On countable cofinality and decomposition of definable thin orderings

Vladimir Kanovei* Vassily Lyubetsky†

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Abstract

We prove that in some cases definable thin sets (including chains) of Borel partial orderings are necessarily countably cofinal. This includes the following cases: analytic thin sets, **ROD** thin sets in the Solovay model, and $\boldsymbol{\Sigma}^1_2$ thin sets in the assumption that $\omega_1^{\mathbf{L}[x]} < \omega_1$ for all reals x. We also prove that definable thin wellorderings admit partitions into definable chains in the Solovay model.

1 Introduction

Studies of maximal chains in partially ordered sets go back to as early as Hausdorff [6, 7], where this issue appeared in connection with Du Bois Reymond's investigations of orders of infinity. Using the axiom of choice, Hausdorff proved the existence of maximal chains (which he called *pantachies*) in any partial ordering. On the other hand, Hausdorff clearly understood the difference between such a pure existence proof and an actual construction of a maximal chain — see *e.g.* [6, p. 110] or comments in [3] — which we would understand nowadays as the existence of *definable* maximal chains.

The following theorem present three cases in which all linear, and even thin suborders of Borel PQOs are necessarily countably cofinal.

Theorem 1. *If* $\preccurlyeq$ *is a Borel PQO on a (Borel) set* $D = \mathtt{dom}(\preccurlyeq) \subseteq \omega^\omega$, $X \subseteq D$*, and* $\preccurlyeq\restriction X$ *is a thin quasi-ordering then* $\langle X; \preccurlyeq\rangle$ *is countably cofinal in each of the following three cases*:

*IITP RAS and MIIT, Moscow, Russia, `kanovei@googlemail.com`. Partial support of RFFI grant 13-01-00006 acknowledged. — *Contact author.*

†IITP RAS, Moscow, Russia, `lyubetsk@iitp.ru`

(i) *if* X *is a* $\mathbf{\Sigma}^1_1$ *set, and in this case moreover there is no* $\preccurlyeq$*-chains in* X *of uncountable cofinality,*

(ii) *if* X *is a* **ROD** *set in the Solovay model,*

(iii) *if* X *is a* $\mathbf{\Sigma}^1_2$ *set, and* $\omega_1^{\mathbf{L}[r]} < \omega_1$ *for every real* r.

Therefore, if, in addition, it is known that $\langle D\,;\leq\rangle$ *is not countably cofinal, then in all three cases* X *is not cofinal in* D.

The additional condition in the theorem, of the uncountable cofinality, holds for many partial orders of interest, *e.g.*, the eventual domination order on sets like ω^ω or $\mathbb{R}^\omega$, or the *rate of growth order* defined on $\mathbb{R}^\omega$ by $x <_{\rm RG} y$ iff $\lim_{n\to\infty}\frac{y(n)}{x(n)} = \infty$ (see a review in [10]). Needless to say that chains, gaps, and similar structures related to these or similar orderings have been subject of extended studies, of which we mention [1, 2, 18, 14] among those in which the definability aspect is considered.

Part (i) of the theorem is proved in Section 3 by reduction to a result (Theorem 3 below) which extends a theorem in [4] to the case of $\mathbf{\Sigma}^1_1$ suborders of a background Borel PQO as in (i). Part (ii) is already known from [13] in the case of linear, rather than thin, **ROD** suborders, but we present here (Section 6) an essentially simplified proof. Part (iii) is proved in Section 7 by a reference to part (ii) and a sequence of absoluteness arguments.

It is a challenging question to figure out whether claims (ii) and (iii) of Theorem 1 remain true in stronger forms similar to the "moreover" form of claim (i). The answer is pretty simple in the affirmative provided we consider only accordingly definable (but not necessarily cofinal) ω_1-sequences in the given set X — that is to say, **ROD** in claim (ii) and $\mathbf{\Sigma}^1_2$ in claim (iii).

The next theorem (our second main result) extends a classical decomposition theorem in [4] to the case of definable sets in the Solovay model.

Theorem 2 (in the Solovay model)**.** *Let* $\preccurlyeq$ *be an OD PQO on* ω^ω, $\approx$ *be the associated equivalence relation, and* $X^* \subseteq \omega^\omega$ *be an OD* $\preccurlyeq$*-thin set. Then* X^* *is covered by the union of all OD* $\preccurlyeq$*-chains* $C \subseteq \omega^\omega$.

The same is true for any definability class $OD(x)$, *where* x *is a real.*

The proof of this theorem as given in Section 12 has a certain semblance of the proof of Theorem 5.1 in [4] in the context of its general combinatorial structure. Yet the proof includes some changes necessary since OD sets in the Solovay model only partially resemble sets in $\varDelta^1_1$ and $\varSigma^1_1$. In particular we'll have to establish some properties of the OD forcing rather different from the properties of the Gandy – Harrington forcing applied in [4], and also prove a tricky compression lemma (Lemma 22) in Sections 8 – 11.

2 Notation

We proceed with notational remarks.

PQO, *partial quasi-order*: reflexive ($x \le x$) and transitive in the domain;

LQO, *linear quasi-order*: PQO and $x \le y \vee y \le x$ in the domain;

LO, *linear order*: LQO and $x \le y \wedge y \le x \Longrightarrow x = y$;

associated equivalence relation: $x \approx y$ iff $x \le y \wedge y \le x$.

associated strict order: $x < y$ iff $x \le y \wedge y \not\le x$.

By default we consider only *non-strict* orderings. All cases of consideration of *strict* PQOs will be explicitly specified.

strict PQO: irreflexive ($x \not< x$) and transitive;

strict LO: strict PQO and the trichotomy $\forall\, x, y\, (x < y \vee y < x \vee x = y)$.

LR (left–right) order preserving map: any map $f : \langle X\,; \le \rangle \to \langle X'\,; \le' \rangle$ such that we have $x \le y \Longrightarrow f(x) \le' f(y)$ for all $x, y \in \mathtt{dom}\, f$;

RL (right–left) order preserving map: a map $f : \langle X\,; \le \rangle \to \langle X'\,; \le' \rangle$ such that we have $x \le y \Longleftarrow f(x) \le' f(y)$ for all $x, y \in \mathtt{dom}\, f$;

sub-order: a restriction of the given PQO to a subset of its domain.

$<_{\tt lex}$, $\leqslant_{\tt lex}$: the lexicographical LOs on sets of the form 2^α, $\alpha \in \mathtt{Ord}$, resp. strict and non-strict.

Let $\langle P\,; \le \rangle$ be a background PQO. A subset $Q \subseteq P$ is:

cofinal in P: iff $\forall\, p \in P\, \exists\, q \in Q\, (p \le q)$;

countably cofinal (in itself): iff there exists a countable set $Q' \subseteq Q$ cofinal in Q;

a chain: iff it consists of 2wise $\le$-comparable elements, *i.e.*, LQO;

an antichain in P: iff it consists of 2wise $\le$-incomparable elements;

a thin set: iff it contains no perfect $\le$-antichains.

Finally if E is an equivalence relation then let

$$
\begin{array}{rcll}
[x]_{\mathsf{E}} & = & \{y \in \mathtt{dom}\,\mathsf{E} : x \mathrel{\mathsf{E}} y\} & \text{(the } \mathsf{E}\text{-}\textit{class} \text{ of } x \in \mathtt{dom}\,\mathsf{E}\text{)},\\
[X]_{\mathsf{E}} & = & \bigcup_{x \in X} [x]_{\mathsf{E}} & \text{(the } \mathsf{E}\text{-}\textit{saturation} \text{ of } X \subseteq \mathtt{dom}\,\mathsf{E}\text{)}.
\end{array}
$$

3 Analytic thin subsets

In this Section, we prove Theorem 1(i) by reference to the following background result:

Theorem 3 (proved in [12]). *Let* $\preccurlyeq$ *be a* $\varDelta^1_1$ *PQO on* ω^ω, $\approx$ *be the associated equivalence relation, and* $X^* \subseteq \omega^\omega$ *be a* $\varSigma^1_1$ *set. Then*

(I) *if* X^* *is* $\preccurlyeq$*-thin then there is an ordinal* $\alpha < \omega_1^{\mathrm{CK}}$ *and a* $\varDelta^1_1$ *LR order preserving map* $F : \langle \omega^\omega ; \preccurlyeq \rangle \to \langle 2^\alpha ; \leqslant_{\mathtt{lex}} \rangle$ *satisfying the following additional requirement: if* $x, y \in X^*$ *then* $x \not\approx y \implies F(x) \neq F(y)$;

(II) *if* X^* *is* $\preccurlyeq$*-thin then* X^* *is covered by the (countable) union of all* $\varDelta^1_1$ $\preccurlyeq$*-chains* $C \subseteq \omega^\omega$.

LR order preserving maps F, satisfying the extra requirement of non-gluing of $\approx$-classes as in (I), were called *linearization maps* in [9].

Any map F as in (I) of the theorem sends any two $\preccurlyeq$-incomparable reals $x, y \in \omega^\omega$ onto a $<_{\mathtt{lex}}$-comparable pair of $F(x), F(y)$, that is, either strictly $F(x) <_{\mathtt{lex}} F(y)$ or strictly $F(y) <_{\mathtt{lex}} F(x)$. On the other hand, if the background set X^* is already a $\preccurlyeq$-chain then F has to be RL order preserving too, that is, $x \preccurlyeq y$ iff $F(x) \leqslant_{\mathtt{lex}} F(y)$ for all $x, y \in X^*$.

Proof (Claim (i) of Theorem 1 modulo Theorem 3). First of all, assume that the given Borel order $\preccurlyeq$ is in fact $\varDelta^1_1$ and the given set $X = X^*$ is $\varSigma^1_1$. The case of $\varDelta^1_1(p)$ and $\varSigma^1_1(p)$ with any fixed real parameter p is accordingly reducible to a corresponding version of Theorem 3.

Let, by Theorem 3(II), $X^* \subseteq \bigcup_n C_n$, where each C_n is a $\varDelta^1_1$ $\preccurlyeq$-chain, and let F and α be given by Theorem 3(I). To check that X^* is countably cofinal, it suffices to show that such is every set $X_n = X^* \cap C_n$. But X_n is a chain, so if it is **not** countably cofinal then there is a strictly $\prec$-increasing sequence $\{x_\alpha\}_{\alpha<\omega_1}$ of elements $x_\alpha \in X_n$. Then $\{F(x_\alpha)\}_{\alpha<\omega_1}$ is accordingly a strictly $<_{\mathtt{lex}}$-increasing sequence in 2^α, which is impossible.

Finally if there is a $\preccurlyeq$-chain in X of uncountable cofinality then a similar argument leads to such a chain in $\langle 2^\alpha ; \leqslant_{\mathtt{lex}} \rangle$, with the same contradiction.

$\square$ (Theorem 1(i))

Theorem 3 itself is an extension of two results in [4] (theorems 3.1 and 5.1). The latter directly correspond to the case of $\varDelta^1_1$ sets X^* in Theorem 3. However the proof of Theorem 3 we manufactured in [12] rather strictly follows the arguments in [4]. See also [9] in matters of the additional requirement in claim (I), which also is presented in [4] implicitly.

4 Remarks and corollaries

Claim (I) of Theorem 3 can be strengthened as follows:

(I′) *if there is no continuous 1-1 LR order preserving map* $F : \langle 2^\omega \,;\leq_0\rangle \to \langle X^* \,;\preccurlyeq\rangle$ *such that* $a \mathrel{\not\mathsf{E}_0} b$ *implies that* $F(a), F(b)$ *are* $\preccurlyeq$*-incomparable, then there is an ordinal* $\alpha < \omega_1^{\mathrm{CK}}$ *and a* Δ^1_1 *LR order preserving map* $F : \langle \omega^\omega \,;\preccurlyeq\rangle \to \langle 2^\alpha \,;\leqslant_{\mathtt{lex}}\rangle$ *satisfying the following additional requirement: if* $x, y \in X^*$ *then* $x \not\approx y \implies F(x) \neq F(y)$.

Here $\leq_0$ is the PQO on 2^ω defined so that $x \leq_0 y$ iff $x \mathrel{\mathsf{E}_0} y$ and either $x = y$ or $x(k) < y(k)$, where k is the largest number with $x(k) \neq y(k)$. [1] The "if" premice in (I′) is an immediate consequence of the $\preccurlyeq$-thinness of X^* as in (I), and hence (I′) really strenthens (I) of Theorem 3.

Claim (I′) is an extension of Theorem 3 in [9]; the latter corresponds to the case of Δ^1_1 sets X^*.

In the category of chains (rather than thin sets), the case of Σ^1_1 sets X^* in Theorem 1(i) is reducible to the case of Δ^1_1 sets simply because any Σ^1_1 chain X can be covered by a Δ^1_1 chain Y. We find such a set Y by means of the following two-step procedure. [2] The set C of all elements, $\preccurlyeq$-comparable with every element $x \in X$, is Π^1_1, and $X \subseteq C$ (as X is a chain). By the Separation theorem, there is a Δ^1_1 set B such that $X \subseteq B \subseteq C$. Now, the set U of all elements in B, comparable with every element in B, is Π^1_1, and we have $X \subseteq B$. Once again, by Separation, there is a Δ^1_1 set Y such that $X \subseteq Y \subseteq U$. By construction, U and Y are chains, as required.

Recall the following well-known earlier result in passing by, originally due to H. Friedman, as mentioned in [5].

Corollary 4 (of Theorem 1(i))**.** *Every Borel LQO* $\leq$ *is countably cofinal, and moreover, there is no strictly increasing* ω_1*-sequences.* □

The next immediate corollary says that maximal chains cannot be analytic provided they are not countably cofinal.

Corollary 5. *If* $\preccurlyeq$ *is a Borel PQO, and every countable set* $D \subseteq \mathtt{dom}(\preccurlyeq)$ *has a strict upper bound, then there is no* $\mathbf{\Sigma}^1_1$ *maximal* $\preccurlyeq$*-chains.* □

Corollary 6 (Harrington and Shelah [5, 15])**.** *If* $\preccurlyeq$ *is a* $\mathbf{\Pi}^1_1$ *LQO on a Borel set then there is no strictly increasing* ω_1*-chains in* $\preccurlyeq$.

[1] $<_0$ orders each E_0-class similarly to the (positive and negative) integers, except for the class $[\omega \times \{0\}]_{\mathsf{E}_0}$ ordered as ω and the class $[\omega \times \{1\}]_{\mathsf{E}_0}$ ordered the inverse of ω.

[2] See a different argument, based on a reflection principle in [4, Corollary 1.5].

Proof. The result was first obtained by a direct and rather complicated argument. But fortunately there is a reduction to the Borel case.

Indeed let $x \prec y$ iff $y \not\preccurlyeq x$, so in fact $R_0 = \prec$ is just the strict LQO associated with $\preccurlyeq$. As $R_0 \subseteq (\preccurlyeq)$, by Separation there is a Borel set B_0, $R_0 \subseteq B_0 \subseteq (\preccurlyeq)$. Let B_0' be the relation of B_0-incomparability, and let R_1 be the PQO-hull of $B_0 \cup B_0'$. Thus R_1 is a LQO and $R_0 \subseteq B_0 \subseteq R_1 \subseteq (\preccurlyeq)$.

Once again, let B_1 is Borel set such that $R_1 \subseteq B_1 \subseteq (\preccurlyeq)$. Define sets B_1' and R_2 as above. And so on.

Finally, after ω steps, the union $R = \bigcup_n B_n = \bigcup_n R_n$ is a Borel LQO and $(\prec) \subseteq R \subseteq (\preccurlyeq)$. Any strictly $\preccurlyeq$-increasing chain is strictly R-increasing as well. It remains to apply Corollary 4. □

5 Near-counterexamples for chains

The following examples show that, even in the particular case of chains instead of thin orderings, Theorem 1(i) is not true any more for different extensions of the domain of $\mathbf{\Sigma}^1_1$ suborders of a Borel partial quasi-orders, such as $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ linear quasi-orders — not necessarily suborders of Borel orderings, as well as $\mathbf{\Delta}^1_2$ and $\mathbf{\Pi}^1_1$ suborders of Borel orderings. In each of these classes, a counterexample of cofinality ω_1 will be defined.

Example 1 ($\mathbf{\Sigma}^1_1$ LQO)**.** Consider a recursive coding of sets of rationals by reals. Let Q_x be the set coded by a real x. Let X_α be the set of all reals x such that the maximal well-ordered initial segment of Q_x has the order type α. We define

$$x \le y \quad \text{iff} \quad \exists\,\alpha\,\exists\,\beta\,(x \in X_\alpha \wedge y \in X_\beta \wedge \alpha \le \beta).$$

Then $\le$ is a Σ^1_1 LQO on ω^ω of cofinality ω_1.

Note that the associated strict order $x < y$, iff $x \le y$ but not $y \le x$, is then more complicated than just $\mathbf{\Sigma}^1_1$, therefore there is no contradiction in this example to the result mentioned in Remark 6. □

Example 2 ($\mathbf{\Pi}^1_1$ LQO)**.** Let $D \subseteq \omega^\omega$ be the Π^1_1 set of codes of (countable) ordinals. Then

$$x \le y \quad \text{iff} \quad x, y \in D \,\wedge\, |x| \le |y|$$

is a Π^1_1 LQO of cofinality ω_1. Note that $\le$ is defined on a non-Borel Π^1_1 set D, and there is no $\mathbf{\Pi}^1_1$ LQO of cofinality ω_1 but defined on a Borel set — by exactly the same argument as in Remark 6. □

Example 3 ($\mathbf{\Pi}^1_1$ LO)**.** To sharpen Example 2, define

$$x \le y \quad \text{iff} \quad x, y \in D \,\wedge\, \big(|x| < |y| \,\vee\, (|x| = |y| \wedge x <_{\texttt{lex}} y)\big);$$

this is a $\mathit{\Pi}^1_1$ LO of cofinality ω_1. $\square$

Example 4 ($\mathbf{\Delta}^1_2$ suborders)**.** Let $\le$ be the eventual domination order on ω^ω. Assuming the axiom of constructibility $\mathbf{V}=\mathbf{L}$, one can define a strictly $\le$-increasing $\mathit{\Delta}^1_2$ ω_1-sequence $\{x_\alpha\}_{\alpha<\omega_1}$ in ω^ω. $\square$

Example 5 ($\mathbf{\Pi}^1_1$ suborders)**.** Define a PQO $\le$ on $(\omega\smallsetminus\{0\})^\omega$ so that

$$x\le y \quad \text{iff} \quad \text{either} \quad x=y \quad \text{or} \quad \lim_{n\to\infty} y(n)\,/\,x(n)=\infty$$

(the "or" option defines the associated strict order $<$). Assuming the axiom of constructibility $\mathbf{V}=\mathbf{L}$, define a strictly increasing $\mathit{\Delta}^1_2$ ω_1-sequence $\{x_\alpha\}_{\alpha<\omega_1}$ in ω^ω. By the Novikov – Kondo – Addison $\mathit{\Pi}^1_1$ Uniformization theorem, there is a $\mathit{\Pi}^1_1$ set $\{\langle x_\alpha,y_\alpha\rangle\}_{\alpha<\omega_1}\subseteq\omega^\omega\times 2^\omega$. Let $z_\alpha(n)=3^{x_\alpha(n)}\cdot 2^{y_\alpha(n)}$, $\forall n$. Then the ω_1-sequence $\{z_\alpha\}_{\alpha<\omega_1}$ is $\mathit{\Pi}^1_1$ and strictly increasing: indeed, factors of the form $2^{y_\alpha(n)}$ are equal 1 or 2 whenever $\alpha\in 2^\omega$. $\square$

6 Definable thin suborders in the Solovay model

Here **we prove Theorem 1**(ii). Arguing in the Solovay model (a model of **ZFC** defined in [16], in which all **ROD** sets of reals are Lebesgue measurable), we assume that $\preccurlyeq$ is a Borel PQO on a Borel set $D\subseteq\omega^\omega$, $X\subseteq D$ is a **ROD** (real-ordinal definable) set, and the set X is a $\preccurlyeq$-thin.

Let $\rho<\omega_1$ be such that $\preccurlyeq$ is a relation in $\mathbf{\Sigma}^0_\rho$.

Prove that the restricted ordering $\langle X;\preccurlyeq\rangle$ *is countably cofinal*, *i.e.*, contains a countable cofinal subset (not necessarily a chain).

It is known that in the Solovay model any **ROD** set in ω^ω is a union of a **ROD** ω_1-sequence of analytic sets. Thus there is a $\subseteq$-increasing **ROD** sequence $\{X_\alpha\}_{\alpha<\omega_1}$ of $\mathbf{\Sigma}^1_1$ sets X_α, such that $X=\bigcup_{\alpha<\omega_1}X_\alpha$. Let $r\in\omega^\omega$ be a real parameter such that in fact the sequence $\{X_\alpha\}_{\alpha<\omega_1}$ is $\mathrm{OD}(r)$.

As the sets X_α are countably $\preccurlyeq$-cofinal by claim (i) of Theorem 1, it suffices to prove that one of X_α is cofinal in X.

Suppose otherwise. Then the sets $D_\alpha=\{z\in D:\exists\,x\in X_\alpha\,(z\preccurlyeq x)\}$ contain $\aleph_1$ different sets and form an $\mathrm{OD}(r)$ sequence. We claim that every set D_α belongs to the same class $\mathbf{\Sigma}^0_\rho$ as the given Borel order $\preccurlyeq$. Indeed let $\{x_n:n\in\omega\}$ be any countable cofinal set in X_α. Then the set $D_\alpha=\{z\in D:\exists\,n\,(z\preccurlyeq x_n)\}$ is $\mathbf{\Sigma}^0_\rho$ by obvious reasons.

We conclude that the Borel class $\mathbf{\Sigma}^0_\rho$ contains $\aleph_1$ pairwise different sets in $\mathrm{OD}(r)$ for one and the same $r\in\omega^\omega$. But this contradicts to a well-known result of Stern [17].

$\square$ (Theorem 1(ii))

7 $\mathbf{\Sigma}^1_2$ thin suborders of Borel PQOs

Here **we prove Theorem 1**(iii). Assume that $\preccurlyeq$ is a Borel PQO on a Borel set $D \subseteq \omega^\omega$, $X \subseteq D$ is a $\mathbf{\Sigma}^1_2$ set, and X is $\preccurlyeq$-thin. We also assume that $\omega_1^{\mathbf{L}[r]} < \omega_1$ for every real r.

Prove that the ordering $\langle X;\preccurlyeq\rangle$ *is countably cofinal.*

Pick a real r such that X is $\Sigma^1_2(r)$ and $\preccurlyeq$ is $\Delta^1_1(r)$. To prepare for an absoluteness argument, fix canonical formulas,

$$\varphi(\cdot,\cdot) \text{ of type } \Sigma^1_2\,, \quad \sigma(\cdot,\cdot,\cdot) \text{ of type } \Sigma^1_1\,, \quad \pi(\cdot,\cdot,\cdot) \text{ of type } \Pi^1_1\,,$$

which define X and $\preccurlyeq$ in the set universe $\mathbf{V}$, so that it is true in $\mathbf{V}$ that

$$x \preccurlyeq y \iff \sigma(r,x,y) \iff \pi(r,x,y) \quad \text{and} \quad x \in X \iff \varphi(r,x)\,.$$

for all $x, y \in \omega^\omega$. We let $X_\varphi = \{x \in \omega^\omega : \varphi(r,x)\}$ and

$$x \leq_{\sigma\pi} y \quad\iff\quad \sigma(r,x,y) \quad\iff\quad \pi(r,x,y)$$

so that $X_\varphi = X$ and $\leq_{\sigma\pi}$ is $\preccurlyeq$ in $\mathbf{V}$, but X_φ and $\leq_{\sigma\pi}$ can be defined in any transitive universe containing r and containing all ordinals (to preserve the equivalence of formulas σ and π).

Let $\mathbf{WO}$ be the canonical Π^1_1 set of codes of (countable) ordinals, and for $w \in \mathbf{WO}$ let $|w| < \omega_1$ be the ordinal coded by w.

Let $X_\varphi = \bigcup_{\alpha<\omega_1} X_\varphi(\alpha)$ be a canonical representation of X_φ as an increasing union of $\mathbf{\Sigma}^1_1$ sets. Thus to define $X_\varphi(\alpha)$ fix a $\Pi^1_1(r)$ set $P \subseteq (\omega^\omega)^2$ such that $X = \{x : \exists\, y\, P(x,y)\}$, fix a canonical $\Pi^1_1(r)$ norm $f : P \to \omega_1$, and let

$$P_\alpha = \{\langle x,y\rangle : f(x,y) < \alpha\} \quad \text{and} \quad X_\varphi(\alpha) = \{x : \exists\, y\, (\langle x,y\rangle \in P_\alpha)\}\,.$$

In our assumptions, the ordinal $\Omega = \omega_1$ is inaccessible in $\mathbf{L}[r]$. Let $\mathscr{P} = \mathrm{Coll}(<\Omega,\omega) \in \mathbf{L}[r]$ be the corresponding Levy collapse forcing. Consider a $\mathscr{P}$-generic extension $\mathbf{V}[G]$ of the universe. Then $\mathbf{L}[r][G]$ is a Solovay-model generic extension of $\mathbf{L}[r]$. The plan is to compare the models $\mathbf{V}$ and $\mathbf{L}[r][G]$. Note that $\mathbf{L}[r]$ is their common part, $\mathbf{V}[G]$ is their common extension, and the three models have the same cardinal $\omega_1^{\mathbf{V}} = \omega_1^{\mathbf{L}[r][G]} = \omega_1^{\mathbf{V}[G]} = \Omega > \omega_1^{\mathbf{L}[r]}$.

Lemma 7. *It is true both in* $\mathbf{V}[G]$ *and* $\mathbf{L}[r][G]$ *that if* $\alpha < \Omega$ *then the set* $X_\varphi(\alpha)$ *is* $\leq_{\sigma\pi}$*-thin.*

This key absoluteness lemma has no analogies in a simpler case of chains (instead of thin sets) earlier considered in [11]. Here we even don't claim the absolutenes of the thinness poperty of the whole set $X_\varphi = \bigcup_{\alpha<\Omega} X_\varphi(\alpha)$!

Proof (Lemma)**.** Note that the thinness of $X_\varphi(\alpha)$ is a Π^1_3 statement with parameters r and any real which codes α. This makes the step $\mathbf{V}[G] \to \mathbf{L}[r][G]$ trivial by Shoenfield, and allows to concentrate on $\mathbf{V}[G]$.

Suppose towards the contrary that there is a perfect tree $T \in \mathbf{V}[G]$, $T \subseteq \omega^{<\omega}$, such that the perfect set $[T] = \{x \in \omega^\omega : \forall\, n\, (x \restriction n \in T)\}$ satisfies

(1) $[T] \subseteq X_\varphi(\alpha)$ and $[T]$ is a $\le_{\sigma\pi}$-antichain

in $\mathbf{V}[G]$. There exist an ordinal $\gamma < \Omega$ and a $\mathbf{Coll}(\omega,\gamma)$-generic map $F \in \mathbf{V}[G]$ such that already $T \in \mathbf{V}[F]$, so that $T = t[F]$, where $t \in \mathbf{V}$, $t \subseteq \mathbf{Coll}(\omega,\gamma) \times \omega^{<\omega}$ is a $\mathbf{Coll}(\omega,\gamma)$-name.

Note that (1) is still true in $\mathbf{V}[F] \subseteq \mathbf{V}[G]$ by Shoenfield, moreover, (1) is true in $\mathbf{L}[z][F]$, where a real $x \in \omega^\omega \cap \mathbf{V}$ codes all of α, γ, r, t. Therefore there is a condition $s \subset F$ (a finite string of ordinals $\xi < \gamma$) which $\mathbf{Coll}(\omega,\gamma)$-forces (1) (with T replaced by the name t) over $\mathbf{L}[z]$.

Now, by the assumptions of Theorem 1(iii), there is a map $F' \in \mathbf{V}$, still $\mathbf{Coll}(\omega,\gamma)$-generic over $\mathbf{L}[z]$ and satisfying $s \subset F'$. Then the tree $T' = t[F']$ belongs to the model $\mathbf{L}[z][F'] \subseteq \mathbf{V}$ and satisfies (1) (in the place of T) in $\mathbf{L}[z][F']$, hence, in $\mathbf{V}$ as well by Shoenfield. But this contradicts to the choice of $X = X_\varphi$. □ (Lemma)

We continue the proof of Theorem 1(iii). It follows from the lemma that all orderings $\langle X_\varphi(\alpha)\,; \le_{\sigma\pi}\rangle$, $\alpha < \Omega$, are countably cofinal in $\mathbf{L}[r][G]$ by Theorem 1(i). However $\mathbf{L}[r][G]$ is a Solovay-model type extension of $\mathbf{L}[r]$. Therefore (see the argument in Section 6) it is true in $\mathbf{L}[r][G]$ that the whole ordering $\langle X_\varphi\,; \le_{\sigma\pi}\rangle$ is countably cofinal, hence there is an ordinal $\alpha < \Omega = \omega_1^{\mathbf{L}[r][G]}$ such that the sentence

(2) the subset $X_\varphi(\alpha)$ is $\le_{\sigma\pi}$-cofinal in the whole set X_φ

is true in $\mathbf{L}[r][G]$. However (2) can be expressed by a Π^1_2 formula with r and an arbitrary code $w \in \mathbf{WO} \cap \mathbf{L}[r][G]$ such that $|w| = \alpha$ — as the only parameters. It follows, by Shoenfield, that (2) is true in $\mathbf{V}[G]$ as well.

Then by exactly the same absoluteness argument (2) is true in $\mathbf{V}$, too. Thus it is true in $\mathbf{V}$ that $X_\varphi(\alpha)$, a $\mathbf{\Sigma}^1_1$ set, is cofinal in the whole set $X = X_\varphi$. But $X_\varphi(\alpha)$ is countably cofinal by Theorem 1(i).

□ (Theorem 1(iii))

8 The Solovay model and OD forcing

Here we begin the proof of Theorem 2. We emulate the proof of Theorem 5.1 in [4] and a similar proof of Theorem 3(II) above (given in [12]), changing the Gandy – Harrington forcing $\mathbb{P}$ with the OD forcing $\mathbf{P}$. There is no direct analogy between the two forcing notions, so we'll both enjoy some simplifications and suffer from some complications.

We start with a brief review of the Solovay model. Let Ω be an ordinal. Let Ω-SM be the following hypothesis:

Ω-SM: $\Omega = \omega_1$, Ω is strongly inaccessible in $\mathbf{L}$, the constructible universe, and the whole universe $\mathbf{V}$ is a generic extension of $\mathbf{L}$ via the Levy collapse forcing $\mathbf{Coll}(\omega, {<}\Omega)$, as in [16].

Assuming Ω-SM, let $\mathbf{P}$ be the set of all **non-empty** OD sets $Y \subseteq \omega^\omega$. We consider $\mathbf{P}$ as a forcing notion (smaller sets are stronger). A set $D \subseteq \mathbf{P}$ is:

- *dense*, iff for every $Y \in \mathbf{P}$ there exists $Z \in D$, $Z \subseteq Y$;
- *open dense*, iff in addition we have $Y \in D \Longrightarrow X \in D$ whenever sets $Y \subseteq X$ belong to $\mathbf{P}$;

A set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic, iff 1) if $X, Y \in G$ then there is a set $Z \in G$, $Z \subseteq X \cap Y$, and 2) if $D \subseteq \mathbf{P}$ is OD and dense then $G \cap D \neq \varnothing$.

Given an OD equivalence relation E on ω^ω, a *reduced product* forcing notion $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ consists of all sets of the form $X \times Y$, where $X, Y \in \mathbf{P}$ and $[X]_{\mathsf{E}} \cap [Y]_{\mathsf{E}} \neq \varnothing$. For instance $X \times X$ belongs to $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ whenever $X \in \mathbf{P}$. The notions of sets dense and open dense in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$, and $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic sets are similar to the case of $\mathbf{P}$

A condition $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ is *saturated* iff $[X]_{\mathsf{E}} = [Y]_{\mathsf{E}}$.

Lemma 8. *If $X \times Y$ is a condition in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ then there is a stronger saturated subcondition $X' \times Y'$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$.*

Proof. Let $X' = X \cap [Y]_{\mathsf{E}}$ and $Y' = Y \cap [X]_{\mathsf{E}}$. □

Proposition 9 (lemmas 14, 16 in [8])**.** *Assume Ω-SM.*

If a set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic then the intersection $\bigcap G = \{x[G]\}$ consists of a single real $x[G]$, called $\mathbf{P}$-generic — its name will be $\dot{\boldsymbol{x}}$.

Given an OD equivalence relation E on ω^ω, if $G \subseteq \mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ is $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic then the intersection $\bigcap G = \{\langle x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]\rangle\}$ consists of a single pair of reals $x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]$, called an $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic pair — their names will be $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$; either of $x_{\mathtt{le}}[G], x_{\mathtt{ri}}[G]$ is separately $\mathbf{P}$-generic. □

As the set $\mathbf{P}$ is definitely uncountable, the existence of $\mathbf{P}$-generic sets does not immediately follow from Ω-SM by a cardinality argument. Yet fortunately $\mathbf{P}$ is *locally countable*, in a sense.

Definition 10 (assuming Ω-SM)**.** A set $X \in \text{OD}$ is *OD-1st-countable* if the set $\mathscr{P}_{\mathsf{OD}}(X) = \mathscr{P}(X) \cap \text{OD}$ of all OD subsets of X is at most countable. □

For instance, assuming Ω-SM, the set $X = \omega^\omega \cap \text{OD} = \omega^\omega \cap \mathbf{L}$ of all OD reals is OD-1st-countable. Indeed $\mathscr{P}_{\mathsf{OD}}(X) = \mathscr{P}(X) \cap \mathbf{L}$, and hence $\mathscr{P}_{\mathsf{OD}}(X)$ admits an OD bijection onto the ordinal $\omega_2^{\mathbf{L}} < \omega_1 = \Omega$.

Lemma 11 (assuming Ω-SM)**.** *If a set $X \in OD$ is OD-1st-countable then the set $\mathscr{P}_{\mathsf{OD}}(X)$ is OD-1st-countable either.*

Proof. There is an ordinal $\lambda < \omega_1 = \Omega$ and an OD bijection $b : \lambda \xrightarrow{\text{onto}} \mathscr{P}_{\mathsf{OD}}(X)$. Any OD set $Y \subseteq \lambda$ belongs to $\mathbf{L}$, hence, the OD power set $\mathscr{P}_{\mathsf{OD}}(\lambda) = \mathscr{P}(\lambda) \cap \mathbf{L}$ belongs to $\mathbf{L}$ and $\mathtt{card}(\mathscr{P}_{\mathsf{OD}}(\lambda)) \le \lambda^+ < \Omega$ in $\mathbf{L}$. We conclude that $\mathscr{P}_{\mathsf{OD}}(\lambda)$ is countable. It follows that $\mathscr{P}_{\mathsf{OD}}(\mathscr{P}_{\mathsf{OD}}(X))$ is countable, as required. □

Lemma 12 (assuming Ω-SM)**.** *If $\lambda < \Omega$ then the set $\textsc{Coh}_\lambda$ of all elements $f \in \lambda^\omega$, $\mathbf{Coll}(\omega, \lambda)$-generic over $\mathbf{L}$, is OD-1st-countable.*

Proof. If $Y \subseteq \textsc{Coh}_\lambda$ is OD and $x \in Y$ then "$\check{x} \in \check{Y}$" is $\mathbf{Coll}(\omega, \lambda)$-forced over $\mathbf{L}$. It follows that there is a set $S \subseteq \lambda^{<\omega} = \mathbf{Coll}(\omega, \lambda)$, $S \in \mathbf{L}$, such that $Y = \textsc{Coh}_\lambda \cap \bigcup_{t \in S} \mathscr{N}_t$, where $\mathscr{N}_t = \{x \in \lambda^{<\omega} : t \subset x\}$, a Baire interval in $\lambda^{<\omega}$. But the collection of all such sets S belongs to $\mathbf{L}$ and has cardinality λ^+ in $\mathbf{L}$, hence, is countable under Ω-SM. □

Let $\mathbf{P}^*$ be the set of all OD-1st-countable sets $X \in \mathbf{P}$. We also define

$$\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^* = \{X \times Y \in \mathbf{P} \times_{\mathsf{E}} \mathbf{P} : X, Y \in \mathbf{P}^*\}.$$

Lemma 13 (assuming Ω-SM)**.** *The set $\mathbf{P}^*$ is dense in $\mathbf{P}$, that is, if $X \in \mathbf{P}$ then there is a condition $Y \in \mathbf{P}^*$ such that $Y \subseteq X$.*

If E is an OD equivalence relation on ω^ω then the set $\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$ is dense in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ and any $X \times Y$ in $\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$ is OD-1st-countable.

Proof. Let $X \in \mathbf{P}$. Consider any $x \in X$. It follows from Ω-SM that there is an ordinal $\lambda < \omega_1 = \Omega$, an element $f \in \textsc{Coh}_\lambda$, and an OD map $H : \lambda^\omega \to \omega^\omega$, such that $x = H(f)$. The set $P = \{f' \in \textsc{Coh}_\lambda : H(f') \in X\}$ is then OD and non-empty (contains f), and hence so is its image $Y = \{H(f') : f' \in P\} \subseteq X$ (contains x). Finally, $Y \in \mathbf{P}^*$ by Lemma 12.

To prove the second claim, let $X \times Y$ be a condition in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$. By Lemma 8 there is a stronger saturated subcondition $X' \times Y' \subseteq X \times Y$. By the first part of the lemma, let $X'' \subseteq X'$ be a condition in $\mathbf{P}^*$, and $Y'' = Y' \cap [X'']_{\mathsf{E}}$. Similarly, let $Y''' \subseteq Y''$ be a condition in $\mathbf{P}^*$, and $X''' = X'' \cap [Y''']_{\mathsf{E}}$. Then $X''' \times Y'''$ belongs to $\mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$. □

Corollary 14 (assuming Ω-SM)**.** *If* $X \in \mathbf{P}$ *then there exists a* $\mathbf{P}$*-generic set* $G \subseteq \mathbf{P}$ *containing* X*. If* $X \times Y$ *is a condition in* $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ *then there exists a* $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$*-generic set* $G \subseteq \mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ *containing* $X \times Y$*.*

Proof. By Lemma 13, assume that $X \in \mathbf{P}^*$. Then the set $\mathbf{P}_{\subseteq X}$ of stronger conditions contains only countably many OD subsets by Lemma 11. □

9 The OD forcing relation

The forcing notion $\mathbf{P}$ will play the same role below as the Gandy – Harrington forcing in [4, 12]. There is a notable technical difference: under Ω-SM, OD-generic sets exist in the ground Solovay-model universe by Corollary 14. Another notable difference is connected with the forcing relation.

Definition 15 (assuming Ω-SM)**.** Let $\varphi(x)$ be an $\mathtt{Ord}$-*formula*, that is, a formula with ordinals as parameters.

A condition $X \in \mathbf{P}$ is said to $\mathbf{P}$-*force* $\varphi(\dot{\boldsymbol{x}})$ iff $\varphi(x)$ is true (in the Solovay-model set universe considered) for any $\mathbf{P}$-generic real x.

If E is an OD equivalence relation on ω^ω then a condition $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ is said to $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-*force* $\varphi(\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}})$ iff $\varphi(x, y)$ is true for any $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$-generic pair $\langle x, y \rangle$. □

Lemma 16 (assuming Ω-SM)**.** *Given an* $\mathtt{Ord}$*-formula* $\varphi(x)$ *and a* $\mathbf{P}$*-generic real* x*, if* $\varphi(x)$ *is true* (in the Solovay-model set universe considered) *then there is a condition* $X \in \mathbf{P}$ *containing* x*, which* $\mathbf{P}$*-forces* $\varphi(\dot{\boldsymbol{x}})$*.*

Let E *be an OD equivalence relation on* ω^ω*. Given an* $\mathtt{Ord}$*-formula* $\varphi(x, y)$ *and a* $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$*-generic pair* $\langle x, y \rangle$*, if* $\varphi(x, y)$ *is true then there is a condition in* $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ *containing* $\langle x, y \rangle$*, which* $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})$*-forces* $\varphi(\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}})$*.*

Proof. To prove the first claim, put $X = \{x' \in \omega^\omega : \varphi(x')\}$. But this argument does not work for $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$. To fix the problem, we propose a longer argument which equally works in both cases — but we present it in the case of $\mathbf{P}$ which is slightly simpler.

Formally the forcing notion $\mathbf{P}$ does not belong to $\mathbf{L}$. But it is order-isomorphic to a certain forcing notion $P \in \mathbf{L}$, namely, the set P of *codes* [3] of OD sets in $\mathbf{P}$. The order between the codes in P, which reflects the relation $\subseteq$ between the OD sets themselves, is expressible in $\mathbf{L}$, too. Furthermore dense OD sets in $\mathbf{P}$ correspond to dense sets in the coded forcing P in $\mathbf{L}$.

[3] A code of an OD set X is a finite sequence of logical symbols and ordinals which correspond to a definition in the form $X = \{x \in \mathbf{V}_\alpha : \mathbf{V}_\alpha \models \varphi(x)\}$.

Now, let x be $\mathbf{P}$-generic and $\varphi(x)$ be true. It is a known property of the Solovay model that there is another $\mathtt{Ord}$-formula $\psi(x)$ such that $\varphi(x)$ iff $\mathbf{L}[x] \models \psi(x)$. Let $g \subseteq P$ be the set of all codes of conditions $X \in \mathbf{P}$ such that $x \in X$. Then g is P-generic over $\mathbf{L}$ by the choice of x, and x is the corresponding generic object, hence there is a condition $p \in g$ which P-forces $\psi(\dot{\boldsymbol{x}})$ over $\mathbf{L}$. Let $X \in \mathbf{P}$ be the OD set coded by p, so $x \in X$. To prove that X OD-forces $\varphi(\dot{\boldsymbol{x}})$, let $x' \in X$ be a $\mathbf{P}$-generic real. Let $g' \subseteq P$ be the P-generic set of all codes of conditions $Y \in \mathbf{P}$ such that $x' \in Y$. Then $p \in g'$, hence $\psi(x')$ holds in $\mathbf{L}[x']$, by the choice of p. Then $\varphi(x')$ holds (in the Solovay-model set universe) by the choice of ψ, as required. □

Corollary 17 (assuming Ω-SM)**.** *Given an* $\mathtt{Ord}$*-formula* $\varphi(x)$*, if* $X \in \mathbf{P}$ *does not* $\mathbf{P}$*-force* $\varphi(\dot{\boldsymbol{x}})$ *then there is a condition* $Y \in \mathbf{P}$, $Y \subseteq X$*, which* $\mathbf{P}$*-forces* $\neg\,\varphi(\dot{\boldsymbol{x}})$*. The same for* $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$*.* □

10 Adding a perfect antichain

The next result will be pretty important.

Lemma 18 (assuming Ω-SM)**.** *Assume that* $\preccurlyeq$ *is an OD PQO on* ω^ω*, and* E_A *is an OD equivalence relation on* ω^ω *for any* $A \in \mathbf{P}$*, such that if* $A \subseteq B$ *then* $x \mathrel{\mathsf{E}_A} y$ *implies* $x \mathrel{\mathsf{E}_B} y$*.*

Suppose that $X^* \in \mathbf{P}$*, and if* $B \in \mathbf{P}$, $B \subseteq X^*$ *then* $B \times B$ *does* ***not*** $(\mathbf{P} \times_{\mathsf{E}_B} \mathbf{P})$*-force that* $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ *are* $\preccurlyeq$*-comparable. Then* X^* *is not* $\preccurlyeq$*-thin.*

Proof (follows 2.9 in [4])**.** Let T be the set of all finite trees $t \subseteq 2^{<\omega}$. If $t \in T$ then let $M(t)$ be the set of all $\subset$-maximal elements of t.

Let Φ be the set of systems $\varphi = \{X_u\}_{u \in t}$ of sets $X_u \in \mathbf{P}^*$, such that $t \in T$ and the following conditions (i) – (iv) are satisfied:

(i) $X_\Lambda \subseteq X^*$ (where Λ is the empty string);

(ii) if $u \subset v \in t$ then $X_v \subseteq X_u$;

(iii) if $u^\wedge 0$ and $u^\wedge 1$ belong to t then $X_{u^\wedge 0} \times X_{u^\wedge 1}$ belongs to $\mathbf{P}^* \times_{\mathsf{E}_{X_u}} \mathbf{P}^*$ and $(\mathbf{P} \times_{\mathsf{E}_{X_u}} \mathbf{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}}$ is $\preccurlyeq$-incomparable to $\dot{\boldsymbol{x}}_{\mathtt{ri}}$;

(iv) compatibility: there is a sequence $\{x_u\}_{u \in M(t)}$ of points $x_u \in X_u$ such that if $u, v \in M(t)$ then $x_u \mathrel{\mathsf{E}_{X_{u \wedge v}}} x_v$, where $u \wedge v$ is the largest string $w \in 2^{<\omega}$ such that $w \subset u$ and $w \subset v$ — it easily follows that then $X_u \times X_v$ is a condition in $\mathbf{P} \times_{\mathsf{E}_{X_{u \wedge v}}} \mathbf{P}$.

Say that a system $\{X_u\}_{u \in t} \in \Phi$ is *saturated* if in addition

(v) for any $v \in M(t)$ and $x \in X_v$ there is a sequence $\{x_u\}_{u\in M(t)}$ as in (iv), such that $x_v = x$.

Say that a system $\{X'_u\}_{u\in t'} \in \Phi$: 1) *weakly extends* another system $\varphi = \{X_u\}_{u\in t}$ if $t \subseteq t'$, $X_u = X'_u$ for all $u \in t \smallsetminus M(t)$, and $X'_u \subseteq X_u$ for all $u \in M(t)$; and 2) *properly extends* $\{X_u\}_{u\in t}$ if $t \subseteq t'$ and $X_u = X'_u$ for all $u \in t$. Thus a weak extension not just adds new sets to a given system φ but also shrinks old sets of the top layer $\varphi = \{X_u\}_{u\in M(t)}$ of φ.

Claim 19. *For any system $\varphi = \{X_u\}_{u\in t} \in \Phi$ there is a **saturated** system $\{X'_u\}_{u\in t}$ in Φ (with the same domain t) which weakly extends φ.*

Proof. If $u \in M(t)$ then simply let X'_u be the set of all points $x \in X_u$ such that $x = x_u$ for some sequence $\{x_u\}_{u\in M(t)}$ as in (iv). □ (claim)

Claim 20. *For any saturated system $\varphi = \{X_u\}_{u\in t} \in \Phi$, if $u \in M(t)$ then there are sets $X_{u^\wedge 0}, X_{u^\wedge 1}$ such that the system φ extended by those sets still belongs to Φ and properly extends φ.*

Proof. As $X_u \in \mathbf{P}$ and $X_u \subseteq X^*$, the condition $X_u \times X_u$ does **not** $(\mathbf{P} \times_{\mathsf{E}_{X_u}} \mathbf{P})$-force that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable. By Corollary 17, pick a stronger condition $U \times V \subseteq X_u \times X_u$ in $\mathbf{P} \times_{\mathsf{E}_{X_u}} \mathbf{P}$ which $(\mathbf{P} \times_{\mathsf{E}_{X_u}} \mathbf{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-incomparable. By Lemmas 13 and 8 we may assume that $U \times V$ belongs to $\mathbf{P}^* \times_{\mathsf{E}_{X_u}} \mathbf{P}^*$ and is E_{X_u}-saturated, so that $[U]_{\mathsf{E}_{X_u}} = [V]_{\mathsf{E}_{X_u}}$. We assert that the sets $X_{u^\wedge 0} = U$ and $X_{u^\wedge 1} = V$ prove the claim. It's enough to check (iv) for the extended system.

Fix any $x \in X_{u^\wedge 0} = U$. Then $x \in X_u$, hence, as the given system is saturated, there is a sequence $\{x_v\}_{v\in M(t)}$ of points $x_v \in X_v$ as in (iv), such that $x_u = x$. On the other hand, as $[U]_{\mathsf{E}_{X_u}} = [V]_{\mathsf{E}_{X_u}}$, there is a point $y \in V = X_{u^\wedge 1}$ such that $x \mathrel{\mathsf{E}_{X_u}} y$. Put $x_{u^\wedge 0} = x$ and $x_{u^\wedge 1} = y$. □ (claim)

If E is an OD equivalence relation and $X \times Y \in \mathbf{P}^* \times_{\mathsf{E}} \mathbf{P}^*$ then the set $\mathscr{D}(\mathsf{E}, X, Y)$ of all sets, *open dense in* $\mathbf{P} \times_{\mathsf{E}} \mathbf{P}$ *below* $X \times Y$ [4], is countable by Lemma 13; fix an enumeration $\mathscr{D}(\mathsf{E}, X, Y) = \{D_n(\mathsf{E}, X, Y) : n \in \omega\}$ such that $D_n(\mathsf{E}, X, Y) \subseteq D_m(\mathsf{E}, X, Y)$ whenever $m < n$.

Claim 21. *Let $n \in \omega$ and $\varphi = \{X_u\}_{u\in 2^{\le n}} \in \Phi$. Then there is a system $\varphi' = \{X'_u\}_{u\in 2^{\le n+1}} \in \Phi$ which weakly extends φ and satisfies the following additional genericity requirement*:

[4] That is, open dense subsets of the restricted forcing $(\mathbf{P} \times_{\mathsf{E}} \mathbf{P})_{\subseteq X\times Y} = \{X' \times Y' \in \mathbf{P} \times_{\mathsf{E}} \mathbf{P} : X' \subseteq X \wedge Y' \subseteq Y\}$.

($*$) *if strings* $u \neq v$ *belong to* 2^{n+1} *and* $w = u \wedge v$ (defined as in (iv)) *then condition* $X_u \times X_v$ *belongs to* $D_n(\mathsf{E}_{X_w}, X_{w^\wedge 0}, X_{w^\wedge 1})$.

Proof. We first extend φ by one layer of sets $X'_{u^\wedge i}$, $u \in 2^n$ and $i = 0, 1$, obtained by consecutive 2^n splitting operations as in Claim 20, followed by the saturating reduction as in Claim 19. This way we get a saturated system $\eta = \{Y_u\}_{u \in 2^{\le n+1}} \in \Phi$ which weakly extends φ.

To fulfill ($*$), let us shrink the sets in the top layer $\{Y_u\}_{u \in 2^{n+1}}$ of η.

Consider any pair of strings $u \neq v$ in 2^{n+1}. Let $w = u \wedge u$, so that $k = \mathtt{dom}\, w < n$, $w \subset u$, $w \subset v$, and $u(k) \neq v(k)$; let, say, $u(k) = 0$, $v(k) = 1$. Condition $Y_{w^\wedge 0} \times Y_{w^\wedge 1}$ belongs to $\mathbf{P}^* \times_{\mathsf{E}_{Y_w}} \mathbf{P}^*$ by (iii) while $Y_u \times Y_v$ belongs to $\mathbf{P} \times_{\mathsf{E}_{Y_w}} \mathbf{P}$ by (iv) and satisfies $Y_u \subseteq Y_{w^\wedge 0}$ and $Y_v \subseteq Y_{w^\wedge 1}$ by (ii). By the density, there is a subcondition $Z_u \times Z_v \subseteq Y_u \times Y_v$ in $D_n(\mathsf{E}_{Y_w}, Y_{w^\wedge 0}, Y_{w^\wedge 1})$; in particular, $Z_u \times Z_v$ still belongs to $\mathbf{P} \times_{\mathsf{E}_{Y_w}} \mathbf{P}$. In addition to Z_u and Z_v, we let $Z_s = Y_s$ for any $s \in 2^{n+1} \smallsetminus \{u, v\}$. Then $\psi = \{Z_s\}_{s \in 2^{\le n+1}}$ is still a system in Φ. By Claim 19, there is a saturated system $\psi' = \{Z'_s\}_{s \in 2^{\le n+1}} \in \Phi$ such that $Z'_s = Z_s = Y_s$ for all $u \in 2^{\le n}$, and $Z'_s \subseteq Z_s$ for all $s \in 2^{n+1}$. Then $Z'_u \subseteq Z_u$ and $Z'_v \subseteq Z_v$ — so that $Z'_u \times Z'_v \in D_n(\mathsf{E}_{Y_w}, Y_{w^\wedge 0}, Y_{w^\wedge 1})$.

Iterating this shrinking construction $2^n(2^n - 1)$ times (the number of pairs $s \neq t$ in 2^n), we get a required system φ'. □ (claim)

Claim 21 allows to define, by induction, sets $X_u \subseteq X'_u \subseteq X^*$ in $\mathbf{P}^*$ ($u \in 2^{<\omega}$) and systems $\varphi_n = \{X_u\}_{u \in 2^{<n}} \cup \{X'_u\}_{u \in 2^n}$, such that, for any n:

(1) φ_n is a saturated system in Φ, weakly extended by φ_{n+1}, and

(2) condition ($*$) of Claim 21 holds.

Show that this leads to a required perfect set.

Suppose that $a \neq b$ are reals in 2^ω, and $w = a \wedge b$, so that $w \subset a$, $w \subset b$, and $a(k) \neq b(k)$, where $k = \mathtt{dom}\, w$; let, say, $a(k) = 0$, $b(k) = 1$. Then the sequence of sets $X_{a \restriction m} \times X_{b \restriction m}$, $m > k$, is ($\mathbf{P} \times_{\mathsf{E}_{X_w}} \mathbf{P}$)-generic by (2), so that the intersection $\bigcap_{m > k} (X_{a \restriction m} \times X_{b \restriction m})$ consists of a single pair of reals $\langle x_a, x_b \rangle$ by Proposition 9. Moreover, x_a, x_b are $\preccurlyeq$-incomparable by (iii). Finally it easily follows from (2) thet the diameters of sets X_n uniformly tend to 0 with $n \to \infty$, and hence the map $a \longmapsto x_a$ is continuous. Thus $P = \{x_a : a \in 2^\omega\}$ is a perfect $\preccurlyeq$-antichain in X^*. □

11 Compression lemma

Let $\Theta = \Omega^+$; the cardinal successor of Ω in both $\mathbf{L}$, the ground model, and its $\mathbf{Coll}(\omega, {<}\Omega)$-generic extension postulated by Ω-SM to be the set universe; in the latter, $\Omega = \omega_1$ and $\Theta = \omega_2$.

Lemma 22 (compression lemma)**.** *Assume that* $\Omega \le \vartheta \le \Theta$ *and* $X \subseteq 2^{\Theta}$ *is the image of* ω^{ω} *via an OD map. Then there is an OD antichain* $A(X) \subseteq 2^{<\Omega}$ *and an OD isomorphism* $f : \langle X\,; \leqslant_{\texttt{lex}} \rangle \xrightarrow{\text{onto}} \langle A(X)\,; \leqslant_{\texttt{lex}} \rangle$ *.*

Note that any antichain $A \subseteq 2^{<\Omega}$ is linearly ordered by $\leqslant_{\texttt{lex}}$!

Proof. If $\vartheta = \Theta$ then, as $\texttt{card}\, X \le \texttt{card}\, \omega^{\omega} = \Omega$, there is an ordinal $\vartheta < \Theta$ such that $x \restriction \vartheta \ne y \restriction \vartheta$ whenever $x \ne y$ belong to X — this reduces the case $\vartheta = \Theta$ to the case $\Omega \le \vartheta < \Theta$. We prove the latter by induction on ϑ.

The nontrivial step is the step $\texttt{cof}\, \lambda = \Omega$, so that let $\vartheta = \bigcup_{\alpha<\Omega} \vartheta_{\alpha}$, for an increasing OD sequence of ordinals ϑ_{α}. Let $I_{\alpha} = [\vartheta_{\alpha}, \vartheta_{\alpha+1})$. Then, by the induction hypothesis, for any $\alpha < \Omega$ the set $X_{\alpha} = \{S \restriction I_{\alpha} : S \in X\} \subseteq 2^{I_{\alpha}}$ is $<_{\text{lex}}$-order-isomorphic to an antichain $A_{\alpha} \subseteq 2^{<\Omega}$ via an OD isomorphism i_{α}, and the map, which sends α to A_{α} and i_{α}, is OD. It follows that the map, which sends each $S \in X$ to the concatenation of all sequences $i_{\alpha}(x \restriction I_{\alpha})$, is an OD $<_{\text{lex}}$-order-isomorphism X onto an antichain in 2^{Ω}. Therefore it suffices to prove the lemma in the case $\vartheta = \Omega$. Thus let $X \subseteq 2^{\Omega}$.

First of all, note that each sequence $S \in X$ is ROD. Lemma 7 in [8] shows that, in this case, we have $S \in \mathbf{L}[S \restriction \eta]$ for an ordinal $\eta < \Omega$. Let $\eta(S)$ be the least such an ordinal, and $h(S) = S \restriction \eta(S)$, so that $h(S)$ is a countable initial segment of S and $S \in \mathbf{L}[h(S)]$. Note that h is still OD.

Consider the set $U = \texttt{ran}\, h = \{h(S) : S \in X\} \subseteq 2^{<\Omega}$. We can assume that every sequence $u \in U$ has a limit length. Then $U = \bigcup_{\gamma<\Omega} U_{\gamma}$, where $U_{\gamma} = U \cap 2^{\omega\gamma}$ ($\omega\gamma$ is the the γ-th limit ordinal). For $u \in U_{\gamma}$, let $\gamma_u = \gamma$.

If $u \in U$ then by construction the set $X_u = \{S \in X : h(S) = u\}$ is OD(u) and satisfies $X_u \subseteq \mathbf{L}[u]$. Therefore, it follows from the known properties of the Solovay model that X_u belongs to $\mathbf{L}[u]$ and is of cardinality $\le \Omega$ in $\mathbf{L}[u]$. Fix an enumeration $X_u = \{S_u(\alpha) : \gamma_u \le \alpha < \Omega\}$ for all $u \in U$. We can assume that the map $\alpha, u \longmapsto S_u(\alpha)$ is OD.

If $u \in U$ and $\gamma_u \le \alpha < \Omega$, then we define a shorter sequence, $s_u(\alpha) \in 3^{\omega\alpha+1}$, as follows.

(i) $s_u(\alpha)(\xi+1) = S_u(\alpha)(\xi)$ for any $\xi < \omega\alpha$.

(ii) $s_u(\alpha)(\omega\alpha) = 1$.

(iii) Let $\delta < \alpha$. If $S_u(\alpha) \restriction \omega\delta = S_v(\delta) \restriction \omega\delta$ for some $v \in U$ (equal to or different from u) then $s_u(\alpha)(\omega\delta) = 0$ whenever $S_u(\alpha) <_{\texttt{lex}} S_v(\delta)$, and $s_u(\alpha)(\omega\delta) = 2$ whenever $S_v(\delta) <_{\text{lex}} S_u(\alpha)$.

(iv) Otherwise (*i.e.*, if there is no such v), $s_u(\alpha)(\omega\delta) = 1$.

To demonstrate that (iii) is consistent, we show that $S_{u'}(\delta) \restriction \omega\delta = S_{u''}(\delta) \restriction \omega\delta$ implies $u' = u''$. Indeed, as by definition $u' \subset S_{u'}(\delta)$ and $u'' \subset S_{u''}(\delta)$,

u' and u'' must be $\subseteq$-compatible: let, say, $u' \subseteq u''$. Now, by definition, $S_{u''}(\delta) \in \mathbf{L}[u'']$, therefore $\in \mathbf{L}[S_{u'}(\delta)]$ because $u'' \subseteq S_{u''}(\delta) \restriction \omega\delta = S_{u'}(\delta) \restriction \omega\delta$, finally $\in \mathbf{L}[u']$, which shows that $u' = u''$ as $S_{u''}(\delta) \in X_{u''}$.

We are going to prove that the map $S_u(\alpha) \longmapsto s_u(\alpha)$ is a $<_{\texttt{lex}}$-order isomorphism, so that $S_v(\beta) <_{\texttt{lex}} S_u(\alpha)$ implies $s_v(\beta) <_{\texttt{lex}} s_u(\alpha)$.

We first observe that $s_v(\beta)$ and $s_u(\alpha)$ are $\subseteq$-incomparable. Indeed assume that $\beta < \alpha$. If $S_u(\alpha) \restriction \omega\beta \neq S_v(\beta) \restriction \omega\beta$ then clearly $s_v(\beta) \not\subseteq s_u(\alpha)$ by (i). If $S_u(\alpha) \restriction \omega\beta = S_v(\beta) \restriction \omega\beta$ then $s_u(\alpha)(\omega\beta) = 0$ or 2 by (iii) while $s_v(\beta)(\omega\beta) = 1$ by (ii). Thus all $s_u(\alpha)$ are mutually $\subseteq$-incomparable, so that it suffices to show that conversely $s_v(\beta) <_{\text{lex}} s_u(\alpha)$ implies $S_v(\beta) <_{\text{lex}} S_u(\alpha)$. Let ζ be the least ordinal such that $s_v(\beta)(\zeta) < s_u(\alpha)(\zeta)$; then $s_u(\alpha) \restriction \zeta = s_v(\beta) \restriction \zeta$ and $\zeta \le \texttt{min}\{\omega\alpha, \omega\beta\}$.

The case when $\zeta = \xi + 1$ is clear: then by definition $S_u(\alpha) \restriction \xi = S_v(\beta) \restriction \xi$ while $S_v(\beta)(\xi) < S_u(\alpha)(\xi)$, so let us suppose that $\zeta = \omega\delta$, where $\delta \le \texttt{min}\{\alpha, \beta\}$. Then obviously $S_u(\alpha) \restriction \omega\delta = S_v(\beta) \restriction \omega\delta$. Assume that one of the ordinals α, β is equal to δ, say, $\beta = \delta$. Then $s_v(\beta)(\omega\delta) = 1$ while $s_u(\alpha)(\omega\delta)$ is computed by (iii). Now, as $s_v(\beta)(\omega\delta) < s_u(\alpha)(\omega\delta)$, we conclude that $s_u(\alpha)(\omega\delta) = 2$, hence $S_v(\beta) <_{\text{lex}} S_u(\alpha)$, as required. Assume now that $\delta < \texttt{min}\{\alpha, \beta\}$. Then easily α and β appear in one and the same class (iii) or (iv) with respect to the δ. However this cannot be (iv) because $s_v(\beta)(\omega\delta) \neq s_u(\alpha)(\omega\delta)$. Hence we are in (iii), so that, for some (unique) $w \in U$. $0 = S_v(\beta) <_{\text{lex}} S_w(\delta) <_{\texttt{lex}} S_u(\alpha) = 2$, as required.

This ends the proof of the lemma, except for the fact that the sequences $s_u(\alpha)$ belong to $3^{<\Omega}$, but improvement to $2^{<\Omega}$ is easy. □

12 Decomposing thin OD sets in the Solovay model

Here we prove Theorem 2. **We assume to the contrary** that the OD set U^* of all reals $x \in X^*$ such that there is no OD $\preccurlyeq$-chain C containing x, is non-empty. If $R \subseteq \omega^\omega$ is an OD set then let $\mathscr{F}_R$ consist of all OD maps $F : \langle \omega^\omega ; \preccurlyeq \rangle \to \langle A ; \leqslant_{\texttt{lex}} \rangle$, where $A \subseteq 2^{<\Omega}$ is an OD antichain, such that

(I) F is LR order preserving, *i.e.*, $x \preccurlyeq y \implies F(x) \leqslant_{\texttt{lex}} F(y)$ — in particular, $x \approx y \implies F(x) = F(y)$ — for all $x, y \in \omega^\omega$;

(II) if $x, y \in R$ are $\preccurlyeq$-incomparable then $F(x) = F(y)$, or equivalently provided (I) holds, $F(x) <_{\texttt{lex}} F(y) \implies x \prec y$ for all $x, y \in R$.

We let

$$x \mathsf{E}_R y \quad \text{iff} \quad \forall F \in \mathscr{F}_R \, (F(x) = F(y)) \, .$$

Note that a function $F \in \mathscr{F}_R$ has to be not just $\approx$-invariant by (I), but also invariant w.r.t. the common equivalence hull of the relation $\approx$ and the (non-equivalence) relation of being $\preccurlyeq$-incomparable, by (II).

Still any E_R is an OD equivalence relation.

If $R \subseteq R'$ then $\mathscr{F}'_R \subseteq \mathscr{F}_R$, and hence $x \mathrel{\mathsf{E}_R} y$ implies $x \mathrel{\mathsf{E}_{R'}} y$.

Lemma 23. *If $R \subseteq \omega^\omega$ is OD, $\mathsf{E}_R \subseteq H \subseteq \omega^\omega \times \omega^\omega$, and H is OD, then there is a function $F \in \mathscr{F}_R$ such that $\forall\, x, y\, (F(x) = F(y) \Longrightarrow H(x,y))$.*

Proof. Clearly $\mathtt{card}\,\mathscr{F}_R = \Theta$ and $\mathscr{F}_R$ admits an OD enumeration $\mathscr{F}_R = \{F_\xi : \xi < \Theta\}$. If $x \in \omega^\omega$ then let $f(x) = F_0(x)^\wedge F_1(x)^\wedge \dots {}^\wedge F_\xi(x)^\wedge \, \dots$ — the concatenation of all sequences $F_\xi(x)$. Then $f : \langle \omega^\omega ; \preccurlyeq \rangle \to \langle W ; \leqslant_{\mathtt{lex}} \rangle$ is a LR order preserving OD map, where $W = \mathtt{ran}\, f = \{f(r) : r \in \omega^\omega\} \subseteq 2^\Theta$, and $f(x) = f(y) \Longrightarrow H(x,y)$ by the construction. By Lemma 22 there is an OD isomorphism $g : \langle W ; \leqslant_{\mathtt{lex}} \rangle \xrightarrow{\text{onto}} \langle A ; \leqslant_{\mathtt{lex}} \rangle$ onto an antichain $A \subseteq 2^{<\Omega}$. The superposition $F(x) = g(f(x))$ proves the lemma. □

Lemma 24. *If $R \subseteq U^*$ is a non-empty OD set then the condition $R \times R$ $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}} \mathrel{\mathsf{E}_R} \dot{\boldsymbol{x}}_{\mathtt{ri}}$.* □

Proof. Otherwise, by Lemma 16, there is a function $F \in \mathscr{F}_R$ and a condition $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P}$ with $X \cup Y \subseteq R$, which $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces $F(\dot{\boldsymbol{x}}_{\mathtt{le}})(\xi) = 0 \neq 1 = F(\dot{\boldsymbol{x}}_{\mathtt{ri}})(\xi)$ for an ordinal $\xi < \Omega$. We may assume that $X \times Y$ is a saturated condition. Then $F(x)(\xi) = 0 \neq 1 = F(y)(\xi)$ for any pair $\langle x, y \rangle \in X \times Y$, so that we have $F(x) \neq F(y)$ and $\neg\, (x \mathrel{\mathsf{E}_R} y)$ whenever $\langle x, y \rangle \in X \times Y$, which contradicts the choice of $X \times Y$ in $\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P}$. □

Lemma 25. *Let $R \subseteq U^*$ be a non-empty OD set. Then $R \times R$ does **not** $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-force that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable.*

Proof. Suppose to the contrary that $R \times R$ forces the comparability. Then by Lemma 16 a subcondition $X \times Y$ either $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ or $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$; $X, Y \subseteq R$ are OD sets and $[X]_{\mathsf{E}_R} \cap [Y]_{\mathsf{E}_R} \neq \varnothing$.

Case A: condition $X \times Y$ $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$. We claim that the OD set $W = \{\langle x, x' \rangle \in X \times X : x \mathrel{\mathsf{E}_R} x' \wedge x' \not\approx x\}$ is empty. Indeed otherwise W is a condition in the forcing $\mathbf{P}^{(2)} = \{P \subseteq \omega^\omega \times \omega^\omega : \varnothing \neq P \in \mathrm{OD}\}$, which is just the 2-dimensional version of $\mathbf{P}$ with the same basic properties. Note that $\mathbf{P}^{(2)}$ adds pairs $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}} \rangle \in W$ of $\mathbf{P}$-generic (separately) reals $\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}' \in X$, and the condition W $\mathbf{P}^{(2)}$-forces that $\dot{\boldsymbol{x}}' \mathrel{\mathsf{E}_R} \dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{x}}' \not\approx \dot{\boldsymbol{x}}$.

If $P \in \mathbf{P}_{\subseteq W}$ then obviously $[\mathtt{dom}\, P]_{\mathsf{E}_R} = [\mathtt{ran}\, P]_{\mathsf{E}_R}$.

Consider a more complex forcing $\mathscr{P}$ of all pairs $P \times Y'$, where $P \in \mathbf{P}^{(2)}$, $P \subseteq W$, $Y' \in \mathbf{P}$, $Y' \subseteq Y$, and $[\mathtt{dom}\, P]_{\mathsf{E}_R} \cap [Y']_{\mathsf{E}_R} \neq \varnothing$. For instance,

$W \times Y \in \mathscr{P}$. Then $\mathscr{P}$ adds a pair $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}} \rangle \in W$ and a separate real $\dot{\boldsymbol{x}} \in Y$ such that the pairs $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}} \rangle$, $\langle \dot{\boldsymbol{x}}_{\mathtt{ri}}, \dot{\boldsymbol{x}} \rangle$ belong to $X \times Y$ and are $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-generic, hence $\dot{\boldsymbol{x}}_{\mathtt{le}} \approx \dot{\boldsymbol{x}} \approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ by the choice of $X \times Y$. On the other hand, $\dot{\boldsymbol{x}}_{\mathtt{le}} \not\approx \dot{\boldsymbol{x}}_{\mathtt{ri}}$ since the pair belongs to W, which is a contradiction.

Thus $W = \varnothing$. Then X is a $\preccurlyeq$-chain: indeed if $x, y \in X$ are $\preccurlyeq$-incomparable then by definition we have $x \mathrel{\mathsf{E}_R} y$, hence $x \approx y$, contradiction. Thus X is an OD $\preccurlyeq$-chain with $\varnothing \neq X \subseteq U^*$, contrary to the definition of U^*.

Case B: condition $X \times Y$ $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. We claim that the OD set $W' = \{\langle x, y \rangle \in X \times Y : x \mathrel{\mathsf{E}_R} y \wedge x \not\prec y\}$ is empty. Suppose towards the contrary that $W' \neq \varnothing$. Let $X' = \mathtt{dom}\, W'$. As $X' \subseteq R$, the condition $X' \times X'$ $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces that $\dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}}$ are $\preccurlyeq$-comparable. Therefore there is a condition $A \times B$ in $\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P}$, with $A \cup B \subseteq X'$, which $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$; for if it forces $\dot{\boldsymbol{x}}_{\mathtt{ri}} \prec \dot{\boldsymbol{x}}_{\mathtt{le}}$ then just consider $B \times A$ instead of $A \times B$, and it cannot force $\dot{\boldsymbol{x}}_{\mathtt{ri}} \approx \dot{\boldsymbol{x}}_{\mathtt{le}}$ by the result in Case A. Let $Z = \{\langle x, y \rangle \in W' : x \in A\}$.

Consider the forcing notion $\mathscr{P}$ of all non-empty OD sets of the form $P \times B'$, where $P \subseteq Z$, $B' \subseteq B$, and $[B']_{\mathsf{E}_R} \cap [\mathtt{dom}\, P]_{\mathsf{E}_R} \neq \varnothing$ (equivalently, $[B']_{\mathsf{E}_R} \cap [\mathtt{ran}\, P]_{\mathsf{E}_R} \neq \varnothing$). For instance, $Z \times B \in \mathscr{P}$. Note that $\mathscr{P}$ adds a pair $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}}_{\mathtt{ri}} \rangle \in Z$ and a separate real $\dot{\boldsymbol{x}} \in B$ such that both pairs $\langle \dot{\boldsymbol{x}}_{\mathtt{le}}, \dot{\boldsymbol{x}} \rangle$ and $\langle \dot{\boldsymbol{x}}_{\mathtt{ri}}, \dot{\boldsymbol{x}} \rangle$ are $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-generic. It follows that $\mathscr{P}$ forces both $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}$ (as this pair belongs to $A \times B$) and $\dot{\boldsymbol{x}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$ (it belongs to $X \times Y$), hence, forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$. On the other hand $\mathscr{P}$ forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \not\prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$ (as this pair belongs to $Z \subseteq W'$), a contradiction.

Thus $W' = \varnothing$; in other words, if $x \in X$, $y \in Y$, and $x \mathrel{\mathsf{E}_R} y$ then $x \prec y$ strictly. The OD set $C = \{x' : \exists x \in X\, (x \mathrel{\mathsf{E}_R} x' \wedge x' \preccurlyeq x)\}$ is downwards $\preccurlyeq$-closed in each E_R-class, $X \subseteq C$, and still $Y \cap C = \varnothing$.

Claim 26. *If $x \in C \cap R$, $y \in R \smallsetminus C$, and $y \mathrel{\mathsf{E}_R} x$, then $x \prec y$.*

Proof. Otherwise, the following OD set

$$H_0 = \{y \in R \smallsetminus C : \exists x \in C \cap R\, (x \mathrel{\mathsf{E}_R} y \wedge x \not\prec y)\} \subseteq R$$

is non-$\varnothing$. As above (Subcase B1), there is a condition $H \times H'$ in $\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P}$, with $H \cup H' \subseteq H_0$, which $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-forces $\dot{\boldsymbol{x}}_{\mathtt{le}} \prec \dot{\boldsymbol{x}}_{\mathtt{ri}}$, and then, by the result in Case B1, $y \prec y'$ holds whenever $\langle y, y' \rangle \in H \times H'$ and $y \mathrel{\mathsf{E}_R} y'$. By construction the OD set

$$C_1 = \{x \in C \cap R : \exists y' \in H'\, (x \mathrel{\mathsf{E}_R} y' \wedge x \not\prec y')\}$$

satisfies $[C_1]_{\mathsf{E}_R} = [H]_{\mathsf{E}_R} = [H']_{\mathsf{E}_R}$, hence $C_1 \times H$ is a condition in $\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P}$. Let $\langle x_1, y \rangle \in C_1 \times H$ be any $(\mathbf{P} \times_{\mathsf{E}_R} \mathbf{P})$-generic pair. Then $x_1 \mathrel{\mathsf{E}_R} y$ by

Lemma 24, and, by the choice of R and the result in Case A, we have $x_1 \prec y$ or $y \prec x_1$. Yet by construction $x_1 \in C$, $y \notin C$, and C is downwards closed in each E_R-class. Thus in fact $x_1 \prec y$. Therefore, for all $y' \in H'$, if $x_1 \mathrel{\mathsf{E}_R} y'$ then $x_1 \prec z \prec y'$, which contradicts to $x_1 \in C_1$. □ (Claim)

We conclude by Lemma 23 that *there is a single function* $F \in \mathscr{F}_R$ *such that if* $x \in C \cap R$, $y \in R \smallsetminus C$, *and* $F(x) = F(y)$, *then* $x \prec y$.

Prove that *the derived function*

$$G(x) = \begin{cases} F(x)^\wedge 0\,, & \text{whenewer} \quad x \in C \\ F(x)^\wedge 1\,, & \text{whenewer} \quad x \in \omega^\omega \smallsetminus C \end{cases}$$

belongs to $\mathscr{F}_R$. First of all, still $G \in \mathscr{F}$ since C is downwards $\preccurlyeq$-closed in each E_R-class. Now suppose that $x, y \in R$ and $G(x) <_{\mathtt{lex}} G(y)$. Then either $F(x) <_{\mathtt{lex}} F(y)$, or $F(x) = F(y)$ and $x \in C$ but $y \notin C$. In the "either" case immediately $x \prec y$ since $F \in \mathscr{F}_R$. In the "or" case we have $x \prec y$ by the choice of F and the definition of G. Thus $G \in \mathscr{F}_R$.

Now pick any pair of reals $x \in X$ and $y \in Y$ with $x \mathrel{\mathsf{E}_R} y$. Then $G(x) = G(y)$ since $G \in \mathscr{F}_R$. But we have $x \in C$ and $y \notin C$ since $X \subseteq C$ and $Y \cap C = \varnothing$ by construction, and in this case surely $G(y) \neq G(x)$ by the definition of G. This contradiction completes the proof of Lemma 25. □

Lemma 25 plus Lemma 18 imply Theorem 2.

□ (Theorem 2)

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