OD elements of countable OD sets in the Solovay model

Vladimir Kanovei*

March 17, 2016

Abstract

It is true in the Solovay model that every countable ordinal-definable set of sets of reals contains only ordinal-definable elements.

1 Introduction

It is known that the existence of a non-empty OD (ordinal-definable) set of reals X with no OD element is consistent with **ZFC**; the set of all non-constructible reals gives a transparent example in many generic models.

Can such a set X *be countable*?

This question was initiated and discussed at the *Mathoverflow* website [1] and at FOM [2] . In particular Ali Enayat (Footnote 2) conjectured that the problem can be solved by the finite-support countable product $\mathbb{P}^{<\omega}$ (see [2]) of the Jensen "minimal Π^1_2 real singleton forcing" $\mathbb{P}$ defined in [4] (see also Section 28A of [3]). We proved in [5] that indeed, in a $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$, the set of all reals $\mathbb{P}$-generic over $\mathbf{L}$ is a countable Π^1_2 set with no OD elements. Moreover there is a modification $\mathbb{P}'$ of $\mathbb{P}$ such that it is true in a $\mathbb{P}'$-generic extension of $\mathbf{L}$ that there is a Π^1_2 E_0-equivalence class containing no OD reals, [7].

On the other hand, one may ask do countable non-empty OD sets without OD elements exist in such a more typical generic extension as the Solovay model? We partially answer this question in the negative.

*IITP RAS and MIIT, Moscow, Russia, kanovei@googlemail.com — contact author.

[1] *Mathoverflow*, March 09, 2010. http://mathoverflow.net/questions/17608.

[2] FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html

Theorem 1.1. *It is true in the Solovay model that every non-empty OD countable or finite set* $\mathscr{X}$ **of sets of reals** *necessarily contains an OD element, and hence, in fact, consists of OD elements.*

The Solovay model here is a model of **ZFC** defined in [8] in which all projective (and generally all ROD, real-ordinal definable) sets of reals are Lebesgue measurable. The case, when $\mathscr{X}$ is a (non-empty OD countable) **set of reals** in this theorem, is well known and is implicitly contained in the proof of the perfect set property by Solovay [8]. Hovever the proofs known for this particular case of sets of reals (as, *e.g.*, in [9] or [6]) do not work even for sets $\mathscr{X} \subseteq \mathscr{P}(\omega^\omega)$ (as in the theorem). In this paper, we present the proof of Theorem 1.1.

2 Notation

We consider the constructible universe **L** as the ground model by default. Suppose that Ω is an inaccessible cardinal.

Blanket assumption 2.1. By a *generic set* we'll always mean a *filter*, that is, both pairwise compatible in itself and containing all weaker conditions.

Definition 2.2. We represent the *Levy – Solovay forcing* associated with Ω is the set **LS** of all partial maps $p : \mathtt{dom}\,p \to \Omega$ such that $\mathtt{dom}\,p \subseteq \Omega\times\omega$ is a finite set and $p(\alpha, n) < \alpha$ whenever $\langle\alpha, n\rangle \in \mathtt{dom}\,p$. Let $|p| = \{\alpha : \exists\, n\, (\langle\alpha, n\rangle \in \mathtt{dom}\,p)\}$.

If $\gamma \le \Omega$ then $\mathbf{LS}_\gamma = \{p \in \mathbf{LS} : |p| \subseteq \gamma\}$; in particular $\mathbf{LS}_\Omega = \mathbf{LS}$.

If $p \in \mathbf{LS}$ and $\alpha < \Omega$ then the *α-component* p_α of p is a map defined on the set $\mathtt{dom}\,p_\alpha = \{n : \langle\alpha, n\rangle \in \mathtt{dom}\,p\} \subseteq \omega$ by $p_\alpha(n) = p(\alpha, n)$. □

If $G \subseteq \mathbf{LS}$ is an **LS**-generic set over **L** then $\mathbf{L}[G]$ is **the Solovay model**, to which Theorem 1.1 refers. The next lemma will be important below.

Lemma 2.3 (reduction to ROD)**.** *It is true in the Solovay model that if* $\mathscr{X}$ *is a non-empty OD countable set and* $X \in \mathscr{X}$ *is ROD then* X *is OD.*

Thus somewhat surprisingly, it turns out that it suffices to prove the existence of a ROD (real-ordinal definable) element $X \in \mathscr{X}$ in Theorem 1.1.

Proof. Arguing in the Solovay model, assume that

$$X = X_{p_0} = \{x : \varphi(x, p_0)\}\,,$$

where φ is a formula with a real parameter $p_0 \in \omega^\omega$ and hidden ordinal parameters. The set $P = \{p \in \omega^\omega : X_p \in \mathscr{X}\}$ is OD and contains p_0, and the equivalence relation, $p \mathrel{\mathsf{E}} q$ iff $X_p = X_q$ on P, is OD as well, and E has at most countably many equivalence classes in P. However it is known that, in the Solovay model, if an OD equivalence relation on ω^ω has at most countably many equivalence classes then all its equivalence classes are OD, [6, 9]. In particular $[p_0]_{\mathsf{E}}$ is OD, and hence the set $X = X_{p_0} = \{x : \exists\, p \in [p_0]_{\mathsf{E}}\, \varphi(x, p_0)\}$ is OD. $\square$

Definition 2.4 (ramified names)**.** We'll use the ordinary ramified system of $\mathbf{LS}$-*names* for differens sets in $\mathbf{L}[G]$, so that $U[\![G]\!]$ will be the G-*interpretation* of a name U (basically, any set) defined by $\in$-rank induction by

$$U[\![G]\!] = \{u[\![G]\!] : \exists\, p \in G\, (\langle p, u\rangle \in U)\}\,.$$

Then, if $G \subseteq \mathbf{LS}$ is generic over $\mathbf{L}$ then $\mathbf{L}[G] = \{U[\![G]\!] : U \in \mathbf{L}\}$. $\square$

Each set $x \in \mathbf{L}$ has a canonical $\mathbf{LS}$-name $\check{x} \in \mathbf{L}$, such that $\check{x}[G] = x$ for any generic set $G \subseteq \mathbf{LS}$. Yet following common practice we shall identify $\check{x}$ with x itself whenever possible.

Definition 2.5 (simple names)**.** To somewhat simplify notation, we'll make use of a simpler system of names particularly for subsets of $\mathbf{LS}$. Let $\mathbf{N} = \mathscr{P}(\mathbf{LS} \times \mathbf{LS})$, and if $t \in \mathbf{N}$ and $G \subseteq \mathbf{LS}$ then $t[G] = \{q : \exists\, p \in G\, (\langle p, q\rangle \in t)\} \subseteq \mathbf{LS}$.

Thus $\mathbf{N}$ consists of all $\mathbf{LS}$-names for subsets of $\mathbf{LS}$.

If $\gamma < \Omega$ then let $\mathbf{N}_\gamma = \mathscr{P}((\mathbf{LS}_\gamma) \times (\mathbf{LS}_\gamma))$, so that any $t \in \mathbf{N}_\gamma$ is a $\mathbf{LS}_\gamma$-name for a subset of $\mathbf{LS}_\gamma$. $\square$

The name $\underline{G} = \{\langle p, p\rangle : p \in \mathbf{LS}\}$ belongs to $\mathbf{N}$, and $\underline{G}[G] = G$.

3 Double names

In many cases below, we'll consider pairs of $\mathbf{LS}$-generic sets $G, G' \subseteq \mathbf{LS}$ over $\mathbf{L}$, such that $\mathbf{L}[G] = \mathbf{L}[G']$; note that this is not a $(\mathbf{LS} \times \mathbf{LS})$-generic pair! Similar pairs will be considered for the forcing notions $\mathbf{LS}_\gamma$ $(\gamma < \Omega)$ instead of $\mathbf{LS}$. The next definition introduces a useful tool related to such pairs.

Definition 3.1. In $\mathbf{L}$, if $\gamma \le \Omega$ then any pair $a = \langle t^a_{\mathtt{lef}}, t^a_{\mathtt{rig}}\rangle$ of names $t^a_{\mathtt{lef}}, t^a_{\mathtt{rig}} \in \mathbf{N}_\gamma$ will be called a *double-name*. Let $\mathbf{DN}_\gamma$ consist of all double-names $a = \langle t^a_{\mathtt{lef}}, t^a_{\mathtt{rig}}\rangle$ such that $t^a_{\mathtt{lef}} \ne \varnothing$, $t^a_{\mathtt{rig}} \ne \varnothing$, and

(1) if $p \in \mathtt{dom}\, t^a_{\mathtt{lef}}$ then p $\mathbf{LS}_\gamma$-forces: (a) $t^a_{\mathtt{lef}}[\underline{G}]$ is $\mathbf{LS}_\gamma$-generic, and
(b) $\underline{G} = t^a_{\mathtt{rig}}[t^a_{\mathtt{lef}}[\underline{G}]]$;

(2) if $p \in \mathtt{dom}\, t^a_{\mathtt{rig}}$ then p $\mathbf{LS}_\gamma$-forces: (a) $t^a_{\mathtt{rig}}[\underline{G}]$ is $\mathbf{LS}_\gamma$-generic, and
(b) $\underline{G} = t^a_{\mathtt{lef}}[t^a_{\mathtt{rig}}[\underline{G}]]$.

Define $\mathbf{DN} = \bigcup_{\gamma<\Omega} \mathbf{DN}_\gamma$; this is different from $\mathbf{DN}_\Omega$. It follows from (1) or (2) that for any $a \in \mathbf{DN}$ there is a *unique* $\gamma = |a| < \Omega$ such that $a \in \mathbf{DN}_\gamma$. □

Note that all sets $\mathbf{N}_\gamma$ and $\mathbf{DN}_\gamma$ belong to $\mathbf{L}$.

Lemma 3.2. *Assume that* $\gamma \le \Omega$ *and* $a \in \mathbf{DN}_\gamma$. *Then*:

(i) *if* $G_{\mathtt{lef}} \subseteq \mathbf{LS}_\gamma$ *is an* $\mathbf{LS}_\gamma$*-generic set and* $G_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \ne \varnothing$ *then* $G_{\mathtt{rig}} = t^a_{\mathtt{lef}}[G_{\mathtt{lef}}]$ *is* $\mathbf{LS}_\gamma$*-generic,* $G_{\mathtt{rig}} \cap \mathtt{dom}\, t^a_{\mathtt{rig}} \ne \varnothing$, *and* $G_{\mathtt{lef}} = t^a_{\mathtt{rig}}[G_{\mathtt{rig}}]$;

(ii) *if* $G_{\mathtt{rig}} \subseteq \mathbf{LS}_\gamma$ *is* $\mathbf{LS}_\gamma$*-generic and* $G_{\mathtt{rig}} \cap \mathtt{dom}\, t^a_{\mathtt{rig}} \ne \varnothing$ *then* $G_{\mathtt{lef}} = t^a_{\mathtt{rig}}[G_{\mathtt{rig}}]$ *is* $\mathbf{LS}_\gamma$*-generic,* $G_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \ne \varnothing$, $G_{\mathtt{rig}} = t^a_{\mathtt{lef}}[G_{\mathtt{lef}}]$. □

Thus each $a \in \mathbf{DN}_\gamma$ induces a bijection between all $\mathbf{LS}_\gamma$-generic sets $G \subseteq \mathbf{LS}_\gamma$ satisfying $G \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \ne \varnothing$ and those satisfying $G \cap \mathtt{dom}\, t^a_{\mathtt{rig}} \ne \varnothing$.

Corollary 3.3. *If* $\gamma \le \Omega$, $a \in \mathbf{DN}_\gamma$, $\langle q, p\rangle \in t^a_{\mathtt{rig}}$, *and* $q \subseteq q' \in \mathbf{LS}_\gamma$ *then there is a condition* $p' \in \mathbf{LS}_\gamma$ *compatible with* p *and such that* $\langle p', q'\rangle \in t^a_{\mathtt{lef}}$.

Proof. Let $G_{\mathtt{rig}} \subseteq \mathbf{LS}_\gamma$ be a generic set containing q', hence containing q as well. Then $G_{\mathtt{lef}} = t^a_{\mathtt{rig}}[G_{\mathtt{rig}}]$ is a $\mathbf{LS}_\gamma$-generic set containing p, and $G_{\mathtt{rig}} = t^a_{\mathtt{lef}}[G_{\mathtt{lef}}]$ by Lemmaio. As $q' \in G_{\mathtt{rig}}$, there is a condition $p' \in G_{\mathtt{lef}}$ such that $\langle p', q'\rangle \in t^a_{\mathtt{lef}}$. As p also belongs to $G_{\mathtt{lef}}$, p, p' are compatible. □

4 Full, regular, equivalent names

Recall that a set $D \subseteq \mathbf{LS}_\gamma$ is *dense* if for any $p \in \mathbf{LS}_\gamma$ there is $q \in D$ with $p \subseteq q$, and is *open* if $(p \in D \wedge p \subseteq q \in \mathbf{LS}_\gamma) \implies q \in D$.

Definition 4.1. Let $\gamma \le \Omega$. A name $t \in \mathbf{N}_\gamma$ is *full* if the set $\mathtt{dom}\, t$ is dense in $\mathbf{LS}_\gamma$. A double-name $a \in \mathbf{DN}_\gamma$ is *full* if such are the names $t^a_{\mathtt{lef}}$ and $t^a_{\mathtt{rig}}$.

A name $t \in \mathbf{N}_\gamma$ is *regular*, if the following holds: if $p, q \in \mathbf{LS}_\gamma$ and p $\mathbf{LS}_\gamma$-forces $q \in t[\underline{G}]$ then $\langle p, q\rangle \in t$. In particular, in this case, if $\langle p, q\rangle \in t$ and $p \subseteq p' \in \mathbf{LS}_\gamma$ then $\langle p', q\rangle \in t$, too. A double-name $a \in \mathbf{DN}_\gamma$ is *regular*, if so are both components $t^a_{\mathtt{lef}}$ and $t^a_{\mathtt{rig}}$. Define the *regular hull*

$$^{\mathsf{rh}}t = \{\langle p, q\rangle \in \mathbf{LS}_\gamma \times \mathbf{LS}_\gamma : p \ \ \mathbf{LS}_\gamma\text{-forces } q \in t[\underline{G}]\}.$$

of any $t \in \mathbf{N}_\gamma$. If $a \in \mathbf{DN}_\gamma$ then let $^{\mathsf{rh}}a = \langle {}^{\mathsf{rh}}t^a_{\mathtt{lef}}, {}^{\mathsf{rh}}t^a_{\mathtt{rig}}\rangle$. □

Lemma 4.2. *Assume that* $\gamma \le \Omega$ *and* $a \in \mathbf{DN}_\gamma$ *is full. Then* $\mathtt{ran}\, t^a_{\mathtt{lef}} = \mathtt{ran}\, t^a_{\mathtt{rig}} = \mathbf{LS}_\gamma$, *and if* $G \subseteq \mathbf{LS}_\gamma$ *is* $\mathbf{LS}_\gamma$*-generic then so are* $t^a_{\mathtt{lef}}[G]$ *and* $t^a_{\mathtt{rig}}[G]$.

Proof. To prove the genericity claim note that if say $\mathtt{dom}\, t^a_{\mathtt{lef}}$ is dense then any generic set $G \subseteq \mathbf{LS}_\gamma$ intersects $\mathtt{dom}\, t^a_{\mathtt{lef}}$, then use Lemma 3.2. To prove the first claim, let $q \in \mathbf{LS}_\gamma$. Consider a generic set $G_{\mathtt{rig}} \subseteq \mathbf{LS}_\gamma$ containing q. Then $G \cap \mathtt{dom}\, t^a_{\mathtt{rig}} \neq \varnothing$, see above. It follows that $G_{\mathtt{lef}} = t^a_{\mathtt{rig}}[G_{\mathtt{rig}}]$ is generic and $G_{\mathtt{rig}} = t^a_{\mathtt{lef}}[G_{\mathtt{lef}}]$ by Lemma 3.2. But $q \in G_{\mathtt{rig}}$, hence $q \in \mathtt{ran}\, t^c_{\mathtt{lef}}$. □

Definition 4.3. Names $s, t \in \mathbf{N}_\gamma$ are *equivalent* if $s[G] = t[G]$ for any generic set $G \subseteq \mathbf{LS}_\gamma$, or equivalently, if any $p \in \mathbf{LS}_\gamma$ $\mathbf{LS}_\gamma$-forces $s[\underline{G}] = t[\underline{G}]$. Double-names $a, b \in \mathbf{DN}_\gamma$ are *equivalent* if $t^b_{\mathtt{lef}}$, $t^b_{\mathtt{rig}}$ are equivalent to resp. $t^a_{\mathtt{lef}}$, $t^a_{\mathtt{rig}}$. □

Lemma 4.4. *Assume that* $\gamma \le \Omega$. *Then*:

(i) *if* $t \in \mathbf{N}_\gamma$ *then* ${}^{\mathsf{rh}}t$ *is regular and equivalent to* t;

(ii) *if* $a \in \mathbf{DN}_\gamma$ *then* ${}^{\mathsf{rh}}a \in \mathbf{DN}_\gamma$, $a \leqslant {}^{\mathsf{rh}}a$, *and* ${}^{\mathsf{rh}}a$ *is equivalent to* a — *therefore the set* $\mathbf{DN}^{\mathtt{reg}}_\gamma = \{b \in \mathbf{DN}_\gamma : b \text{ is regular}\}$ *is dense in* $\mathbf{DN}_\gamma$;

(iii) *if* $a, b \in \mathbf{DN}_\gamma$ *then* a *is equivalent to* b *iff* ${}^{\mathsf{rh}}a = {}^{\mathsf{rh}}b$.

Proof. (i) To establish the equivalence, assume that $G \subseteq \mathbf{LS}_\gamma$ is generic and $q \in {}^{\mathsf{rh}}t[G]$. Then there is $p \in G$ such that $\langle p, q\rangle \in {}^{\mathsf{rh}}t$. By definition p $\mathbf{LS}_\gamma$-forces $q \in t[\underline{G}]$. But then $q \in t[G]$, as required. To establish the regularity, assume that $p, q \in \mathbf{LS}_\gamma$, and p $\mathbf{LS}_\gamma$-forces $q \in {}^{\mathsf{rh}}t[\underline{G}]$ — therefore p $\mathbf{LS}_\gamma$-forces $q \in t[\underline{G}]$ by the equivalence already proved. Then by definition $\langle p, q\rangle \in {}^{\mathsf{rh}}t$.

(ii) follows from (i). The direction $\Longleftarrow$ in (iii) immediately follows from (ii). To prove the opposite direction, it suffices to show that if names $s, t \in \mathbf{N}_\gamma$ are equivalent then ${}^{\mathsf{rh}}s = {}^{\mathsf{rh}}t$. Assume that $\langle p, q\rangle \in {}^{\mathsf{rh}}s$. By definition p $\mathbf{LS}_\gamma$-forces $q \in s[\underline{G}]$. Then, as s, t are equivalent, p also forces $q \in t[\underline{G}]$. It follows that $\langle p, q\rangle \in {}^{\mathsf{rh}}t$, as required. □

Example 4.5. If $\gamma < \Omega$ then let $t_\gamma = \{\langle p, q\rangle : p, q \in \mathbf{LS}_\gamma \wedge q \subseteq p\}$ and $\mathbf{id}[\gamma] = \langle t_\gamma, t_\gamma\rangle$. Then $\mathbf{id}[\gamma] \in \mathbf{DN}_\gamma$ is a full regular double-name and $t^{\mathbf{id}[\gamma]}_{\mathtt{lef}}[G] = t^{\mathbf{id}[\gamma]}_{\mathtt{lef}}[G] = G$ for any $\mathbf{LS}_\gamma$-generic set $G \subseteq \mathbf{LS}_\gamma$: the *identity* name. □

5 Double-name representation theorem

The next theorem shows that the double-name tool adequately represents the case of a pair of $\mathbf{LS}$-generic sets $G, G' \subseteq \mathbf{LS}$ such that $\mathbf{L}[G] = \mathbf{L}[G']$.

Theorem 5.1. *Assume that* $\gamma \le \Omega$, $G_{\mathtt{lef}}, G_{\mathtt{rig}} \subseteq \mathbf{LS}_\gamma$ *are* $\mathbf{LS}_\gamma$*-generic sets over* $\mathbf{L}$, *and* $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G_{\mathtt{rig}}]$. *Then there is a full regular double-name* $c \in \mathbf{DN}_\gamma$ *such that* $G_{\mathtt{rig}} = t^c_{\mathtt{lef}}[G_{\mathtt{lef}}]$, $G_{\mathtt{lef}} = t^c_{\mathtt{rig}}[G_{\mathtt{rig}}]$, *and* $t^c_{\mathtt{lef}} = t^c_{\mathtt{rig}}$.

Proof. If $G_{\mathtt{lef}} = G_{\mathtt{rig}}$ then it suffices to define c by $t^c_{\mathtt{lef}} = t^c_{\mathtt{rig}} = \mathbf{id}[\gamma]$. Therefore assume that $G_{\mathtt{lef}} \neq G_{\mathtt{rig}}$. Then there exist conditions $p_{\mathtt{lef}} \in G_{\mathtt{lef}}$ and $p_{\mathtt{rig}} \in G_{\mathtt{rig}}$ incompatible in $\mathbf{LS}_\gamma$. By a basic forcing theorem, there exist names $s_{\mathtt{lef}}, s_{\mathtt{rig}} \in \mathbf{N}_\gamma$ such that $G_{\mathtt{rig}} = s_{\mathtt{lef}}[G_{\mathtt{lef}}]$, $G_{\mathtt{lef}} = s_{\mathtt{rig}}[G_{\mathtt{rig}}]$, and every condition $p \in \mathtt{dom}\, s_{\mathtt{lef}}$ satisfies $p_{\mathtt{lef}} \subseteq p$ while every condition $q \in \mathtt{dom}\, s_{\mathtt{rig}}$ satisfies $p_{\mathtt{rig}} \subseteq q$. It is not true immediately that $\langle s_{\mathtt{lef}}, s_{\mathtt{rig}} \rangle \in \mathbf{DN}_\gamma$; we need to somewhat modify the names by shrinking.

We can wlog assume that $s_{\mathtt{lef}}$ and $s_{\mathtt{rig}}$ are regular; as otherwise we can replace them by resp. ${}^{\mathsf{rh}}s_{\mathtt{lef}}$ and ${}^{\mathsf{rh}}s_{\mathtt{rig}}$ and use Lemma 4.4(i).

Define $a = \langle t^a_{\mathtt{lef}}, t^a_{\mathtt{rig}} \rangle$, where $t^a_{\mathtt{lef}}$ consists of all pairs $\langle p, q \rangle \in s_{\mathtt{lef}}$ such that

$$p \ \mathbf{LS}_\gamma\text{-forces that } s_{\mathtt{lef}}[\underline{G}] \text{ is } \mathbf{LS}_\gamma\text{-generic and } \underline{G} = s_{\mathtt{rig}}[s_{\mathtt{lef}}[\underline{G}]],$$

and $t^a_{\mathtt{rig}}$ consists of all pairs $\langle q, p \rangle \in s_{\mathtt{rig}}$ such that

$$q \ \mathbf{LS}_\gamma\text{-forces that } s_{\mathtt{rig}}[\underline{G}] \text{ is } \mathbf{LS}_\gamma\text{-generic and } \underline{G} = s_{\mathtt{lef}}[s_{\mathtt{rig}}[\underline{G}]];$$

then $\varnothing \neq t^a_{\mathtt{lef}} \subseteq s_{\mathtt{lef}}$ and $\varnothing \neq t^a_{\mathtt{rig}} \subseteq s_{\mathtt{rig}}$.

We claim that $a \in \mathbf{DN}_\gamma$, and still $G_{\mathtt{rig}} = t^a_{\mathtt{lef}}[G_{\mathtt{lef}}]$ and $G_{\mathtt{lef}} = t^a_{\mathtt{rig}}[G_{\mathtt{rig}}]$.

Lemma 5.2. *If $H_{\mathtt{lef}}$ is an $\mathbf{LS}_\gamma$-generic set and $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \neq \varnothing$ then $t^a_{\mathtt{lef}}[H_{\mathtt{lef}}] = s_{\mathtt{lef}}[H_{\mathtt{lef}}]$. Similarly if $H_{\mathtt{rig}}$ is an $\mathbf{LS}_\gamma$-generic set and $H_{\mathtt{rig}} \cap \mathtt{dom}\, t^a_{\mathtt{rig}} \neq \varnothing$ then $t^a_{\mathtt{rig}}[H_{\mathtt{rig}}] = s_{\mathtt{lef}}[H_{\mathtt{rig}}]$.*

Proof (lemma)**.** By construction $t^a_{\mathtt{lef}}[H_{\mathtt{lef}}] \subseteq= s_{\mathtt{lef}}[H_{\mathtt{lef}}]$. Consider any $q \in s_{\mathtt{lef}}[H_{\mathtt{lef}}]$, so that there is $p \in H_{\mathtt{lef}}$ with $\langle p, q \rangle \in s_{\mathtt{lef}}$. On the other hand, as $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \neq \varnothing$, there is a condition $p' \in H_{\mathtt{lef}}$ with $p \subseteq p'$ which $\mathbf{LS}_\gamma$-forces that $s_{\mathtt{lef}}[\underline{G}]$ is $\mathbf{LS}_\gamma$-generic and $\underline{G} = s_{\mathtt{rig}}[s_{\mathtt{lef}}[\underline{G}]]$. Then $\langle p', q \rangle \in t^a_{\mathtt{lef}}$ by the regularity assumption, and we have $q \in t^a_{\mathtt{lef}}[H_{\mathtt{lef}}]$. □ (Lemma)

Now to check 3.1(1) for a let $H_{\mathtt{lef}}$ be an $\mathbf{LS}_\gamma$-generic set and $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \neq \varnothing$. Then $t^a_{\mathtt{lef}}[H_{\mathtt{lef}}] = s_{\mathtt{lef}}[H_{\mathtt{lef}}]$ by the lemma. Therefore $H_{\mathtt{rig}} = t^a_{\mathtt{lef}}[H_{\mathtt{lef}}]$ is $\mathbf{LS}_\gamma$-generic and $H_{\mathtt{lef}} = s_{\mathtt{rig}}[H_{\mathtt{rig}}]$ by the definition of $t^a_{\mathtt{lef}}$. Thus $s_{\mathtt{rig}}[H_{\mathtt{rig}}]$ is generic and $s_{\mathtt{lef}}[s_{\mathtt{rig}}[H_{\mathtt{rig}}]] = H_{\mathtt{rig}}$ by construction. This is forced by some $q \in H_{\mathtt{rig}}$. On the other hand, as $H_{\mathtt{lef}} = s_{\mathtt{rig}}[H_{\mathtt{rig}}] \neq \varnothing$, there exists some $q' \in H_{\mathtt{rig}} \cap \mathtt{dom}\, s_{\mathtt{rig}}$. We can assume that $q' \subseteq q$. Then $q \in \mathtt{dom}\, s_{\mathtt{rig}}$, too, by the regularity assumption, and hence $q \in \mathtt{dom}\, t^a_{\mathtt{rig}}$, and $H_{\mathtt{rig}} \cap \mathtt{dom}\, t^a_{\mathtt{rig}} \neq \varnothing$. We conclude that $t^a_{\mathtt{rig}}[H_{\mathtt{rig}}] = s_{\mathtt{rig}}[H_{\mathtt{rig}}] = H_{\mathtt{lef}}$, by the lemma. Finally $t^a_{\mathtt{rig}}[t^a_{\mathtt{lef}}[H_{\mathtt{lef}}]] = H_{\mathtt{lef}}$; this ends the verification of 3.1(1) for a.

Thus $a \in \mathbf{DN}_\gamma$. In addition, by the choice of $s_{\mathtt{lef}}$ and $s_{\mathtt{rig}}$, some $p \in G_{\mathtt{lef}}$ forces that "$s_{\mathtt{lef}}[\underline{G}]$ is generic and $\underline{G} = s_{\mathtt{rig}}[s_{\mathtt{lef}}[G]]$". Then $p \in \mathtt{dom}\, s_{\mathtt{lef}}$, $p \in \mathtt{dom}\, t^a_{\mathtt{lef}}$, $G_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \neq \varnothing$, and $t^a_{\mathtt{lef}}[G_{\mathtt{lef}}] = s_{\mathtt{lef}}[G_{\mathtt{lef}}] = G_{\mathtt{rig}}$, as above. Similarly we have $G_{\mathtt{lef}} = t^a_{\mathtt{rig}}[G_{\mathtt{rig}}]$.

To fix the regularity condition of the theorem, let $b = {}^{\mathsf{rh}}a$; then still $b \in \mathbf{DN}_\gamma$, $G_{\mathtt{rig}} = t^b_{\mathtt{lef}}[G_{\mathtt{lef}}]$, $G_{\mathtt{lef}} = t^b_{\mathtt{rig}}[G_{\mathtt{rig}}]$, and b is regular, by Lemma 4.4.

It is not necessarily true, of course, that sets $\mathtt{dom}\, t^b_{\mathtt{lef}}$ and $\mathtt{dom}\, t^b_{\mathtt{rig}}$ are dense. To fix this shortcoming, we define

$$W = \{p \in \mathbf{LS}_\gamma : \forall\, q \in \mathtt{dom}\, t^b_{\mathtt{lef}} \cup \mathtt{dom}\, t^b_{\mathtt{rig}}\, (p \text{ is incompatible with } q)\}$$

and let $c = \langle t^c_{\mathtt{lef}}, t^c_{\mathtt{rig}} \rangle$, where $t^c_{\mathtt{lef}} = t^c_{\mathtt{rig}} = t^b_{\mathtt{lef}} \cup t^b_{\mathtt{rig}} \cup \{\langle p, q\rangle : p \in W \wedge q \subseteq p\}$.

The set $\mathtt{dom}\, t^c_{\mathtt{lef}} = \mathtt{dom}\, t^c_{\mathtt{rig}} = \mathtt{dom}\, t^b_{\mathtt{lef}} \cup \mathtt{dom}\, t^b_{\mathtt{rig}} \cup W$ is dense in $\mathbf{LS}_\gamma$ by construction. We claim that $c \in \mathbf{DN}_\gamma$. Indeed let $H_{\mathtt{lef}} \subseteq \mathbf{LS}_\gamma$ be an $\mathbf{LS}_\gamma$-generic set. Then $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^c_{\mathtt{lef}} \neq \varnothing$. But $\mathtt{dom}\, t^c_{\mathtt{lef}} = \mathtt{dom}\, t^b_{\mathtt{lef}} \cup \mathtt{dom}\, t^b_{\mathtt{rig}} \cup W$.

Case 1: $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^b_{\mathtt{lef}} \neq \varnothing$. Then $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^b_{\mathtt{rig}} = \varnothing$ since if $p' \in \mathtt{dom}\, t^b_{\mathtt{lef}}$ and $q' \in \mathtt{dom}\, t^b_{\mathtt{lef}}$ then p', q' are incompatible by the original choice of $p_{\mathtt{lef}}, p_{\mathtt{rig}}$. We also have $H_{\mathtt{lef}} \cap W = \varnothing$ by obvious reasons. It follows that $t^c_{\mathtt{lef}}[H_{\mathtt{lef}}] = t^b_{\mathtt{lef}}[H_{\mathtt{lef}}]$, and hence $H_{\mathtt{rig}} = t^c_{\mathtt{lef}}[H_{\mathtt{lef}}]$ is an $\mathbf{LS}_\gamma$-generic set and $H_{\mathtt{lef}} = t^b_{\mathtt{rig}}[H_{\mathtt{rig}}]$, because $b \in \mathbf{DN}_\gamma$. In particular $H_{\mathtt{rig}} \cap \mathtt{dom}\, t^b_{\mathtt{lef}} \neq \varnothing$, so that $t^c_{\mathtt{rig}}[H_{\mathtt{rig}}] = t^b_{\mathtt{rig}}[H_{\mathtt{rig}}]$, as above.

Case 2: $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^b_{\mathtt{rig}} \neq \varnothing$, similar.

Case 3: $H_{\mathtt{lef}} \cap W \neq \varnothing$. Then $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^b_{\mathtt{lef}} = H_{\mathtt{lef}} \cap \mathtt{dom}\, t^b_{\mathtt{rig}} = \varnothing$ as above. It follows that $t^c_{\mathtt{lef}}[H_{\mathtt{lef}}] = t^c_{\mathtt{rig}}[H_{\mathtt{lef}}] = H_{\mathtt{lef}}$.

Thus indeed $c \in \mathbf{DN}_\gamma$, $t^c_{\mathtt{lef}} = t^c_{\mathtt{rig}}$, the set $\mathtt{dom}\, t^c_{\mathtt{lef}} = \mathtt{dom}\, t^c_{\mathtt{rig}}$ is open dense in $\mathbf{LS}_\gamma$, and the arguments above (Case 1) also imply that $G_{\mathtt{rig}} = t^c_{\mathtt{lef}}[G_{\mathtt{lef}}]$, $G_{\mathtt{lef}} = t^c_{\mathtt{rig}}[G_{\mathtt{rig}}]$. Moreover, c inherits the regularity of b. □

6 Extensions

Definition 6.1 (extension)**.** Suppose that a, b are double-names. We say that b *extends* a, in symbol $a \leqslant b$, if just $t^a_{\mathtt{lef}} \subseteq t^b_{\mathtt{lef}}$ and $t^a_{\mathtt{rig}} \subseteq t^b_{\mathtt{rig}}$. □

Lemma 6.2 (in $\mathbf{L}$)**.** *If* $\beta < \gamma \le \Omega$ *and* $a \in \mathbf{DN}_\beta$*, then there is a double-name* $b \in \mathbf{DN}_\gamma$ *which extends* a*.*

Proof. Let $t^b_{\mathtt{lef}}$ consist of all pairs $\langle p \cup r, q \cup r\rangle$, where $\langle p, q\rangle \in t^a_{\mathtt{lef}}$ and r is a condition in $\mathbf{LS}_\gamma$ satisfying $|r| \subseteq \gamma \smallsetminus \beta$; let $t^b_{\mathtt{rig}}$ be defined the same way.

This can be explained as follows. Suppose that $G_{\mathtt{lef}} \subseteq \mathbf{LS}_\gamma$ is a $\mathbf{LS}_\gamma$-generic set containing $p_{\mathtt{lef}}$. Then the factors $G'_{\mathtt{lef}} = G_{\mathtt{lef}} \cap \mathbf{LS}_\beta$ and $G''_{\mathtt{lef}} = G_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma\smallsetminus\beta}$ are resp. $\mathbf{LS}_\beta$-generic and $\mathbf{LS}_{\gamma\smallsetminus\beta}$-generic, and $G_{\mathtt{lef}}$ can be identified with $G'_{\mathtt{lef}} \times G''_{\mathtt{lef}}$ by the product forcing theorem. Then by definition the set $G_{\mathtt{rig}} = t^b_{\mathtt{lef}}[G_{\mathtt{lef}}]$ has the form $G'_{\mathtt{rig}} \times G''_{\mathtt{rig}}$, where $G'_{\mathtt{rig}} = t^a_{\mathtt{lef}}[G'_{\mathtt{lef}}]$ while simply $G''_{\mathtt{rig}} = G''_{\mathtt{lef}}$. The genericity of $G_{\mathtt{rig}}$ easily follows. □

Definition 6.3 (restriction)**.** Let $\alpha < \beta \le \Omega$. If $t \in \mathbf{LS}_\beta$ then define $t{\restriction}\alpha = t \cap (\mathbf{LS}_\alpha \times \mathbf{LS}_\alpha)$; $t{\restriction}\alpha \in \mathbf{N}_\alpha$. If $a \in \mathbf{DN}_\beta$, then let $a{\restriction}\alpha = \langle t^a_{\mathtt{lef}}{\restriction}\alpha, t^a_{\mathtt{rig}}{\restriction}\alpha\rangle$. □

It is not asserted that *always* $a{\restriction}\alpha \in \mathbf{DN}_\alpha$!

Lemma 6.4. *If, in* $\mathbf{L}$, $\alpha < \beta \le \Omega$, $a \in \mathbf{DN}_\alpha$, $b \in \mathbf{DN}_\beta$, *and* $a \leqslant b$, *then*

(i) *if* $G_{\mathtt{lef}} \subseteq \mathbf{LS}_\beta$ *is an* $\mathbf{LS}_\beta$*-generic set then* (a) $H_{\mathtt{lef}} = G_{\mathtt{lef}} \cap \mathbf{LS}_\alpha$ *is* $\mathbf{LS}_\alpha$*-generic, and* (b) *if* $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \ne \varnothing$ *then* $t^a_{\mathtt{lef}}[H_{\mathtt{lef}}] = t^b_{\mathtt{lef}}[G_{\mathtt{lef}}] \cap \mathbf{LS}_\alpha$;

(ii) *if* $G_{\mathtt{rig}} \subseteq \mathbf{LS}_\beta$ *is an* $\mathbf{LS}_\beta$*-generic set then* (a) $H_{\mathtt{rig}} = G_{\mathtt{rig}} \cap \mathbf{LS}_\alpha$ *is* $\mathbf{LS}_\alpha$*-generic, and* (b) *if* $G_{\mathtt{rig}} \cap \mathtt{dom}\, t^a_{\mathtt{rig}} \ne \varnothing$ *then* $t^a_{\mathtt{rig}}[H_{\mathtt{rig}}] = t^b_{\mathtt{rig}}[G_{\mathtt{rig}}] \cap \mathbf{LS}_\alpha$;

(iii) $c = b{\restriction}\alpha$ *belongs to* $\mathbf{DN}_\alpha$ *and* $a \leqslant c \leqslant b$.

Proof. (i)(a) That $H_{\mathtt{lef}}$ is generic holds by the product forcing theorem.

(i)(b) If $H_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \ne \varnothing$ then $G_{\mathtt{lef}} \cap \mathtt{dom}\, t^b_{\mathtt{lef}} \ne \varnothing$, and hence the sets $G_{\mathtt{rig}} = t^b_{\mathtt{lef}}[G_{\mathtt{lef}}]$ and $H_{\mathtt{rig}} = t^a_{\mathtt{lef}}[H_{\mathtt{lef}}]$ are generic sets in resp. $\mathbf{LS}_\beta$ and $\mathbf{LS}_\alpha$ by Lemma 3.2, and $H_{\mathtt{rig}} \subseteq G_{\mathtt{rig}}$ since $a \leqslant b$. Therefore $H_{\mathtt{rig}} \subseteq H'_{\mathtt{rig}} = G_{\mathtt{rig}} \cap \mathbf{LS}_\alpha$. However $H'_{\mathtt{rig}}$ is $\mathbf{LS}_\alpha$-generic by the product forcing. Thus both $H_{\mathtt{rig}} \subseteq H'_{\mathtt{rig}}$ are generic sets, hence easily $H_{\mathtt{rig}} = H'_{\mathtt{rig}}$ as required.

(iii) To check 3.1(1)(a) for some $p \in \mathtt{dom}\, t^c_{\mathtt{lef}}$, consider any $\mathbf{LS}_\alpha$-generic set $H_{\mathtt{lef}} \subseteq \mathbf{LS}_\alpha$ containing p and extend it to a $\mathbf{LS}_\beta$-generic set $G_{\mathtt{lef}} \subseteq \mathbf{LS}_\alpha$ so that $H_{\mathtt{lef}} = G_{\mathtt{lef}} \cap \mathbf{LS}_\alpha$. The (generic by Lemma 3.2) sets $H_{\mathtt{rig}} = t^a_{\mathtt{lef}}[H_{\mathtt{lef}}]$ and $G_{\mathtt{rig}} = t^b_{\mathtt{lef}}[G_{\mathtt{lef}}]$ satisfy $H_{\mathtt{rig}} = G_{\mathtt{rig}} \cap \mathbf{LS}\beta$ by (i). On the other hand $H_{\mathtt{rig}} \subseteq t^c_{\mathtt{lef}}[H_{\mathtt{lef}}] \subseteq G_{\mathtt{rig}} \cap \mathbf{LS}_\beta$, hence $t^c_{\mathtt{lef}}[H_{\mathtt{lef}}] = H_{\mathtt{rig}}$ is generic, as required. The verification of 3.1(1)(b) also is very simple. □

Lemma 6.5. *In* $\mathbf{L}$, *assume that* $\alpha < \beta \le \Omega$. *Then*:

(i) *if* $s \in \mathbf{N}_\alpha$, $t \in \mathbf{N}_\beta$, *and* $s \subseteq t$, *then* ${}^{\mathsf{rh}}s \subseteq {}^{\mathsf{rh}}t$;

(ii) *therefore if* $a \in \mathbf{DN}_\alpha$, $b \in \mathbf{DN}_\beta$, *and* $a \leqslant b$, *then* ${}^{\mathsf{rh}}a \leqslant {}^{\mathsf{rh}}b$;

(iii) *if* $b \in \mathbf{DN}_\beta$ *is regular and* $a = b{\restriction}\alpha \in \mathbf{DN}_\alpha$ *then* a *is regular, too.*

Proof. (i) Suppose that $\langle p', q\rangle \in {}^{\mathsf{rh}}s$, *i.e.*, $p', q \in \mathbf{LS}_\alpha$ and there is a condition $p \subseteq p'$ which $\mathbf{LS}_\alpha$-forces that $q \in t[\underline{G}]$. Prove that p also $\mathbf{LS}_\beta$-forces $q \in t[\underline{G}]$. Let a set $G_{\mathtt{lef}} \subseteq \mathbf{LS}_\beta$ be a set $\mathbf{LS}_\beta$-generic over $\mathbf{L}$ and containing p; prove that $q \in G_{\mathtt{rig}} = t[G_{\mathtt{lef}}]$. The set $H_{\mathtt{lef}} = G_{\mathtt{lef}}{\restriction}\mathbf{LS}_\alpha$ is $\mathbf{LS}_\alpha$-generic by Lemma 6.4 and still $p \in H_{\mathtt{lef}}$, hence $q \in s[H_{\mathtt{lef}}] \subseteq t[G_{\mathtt{lef}}] = G_{\mathtt{rig}}$, as required.

(iii) Assume that $p, q, p' \in \mathbf{LS}_\alpha$, $p \subseteq p'$ and p $\mathbf{LS}_\alpha$-forces $q \in t^a_{\mathtt{lef}}[\underline{G}]$; we have to prove that $\langle p', q\rangle \in t^a_{\mathtt{lef}}$. As $a = b{\restriction}\alpha$, it suffices to show that $\langle p', q\rangle \in t^b_{\mathtt{lef}}$. The same argument based on Lemma 6.4 shows that p also $\mathbf{LS}_\beta$-forces $q \in t^a_{\mathtt{lef}}[\underline{G}]$. Therefore $\langle p', q\rangle \in t^b_{\mathtt{lef}}$ since b is regular. □

7 Increasing sequences

Suppose that a set $\Gamma \subseteq \mathbf{DN}$ is pairwise $\leqslant$-compatible. Then define the double-name $A = \bigvee \Gamma$ by $t^A_{\mathtt{lef}} = \bigcup_{a\in\Gamma} t^a_{\mathtt{lef}}$, $t^A_{\mathtt{rig}} = \bigcup_{a\in\Gamma} t^a_{\mathtt{rig}}$.

Lemma 7.1 (in $\mathbf{L}$). (i) *If* $\lambda < \Omega$ *is a limit ordinal and* $\{a_\xi\}_{\xi<\lambda}$ *is a* $\leqslant$*-increasing sequence in* $\mathbf{DN}$ *then* $A = \bigvee\{a_\xi : \xi < \lambda\}$ *belongs to* $\mathbf{DN}$;

(ii) *therefore the set* $\mathbf{DN} = \bigcup_{\gamma<\Omega} \mathbf{DN}_\gamma$ *is* Ω*-closed in the sense of* $\leqslant$;

(iii) *if* $\{a_\xi\}_{\xi<\Omega}$ *is a strictly* $\leqslant$*-increasing sequence in* $\mathbf{DN}$ *then the double-name* $A = \bigvee\{a_\xi : \xi < \lambda\}$ *belongs to* $\mathbf{DN}_\Omega$.

Proof. (i) Suppose that $\{\gamma_\xi\}_{\xi<\lambda}$ is a strictly increasing sequence of ordinals $\gamma_\xi < \Omega$, and double-names $a_\xi = \langle t^\xi_{\mathtt{lef}}, t^\xi_{\mathtt{rig}}\rangle \in \mathbf{DN}_{\gamma_\xi}$ form a strictly $\leqslant$-increasing sequence: if $\xi < \eta < \lambda$ then $t^\xi_{\mathtt{lef}} \subseteq t^\eta_{\mathtt{lef}}$ and $t^\xi_{\mathtt{rig}} \subseteq t^\eta_{\mathtt{rig}}$. Let $t^A_{\mathtt{lef}} = \bigcup_{\xi<\lambda} t^\xi_{\mathtt{lef}}$, $t^A_{\mathtt{rig}} = \bigcup_{\xi<\lambda} t^\xi_{\mathtt{rig}}$, and $\gamma = \sup_{\xi<\lambda}\gamma_\xi$. We claim that $A = \langle t^A_{\mathtt{lef}}, t^A_{\mathtt{rig}}\rangle \in \mathbf{DN}_\gamma$.

Let's verify 3.1(1). Assume that $G_{\mathtt{lef}} \subseteq \mathbf{LS}_\gamma$ is a generic set containing some $p \in \mathtt{dom}\, t^A_{\mathtt{lef}}$; we have to prove that $G_{\mathtt{rig}} = t^A_{\mathtt{lef}}[G_{\mathtt{lef}}]$ is $\mathbf{LS}_\gamma$-generic and $G_{\mathtt{lef}} = t^A_{\mathtt{rig}}[G_{\mathtt{rig}}]$. Note first of all that each set $G^\xi_{\mathtt{lef}} = G_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma_\xi}$, $\xi < \lambda$, is $\mathbf{LS}_{\gamma_\xi}$-generic by the product forcing theorem, and p belongs to some $\mathtt{dom}\, t^{a_\zeta}_{\mathtt{lef}}$, $\zeta < \Omega$. We can assume that $\zeta = 0$ (otherwise simply cut all double-names a_ξ, $\xi < \zeta$). Then $p \in \mathtt{dom}\, t^0_{\mathtt{lef}}$, therefore $p \in \mathtt{dom}\, t^\xi_{\mathtt{lef}}$ for all $\xi < \Omega$. It follows that each set $G^\xi_{\mathtt{rig}} = t^\xi_{\mathtt{lef}}[G^\xi_{\mathtt{lef}}] \subseteq \mathbf{LS}_{\gamma_\xi}$ is $\mathbf{LS}_{\gamma_\xi}$-generic, $G^\xi_{\mathtt{rig}} \cap \mathtt{dom}\, t^\xi_{\mathtt{rig}} \neq \varnothing$, and $G^\xi_{\mathtt{lef}} = t^\xi_{\mathtt{rig}}[G^\xi_{\mathtt{rig}}]$, by Lemma 3.2. And as $G_{\mathtt{rig}} = \bigcup_{\xi<\lambda} G^\xi_{\mathtt{rig}}$, we conclude that at least $G_{\mathtt{rig}}$ is a filter in $\mathbf{LS}_\gamma$ and $G_{\mathtt{lef}} = t^A_{\mathtt{rig}}[G_{\mathtt{rig}}]$, that is, 3.1(1)(b).

To continue with 3.1(1)(a), we prove the $\mathbf{LS}_\gamma$-genericity of $G_{\mathtt{rig}}$.

Let $D \subseteq \mathbf{LS}_\gamma$ be a dense subset of $\mathbf{LS}_\gamma$, in $\mathbf{L}$. Assume towards the contrary that $G_{\mathtt{rig}} \cap D = \varnothing$. Then there is a condition $p \in G_{\mathtt{lef}}$ which $\mathbf{LS}_\gamma$-forces that $t^A_{\mathtt{lef}}[\underline{G}] \cap D = \varnothing$. Then $p \in G^\xi_{\mathtt{lef}}$ for some $\xi < \lambda$, and there is a condition $q \in G^\xi_{\mathtt{rig}}$ which puts p in $G^\xi_{\mathtt{lef}} = t^\xi_{\mathtt{rig}}[G^\xi_{\mathtt{rig}}]$ in the sense that $\langle q, p\rangle \in t^\xi_{\mathtt{rig}}$. As D is dense, there is some $q' \in D$ with $q \subseteq q'$. Then q' belongs to some $\mathbf{LS}_{\gamma_\eta}$, $\xi < \eta < \lambda$. By Corollary 3.3, there is a condition $p' \in \mathbf{LS}_{\gamma_\eta}$, compatible with p and such that $\langle p', q'\rangle \in t^\eta_{\mathtt{lef}}$. Then p' $\mathbf{LS}_\gamma$-forces $q' \in t^\eta_{\mathtt{lef}}[\underline{G}] \cap D$, while p, a compatible condition, forces the opposite, which is a contradiction.

(iii) Pretty similar argument. □

Corollary 7.2 (in $\mathbf{L}$). *Assume that* $c \in \mathbf{DN}_\Omega$. *Then*

(i) *the set* $\Xi = \{\gamma < \Omega : c\restriction\gamma \in \mathbf{DN}_\gamma\}$ *is a club in* Ω;

(ii) *if* c *is full* (Definition 4.1) *then* $\Xi' = \{\gamma \in \Xi : c\restriction\gamma$ is full$\}$ *is a club*;

(iii) *if* $\Xi'' = \{\gamma \in \Xi : c\restriction\gamma \text{ is regular}\}$ *is unbounded in* Ω *then* $\Xi'' = \Xi$.

Proof. (i) That Ξ is closed follows from Lemma 7.1(i). To prove that Ξ is unbounded, let $\alpha < \Omega$ and find a larger ordinal $\beta \in \Xi$.

Recall that to decide a sentence Φ means to force Φ or to force $\neg\,\Phi$.

By basic forcing theorems, if $p \in \mathbf{LS}$ then the set

$$D_p = \{p \in \mathbf{LS} : p \text{ decides } q \in t^c_{\mathtt{lef}}[\underline{G}] \text{ and decides } q \in t^c_{\mathtt{rig}}[\underline{G}]\}$$

is dense in $\mathbf{LS}$, therefore by the ccc property of $\mathbf{LS}$ there is an ordinal β, $\alpha < \beta < \Omega$, such that D_p is dense in $\mathbf{LS}_\beta$ for all $p \in \mathbf{LS}_\beta$. Then $\beta \in \Xi$.

(ii) easily follows from (i). To prove (iii) apply Lemma 6.5(iii). $\square$

8 Superpositions

Assume that $\gamma \le \Omega$ and $a, c \in \mathbf{DN}_\gamma$. Define

$$\begin{aligned} t^{a\cdot c}_{\mathtt{lef}} &= \{\langle p', q\rangle \in \mathbf{LS}_\gamma \times \mathbf{LS}_\gamma : \exists\, p \in \mathbf{LS}_\gamma\, (\langle p', p\rangle \in t^c_{\mathtt{lef}} \wedge \langle p, q\rangle \in t^a_{\mathtt{lef}})\},\\ t^{a\cdot c}_{\mathtt{rig}} &= \{\langle q, p'\rangle \in \mathbf{LS}_\gamma \times \mathbf{LS}_\gamma : \exists\, p \in \mathbf{LS}_\gamma\, (\langle q, p\rangle \in t^a_{\mathtt{rig}} \wedge \langle p, p'\rangle \in t^c_{\mathtt{rig}})\}. \end{aligned}$$

and $a \cdot c = \langle t^{a\cdot c}_{\mathtt{lef}}, t^{a\cdot c}_{\mathtt{rig}}\rangle$.

Lemma 8.1. *If* $\gamma \le \Omega$, $a, c \in \mathbf{DN}_\gamma$, *and* $G \subseteq \mathbf{LS}_\gamma$, *then* $t^{a\cdot c}_{\mathtt{lef}}[G] = t^a_{\mathtt{lef}}[t^c_{\mathtt{lef}}[G]]$ *and* $t^{a\cdot c}_{\mathtt{rig}}[G] = t^c_{\mathtt{rig}}[t^a_{\mathtt{rig}}[G]]$.

Proof. Assume that $q \in t^{a\cdot c}_{\mathtt{lef}}[G]$. Then there is a pair $\langle p', q\rangle \in t^{a\cdot c}_{\mathtt{lef}}$ with $p' \in G$. By definition there is a condition p such that $\langle p', p\rangle \in t^c_{\mathtt{lef}}$ and $\langle p, q\rangle \in t^a_{\mathtt{lef}}$. Then $p \in t^c_{\mathtt{lef}}[G]$ and hence $q \in t^a_{\mathtt{lef}}[t^c_{\mathtt{lef}}[G]]$. To prove the converse assume that $q \in t^a_{\mathtt{lef}}[t^c_{\mathtt{lef}}[G]]$. Then there is a pair $\langle p, q\rangle \in t^a_{\mathtt{lef}}$ with $p \in t^c_{\mathtt{lef}}[G]$, and further there is a pair $\langle p', p\rangle \in t^c_{\mathtt{lef}}$ with $p' \in G$. Then p witnesses that $\langle p', q\rangle \in t^{a\cdot c}_{\mathtt{lef}}$, and hence $q \in t^{a\cdot c}_{\mathtt{lef}}[G]$. $\square$

Corollary 8.2. *Assume that* $\gamma < \Omega$ *and* $a, b, c \in \mathbf{LS}_\gamma$. *If* a, b *are equivalent (in the sense of Definition 4.3) then so are* $a \cdot c$ *and* $b \cdot c$. $\square$

Lemma 8.3. *If* $\gamma \le \Omega$ *and* $a, c \in \mathbf{DN}_\gamma$ *then the following are equivalent*:

(1) $\mathtt{ran}\, t^c_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \ne \varnothing$, (2) $\mathtt{ran}\, t^a_{\mathtt{rig}} \cap \mathtt{dom}\, t^c_{\mathtt{rig}} \ne \varnothing$, (3) $a \cdot c \in \mathbf{DN}_\gamma$.

Proof. Let $\mathtt{ran}\, t^c_{\mathtt{lef}} \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \ne \varnothing$. To prove (3) consider an $\mathbf{LS}_\gamma$-generic set $G' \subseteq \mathbf{LS}_\gamma$, and let $p' \in G' \cap \mathtt{dom}\, t^{a\cdot c}_{\mathtt{lef}}$. Then $p' \in \mathtt{dom}\, t^c_{\mathtt{lef}}$, hence $G = t^b_{\mathtt{lef}}[G']$ is an $\mathbf{LS}_\gamma$-generic set by Lemma 3.2. As $p' \in \mathtt{dom}\, t^{a\cdot b}_{\mathtt{lef}}$, $G \cap \mathtt{dom}\, t^a_{\mathtt{lef}} \ne \varnothing$. It follows that $H = t^a_{\mathtt{lef}}[G]$ is an $\mathbf{LS}_\gamma$-generic set. Finally $H = t^{a\cdot c}_{\mathtt{lef}}[G']$ by Lemma 8.1.

This argument also proves that $G' = t^{a\cdot c}_{\mathtt{rig}}[H]$. Thus (1) $\Longrightarrow$ (3).

That (3) $\Longrightarrow$ (1) is obvious. $\square$

Corollary 8.4. *If* $\gamma \le \Omega$, $a, c \in \mathbf{DN}_\gamma$, *and* c *is full (in the sense of Definition 4.1) then* $a \cdot c \in \mathbf{DN}_\gamma$.

Proof. By Lemma 4.2, $\mathtt{ran}\, t^c_{\mathtt{lef}} = \mathtt{ran}\, t^c_{\mathtt{rig}} = \mathbf{LS}_\gamma$. Now use Lemma 8.3. □

Thus if $c \in \mathbf{DN}_\gamma$ is a full double-name then $a \mapsto a \cdot c$ is a map $\mathbf{DN}_\gamma \to \mathbf{DN}_\gamma$. In this case, consider the *inverse* double-name $c^{-1} = \langle t^c_{\mathtt{rig}}, t^c_{\mathtt{lef}} \rangle$, let $a \in \mathbf{DN}_\gamma$, and compare a with $a' = a \cdot c \cdot c^{-1}$. On the one hand, we have $t^{a'}_{\mathtt{lef}}[G] = t^a_{\mathtt{lef}}[t^c_{\mathtt{lef}}[t^{c^{-1}}_{\mathtt{lef}}[G]]]$ for any $\mathbf{LS}_\gamma$-generic set G by Lemma 8.1. It follows that $t^{a'}_{\mathtt{lef}}[G] = t^a_{\mathtt{lef}}[t^c_{\mathtt{lef}}[t^c_{\mathtt{rig}}[G]]] = t^a_{\mathtt{lef}}[G]$ since the successive action of $t^c_{\mathtt{lef}}$ and $t^c_{\mathtt{rig}}$ is the identity by Lemma 3.2. Similarly $t^{a'}_{\mathtt{rig}}[G] = t^a_{\mathtt{rig}}[G]$. Therefore a and a' are equivalent, and hence ${}^{\mathsf{rh}}a = {}^{\mathsf{rh}}a'$ by Lemma 4.4, but generally speaking we cannot assert that straightforwardly $a = a'$.

To fix this problem, define the modified action $a * c = {}^{\mathsf{rh}}(a \cdot c)$.

Lemma 8.5. *Let* $\gamma < \Omega$ *and let* $c \in \mathbf{DN}_\gamma$ *be a full double-name. If* $a \in \mathbf{DN}_\gamma$ *is regular (that is,* $a = {}^{\mathsf{rh}}a$*) then* $b = a * c \in \mathbf{DN}_\gamma$, b *is regular, and* $a = b * c^{-1}$.

Proof. That $b \in \mathbf{DN}_\gamma$ follows from Corollary 8.4. The regularity holds by Lemma 4.4. To prove $a = b * c^{-1}$, note that both a and $b * c^{-1}$ are regular double-names, and hence it suffices, by Lemma 4.4, to prove that a and $b * c^{-1}$ are equivalent. However, still by Lemma 4.4, $b * c^{-1}$ is equivalent to $b \cdot c^{-1}$, and $b = a * c$ is equivalent to $a \cdot c$, hence overall $b * c^{-1}$ is equivalent to $a \cdot c \cdot c^{-1}$ by Corollary 8.2. Finally a is equivalent to $a \cdot c \cdot c^{-1}$, see above. □

Lemma 8.6. *Assume that* $\gamma < \delta \le \Omega$, $c \in \mathbf{DN}_\gamma$ *and* $d \in \mathbf{DN}_\delta$ *are full double-names,* $c = d \restriction \gamma$, *and* $a \in \mathbf{DN}_\gamma$, $b \in \mathbf{DN}_\delta$. *Then*

(i) *if* $a \leqslant b$ *then* $a \cdot c \leqslant b \cdot d$;

(ii) *if* a, b *are regular then* $a \leqslant b$ *iff* $a * c \leqslant b * d$.

Proof. (i) is clear since $a \cdot c$ is monotone on both a and c. As for (ii), the implication $\Longrightarrow$ holds by (i) and Lemma 6.5 while to prove the inverse make use of Lemma 8.5. □

9 Generic double-names and product forcing

By Lemma 7.1, we can consider the set $\mathbf{DN} = \bigcup_{\gamma<\Omega} \mathbf{DN}_\gamma$ ordered by $\leqslant$ as an Ω-closed forcing notion in $\mathbf{L}$ ($\leqslant$-bigger double-names are stronger conditions). Suppose that $\Gamma \subseteq \mathbf{DN}$ is a $\mathbf{DN}$-generic set over $\mathbf{L}$. Then a double-name $A = \bigvee \Gamma \in \mathbf{L}[\Gamma]$ can be defined as in Section 7; we call such double-names $A = \bigvee \Gamma$ *generic over* $\mathbf{L}$ (together with the background generic sets Γ).

Let $\underline{\Gamma}$ and $\underline{A}$ be canonical $\mathbf{DN}$-names of resp. Γ and $A = \bigvee \Gamma$.

Remark 9.1. As $\mathbf{L}$ is our default ground model unless otherwise specified, the sets Γ and $A = \bigvee \Gamma$ do not belong to $\mathbf{L}$, however all reals and generally all sets $x \subseteq \gamma < \Omega$ in $\mathbf{L}[\Gamma]$ belong to $\mathbf{L}$ by Lemma 7.1. It follows that the definition of $\mathbf{DN}_\gamma$ $(\gamma < \Omega)$ in $\mathbf{L}$ is absolute for $\mathbf{L}[\Gamma]$. That is, if $a \in \mathbf{DN}_\gamma$ in $\mathbf{L}$ then it is true in $\mathbf{L}[\Gamma]$ that $a \in \mathbf{DN}_\gamma$. And conversely, if $a \in \mathbf{L}[\Gamma]$ and it is true in $\mathbf{L}[\Gamma]$ that $a \in \mathbf{DN}_\gamma$ then $a \in \mathbf{L}$ and it is true in $\mathbf{L}$ that $a \in \mathbf{DN}_\gamma$. □

Corollary 9.2. *Assume that* Γ *is* $\mathbf{DN}$*-generic over* $\mathbf{L}$ *and* $A = \bigvee \Gamma$. *Then*

(i) *it holds in* $\mathbf{L}[\Gamma]$ *that* A *belongs to* $\mathbf{DN}_\Omega$;

(ii) *if* $G_{\mathtt{lef}}$ *is* $\mathbf{LS}$*-generic over* $\mathbf{L}[\Gamma]$, *and* $G_{\mathtt{lef}} \cap \operatorname{\mathtt{dom}} t^A_{\mathtt{lef}} \neq \varnothing$, *then* $G_{\mathtt{rig}}$ *is* $\mathbf{LS}$*-generic over* $\mathbf{L}[\Gamma]$ *and* $G_{\mathtt{lef}} = t^A_{\mathtt{rig}}[G_{\mathtt{rig}}]$;

(iii) *if* $a \in \mathbf{DN}$, $a \subseteq A$, *and* $\gamma = |a|$ *then* $A \restriction \gamma \in \mathbf{DN}_\gamma \cap \Gamma$ *and* $a \leqslant A \restriction \gamma \leqslant A$.

Proof. (i) Remark 9.1 allows simply to refer to Lemma 7.1.

(ii) Make use of Lemma 3.2.

(iii) To prove that $a' = A \restriction \gamma \in \mathbf{DN}_\gamma$ and $a \leqslant a' \leqslant A$ refer to Lemma 6.4(iii). To prove that $a' \in \Gamma$ note that by Lemma 7.1 there is some $c \in \Gamma$ which decides each $b \in \mathbf{LS}_\gamma$ to belong or not to belong to Γ; then $a' \subseteq c$. □

10 The first ingredient

Generic double-names and forcing with $\mathbf{LS} \times \mathbf{DN}$ enable us to carry out the first main step towards Theorem 1.1.

In $\mathbf{L}$, let $\mathbf{H}\Omega$ be the set of all sets x such that the transitive closure $\mathtt{TC}(x)$ has cardinality $\mathtt{card}(\mathtt{TC}(x)) < \Omega$ strictly.

Blanket assumption 10.1. Thus suppose that $G_0 \subseteq \mathbf{LS}$ is a $\mathbf{LS}$-generic set over $\mathbf{L}$, let $\mathscr{X} \in \mathbf{L}[G_0]$, and it is true in $\mathbf{L}[G_0]$ that $\mathscr{X}$ is a countable OD non-empty set of sets of reals. There is a formula $\varphi(\cdot, \pi)$ with some $\pi \in \mathrm{Ord}$ as the only parameter, such that it is true in $\mathbf{L}[G_0]$ that $\mathscr{X}$ is the only set x satisfying $\varphi(x, \pi)$.

There is a sequence $\mathbb{u} = \{U_n\}_{n \in \omega} \in \mathbf{L}$ of names $U_n \in \mathbf{L}$, such that $\mathscr{X} = \mathbb{u}[\![G_0]\!] := \{U_n[\![G_0]\!] : n \in \omega\}$. Each U_n can be assumed to be an $\mathbf{LS}$-name of a set of reals, that is, in $\mathbf{L}$, $U_n \subseteq \mathbf{LS} \times \mathbb{T}$, where T is the set of all $\mathbf{LS}$-names for reals. Furthermore, according to the Ω-cc property of the forcing $\mathbf{LS}$, each $\mathbf{LS}$-name for a real can be assumed to be a set in $\mathbf{H}\Omega$. Therefore we shall wlog assume that $U_n \subseteq \mathbf{H}\Omega$ for all n.

Anyway there is a condition $\bar{p} \in G_0$ which $\mathbf{LS}$-forces over $\mathbf{L}$ that "$\mathbb{u}[\![\underline{G}]\!]$ is the only set x satisfying $\varphi(x, \pi)$, and $\mathbb{u}[\![\underline{G}]\!]$ is a set of sets of reals". Let $\bar{\gamma} < \Omega$ be the least ordinal satisfying $\bar{p} \in \mathbf{LS}_{\bar{\gamma}}$. □

Let a $\bar{p}$*-pair* be any pair $\langle p, a\rangle \in \mathbf{LS}\times\mathbf{DN}$ such that $\bar{p}\subseteq p\in \mathtt{dom}\, t^a_{\mathtt{lef}}$ and p $\mathbf{LS}_\gamma$-forces that $\bar{p}\in t^a_{\mathtt{lef}}[\underline{G}]$, where $\gamma = |a|$.

Remark 10.2. Let $\bar{a} = \mathbf{id}[\bar{\gamma}]$. Then $\langle\bar{p},\bar{a}\rangle$ is a $\bar{p}$-pair; $\bar{p}$ $\mathbf{LS}_{\bar{\gamma}}$-forces that $t^{\bar{a}}_{\mathtt{lef}}[\underline{G}] = \underline{G}$. □

Lemma 10.3. *Let $\langle p, a\rangle \in \mathbf{LS}\times\mathbf{DN}$ be a $\bar{p}$-pair, $q\in\mathbf{LS}$, $b\in\mathbf{DN}$, $p\subseteq q$, $a\leqslant b$. There is a double-name $c\in\mathbf{DN}$ such that $b\leqslant c$ and $\langle q, c\rangle$ is a $\bar{p}$-pair.*

Proof. If $q\in\mathbf{LS}_\gamma$, where $\gamma = |b|$, then to define c add to $t^b_{\mathtt{lef}}$ all pairs $\langle q, r\rangle$ such that already $\langle p, r\rangle\in b$. We claim that $\langle q, c\rangle$ is a $\bar{p}$-pair. Indeed if $G_{\mathtt{lef}}\subseteq\mathbf{LS}_\gamma$ is generic then easily (*) $t^c_{\mathtt{lef}}[G_{\mathtt{lef}}] = t^b_{\mathtt{lef}}[G_{\mathtt{lef}}]$, hence $c\in\mathbf{DN}_\gamma$. Further $\bar{p}\subseteq p\subseteq q\in\mathtt{dom}\, t^c_{\mathtt{lef}}$ by construction. Finally q $\mathbf{LS}_\gamma$-forces that $\bar{p}\in t^a_{\mathtt{lef}}[\underline{G}]$ because so does p, and we can replace $t^a_{\mathtt{lef}}$ by $t^c_{\mathtt{lef}}$ since $a\subseteq b\subseteq c$.

If $q\notin\mathbf{LS}_\gamma$ then still $q\in\mathbf{LS}_\delta$ for some δ, $\gamma<\delta<\Omega$. Use Lemma 6.2 to get a double-name $b'\in\mathbf{DN}_\delta$ with $b\leqslant b'$, and argue as in the first case. □

Theorem 10.4. *Suppose that $G_{\mathtt{lef}}\times\Gamma$ is a $\mathbf{LS}\times\mathbf{DN}$-generic set over $\mathbf{L}$, $A = \bigvee\Gamma$, and $\langle p, a\rangle\in G_{\mathtt{lef}}\times\Gamma$ is a $\bar{p}$-pair. Then*

(i) *$p,\bar{p}\in G_{\mathtt{lef}}$, $\bar{p}\in G_{\mathtt{rig}} = t^A_{\mathtt{lef}}[G_{\mathtt{lef}}]$, and $G_{\mathtt{rig}}$ is $\mathbf{LS}$-generic over $\mathbf{L}[\Gamma]$;*

(ii) *$\mathbb{u}[\![G_{\mathtt{lef}}]\!] = \mathbb{u}[\![G_{\mathtt{rig}}]\!]$ — in other words, any $\bar{p}$-pair $\langle p, a\rangle$ $(\mathbf{LS}\times\mathbf{DN})$-forces $\mathbb{u}[\![\underline{G}]\!] = \mathbb{u}[\![t^A_{\mathtt{lef}}[\underline{G}]]\!]$ over $\mathbf{L}$.*

Proof. (i) To prove the genericity apply Corollary 9.2.

To prove (ii) suppose otherwise. Then there is a pair $\langle q, b\rangle$ in $\mathbf{LS}\times\mathbf{DN}$ with $p\subseteq q$, $a\leqslant b$, which $(\mathbf{LS}\times\mathbf{DN})$-forces $\mathbb{u}[\![\underline{G}]\!]\neq\mathbb{u}[\![t^A_{\mathtt{lef}}[\underline{G}]]\!]$, that is

(†) if $G_{\mathtt{lef}}\times\Gamma$ is a $(\mathbf{LS}\times\mathbf{DN})$-generic set over $\mathbf{L}$ containing $\langle q, b\rangle$, $A = \bigvee\Gamma$, and $G_{\mathtt{rig}} = t^A_{\mathtt{lef}}[G_{\mathtt{lef}}]$, then $\mathbb{u}[\![G_{\mathtt{lef}}]\!]\neq\mathbb{u}[\![G_{\mathtt{rig}}]\!]$.

Let $\mathcal{L}\in\mathbf{L}$ be an elementary submodel of a large model, such that $\mathbf{H}\Omega\subseteq\mathcal{L}$, Ω and π belong to $\mathcal{L}$, $\mathtt{card}(\mathcal{L}) = \Omega$ in $\mathbf{L}$, and $\mathcal{L}$ is an elementary submodel of $\mathbf{L}$ v.r.t. all $\varSigma_{100}$ formulas. Let $\mathcal{L}'\in\mathbf{L}$ be the Mostowski collapse of $\mathcal{L}$; still $\mathtt{card}(\mathcal{L}') = \Omega$ in $\mathbf{L}$. Note that $\mathcal{L}'$ is a transitive model of Zermelo with choice, and the collapse map $\phi:\mathcal{L}\xrightarrow{\text{onto}}\mathcal{L}'$ is the identity on $\mathbf{H}\Omega$, hence even on $\mathscr{P}(\mathbf{H}\Omega)\cap\mathcal{L}$. In particular, $\phi(\Omega) = \Omega$, $\phi(\mathbb{u}) = \mathbb{u}$, $\phi(U_n) = U_n$ for all n, $\phi(\mathbf{LS}) = \mathbf{LS}$, $\phi(\mathbf{DN}) = \mathbf{DN}$, $\mathbf{H}\Omega\subseteq\mathcal{L}'$, and even $\mathscr{P}(\mathbf{H}\Omega)\cap\mathcal{L}\subseteq\mathcal{L}'$.

By the elementary submodel property, $\langle q, b\rangle$ still $(\mathbf{LS}\times\mathbf{DN})$-forces over $\mathcal{L}'$ that $\mathbb{u}[\![\underline{G}]\!]\neq\mathbb{u}[\![t^A_{\mathtt{lef}}[\underline{G}]]\!]$ — that is

(‡) if $G_{\mathtt{lef}}\times\Gamma$ is a $(\mathbf{LS}\times\mathbf{DN})$-generic set over $\mathcal{L}'$ containing $\langle q, b\rangle$, $A = \bigvee\Gamma$, and $G_{\mathtt{rig}} = t^A_{\mathtt{lef}}[G_{\mathtt{lef}}]$, then $\mathbb{u}[\![G_{\mathtt{lef}}]\!]\neq\mathbb{u}[\![G_{\mathtt{rig}}]\!]$.

To infer a contradiction, note that since $\mathtt{card}(\mathcal{L}') = \Omega$ in $\mathbf{L}$, by Lemma 7.1 there exists a set $\Gamma \in \mathbf{L}$, $\mathbf{DN}$-generic over $\mathcal{L}'$ and containing b, hence containing a as well. We underline that $\Gamma \in \mathbf{L}$, and then $A = \bigvee \Gamma$ belongs to $\mathbf{L}$, too. Let $G_{\mathtt{lef}} \subseteq \mathbf{LS}$ be a set $\mathbf{LS}$-generic over $\mathbf{L}$, hence over $\mathcal{L}'[\Gamma]$ as well, and containing q, and then containing p. Then the set $G_{\mathtt{rig}} = t^A_{\mathtt{lef}}[G_{\mathtt{lef}}]$ is $\mathbf{LS}$-generic over $\mathbf{L}$ and over $\mathcal{L}'[\Gamma]$ by Lemma 3.2, and in addition, $\mathbb{u}[\![G_{\mathtt{lef}}]\!] \neq \mathbb{u}[\![G_{\mathtt{rig}}]\!]$ by (‡).

Recall that $\langle p, a\rangle$ also belongs to $G_{\mathtt{lef}} \times A$. Therefore $\bar{p} \in G_{\mathtt{lef}} \cap G_{\mathtt{rig}}$ by (i). Thus $G_{\mathtt{lef}}$ and $G_{\mathtt{rig}}$ are $\mathbf{LS}$-generic sets over $\mathbf{L}$ and both contain $\bar{p}$, $\mathbb{u}[G_{\mathtt{lef}}]$ is the only set x satisfying $\varphi(x,\pi)$ in $\mathbf{L}[G_{\mathtt{lef}}]$ while $\mathbb{u}[G_{\mathtt{rig}}]$ is the only set x satisfying $\varphi(x,\pi)$ in $\mathbf{L}[G_{\mathtt{rig}}]$. However $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G_{\mathtt{rig}}]$ (because $G_{\mathtt{rig}} = t^A_{\mathtt{lef}}[G_{\mathtt{lef}}]$, $G_{\mathtt{lef}} = t^A_{\mathtt{rig}}[G_{\mathtt{rig}}]$, and $A \in \mathbf{L}$), while on the other hand $\mathbb{u}[\![G_{\mathtt{lef}}]\!] \neq \mathbb{u}[\![G_{\mathtt{rig}}]\!]$, which is a contradiction. □

11 Stabilizing pairs and second ingredient

Let a *stabilizing $\bar{p}$-pair* be any $\bar{p}$-pair $\langle \hat{p}, \hat{a}\rangle \in \mathbf{LS} \times \mathbf{DN}$ which, for some n, $(\mathbf{LS} \times \mathbf{DN})$-forces $U_0[\![\underline{G}]\!] = U_n[\![t^A_{\mathtt{lef}}[\underline{G}]]\!]$ over $\mathbf{L}$.

Corollary 11.1. *If $G_{\mathtt{lef}}$ is an $\mathbf{LS}$-generic set over $\mathbf{L}$ containing $\bar{p}$, then there is a stabilizing $\bar{p}$-pair $\langle \hat{p}, \hat{a}\rangle \in \mathbf{LS} \times \mathbf{DN}$ with $\hat{p} \in G_{\mathtt{lef}}$.*

Proof. Let $\bar{a} = \mathbf{id}[\bar{\gamma}]$, see Remark 10.2. Let $\Gamma \subseteq \mathbf{DN}$ be a set $\mathbf{DN}$-generic over $\mathbf{L}[G_{\mathtt{lef}}]$ and containing $\bar{a}$, so that $G_{\mathtt{lef}} \times \Gamma$ is $(\mathbf{LS} \times \mathbf{DN})$-generic. Let $A = \bigvee \Gamma$. Then the set $G_{\mathtt{rig}} = t^A_{\mathtt{lef}}[G_{\mathtt{lef}}]$ satisfies $\mathbb{u}[\![G_{\mathtt{lef}}]\!] = \mathbb{u}[\![G_{\mathtt{rig}}]\!]$ by Theorem 10.4. Therefore there is a number $n \in \omega$ such that $U_0[\![G_{\mathtt{lef}}]\!] = U_n[\![G_{\mathtt{rig}}]\!]$. Then there is a stronger pair $\langle \hat{p}, \hat{a}\rangle \in G_{\mathtt{lef}} \times \Gamma$ ($\bar{p} \subseteq \hat{p}$ and $\bar{a} \leqslant \hat{a}$) which $(\mathbf{LS} \times \mathbf{DN})$-forces $U_0[\![\underline{G}]\!] = U_n[\![t^A_{\mathtt{lef}}[\underline{G}]]\!]$. We can assume that $\langle \hat{p}, \hat{a}\rangle$ is a $\bar{p}$-pair, by Lemma 10.3. □

Proposition 11.2. *Let $\langle \hat{p}, \hat{a}\rangle \in \mathbf{LS} \times \mathbf{DN}$ be a stabilizing $\bar{p}$-pair. Assume that $G_{\mathtt{lef}} \times \Gamma$, $G'_{\mathtt{lef}} \times \Gamma'$ are sets $(\mathbf{LS} \times \mathbf{DN})$-generic over $\mathbf{L}$ and containing $\langle \hat{p}, \hat{a}\rangle$, $A = \bigvee \Gamma$, $A' = \bigvee \Gamma'$, and $t^A_{\mathtt{lef}}[G_{\mathtt{lef}}] = t^{A'}_{\mathtt{lef}}[G'_{\mathtt{lef}}]$. Then $U_0[\![G_{\mathtt{lef}}]\!] = U_0[\![G'_{\mathtt{lef}}]\!]$.*

Proof. By definition, $U_0[\![G_{\mathtt{lef}}]\!] = U_n[\![t^A_{\mathtt{lef}}[G_{\mathtt{lef}}]]\!]$ and $U_0[\![G'_{\mathtt{lef}}]\!] = U_n[\![t^{A'}_{\mathtt{lef}}[G'_{\mathtt{lef}}]]\!]$ for one and the same n. □

The second ingredient in the proof of Theorem 1.1 will be the following:

Theorem 11.3. *Assume that $\langle \hat{p}, \hat{a}\rangle \in \mathbf{LS} \times \mathbf{DN}$ is a stabilizing $\bar{p}$-pair, $\hat{\gamma} < \Omega$, $\hat{a} \in \mathbf{DN}_{\hat{\gamma}}$, $\hat{p} \in \mathbf{LS}_{\hat{\gamma}}$, $G_{\mathtt{lef}}, G'_{\mathtt{lef}} \subseteq \mathbf{LS}$ are $\mathbf{LS}$-generic sets over $\mathbf{L}$ containing $\hat{p}$, $G_{\mathtt{lef}} \cap \mathbf{LS}_{\hat{\gamma}} = G'_{\mathtt{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$, and $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G'_{\mathtt{lef}}]$. Then $U_0[\![G_{\mathtt{lef}}]\!] = U_0[\![G'_{\mathtt{lef}}]\!]$.*

Let's show how this implies Theorem 1.1. The proof of Theorem 11.3 itself will follow in the next sections.

Proof (Theorem 1.1 from Theorem 11.3)**.** We argue in the assumptions and notation of 10.1. Let $G_{\texttt{lef}} = G_0$, so that $\bar{p} \in G_{\texttt{lef}}$ by 10.1. Then by Corollary 11.1, there is a stabilizing $\bar{p}$-pair $\langle \hat{p}, \hat{a} \rangle \in \mathbf{LS} \times \mathbf{DN}$ such that $\hat{p} \in G_{\texttt{lef}}$. Pick $\hat{\gamma} < \Omega$ such that $\hat{a} \in \mathbf{DN}_{\hat{\gamma}}$ and $\hat{p} \in \mathbf{LS}_{\hat{\gamma}}$. Consider, in $\mathbf{L}[G_{\texttt{lef}}]$, the set $\mathscr{G}$ of all sets $G \subseteq \mathbf{LS}$, $\mathbf{LS}$-generic over $\mathbf{L}$ and satisfying $\mathbf{L}[G] = \mathbf{L}[G_{\texttt{lef}}]$, $\hat{p} \in G$, and $G \cap \mathbf{LS}_{\hat{\gamma}} = G_{\texttt{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$. In particular $G_{\texttt{lef}} \in \mathscr{G}$. The only essential parameter of the definition of $\mathscr{G}$ which is not immediately OD — is $G_{\texttt{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$. However $G_{\texttt{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$ itself, as basically any subset of any $\mathbf{LS}_{\gamma}$, $\gamma < \Omega$, is ROD in the Solovay model. We conclude that $\mathscr{G}$ is ROD in $\mathbf{L}[G_{\texttt{lef}}]$.

On the other hand, suppose that $G \in \mathscr{G}$. Then $U_0[\![G_{\texttt{lef}}]\!] = U_0[\![G]\!]$ by Theorem 11.3. Therefore the set $U_0[\![G_{\texttt{lef}}]\!]$ can be defined as $U_0[\![G]\!]$ for some / every $G \in \mathscr{G}$. This witnesses that $U_0[\![G_{\texttt{lef}}]\!]$ is ROD in $\mathbf{L}[G_{\texttt{lef}}]$, because so is $\mathscr{G}$ by the above. Thus the set $\mathscr{X} = \mathbb{U}[\![G_{\texttt{lef}}]\!]$ contains a ROD element. It follows that $\mathscr{X}$ contains an OD element, by Lemma 2.3, as required.

□ (Thm 1.1 mod Thm 11.3)

12 Final

Here we prove Theorem 11.3 and finally prove Theorem 1.1. **We argue in the assumptions and notation of Theorem 11.3.** That is,

(1) $\langle \hat{p}, \hat{a} \rangle \in \mathbf{LS} \times \mathbf{DN}$ is a stabilizing $\bar{p}$-pair, $\hat{\gamma} < \Omega$, $\hat{a} \in \mathbf{DN}_{\hat{\gamma}}$, $\hat{p} \in \mathbf{LS}_{\hat{\gamma}}$, the sets $G_{\texttt{lef}}, G'_{\texttt{lef}} \subseteq \mathbf{LS}$ are $\mathbf{LS}$-generic over $\mathbf{L}$ and both contain $\hat{p}$, and in addition $G_{\texttt{lef}} \cap \mathbf{LS}_{\hat{\gamma}} = G'_{\texttt{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$, $\mathbf{L}[G_{\texttt{lef}}] = \mathbf{L}[G'_{\texttt{lef}}]$.

In this assumption, we have to prove that $U_0[\![G_{\texttt{lef}}]\!] = U_0[\![G'_{\texttt{lef}}]\!]$. Working towards this goal, our plan will be to find:

($*$) sets $\Gamma, \Gamma' \subseteq \mathbf{DN}$, $\mathbf{DN}$-generic over $\mathbf{L}[G_{\texttt{lef}}] = \mathbf{L}[G'_{\texttt{lef}}]$, containing $\hat{a}$, and satisfying $t^{A}_{\texttt{lef}}[G_{\texttt{lef}}] = t^{A'}_{\texttt{lef}}[G'_{\texttt{lef}}]$, where $A = \bigvee \Gamma$ and $A' = \bigvee \Gamma'$;

then the products $G_{\texttt{lef}} \times \Gamma$ and $G'_{\texttt{lef}} \times \Gamma'$ will be $(\mathbf{LS} \times \mathbf{DN})$-generic over $\mathbf{L}$ and containing $\langle \hat{p}, \hat{a} \rangle$, so that $U_0[\![G_{\texttt{lef}}]\!] = U_0[\![G'_{\texttt{lef}}]\!]$ follows by Proposition 11.2, accomplishing the proof of Theorem 11.3.

By Theorem 5.1 there is a double-name $C \in \mathbf{DN}_{\Omega}$ in $\mathbf{L}$, such that

(2) C is full, $t^{C}_{\texttt{lef}} = t^{C}_{\texttt{rig}}$, $G_{\texttt{lef}} = t^{C}_{\texttt{lef}}[G'_{\texttt{lef}}]$, and $G'_{\texttt{lef}} = t^{C}_{\texttt{rig}}[G_{\texttt{lef}}]$.

As $G_{\texttt{lef}} \cap \mathbf{LS}_{\hat{\gamma}} = G'_{\texttt{lef}} \cap \mathbf{LS}_{\hat{\gamma}}$, we can further assume that

(3) the restricted double-name $C \restriction \hat{\gamma}$ coincides with $\mathbf{id}[\hat{\gamma}]$ of Example 4.5, so that $C \restriction \hat{\gamma} \in \mathbf{LS}_{\hat{\gamma}}$ is full and regular, and $t^{C \restriction \hat{\gamma}}_{\texttt{lef}}[G] = t^{C \restriction \hat{\gamma}}_{\texttt{rig}}[G] = G$ for all G.

Let Γ be any set $\Gamma \subseteq \mathbf{DN}$ with $\hat{a} \in \Gamma$, $\mathbf{DN}$-generic over $\mathbf{L}[G_{\mathtt{lef}}]$. Then $A = \bigvee \Gamma \in \mathbf{DN}_\Omega$ in $\mathbf{L}[\Gamma]$ by Corollary 9.2, and $\bar{p} \subseteq \hat{p} \in \mathtt{dom}\, A$ since $\hat{a} \in \Gamma$.

Corollary 12.1. (i) *The set* $X = \{\gamma < \Omega : A \restriction \gamma \in \mathbf{DN}_\gamma\} \in \mathbf{L}[\Gamma]$ *is a club in* Ω, *and if* $\gamma \in X$ *then* $A \restriction \gamma$ *is regular*;

(ii) *the set* $Y = \{\gamma < \Omega : C \restriction \gamma \in \mathbf{DN}_\gamma \text{ and } C \restriction \gamma \text{ is full}\} \in \mathbf{L}$ *is a club in* Ω;

(iii) *therefore* $Z = \{\gamma \in X \cap Y : \hat{\gamma} \le \gamma\}$ *is a club, and in addition* $\hat{\gamma} \in Z$.

Proof. To prove (i) and (ii) apply Corollary 7.2; the unboundedness condition in 7.2(iii) follows from the genericity of Γ and the density of the set of all regular double-names $a \in \mathbf{DN}$ by Lemma 4.4(ii).

Claim $\hat{\gamma} \in Z$ in (iii) follows from (3). $\square$

Now suppose that $\gamma \in Y$, hence $C \restriction \gamma \in \mathbf{DN}_\gamma$ is full. Let $a \in \mathbf{DN}_\gamma$ be regular. Define $a * C = a * (C \restriction \gamma)$ (see Section 8).

Lemma 12.2. *The map* $a \mapsto a * C$ *is a* $\leqslant$*-preserving bijection of the set* $\mathbf{DN}^Y_{\mathtt{reg}} = \{a \in \mathbf{DN} : a \text{ is regular} \wedge |a| \in Y\}$ *onto itself, satisfying* $a * C * C = a$.

Proof. If $a \in \mathbf{DN}^Y_{\mathtt{reg}}$ and $\gamma = |a|$ then $a * C = a * (C \restriction \gamma)$ belongs to $\mathbf{DN}_\gamma$ and is regular by Lemma 8.5, hence $a * C \in \mathbf{DN}^Y_{\mathtt{reg}}$. If $\delta > \gamma$ is a bigger ordinal still in Y, and $b \in \mathbf{DN}^Y_{\mathtt{reg}}$, $\delta = |b|$, then $a \leqslant b$ iff $a * C \leqslant b * C$ by Lemma 8.6(ii). Finally $a * C * C = a$ holds still by Lemma 8.5, because $C^{-1} = C$ (that is, $t^C_{\mathtt{lef}} = t^C_{\mathtt{rig}}$) by (2). $\square$

In particular, if $\gamma \in Z$ then $A \restriction \gamma \in \mathbf{DN}^Y_{\mathtt{reg}}$, and hence $(A \restriction \gamma) * C \in \mathbf{DN}^Y_{\mathtt{reg}}$ is a regular double-name. Thus $\{(A \restriction \gamma) * C\}_{\gamma \in Z} \in \mathbf{L}[\Gamma]$ is a $\leqslant$-increasing sequence of regular double-names. The following is a key fact.

Lemma 12.3. *The sequence* $\{(A \restriction \gamma) * C\}_{\gamma \in Z}$ *is* $\mathbf{DN}$*-generic over* $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G'_{\mathtt{lef}}]$, *in the sense that if a set* $D' \subseteq \mathbf{DN}$, $D' \in \mathbf{L}[G_{\mathtt{lef}}]$, *is open dense in* $\mathbf{DN}$ *then there is an ordinal* $\gamma \in Z$ *such that* $(A \restriction \gamma) * C \in D'$.

Proof. The set $\Delta' = D' \cap \mathbf{DN}^Y_{\mathtt{reg}}$ belongs to $\mathbf{L}[G_{\mathtt{lef}}]$ and still is dense in $\mathbf{DN}$ by Lemma 4.4(ii). Therefore its C-image $\Delta = \{a * C : a \in \Delta'\}$ still belongs to $\mathbf{L}[G_{\mathtt{lef}}]$ and is dense in $\mathbf{DN}$ by Lemma 12.2. It follows by the genericity of Γ that $A \restriction \gamma \in \Delta$ for some $\gamma \in Z$. Then $a = (A \restriction \gamma) * C \in \Delta'$, since $a * C = A \restriction \gamma$ by Lemma 12.2. $\square$

Corollary 12.4. *The set* $\Gamma' = \{a \in \mathbf{DN} : \exists\, \gamma \in Z\, (a \leqslant (A \restriction \gamma) * C)\}$ *is* $\mathbf{DN}$*-generic over* $\mathbf{L}[G_{\mathtt{lef}}] = \mathbf{L}[G'_{\mathtt{lef}}]$. $\square$

Let us check the other intended properties of Γ' as in $(*)$.

To see that $\hat{a} \in \Gamma'$, recall that $\hat{a} \in \Gamma \cap \mathbf{DN}_{\hat{\gamma}}$. It follows by Corollary 9.2(iii) that $\hat{a} \leqslant a = A \restriction \hat{\gamma}$. However $\hat{\gamma} \in Z$ by Corollary 12.1(iii). We conclude that $\hat{a} * C \in \Gamma'$. Finally $\hat{a} * C = \hat{a} * (C \restriction \hat{\gamma}) = \hat{a}$ since $C \restriction \hat{\gamma} = \mathbf{id}[\hat{\gamma}]$ by (3). Thus $\hat{a} \in \Gamma'$, as required.

Finally prove that $t^{A}_{\mathtt{lef}}[G_{\mathtt{lef}}] = t^{A'}_{\mathtt{lef}}[G'_{\mathtt{lef}}]$, where $A = \bigvee \Gamma$ and $A' = \bigvee \Gamma'$. It suffices to show that if $\gamma \in Z$ then

$$t^{A \restriction \gamma}_{\mathtt{lef}}[G_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma}] = t^{A' \restriction \gamma}_{\mathtt{lef}}[G'_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma}] \,. \tag{5}$$

However by construction $A' \restriction \gamma = (A \restriction \gamma) * C = (A \restriction \gamma) * (C \restriction \gamma)$, and on the other hand $t^{(A \restriction \gamma)*(C \restriction \gamma)}_{\mathtt{lef}}[G] = t^{A \restriction \gamma}_{\mathtt{lef}}[t^{C \restriction \gamma}_{\mathtt{lef}}[G]]$ for all G by Lemma 8.1, therefore (5) is equivalent to

$$t^{A \restriction \gamma}_{\mathtt{lef}}[G_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma}] = t^{A \restriction \gamma}_{\mathtt{lef}}[t^{C \restriction \gamma}_{\mathtt{lef}}[G'_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma}]] \,,$$

which obviously follows from

$$G_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma} = t^{C \restriction \gamma}_{\mathtt{lef}}[G'_{\mathtt{lef}} \cap \mathbf{LS}_{\gamma}] \,,$$

and this is a corollary of the equality $G_{\mathtt{lef}} = t^{C}_{\mathtt{lef}}[G'_{\mathtt{lef}}]$ in (2) by Lemma 6.4(i)(b).

□ (Theorem 11.3)

This also completes the proof of Theorem 1.1 (see the end of Section 11).

□ (Theorem 1.1)

13 Conclusive remarks

Question 13.1. Is Theorem 1.1 true for ***arbitrary*** sets $\mathscr{X}$, not necessarily sets of reals? In this general case, the proof given above fails in the proof of Theorem 10.4, since it is not true anymore that $U_n \subseteq \mathbf{H}\Omega$ and $\phi(U_n) = U_n$. □

It follows from Theorem 1.1 that, in the Solovay model, any OD set $\mathscr{X}$ of sets of reals containing non-OD elements is *uncountable*. If moreover $\mathscr{X}$ is a set **of reals** then in fact $\mathscr{X}$ contains a perfect subset and hence has cardinality $\mathfrak{c}$ by a profound theorem in [8]. Does this stronger result reasonably generalize to sets of sets of reals and more complex sets?

Conjecture 13.2. It is true in the Solovay model that if $\mathscr{X}$ is an OD set then

(I) if $\mathscr{X}$ contains only OD elements then it is OD-wellorderable;

(II) if $\mathscr{X}$ contains only ROD elements, among them at leat one non-OD element, then $\mathscr{X}$ includes a ROD-image of the continuum 2^{ω};

(III) if $\mathscr{X}$ contains a non-ROD element then $\mathscr{X}$ has cardinality $\geq 2^{\mathfrak{c}}$.

The set of all **LS**-generic sets over **L** is a less trivial example of a set of type (III) in the Solovay model. □

A proof of (III) would be an alternative (and perhaps simpler) proof of Theorem 1.1 of this paper.

It remains to note that Caicedo and Ketchersid [1] obtained a somewhat similar trichotomy result in in a strong determinacy assumption.

References

[1] Andrés Eduardo Caicedo and Richard Ketchersid. A trichotomy theorem in natural models of AD^+. In *Set theory and its applications. Annual Boise extravaganza in set theory, Boise, ID, USA, 1995–2010*, pages 227–258. Providence, RI: American Mathematical Society (AMS), 2011.

[2] Ali Enayat. On the Leibniz-Mycielski axiom in set theory. *Fundam. Math.*, 181(3):215–231, 2004.

[3] Thomas Jech. *Set theory.* Berlin: Springer, the third millennium revised and expanded edition, 2003.

[4] Ronald Jensen. Definable sets of minimal degree. Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968, pp. 122-128, 1970.

[5] V. Kanovei and V. Lyubetsky. A countable definable set of reals containing no definable elements. *ArXiv e-prints, 1408.3901*, August 2014.

[6] Vladimir Kanovei. An Ulm-type classification theorem for equivalence relations in Solovay model. *J. Symb. Log.*, 62(4):1333–1351, 1997.

[7] Vladimir Kanovei and Vassily Lyubetsky. A definable E_0 class containing no definable elements. *Arch. Math. Logic*, 54(5-6):711–723, 2015.

[8] R.M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. Math. (2)*, 92:1–56, 1970.

[9] Jacques Stern. On Lusin's restricted continuum problem. *Ann. Math. (2)*, 120:7–37, 1984.

Index