq \mathbf{WO} \cap \mathbf{L}$ are $\Pi_2^{\mathbb{HC}(H)}$ together with A_H and B_H, and hence Π^1_3, and $X \cap Y = \varnothing$. Suppose towards the contrary that $X', Y' \subseteq 2^\omega$ are disjoint sets in $\mathbf{\Sigma}^1_3$, hence in $\mathbf{\Sigma}_2^{\mathbb{HC}(H)}$, such that $X \subseteq X'$ and $Y \subseteq Y'$. Then

$$A = \{\xi < \omega_1^{\mathbf{L}} : w_\xi \in X'\} \quad \text{and} \quad B = \{\xi < \omega_1^{\mathbf{L}} : w_\xi \in Y'\}$$

are disjoint sets in $\mathbf{\Sigma}_2^{\mathbb{HC}(H)}$, and we have $A_H \subseteq A$ and $B_H \subseteq B$ by construction, contrary to Theorem 8.2. □

The proof of Theorem 8.2 involves the following result which will be established in the next section. Theorem 8.4 esentially says that the coding structure in $\mathbf{L}[G]$ described in Section 7 survives a further Cohen-generic extension.

Theorem 8.4 (Cohen-generic stability)**.** *In the assumptions of Definition 8.1:*

(i) *if* $\xi < \omega_1^{\mathbf{L}}$ *and* $x \in \mathbf{L}[G][H] \cap 2^\omega$ *then* $x \in \{x_{\xi k} : k < \omega\}$ *iff* x *is a* $\mathbb{P}_\xi$*-generic real over* $\mathbf{L}$;

(ii) *if* $\xi < \omega_1^{\mathbf{L}}$ *and* $k < \omega$ *then* $x_{\xi k} \notin \mathbf{L}[\langle x_{\eta\ell}\rangle_{\langle \eta,\ell\rangle \neq \langle \xi,k\rangle}][H]$;

(iii) *if* $\xi < \omega_1^{\mathbf{L}}$ *and* $k < \omega$ *then* $x_{\xi k}$ *is not* $OD(\langle x_{\eta\ell}\rangle_{\eta\neq\xi,\, k<\omega}, H)$ *in* $\mathbf{L}[G][H]$*.*

Proof (Theorem 8.2 modulo Theorem 8.4)**.** That $A_H \cap B_H = \varnothing$ is clear. To see that, say, A_H is $\varPi_2^{\mathbb{HC}(H)}$ in $\mathfrak{N}_H$, prove that the equality

$$A_H = \{\nu < \omega_1 : \neg\, \exists\, x\, P(2\nu+1, x)\}$$

holds in $\mathfrak{N}_H$, where P is the $\varPi_1^{\mathbb{HC}}$ set of Corollary 7.2. (For B_H it would be $P(2\nu, x)$ in the displayed formula.)

First suppose that $\nu < \omega_1^{\mathbf{L}}$, $\xi = 2\nu+1$, $x \in \mathfrak{N}_H \cap \omega^\omega$, and $P(\xi, x)$ holds in $\mathfrak{N}_H$; prove that $\nu \notin A_H$. By definition x is $\mathbb{P}_\xi$-generic over $\mathbf{L}$. Then $x = x_{\xi k}$ for some k by Theorem 8.4(i). Therefore $k = 0$ and ξ has to belong to Ξ_H by Theorem 8.4(ii). But then $\nu \in B_H \cup D_H$, so $\nu \notin A_H$, as required.

To prove the converse, suppose that $\nu \notin A_H$, so that $\nu \in B_H \cup D_H$. Then $\xi = 2\nu+1 \in \Xi_H$, and hence $x = x_{\xi 0} \in \mathfrak{N}_H$. We conclude that $\langle \xi, x\rangle = \langle 2\nu+1, x\rangle \in P$ by Lemma 7.4, as required.

Finally, **to prove the non-separability**, suppose towards the contrary that, in $\mathfrak{N}_H$, A_H and B_H are separable by a pair of disjoint $\mathbf{\Sigma}_2^{\mathbb{HC}}$ sets $A, B \subseteq \omega_1 = \omega_1^{\mathbf{L}}$. These sets are defined in the set $\mathbb{HC}(H) = (\mathbb{HC})^{\mathfrak{N}_H}$ by $\varPi_2$ formulas, resp., $\varphi(a,\xi)$, $\psi(b,\xi)$, with real parameters $a, b \in \mathfrak{N}_H \cap 2^\omega$. Let $\lambda < \omega_1^{\mathbf{L}}$ be a limit ordinal such that $a, b \in \mathbf{L}[\langle x_{\xi 0}\rangle_{\xi \in \Xi_H \cap \lambda}]$, and let $\sigma, \tau \in \mathbf{L}[G]$ be $\mathbb{Q}$-real names such that $a = \sigma[H]$ and $b = \tau[H]$, which depend on $\langle x_{\xi 0}\rangle_{\xi\in\Xi_H\cap\lambda}$ only.

If $K \subseteq \mathbb{Q}$ is a set $\mathbb{Q}$-generic over $\mathbf{L}[G]$ (*e.g.*, $K = H$), then let

$$A_K^* = \{\xi < \omega_1^{\mathbf{L}} : \varphi(\sigma[K], \xi)^{\mathbb{HC}(K)}\}\,,\ \ B_K^* = \{\xi < \omega_1^{\mathbf{L}} : \psi(\tau[K], \xi)^{\mathbb{HC}(K)}\}\,,$$

so that by definition $A_H \subseteq A = A_H^*$, $B_H \subseteq B = B_H^*$, and $A_H^* \cap B_H^* = \varnothing$. Fix a condition $q_0 \in H$ which forces, over $\mathbf{L}[G]$, that $A_{\mathbf{h}} \subseteq A_{\mathbf{h}}^*$, $B_{\mathbf{h}} \subseteq B_{\mathbf{h}}^*$, and $A_{\mathbf{h}}^* \cap B_{\mathbf{h}}^* = \varnothing$, where $\mathbf{h}$ is the canonical name for H. We may assume that $\mathtt{dom}\, q_0 \subseteq \lambda$ as well, for otherwise just increase λ.

Now let ξ_0 be any ordinal with $\lambda \le \xi_0 < \omega_1$. Consider three sets $H_0, H_1, H_2 \subseteq \mathbb{Q}$, generic over $\mathbf{L}[G]$ and containing q_0, whose generic maps $F_{H_i} : \omega_1^{\mathbf{L}} \to 3$ satisfy $F_{H_i}(\xi_0) = i$ and $F_{H_0}(\xi) = F_{H_1}(\xi) = F_{H_2}(\xi)$ for all $\xi \neq \xi_0$.

Then $\sigma[H_0] = \sigma[H_2]$, $\tau[H_0] = \tau[H_2]$, and $\Xi_{H_2} = \Xi_{H_0} \cup \{2\xi_0+1\}$, hence, $\mathfrak{N}_{H_0} \subseteq \mathfrak{N}_{H_2}$. It follows by Shoenfield that $A_{H_0}^* \subseteq A_{H_2}^*$ and $B_{H_0}^* \subseteq B_{H_2}^*$, hence

$$A_{H_2} \subseteq A_{H_0} \subseteq A_{H_0}^* \subseteq A_{H_2}^*\,,\ \ B_{H_2} = B_{H_0} \subseteq B_{H_0}^* \subseteq B_{H_2}^*\,,\ \ A_{H_2}^* \cap B_{H_2}^* = \varnothing$$

by the choice of q_0. We conclude that $\xi_0 \in A^*_{H_2}$, just because $\xi_0 \in A_{H_0}$ by the choice of H_0. And we have $\xi_0 \in B^*_{H_2}$ by similar reasons. Thus $A^*_{H_2} \cap B^*_{H_2} \neq \varnothing$, contrary to the above. The contradiction ends the proof.

□ (Theorems 8.2 and 1.4 modulo Theorem 8.4)

9 The proof of the Cohen-generic stability theorem

Here **we prove Theorem 8.4.** We concentrate on Claim (i) of the theorem since claims (ii), (iii) are established by the same routine product-forcing arguments outlined in the proof of Lemma 7.5.

First of all, let us somewhat simplify the task. It is known that every real in a $\mathbb{Q}$-generic extension belongs to a simple $2^{<\omega}$-generic extension (that is, a Cohen-generic one) of the same model. That is, it suffices to prove this:

Lemma 9.1 (in the assumptions of Definition 7.3)**.** *If* $a \in 2^\omega$ *is* $2^{<\omega}$*-generic over* $\mathbf{L}[G]$*,* $\xi < \omega_1^{\mathbf{L}}$*, and a real* $x \in \mathbf{L}[G][a] \cap 2^\omega$ *is* $\mathbb{P}_\xi$*-generic over* $\mathbf{L}[G]$ *then* $x = x_{\xi k}$ *for some* k*.*

Proof. Coming back to Definition 6.1, we conclude that the sequence Φ there is generic not only over $\mathfrak{M}_\lambda$ but also over $\mathfrak{M}_\lambda[a]$ by the product forcing theorems. It follows that Lemma 6.4 also is true in $\mathbf{L}[a]$ for all sets $D \in \mathfrak{M}_\alpha[a]$, and so are Lemma 6.6 (for models $\mathbf{L}_{\omega_1}[a]$ and $\mathbf{L}_\alpha[a]$) and Corollaries 6.5 and 6.7. This enables us to prove Lemma 7.4 for all reals $x \in \mathbf{L}[G][a]$, and we are done. □

□ (Theorem 8.4)

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