A generic property of Solovay's set Σ

Vladimir Kanovei[*] Vassily Lyubetsky[†]

November 2, 2016

Abstract

We prove that Solovay's set Σ is generic over the ground model via a forcing notion whose order relation $\subseteq$-extends the given order relation.

1 Introduction

Solovay's paper *A model of set theory in which every set of reals is Lebesgue measurable* [6] belongs to the classics of forcing. Its main reasult is the construction of the said model, and a related model in which only definable sets are claimed to be Lebesgue measurable, but the axiom of choice holds, unlike the titular model, where it fails by necessity. During the course of the paper, Solovay invented several cornerstone methods in the forcing practice. One of those is an "important lemma" of [6, Section 4.4], which asserts, roughly, that a generic extension of a ground model also is a generic extension of any intermediate submodel via a forcing notion Σ which is a subset of the original forcing notion $\mathbb{P}$.

We prove in this note (Theorem 3), that, in the context of the Solovay construction, the set Σ itself is generic over the ground model, via a forcing notion closely related to $\mathbb{P}$ in the sense that its domain is equal to the domain of the original forcing $\mathbb{P}$ but the partial order relation $\subseteq$-extends the given partial order relation $\leq_{\mathbb{P}}$.

As an application, we present a short proof of the following result (Theorem 4): any subextension of a Cohen-generic extension is a Cohen-generic extension itself (or else trivial), and the whole extension is a Cohen-generic extension of the subextension (or else trivial). This is a folklore result known probably from the first years of forcing, but in fact it does not seem to have ever been published with proof.

[*]IITP RAS and MIIT, Moscow, Russia, kanovei@googlemail.com — contact author.
[†]IITP RAS, Moscow, Russia, lyubetsk@iitp.ru

2 The intermediate forcing Σ

The exact content of Solovay's result is as follows:

Proposition 1. *Suppose that $\mathbb{P} \in \boldsymbol{V}$ is a forcing notion in the set universe $\boldsymbol{V}$, $G \subseteq \mathbb{P}$ a $\mathbb{P}$-generic filter over $\boldsymbol{V}$, $t \in \boldsymbol{V}$ a $\mathbb{P}$-name, $X = t[G] \subseteq \boldsymbol{V}$*[1]*. Then there is a set $\Sigma = \Sigma(X, t) \in \boldsymbol{V}[G]$, $\Sigma \subseteq \mathbb{P}$ such that*

- (I) *Σ is closed weakwards in $\mathbb{P}$, so that if $q \in \Sigma$, $p \in \mathbb{P}$, and $p \leq q$, then $q \in \Sigma$;*[2]
- (II) *$\boldsymbol{V}[\Sigma] = \boldsymbol{V}[X]$;*
- (III) *$G \subseteq \Sigma$ and G is Σ-generic over $\boldsymbol{V}[X]$;*
- (IV) *therefore $\boldsymbol{V}[G]$ is a Σ-generic extension of $\boldsymbol{V}[X] = \boldsymbol{V}[\Sigma]$;*
- (V) *if a set $G' \subseteq \Sigma$ is Σ-generic over $\boldsymbol{V}[X]$ then G' is $\mathbb{P}$-generic over $\boldsymbol{V}$ and still $t[G'] = X$.* ←st5

Proof (sketch, see detailed arguments in [6], 4.4)**.** Let $\Sigma = \mathbb{P} \smallsetminus \bigcup_{\xi<\vartheta} A_\xi$, where the sequence of sets $A_\xi \subseteq \mathbb{P}$ is defined in $\boldsymbol{V}[G]$ as follows:

1. (1) A_0 consists of all conditions $p \in \mathbb{P}$ which either force $\breve{x} \in t$ for some $x \in \boldsymbol{V} \smallsetminus X$, or force $\breve{x} \notin t$ for some $x \in X$.[3] ←si1
2. (2) $A_{\xi+1}$ consists of all conditions $p \in \mathbb{P}$ such that there is a dense set $D \in \boldsymbol{V}$ in $\mathbb{P}$ satisfying: if $q \in D$ and $p \leq q$ then $q \in A_\xi$.
3. (3) $A_\lambda = \bigcup_{\xi<\lambda} A_\xi$ whenever λ is a limit ordinal.

To conclude, each condition $p \in A_0$ directly contradicts the assumption that t is a name for X, by (1), and this contradiction prevails by (2) and (3) for all bigger ξ in more and more indirect way. This $\subseteq$-increasing sequence of sets $A_\xi \subseteq \mathbb{P}$ stabilizes on a limit ordinal $\vartheta \in \boldsymbol{V}$, and we have got the sets $A = \bigcup_{\xi<\vartheta} A_\xi$ and $\Sigma = \mathbb{P} \smallsetminus A$. $\square$

Further studies of Grigorieff [2] and others on intermediate submodels of generic extensions demonstrated that not only $\boldsymbol{V}[G]$ is a generic extension of $\boldsymbol{V}[X]$ by (III), but $\boldsymbol{V}[X]$ itself is a generic extension of $\boldsymbol{V}$.[4] Theorem 3 below, our main result, asserts that *the set $\Sigma(X, t)$ itself is a generic filter over $\boldsymbol{V}$*, via a forcing notion closely related to $\mathbb{P}$.

[1] $t[G]$ is the *G-valuation*, also called *interpretation*, of the name t; $t[G] \in \boldsymbol{V}[G]$.

[2] We always assume that $p \leq q$ means that q is a stronger condition.

[3] If $x \in \boldsymbol{V}$ then $\breve{x}$ is a canonical name for x.

[4] See also [1, Fact 11], [3, 15.43], [4, Proposition 10.10], [5], or [7, Section 1], among other references.

3 The genericity of the set Σ

Arguing in the context of Proposition 1, we define a new order $\le_t$ on $\mathbb{P}$, which extends the original order $\le\ =\ \le_{\mathbb{P}}$, as follows: $p \le_t q$ iff q $\mathbb{P}$-forces over $\boldsymbol{V}$ that $\breve{p} \in \Sigma(\breve{t}[\underline{G}], \breve{t})$. In other words, for $p \le_t q$ it is necessary and sufficient that we have $p \in \Sigma(X, t)$ whenever $G \subseteq \mathbb{P}$ is generic over $\boldsymbol{V}$, $X = t[G]$, and $q \in G$.

Lemma 2. *$\le_t$ is a partial (pre)order relation on $\mathbb{P}$ which belongs to $\boldsymbol{V}$ and extends the given order $\le\ =\ \le_{\mathbb{P}}$, so that $\le_{\mathbb{P}}\ \subseteq\ \le_t$ (or equivalently, $p \le_{\mathbb{P}} q$ implies $p \le_t q$).*

Proof. Suppose that $p \le_t q \le_t r$. To prove $p \le_t r$, assume that $G \subseteq \mathbb{P}$ is generic over $\boldsymbol{V}$, $r \in G$, and $X = t[G]$; we have to prove that $p \in \Sigma(X, t)$.

By definition, $q \in \Sigma(X, t)$. Pick a set $G' \subseteq \Sigma(X, t)$ $\Sigma(X, t)$-generic over $\boldsymbol{V}[X]$ and containing q. Then G' is $\mathbb{P}$-generic over $\boldsymbol{V}$ and still $t[G'] = X$, by (V). Therefore $p \in \Sigma(X, t)$, because $p \le_t q$, and we are done.

Finally, suppose that $p \le q$ and prove $p \le_t q$. Assume that $G \subseteq \mathbb{P}$ is generic over $\boldsymbol{V}$, $q \in G$, and $X = t[G]$. To prove that $p \in \Sigma(X, t)$, note first of all that $q \in \Sigma(X, t)$ by (III); then $p \in \Sigma(X, t)$ by (I). □

Theorem 3 (in the assumptions of Proposition 1)**.** *Suppose that $G \subseteq \mathbb{P}$ is generic over $\boldsymbol{V}$ and $X = t[G]$. Then the set $\Sigma = \Sigma(X, t)$ itself is a generic filter over $\boldsymbol{V}$ in the forcing $\mathbb{P}_t = \langle \mathbb{P}\,; \le_t \rangle$.*

Proof. Suppose that $p \in \mathbb{P}$, $q \in \Sigma$, and $p \le_t q$. To prove $p \in \Sigma$, consider any set $G' \subseteq \Sigma$, Σ-generic over $\boldsymbol{V}[X]$ and containing q. Then G' is $\mathbb{P}$-generic over $\boldsymbol{V}$, and $t[G'] = X$, by (V). Now we have $p \in \Sigma$ since $p \le_t q$.

Prove that any two conditions $p, q \in \Sigma$ are $\le_t$-compatible in the set Σ. By genericity there is a condition $r \in G$ which forces both $\breve{p} \in \Sigma(\breve{t}[\underline{G}], \breve{t})$ and $\breve{q} \in \Sigma(\breve{t}[\underline{G}], \breve{t})$. Then by definition $p \le_t r$ and $q \le_t r$, and on the other hand $r \in \Sigma$ by (III).

Finally prove the genericity itself. Suppose that a set $D \subseteq \mathbb{P}$ is $\le_t$-dense (not necessarily dense w.r.t. the original order). Assume towards the contrary that some $p \in G$ forces $\breve{D} \cap \Sigma(\breve{t}[\underline{G}], \breve{t}) = \varnothing$. By density, there is a condition $q \in D$, $p \le_t q$.

Consider a set $G' \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\boldsymbol{V}$ and containing q, and let $X' = t[G']$. Then $q \in G' \subseteq \Sigma' = \Sigma(X', t)$, and hence $p \in \Sigma'$ by the above.

Further, consider a set $G'' \subseteq \Sigma'$, Σ'-generic over $\boldsymbol{V}[X']$ and containing p. Then G'' is $\mathbb{P}$-generic over $\boldsymbol{V}$, and $t[G''] = X'$, by (V). Thus $q \in D \cap \Sigma(t[G''], t)$ and $p \in G''$, contrary to the choice of p. □

4 Intermediate submodels of Cohen-generic extensions

Now we are getting the following for free.

Theorem 4 (forcing folklore)**.** *Assume that* $a \in 2^\omega$ *is a Cohen-generic real over the set universe* $\boldsymbol{V}$*, and* $X \in \boldsymbol{V}[a]$, $X \subseteq \boldsymbol{V}$. *Then*

(i) *either* $X \in \boldsymbol{V}$ *or* $\boldsymbol{V}[X]$ *is a Cohen-generic extension of* $\boldsymbol{V}$*;*

(ii) *either* $\boldsymbol{V}[X] = \boldsymbol{V}[a]$ *or* $\boldsymbol{V}[a]$ *is a Cohen-generic extension of* $\boldsymbol{V}[X]$*.*

Proof. Note that the Cohen forcing is countable. Therefore both Σ in Proposition 1 and $\mathbb{Q}$ in Theorem 3 are countable forcing notions. □

References

[1] G. Fuchs, J. D. Hamkins, J. Reitz. Set-theoretic geology. *Annals of Pure and Applied Logic*, 2015, 166, 4, pp. 464–501.

[2] S. Grigorieff. Intermediate submodels and generic extensions of set theory. *Ann. Math.*, 1975, 101, pp. 447–490.

[3] Thomas Jech. *Set theory*, The third millennium revised and expanded. Springer-Verlag, Berlin-Heidelberg-New York, 2003, xiii + 769 pp.

[4] Akihiro Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings*, 2nd ed. Springer, Berlin, 2003, xiii + 536 pp.

[5] V.G. Kanovei and V.A. Lyubetsky. Generalization of one construction by Solovay. *Siberian Math. J.*, 2015, 56, 6, pp. 1072–1079.

[6] R. M. Solovay. A model of set theory in which every set of reals is Lebesgue measurable. *Ann. Math.*, 1970, 92, pp. 1–56.

[7] J. Zapletal. Terminal notions in set theory. *Ann. Pure Appl. Logic*, 2001, 109, 1-2, pp. 89–116.