

On Harrington's model in which Separation holds but Reduction fails at the 3rd projective level

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Abstract

In a handwritten note of 1975, Leo Harrington sketched a construction of a model of **ZFC** (no large cardinals or anything beyond **ZFC**!) in which Π_3^1 -Separation holds but Σ_3^1 -Reduction fails. The result has never appeared in a journal or book publication except for a few of old references.
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1 Introduction

The separation property for a pointclass K , or simply K -Separation, is the assertion that any two disjoint sets X, Y in K (in the same Polish space) can be separated by a set in $K \cap K^c$, where K^c is the pointclass of complements of sets in K . The reduction property for a pointclass K , or simply K -Reduction, is the assertion that for any two sets X, Y in K (in the same Polish space) there exist *disjoint* sets $X' \subseteq X, Y' \subseteq Y$ in the same class K , such that $X' \cup Y' = X \cup Y$.

It is known classically from studies of Luzin [14, 13], Novikov [18, 19], Kuratowski [12] that Separation holds for projective classes Σ_1^1 (analytic sets) and Π_2^1 , but fails for Π_1^1 (coanalytic sets) and Σ_2^1 , while Reduction holds for Π_1^1 and Σ_2^1 , but fails for Σ_1^1 and Π_2^1 , and generally K -Reduction implies K^c -Separation by a simple argument.

As for the higher projective classes, Addison [1, 2] proved that the axiom of constructibility $\mathbf{V} = \mathbf{L}$ implies that Separation holds for projective classes Π_n^1 , $n \geq 3$, but fails for Σ_n^1 , $n \geq 3$ while Reduction holds for Σ_n^1 , $n \geq 3$, but fails for Π_n^1 , $n \geq 3$. On the other hand, by Martin [15], the axiom of projective determinacy **PD** implies that, similarly to projective level 1, Σ_n^1 -Separation

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and Π_n^1 -Reduction hold for all odd numbers $n \geq 3$, and, similarly to projective level 2, Π_n^1 -Separation and Σ_n^1 -Reduction hold for all even numbers $n \geq 4$.

Apparently not much is known on Separation and Reduction for higher projective classes in generic models. In a handwritten note [5, Part C] (1975), Leo Harrington sketched a construction of a model of **ZFC** in which Π_3^1 -Separation holds but Σ_3^1 -Reduction fails. The model was a generic extension of \mathbf{L} by means of the almost-disjoint coding of [8], with no reference to determinacy, large cardinals or anything beyond **ZFC**. The result has never appeared in a journal or book publication except for a few rather old references.¹ Another result sketched in [5], namely a model in which both Σ_3^1 -Separation and Π_3^1 -Separation fail, has recently been reproved in [9] on the base of different technique.

This paper is devoted to the first mentioned result. We prove

Theorem 1.1 (Harrington [5], Part C). *There exists a set-generic extension of \mathbf{L} , in which Π_3^1 -Separation holds but Σ_3^1 -Reduction fails.*²

In the proof, we'll follow, more or less, the flow of Harrington's arguments, filling in details and claims wherever (we find it) necessary. We'll try to preserve even Harrington wording wherever possible. Of most notable deviations, we change Harrington's Boolean-valued approach to the poset forcing approach, as we observed that the non-absoluteness of the RO operation causes problems in understanding of the behaviour of certain BAs in different models. Of notable additional details, we adjoined some amount of definitions and results related to intermediate submodels of generic extensions, necessary to fully understand the arguments but near completely avoided (or just hinted) in [5].

As a comparably rather simple application of some of Harrington's methods, we sketch the proof of claim (I) of the following result in the end of the paper.

Theorem 1.2. *If $n \geq 3$ then Σ_n^1 -Reduction holds in the following models:*

- (I) *the extension of \mathbf{L} by any number of Cohen-generic reals;*
- (II) *the extension of \mathbf{L} by any number of Solovay-random reals;*
- (III) *the Solovay model = the Levy extension of \mathbf{L} via an inaccessible cardinal.*

¹ Hinman [6, p. 230, end of Section V.3] communicates a much more general consistency result related to the principles of Separation and Reduction, absent even in [5], citing a paper of Harrington entitled "*Consistency and independence results in descriptive set theory*", to appear in *Ann. of Math.*", which has apparently never materialized. Moschovakis [17, Theorem 5B.3 on p. 214] mentions another Harrington's model, present in [5], in which Separation fails for both Π_3^1 and Σ_3^1 . Another similar reference, to Harrington's models in which Σ_n^1 -Separation and Π_n^1 -Separation both fail for a given n , see Mathias [16, p. 166], Notes on subsequent development, a comment on P 3110.

² The following is Harrington's comment to the proof of the theorem in the end of [5, Part C]. *The above proof was directly inspired by a result of Sami, namely: there is a model of **ZFC** in which $\text{Sep}(\Pi_3^1, \Delta_3^1)$ holds but $\text{Red}(\Sigma_3^1, \Sigma_3^1)$ fails for sets of reals.* (Note the lightface Σ_3^1 in the second part.) Apparently such a result has never been published by Ramez Sami. A possible source of Sami's result could be the unpublished PhD Thesis [20], but it is unavailable.

2 Almost disjoint preliminaries

Some definitions related to the almost disjoint forcing of [8].

- $\mathbb{C} = 2^{<\omega}$ (the Cohen forcing).
- Λ (the empty string) is the weakest condition in \mathbb{C} .
- $\omega^{<\omega} = \{s_j : j < \omega\}$ is a fixed recursive enumeration.
- if $f \in \omega^\omega$ then $S(f) = \{j < \omega : s_j \subset f\}$.
- **ZFC**⁻ is **ZFC** without the Power Sets axiom,
T is **ZFC**⁻ plus **V = L** and “all sets are countable”.
- **HC** = all hereditarily-countable sets.
- $\langle \xi_\alpha, n_\alpha \rangle$ is the α th element of the set $\omega_1 \times \omega$, ordered lexicographically.

Definition 2.1. Reals $f_\alpha \in \omega^\omega$ are defined in **L** by induction on $\alpha < \omega_1$. Suppose that f_γ are defined for all $\gamma < \alpha$. Let $\mathbf{L}_{\mu(\alpha)} \models \mathbf{T}$ be the smallest model containing the sequence $\gamma \mapsto f_\gamma$ of already defined reals. Let f_α be the Goedel-least real $f \in \omega^\omega$, Cohen-generic over $\mathbf{L}_{\mu(\alpha)}$ and satisfying $s_{n_\alpha} \subset f$.

If $\xi = \xi_\alpha$ and $n = n_\alpha$ then let $f_{\xi n} := f_\alpha$, hence $s_n \subset f_{\xi n}$ always holds. \square

If $F \subseteq \omega^\omega$ then **JS**(F) is the corresponding almost-disjoint forcing, which consists of all pairs $\langle u, S \rangle$, where $u \subseteq \omega$ and $S \subseteq S(F) = \{S(f) : f \in F\}$ are finite sets, ordered so that $\langle t, S' \rangle \leq \langle t', S' \rangle$ (the smaller condition is stronger) iff $t' \subseteq t$, $S' \subseteq S$, and $u \cap A = u' \cap A$ for all $A \in S'$.

- If $g \in 2^\omega$ then let $F_g = \{f_{\xi i} : \xi < \omega_1, i < \omega, g(i) = 0\}$.
- If $e \in 2^{<\omega}$ then let $F_e = \{f_{\xi i} : \xi < \omega_1, i < \text{lh}(e), e(i) = 0\}$.
- Define $T(g, a)$ iff $\forall \xi < \omega_1 \forall i \in \omega (S(f_{\xi i}) \cap a \text{ is finite} \iff g(i) = 0)$.

Lemma 2.2 (see [8]). *If $g \in 2^\omega$ in a set universe **V** then the forcing **JS**(F_g) adjoins a real $a \subseteq \omega$ satisfying $T(g, a)$.* \square

Definition 2.3. Let $Q \in \mathbf{L}$ be the forcing notion responsible for the following two-step generic extension: 1st, we extend a ground set universe **V** by a real $g \in 2^\omega$ Cohen-generic over **V**, and 2nd, we extend $\mathbf{V}[g]$ by **JS**(F_g).

Thus Q consists of all triples $p = \langle e, u, S \rangle$, where $e \in 2^{<\omega} = \mathbb{C}$ (a Cohen condition) while $\langle u, S \rangle \in \mathbf{JS}(F_e)$. The order is defined so that $p = \langle e, u, S \rangle \leq p' = \langle e', u', S' \rangle$ (p is stronger) iff $e' \subseteq e$ and $\langle u, S \rangle \leq \langle u', S' \rangle$ in $\mathbf{JS}(F_e)$. Note that $1 = \langle \Lambda, \emptyset, \emptyset \rangle \in Q$ is the largest (and weakest) element in Q .

Let $\mathcal{Q} = Q^\omega$ (a finite-support product), with the product order $\leq := \leq_{\mathcal{Q}}$; $p \leq q$ still means that p is stronger. Thus $\mathcal{Q} = \langle \mathcal{Q}; \leq \rangle$ is a forcing in **L**. Its largest (= weakest) element $\mathbf{1} \in \mathcal{Q}$ is defined by $\mathbf{1}(k) = 1, \forall k$. \square

Lemma 2.4 (definability). *The sequences $\langle f_\alpha \rangle_{\alpha < \omega_1}$ and $\langle f_{\xi n} \rangle_{\xi < \omega_1, n < \omega}$ are $\Delta_1^{\mathbf{L}^{\omega_1}}$, hence Σ_1^{HC} . The sets Q and \mathcal{Q} are $\Delta_1^{\mathbf{L}^{\omega_1}}$, hence Σ_1^{HC} . The relations of compatibility and incompatibility in Q and \mathcal{Q} are $\Delta_1^{\mathbf{L}^{\omega_1}}$, hence Σ_1^{HC} . \square*

Proof. To circumvent the naturally required \forall over the given po set in the definition of incompatibility, define the binary operation \wedge on Q as follows. If $p = \langle e, u, S \rangle$ and $p' = \langle e', u', S' \rangle$ belong to Q then put $p \wedge q = \langle e \wedge e', u \cup u', S \cup S' \rangle$, where $e \wedge e' = e$, or $= e'$, or $= \Lambda$ (the empty string) in cases resp. $e' \subseteq e$, $e \subseteq e'$, or e, e' are incomparable in $\mathbb{C} = 2^{<\omega}$. Extend \wedge to \mathcal{Q} componentwise. Then, both in Q and in \mathcal{Q} , conditions p, q are incompatible, in symbol $p \perp q$, iff $p \wedge q \not\leq p$ or $p \wedge q \not\leq q$. This yields the result required. \square

Remark 2.5. (A) The forcing \mathcal{Q} adjoins sequences of the form $A = \langle g_n, a_n \rangle_{n < \omega}$, where each pair $\langle g_n, a_n \rangle$ is Q -generic, hence $g_n \in 2^\omega$ is Cohen-generic, $a_n \subseteq \omega$, and $T(g_n, a_n)$ holds. If $G \subseteq \mathcal{Q}$ is generic over a set universe \mathbf{V} , and $A = A[G] = \langle g_n[G], a_n[G] \rangle_{n < \omega}$ is the corresponding sequence as above, then $\mathbf{V}[G] = \mathbf{V}[A[G]]$.

(B) Any A of such a form can be converted to a real $r(A) \subseteq \omega$ by means of any recursive bijection between $(2^\omega \times \mathcal{P}(\omega))^\omega$ and $\mathcal{P}(\omega)$, thus essentially \mathcal{Q} adds a real, so that if $G \subseteq \mathcal{Q}$ is generic then $r(A[G]) \subseteq \omega$ and $\mathbf{V}[G] = \mathbf{V}[r(A[G])]$.

(C) The forcing notions $\mathbf{JS}(F_g)$, Q , and any finite-support product of Q , in particular $\mathcal{Q} = Q^\omega$ and any \mathcal{Q}^λ , satisfy CCC, see e.g. [8, Lemma 1 in 4.6]. \square

The next lemma is established in [8] in a somewhat different but pretty similar case, as a theorem in Section 4.8, pp. 95–97, so we skip the proof.

Lemma 2.6. *If $g \in 2^\omega$ is Cohen-generic over \mathbf{L} and $B = \langle \langle g_n, a_n \rangle \rangle_{n < \omega}$ is \mathcal{Q} -generic over $\mathbf{L}[g]$ then it holds in $\mathbf{L}[g, B]$ that $\neg \exists a T(g, a)$. \square*

Lemma 2.7. *\mathcal{Q} is homogeneous in the sense that if $p, q \in \mathcal{Q}$ then there is a total order automorphism h of \mathcal{Q} such that p and $h(q)$ are compatible. Therefore*

- (i) *if φ is a formula with names of elements of the ground universe \mathbf{V} as parameters, and some $p \in \mathcal{Q}$ forces φ then \mathcal{Q} (i.e., every $q \in \mathcal{Q}$) forces φ ;*
- (ii) *if $\varphi(\cdot)$ is a formula with names of elements \mathbf{V} as parameters, t is a \mathcal{Q} -name, and some $p \in \mathcal{Q}$ forces $\varphi(t)$, then there is another \mathcal{Q} -name t' such that \mathcal{Q} forces $\varphi(t')$ and p forces $t = t'$.*

Proof. As the supports $|p|, |q| \subseteq \omega$ are finite, there is a permutation π of ω such that the π -image of $|p|$ does not intersect $|q|$. Such a π induces h as required. Claims (i) and (ii) are well-known consequences of the homogeneity. \square

3 On intermediate models

Given a forcing notion $P = \langle P; \leq \rangle$ in a ground set universe \mathbf{V} , if a set $X \subseteq \mathbf{V}$ belongs to a P -generic extension $\mathbf{V}[G]$ of \mathbf{V} , then the submodel $\mathbf{V}[X] \subseteq \mathbf{V}[G]$ is a generic extension of \mathbf{V} . (But it is not asserted that the set X itself is generic over \mathbf{V} !) This issue has been exhaustively studied in terms of boolean-valued forcing (see *e.g.* Lemma 69 in [7]), which we avoid here. Instead we make use of the classical Σ -construction by Solovay [21], rendered here only for the case $P = \mathcal{Q}$ and $X \subseteq \mathbf{V}$. (See [22, Section 1] or [11] for the treatment in the case when X is not a subset of the ground set universe \mathbf{V} .) Basically, the results below hold for any $P \in \mathbf{L}$, and if $P \notin \mathbf{L}$ then the results also hold with P as a uniform parameter.

Definition 3.1 (Solovay [21]). Assume that $t \in \mathbf{V}$, $t \subseteq \mathcal{Q} \times \mathbf{V}$. (\mathbf{V} being a ground set universe.) Let $X \subseteq \mathbf{V}$ be a set in a generic extension of \mathbf{V} . We define $\Sigma(X, t) = \mathcal{Q} \setminus \bigcup_{\alpha < \vartheta} W_\alpha$, where the increasing sequence of sets $W_\alpha \subseteq \mathcal{Q}$ is defined in $\mathbf{V}[X]$ by induction, and an ordinal ϑ is determined in the course of construction.

- (1) W_0 consists of all conditions $p \in \mathcal{Q}$ such that **either** there is a set $x \in X$ such that p is incompatible in \mathcal{Q} with any condition q satisfying $\langle q, x \rangle \in t$, **or** there is $x \notin X$ and a weaker condition $q \geq p$ such that $\langle q, x \rangle \in t$.
- (2) $W_{\alpha+1}$ consists of all conditions $p \in \mathcal{Q}$ such that there is a dense set $D \in \mathbf{V}$, $D \subseteq \mathcal{Q}$ satisfying: if $q \in D$ and $q \leq p$ then $q \in W_\alpha$.
- (3) If λ is limit then $W_\lambda = \bigcup_{\alpha < \lambda} W_\alpha$.

Note that $\alpha < \beta \implies W_\alpha \subseteq W_\beta$, hence there is an ordinal ϑ satisfying $W_\vartheta = W_{\vartheta+1}$, and then $W_\xi = W_\vartheta$ for all $\xi > \vartheta$. Finally, let $W = \bigcup_{\alpha < \vartheta} W_\alpha$. The set $\Sigma(X, t) = \mathcal{Q} \setminus W$ contains all conditions p which, roughly speaking, are compatible with the assumption that $X = t[G]$ for a \mathcal{Q} -generic set G containing p . \square

A set $D \subseteq \mathcal{Q}$ is *dense* iff $\forall p \in \mathcal{Q} \exists q \in D (q \leq p)$, and *open dense* if in addition $(q \in D \wedge q \leq p) \implies p \in D$. The set $t[G] = \{x : \exists p \in G (\langle p, x \rangle \in t)\}$ is the *G-valuation* of t . The next lemma evaluates the length ϑ of the construction of 3.1.

Lemma 3.2. *Under the assumptions of 3.1, $\vartheta \leq \omega_1$.*

Proof. To prove $W_{\omega_1+1} = W_{\omega_1}$, let $p \in \mathcal{Q}$, hence there is a dense set $D \in \mathbf{V}$, $D \subseteq \mathcal{Q}$, satisfying $(q \in D \wedge q \leq p) \implies q \in W_{\omega_1}$. The set

$$D' = \{q' \in D : q' \leq p \vee q', p \text{ are incompatible}\}$$

is still dense and satisfies (*) $(q' \in D' \wedge q' \leq p) \implies q' \in W_{\omega_1}$.

Let $A \subseteq D'$ be a maximal antichain; A is countable by 2.5(C), hence there is $\alpha < \omega_1$ such that $(\dagger) A \cap W = A \cap W_\alpha$. By the maximality of A , the set

$$D'' = \{q'' \in \mathcal{Q} : \exists r \in A (q'' \leq r)\}$$

is dense, and even *open* dense. We claim that $(\ddagger) (q'' \in D'' \wedge q'' \leq p) \implies q'' \in W_\alpha$; this implies $p \in W_\alpha \subseteq W_{\omega_1}$, ending the proof of the lemma. Thus prove (\ddagger) .

By definition, $q'' \leq r$ for some $r \in A$. Thus r, p are compatible. Therefore, as $A \subseteq D'$, we have $r \leq p$. To conclude, $r \leq p$ and $r \in D'$. It follows that $r \in W_{\omega_1}$ by $(*)$, hence $r \in W_\alpha$ by (\dagger) . It follows that $q'' \in W_\alpha$. (Indeed, by induction, each set W_α satisfies $q'' \leq r \in W_\alpha \implies q'' \in W_\alpha$.) As required. \square

Theorem 3.3 (Solovay [21]). *Under the assumptions of 3.1, suppose that a set $G \subseteq \mathcal{Q}$ is \mathcal{Q} -generic over \mathbf{V} , and $X = t[G]$. Then*

- (i) $G \subseteq \Sigma = \Sigma(X, t)$ and $X = t[\Sigma]$ — hence $\mathbf{V}[\Sigma] = \mathbf{V}[X]$,
- (ii) G is Σ -generic over the intermediate model $\mathbf{V}[\Sigma] = \mathbf{V}[X] \subseteq \mathbf{V}[G]$ — hence $\mathbf{V}[G]$ is a set generic extension of $\mathbf{V}[X]$;
- (iii) if $G' \subseteq \Sigma$ is Σ -generic over $\mathbf{V}[\Sigma] = \mathbf{V}[X]$ then G' is \mathcal{Q} -generic over \mathbf{V} and $t[G'] = X$. \square

Corollary 3.4. *Under the assumptions of Theorem 3.3, let $\varphi(\cdot)$ be a formula with \mathcal{Q} -names for sets in \mathbf{V} allowed as parameters. Then $\varphi(X)$ is true in $\mathbf{V}[X]$ iff there is a condition $p \in \Sigma$ that \mathcal{Q} -forces $\mathbf{V}[t[\underline{G}]] \models \varphi(t[\underline{G}])$ over \mathbf{V} .*

Proof. To prove \Leftarrow , the less trivial direction, assume that a condition $p \in \Sigma$ \mathcal{Q} -forces $\mathbf{V}[t[\underline{G}]] \models \varphi(t[\underline{G}])$ over \mathbf{V} . Consider a set $G' \subseteq \Sigma$, Σ -generic over $\mathbf{V}[X]$ and containing p . Then G' is \mathcal{Q} -generic over \mathbf{V} and $t[G'] = X$. It follows that $\varphi(X)$ is true in $\mathbf{V}[X]$ by the choice of p . \square

Definition 3.5 (see [10]). Let $t \in \mathbf{V}$, $t \subseteq \mathcal{Q} \times \mathbf{V}$. Define, in \mathbf{V} , the order relation \leq_t on \mathcal{Q} as follows: $p \leq_t q$, iff p \mathcal{Q} -forces over \mathbf{V} that $\dot{q} \in \Sigma(\dot{t}[\underline{G}], \dot{t})$. Let $\mathcal{Q}_t = \langle \mathcal{Q}; \leq_t \rangle$. \square

Under the hypothesis that for any $p \in P$ there is a P -generic set $G \subseteq P$ containing p , the relation $p \leq_t q$ is equivalent to the following: if G is a set \mathcal{Q} -generic over \mathbf{V} and containing p then $q \in \Sigma(X, t)$.

The next theorem contains the main application of the orders \leq_t .

Theorem 3.6. *Suppose that $t \in \mathbf{V}$, $t \subseteq \mathcal{Q} \times \mathbf{V}$. Then*

- (i) \leq_t is a partial order relation on \mathcal{Q} satisfying $p \leq q \implies p \leq_t q$,
- (ii) if a set $G \subseteq \mathcal{Q}$ is \mathcal{Q} -generic over \mathbf{V} and $X = t[G]$, then the set $\Sigma(X, t)$ is \mathcal{Q}_t -generic over \mathbf{V} and $\mathbf{V}[X] = \mathbf{V}[\Sigma(X, t)]$,

(iii) if a set $\Sigma \subseteq \mathcal{Q}$ is \mathcal{Q}_t -generic over \mathbf{V} , and a set $G' \subseteq \Sigma$ is Σ -generic over $\mathbf{V}[\Sigma]$, then G' is \mathcal{Q} -generic over \mathbf{V} and $\Sigma \in \mathbf{V}[G']$.

Proof. Claims (i), (ii) are established in [10]. To prove (iii), suppose towards the contrary that $p \in \mathcal{Q}$ forces the negation, that is, if $\Sigma \subseteq \mathcal{Q}$ is \mathcal{Q}_t -generic over \mathbf{V} , $G' \subseteq \Sigma$ is Σ -generic over $\mathbf{V}[\Sigma]$, and $p \in G'$, then either G' is **not** \mathcal{Q} -generic over \mathbf{V} or $\Sigma \in \mathbf{V}[G']$. Consider a set $G' \subseteq \mathcal{Q}$, containing p and \mathcal{Q} -generic over \mathbf{V} . Let $X' = t[G']$. The set $\Sigma' = \Sigma(X', t)$ is \mathcal{Q}_t -generic over \mathbf{V} by (ii), G' is Σ -generic over $\mathbf{V}[\Sigma']$ by Theorem 3.3(ii), and $X' = t[G'] \in \mathbf{V}[G']$, hence $\Sigma' = \Sigma(X', t) \in \mathbf{V}[G']$ as well. Finally, $p \in G' \subseteq \Sigma'$, which contradicts the choice of p . \square

Lemma 3.7. Let $G \subseteq \mathcal{Q}$ be \mathcal{Q} -generic over \mathbf{V} , and $x \subseteq \omega$ be a real in $\mathbf{V}[G]$. Then there is a countable set $t \in \mathbf{V}$, $t \subseteq \mathcal{Q} \times \omega$ such that $x = t[G]$ and $\mathbf{V}[x]$ is a \mathcal{Q}_t -generic extension of \mathbf{V} . (But x itself is not asserted to be a generic.)

Proof. By basic forcing theory, there is a set $t \in \mathbf{V}$, $t \subseteq \mathcal{Q} \times \omega$, satisfying $x = t[G]$, and by CCC (see 2.5(C)) there is a countable such t . Now use Theorem 3.6(ii). \square

Remark 3.8. Under the assumptions of the theorem, \mathcal{Q}_t satisfies CCC (in \mathbf{V}). Indeed, as $(\leq) \subseteq (\leq_t)$, any \leq_t -antichain is a \leq -antichain as well. \square

Remark 3.9. (A) Let the *Harrington fan* $\text{HF}(\mathcal{Q})$ consist of all forcing notions of the form $\mathcal{Q}_t = \langle \mathcal{Q}; \leq_t \rangle$, where $t \subseteq \mathcal{Q} \times \omega$ is at most countable.

(B) Coming back to Remark 2.5, let $\tau \in \mathbf{L}$ be a canonical \mathcal{Q} -name of the real $r(A[G])$ as in 2.5(B), so that if G is \mathcal{Q} -generic then $\tau[G] = r(A[G])$ and hence $\mathbf{V}[G] = \mathbf{V}[\tau[G]]$, and moreover if $G \neq G'$ then $\tau[G] \neq \tau[G']$. This implies that the order \leq_τ on \mathcal{Q} coincides with \leq , but by means of a rather lengthy argument, which includes the verification of the separativity of the forcing notion \mathcal{Q} . In order to circumvent these complications, **it will be outright assumed that the partial order \leq_τ on \mathcal{Q} coincides with \leq** , for this particular \mathcal{Q} -name τ , and accordingly if a set $G \subseteq \mathcal{Q}$ is generic and $\tau[G] = r(A[G]) = \mathbf{r} \subseteq \omega$ then $\Sigma(\mathbf{r}, \tau) = G$, so 3.3(i) still holds. With this amendment, we have $\mathcal{Q} \in \text{HF}(\mathcal{Q})$.

(C) In the notation of Remark 2.5(A), let $\mathbf{c} \in \mathbf{L}$ be the canonical \mathcal{Q} -name for the set $\{k < \omega : g_0(k) = 0\}$, so that \mathcal{Q} forces that $\mathbf{c}[G] \subseteq \omega$ is Cohen-generic over the ground universe. Then the forcing $\mathcal{Q}_{\mathbf{c}} \in \text{HF}(\mathcal{Q})$ adds a Cohen real. \square

4 Absoluteness of the Σ construction

We have to consider a subtle issue related to the construction of $\Sigma(X, t)$, namely, its formal dependence of the choice of \mathbf{V} in (2) of Definition 3.1. The next lemma shows that the dependence can be eliminated in a really important case.

Lemma 4.1. *Under the assumptions of Definition 3.1, suppose that, in addition, \mathbf{V} is a set-generic extension of $\mathbf{L}[t]$. Then $\Sigma^{\mathbf{V}}(X, t) = \Sigma^{\mathbf{L}[t]}(X, t)$.*

Proof. Assume that $\Pi \in \mathbf{L}[t]$ is a forcing notion, and $\mathbf{V} = \mathbf{L}[t][H]$, where H is Π -generic over $\mathbf{L}[t]$. Then H is Π -generic over $\mathbf{L}[t][X]$ by the product forcing theorem. Prove by induction that $W_\alpha^{\mathbf{V}} = W_\alpha^{\mathbf{L}[t]}$.

It suffices to handle the inductive step $\alpha \rightarrow \alpha + 1$ in 3.1(2). Thus suppose that $W_\alpha^{\mathbf{V}} = W_\alpha^{\mathbf{L}[t]} = W_\alpha$ and prove $W_{\alpha+1}^{\mathbf{V}} = W_{\alpha+1}^{\mathbf{L}[t]}$. As $\mathbf{L}[t] \subseteq \mathbf{V}$, we have $W_{\alpha+1}^{\mathbf{L}[t]} \subseteq W_{\alpha+1}^{\mathbf{V}}$. To prove the opposite inclusion, suppose that $p_0 \in W_{\alpha+1}^{\mathbf{V}}$, and this is witnessed by a dense set $D \in \mathbf{V}$, $D \subseteq \mathcal{Q}$, as in 3.1(2). The goal is to prove $p_0 \in W_{\alpha+1}^{\mathbf{L}[t]}$. We have $D = \tau[H]$, where $\tau \in \mathbf{L}[t]$, $\tau \subseteq \Pi \times \mathcal{Q}$ (a Π -name of a subset of \mathcal{Q}). There is a condition $\pi_0 \in H$, which Π -forces, over $\mathbf{L}[t][X]$, that

$$\text{“}\tau[\underline{H}] \text{ is dense and } \forall p \in \tau[\underline{H}] (p \leq p_0 \implies p \in W_\alpha^\sim)\text{”}, \quad (\dagger)$$

where $W_\alpha^\sim = \Pi \times W_\alpha$ is the canonical Π -name for the set $W_\alpha \in \mathbf{L}[t][X]$.

We can wlog assume that $(\ddagger) \langle \pi, p \rangle \in \tau \wedge \pi' \in \Pi \wedge \pi' \leq_\Pi \pi \implies \langle \pi', p \rangle \in \tau$.

We notice that the sets $D'_1 = \{p \in \mathcal{Q} : p \leq p_0 \wedge \exists \pi \in \Pi (\pi \leq_\Pi \pi_0 \wedge \langle \pi, p \rangle \in \tau)\}$, $D'_2 = \{p \in \mathcal{Q} : p_0, p \text{ are incompatible}\}$, and $D' = D'_1 \cup D'_2$ belong to $\mathbf{L}[t]$.

We claim that D' is dense in \mathcal{Q} . Indeed let $p \in \mathcal{Q}$. If p is incompatible with p_0 then immediately $p \in D'$. If otherwise, then we can assume that $p \leq p_0$. As π_0 forces (\dagger) , there is a condition $\pi \in H$, $\pi \leq_\Pi \pi_0$, and some $p' \in \mathcal{Q}$, $p' \leq p$, such that π forces $\pi' \in \tau[\underline{H}]$ — that is, $\langle \pi, p' \rangle \in \tau$ by (\ddagger) . Then $p' \in D'$, as required.

We finally claim that if $p \in D'$ and $p \leq p_0$ then $p \in W_\alpha$. Indeed, $p \notin D'_2$, hence, $p \in D'_1$. Let this be witnessed by $\pi \in \Pi$, $\pi \leq_\Pi \pi_0$. Then π obviously forces $p \in \tau[\underline{H}]$, and hence, as π also forces (\dagger) , we conclude that $p \in W_\alpha$, as required.

Thus D' witnesses that $p_0 \in W_{\alpha+1}^{\mathbf{L}[t]}$, as required. \square

Corollary 4.2. *Under the assumptions of Lemma 4.1, if $\mathbf{V}_1 = \mathbf{L}[t, Y_1]$ and $\mathbf{V}_2 = \mathbf{L}[t, Y_2]$, where $Y_1, Y_2 \in \mathbf{V}$ and $Y_1 \cup Y_2 \subseteq \mathbf{L}[t]$, then*

- (i) $\Sigma^{\mathbf{V}_1}(X, t) = \Sigma^{\mathbf{V}_2}(X, t) = \Sigma^{\mathbf{V}}(X, t) = \Sigma^{\mathbf{L}[t]}(X, t)$.
- (ii) $\leq_t^{\mathbf{V}_1} = \leq_t^{\mathbf{V}_2} = \leq_t^{\mathbf{V}} = \leq_t^{\mathbf{L}[t]}$.

Proof. If \mathbf{V} is a set-generic extension of $\mathbf{L}[t]$ then any subextension $\mathbf{L}[t, Y]$, where $Y \in \mathbf{V}$, $Y \subseteq \mathbf{L}[t]$, is a set-generic extension of $\mathbf{L}[t]$ as well by [4] or Theorem 3.6. This implies (i) by Lemma 4.1, and then (ii) also follows by a routine argument. \square

Blanket agreement 4.3. Because of Corollary 4.2, we'll freely use the notation $\Sigma(X, t)$ and \leq_t below without reference to the ground universe; indeed, the

universes considered will always be subuniverses of a fixed generic extension of \mathbf{L} .

Corollary 4.4. *Under the assumptions of Lemma 4.1, the relation \leq_t belongs to $\mathbf{L}[t]$, and we have $\mathbf{L}[t, X] = \mathbf{L}[t, \Sigma(X, t)]$. \square*

5 Definability of the Σ construction

Consider the sets $\mathbb{E} = \{A \in \mathbf{L} : A \text{ is a maximal antichain in } \mathcal{Q}\}$ and

$$\mathbb{E} = \{\langle X, t, p \rangle : X \subseteq \omega \wedge t \in \mathbf{L}_{\omega_1}, t \subseteq \mathcal{Q} \times \omega \wedge p \in \Sigma(X, t)\}.$$

We have $\mathbb{E} \subseteq \mathbf{L}_{\omega_1}$, and also $\mathbb{E} \subseteq \mathbf{L}_{\omega_1}$ since \mathcal{Q} is CCC.

Lemma 5.1. *\mathbb{E} is $\Sigma_1^{\mathbf{L}_{\omega_1}}$.*

Proof. See Section 2 on \mathbf{ZFC}^- and \mathbf{T} . Let \mathbf{T}^+ be the theory \mathbf{ZFC}^- plus the axiom saying that every set belongs to a countable model $\mathbf{L}_\alpha \models \mathbf{T}$. We claim that the following are equivalent: (1) $A \in \mathbb{E}$,

- (2) $\exists \lambda < \omega_1 (\mathbf{L}_\lambda \models \mathbf{T}^+ \wedge A \in \mathbf{L}_\lambda \wedge A \text{ is a maximal antichain in } \mathcal{Q} \cap \mathbf{L}_\lambda)$,
- (3) $\forall \lambda < \omega_1 (\mathbf{L}_\lambda \models \mathbf{T}^+ \wedge A \in \mathbf{L}_\lambda \implies A \text{ is a maximal antichain in } \mathcal{Q} \cap \mathbf{L}_\lambda)$.

If this is established then Lemma 2.4 leads to the definability result required.

It remains to prove the claim. (1) \implies (3) is obvious. To prove (3) \implies (2), consider any countable elementary submodel \mathbf{L}_λ of \mathbf{L}_{ω_1} containing A . Then \mathbf{L}_λ is a model of \mathbf{T}^+ (as so is \mathbf{L}_{ω_1}). Thus A is a maximal antichain in $\mathcal{Q} \cap \mathbf{L}_\lambda$.

Now prove (2) \implies (1), the nontrivial implication. Suppose that $\lambda < \omega_1$, $A \in \mathbf{L}_\lambda \models \mathbf{T}^+$, and A is a maximal antichain in $\mathcal{Q} \cap \mathbf{L}_\lambda$. Then A is an antichain in \mathcal{Q} , since being antichain means that $p \wedge q \not\leq p$ or $\not\leq q$ for any $p \neq q$ in A , and \wedge is an absolute operation (see the proof of Lemma 2.4). It remains to prove that A is a maximal antichain.

Suppose to the contrary that some $p \in \mathcal{Q} \setminus \mathbf{L}_\lambda$ is incompatible with every $q \in A$. By definition there are finitely many reals of the form f_{ξ_n} with $\xi \geq \lambda$, occurring in p , so we may write $p = p(f_{\xi_0, n_0}, \dots, f_{\xi_k, n_k})$, $k < \omega$, $\xi_i \geq \lambda$ for all i . (The substitution form $p(f_{\xi_0, n_0}, \dots, f_{\xi_k, n_k})$ is naturally chosen so that $p(f_{\eta_0, n_0}, \dots, f_{\eta_k, n_k}) \in \mathcal{Q}$ for any other string of ordinals $\eta_0, \dots, \eta_k < \omega_1$.) The reals f_{ξ_n} with $\xi < \lambda$ may occur as well, but they belong to \mathbf{L}_λ and are not to be explicitly mentioned. But anyway there is an ordinal $\nu < \lambda$ such that all f_{ξ_n} with $\xi < \lambda$, occurring in p , actually satisfy $\xi < \nu$, and in addition $A \in \mathbf{L}_\nu \models \mathbf{T}$.

Note that by construction the string of reals $\langle f_{\xi_0, n_0}, \dots, f_{\xi_k, n_k} \rangle \in (\omega^\omega)^{n+1}$ is Cohen generic over \mathbf{L}_λ , hence over \mathbf{L}_ν as well. Therefore the property

$$\text{“}p(f_{\xi_0, n_0}, \dots, f_{\xi_k, n_k}) \text{ is incompatible with every } q \in A\text{”}$$

is forced over \mathbf{L}_ν , in the sense that there exist strings $e_0, \dots, e_k \in \omega^{<\omega}$, such that $e_i \subset f_{\xi_i, n_i}$, $\forall i$, and if $\langle y_0, \dots, y_n \rangle$ is Cohen generic over \mathbf{L}_ν with $e_i \subset y_i$, $\forall i$, then still $p(y_0, \dots, y_k)$ is incompatible with every $q \in A$.

It remains to note that, since $\nu < \lambda$, there exist intermediate ordinals $\eta_0, \dots, \eta_k \in \lambda \setminus \nu$ such that the reals $y_i = f_{\eta_i, n_i}$ satisfy $e_i \subset y_i$, $\forall i$. Then $\langle y_0, \dots, y_n \rangle$ is Cohen generic over \mathbf{L}_ν , hence $p' = p(y_0, \dots, y_k)$ is incompatible with each $q \in A$ by the above. And on the other hand $p' \in \mathcal{Q} \cap \mathbf{L}_\lambda$, a contradiction. \square

Lemma 5.2. Σ is definable in $\text{HC} =$ hereditarily-countable sets by a conjunction of the Σ_1 formula “ $t \in \mathbf{L}_{\omega_1}$ ” and a Π_1 formula $\sigma(X, t, p)$.

Proof. Assume that $X \subseteq \omega$, $t \in \mathbf{L}$, $t \subseteq \mathcal{Q} \times \omega$, and $p \in \mathcal{Q}$. Let a *maximal p -antichain* be any maximal antichain $A \subseteq \mathcal{Q}$ such that if $q \in A$ then either q, p are incompatible or $q \leq p$. Come back to the sets W_α in 3.1. As $t \in \mathbf{L}$, Lemma 4.1 allows us to consider only dense sets $D \in \mathbf{L}$ in 3.1(2).

If $\alpha < \omega_1$ then let an α -ladder be any sequence $\langle W'_\xi, \mathcal{A}_\xi \rangle_{\xi \leq \alpha}$ such that each $W'_\xi \subseteq \mathcal{Q}$ is at most countable, each $\mathcal{A}_\xi \subseteq \mathbb{E}$ is at most countable, and

- (1') $W'_0 \subseteq W_0$ (the latter defined as in 3.1(1));
- (2') if $\xi + 1 \leq \alpha$ and $p \in W'_{\xi+1}$ then there is a maximal p -antichain $A \in \mathcal{A}_\xi$ such that $r \in W'_\xi$ holds for all $r \in A$, $r \leq p$;
- (3') if ξ is limit then $W'_\xi = \bigcup_{\eta < \xi} W'_\eta$.

We assert that if $p \in \mathcal{Q}$ and $\alpha < \omega_1$ then:

- (*) $p \in W_\alpha$ iff there is an α -ladder $\langle W'_\xi, \mathcal{A}_\xi \rangle_{\xi \leq \alpha}$ such that $p \in W'_\alpha$.

If this is established then $\bigcup_{\alpha < \omega_1} W_\alpha$ becomes a Σ_1^{HC} set. (The incompatibility in 3.1(1), to which (1') refers, is handled by Lemma 2.4.) Then $\Sigma(X, t) = \mathcal{Q} \setminus \bigcup_{\alpha < \omega_1} W_\alpha$ becomes a Π_1^{HC} set, and the lemma easily follows. Note that the union needn't exceed ω_1 by Lemma 3.2.

In the direction \Leftarrow of (*), we prove by induction that $W'_\xi \subseteq W_\xi$. The nontrivial step is (2'). Let $p \in W'_{\xi+1}$, and let this be witnessed by $A \in \mathcal{A}_\xi$ in the sense of (2'). As $A \in \mathbb{E}$, the set $D = \{q \in \mathcal{Q} : \exists r \in A (q \leq r)\}$ is dense and $D \in \mathbf{L}$. It remains to prove that if $q \in D$, $q \leq p$, then $q \in W'_\xi$, see 3.1(2). Indeed, by construction there is $r \in A$ with $q \leq r$. But A is a p -antichain, hence either r, p are incompatible or $r \leq p$. However $q \leq p$, $q \leq r$, excluding the ‘either’ case. Thus $r \leq p$. It follows by the choice of A that $r \in W'_\xi$. Thus $r \in W_\xi$ by the inductive hypothesis. We conclude that $q \in W_\xi$ as well, since $q \leq p$.

We prove \Rightarrow in (*) by induction on α . The nontrivial step is still (2'). Thus suppose that $p \in W_{\alpha+1}$, and let this be witnessed by a dense set $D \in \mathbf{L}$,

$D \subseteq \mathcal{Q}$ in the sense of 3.1(2). Let D' consist of all $q \in D$ such that $q \leq p$ or q is incompatible with p ; then $D' \in \mathbf{L}$ is still dense and witnesses $p \in W_{\alpha+1}$. Consider a maximal antichain $A \subseteq D'$ in \mathbf{L} , so that $A \in \mathbb{E}$. Then A is a maximal p -antichain by the definition of D' , and if $q \in A$, $q \leq p$ then $q \in W_\alpha$ by the choice of D , hence, by the inductive hypothesis, there is an α -ladder $\langle W_\xi(q), \mathcal{A}_\xi(q) \rangle_{\xi \leq \alpha}$ satisfying $q \in W_\xi(q)$. To accomplish the proof of \implies in (*), define an $(\alpha + 1)$ -ladder by $W'_\xi = \bigcup_{q \in A, q \leq p} W_\xi(q)$ and $\mathcal{A}'_\xi = \bigcup_{q \in A, q \leq p} \mathcal{A}_\xi(q)$ for $\xi \leq \alpha$, and separately $W'_{\alpha+1} = \{p\}$ and $\mathcal{A}'_{\alpha+1} = \{A\}$. \square

6 The model

Here we start the proof of Theorem 1.1. The key idea of [5, Part C] consists in making use of the ω_1 -long iterated extension of \mathbf{L} , where the forcing at each step is the finite-support product of all elements of the fan $\text{HF}(\mathcal{Q})$ defined within the extension obtained at the previous step of the iteration. We are going to define such an extension as a submodel of a more elementary background set universe \mathfrak{M} .

To define the latter, we consider the forcing notion $\mathcal{Q}^{\omega_1 \times \omega_1} \in \mathbf{L}$ (finite support). As $\mathcal{Q} = Q^\omega$, the forcing $\mathcal{Q}^{\omega_1 \times \omega_1}$ is order-isomorphic to Q^{ω_1} , of course. The forcing $\mathcal{Q}^{\omega_1 \times \omega_1} \in \mathbf{L}$ naturally adjoins an array of mutually \mathcal{Q} -generic sets $G^{\nu\gamma} \subseteq \mathcal{Q}$, $\nu, \gamma < \omega_1$ to \mathbf{L} . We let \mathfrak{M} , *the background model*, be the extension $\mathbf{L}[\langle G^{\nu\gamma} \rangle_{\nu, \gamma < \omega_1}]$.

If $u \in \mathbf{L}$, $u \subseteq \omega_1 \times \omega_1$, then let $\mathfrak{M} \upharpoonright_u = \mathbf{L}[\langle G^{\mu\delta} \rangle_{\langle \mu, \delta \rangle \in u}]$. In particular, if $\nu, \gamma < \omega_1$ then put $\mathfrak{M}_\nu = \mathfrak{M} \upharpoonright_{\nu \times \omega_1} = \mathbf{L}[\langle G^{\mu\delta} \rangle_{\mu < \nu, \delta < \omega_1}]$ and $\mathfrak{M}_{\nu\gamma} = \mathfrak{M} \upharpoonright_{\nu \times \gamma}$.

Lemma 6.1 (by 2.5(C)). *\mathfrak{M} preserves all \mathbf{L} -cardinals. If $x \in \mathfrak{M}$ is a real then x belongs to some $\mathfrak{M}_{\nu\gamma}$, $\nu, \gamma < \omega_1$. Every $\mathfrak{M}_{\nu\gamma}$ is a \mathcal{Q} -generic extension of \mathbf{L} .* \square

The actual model for Theorem 1.1 will be a certain subuniverse $\mathfrak{N} \subseteq \mathfrak{M}$.

Definition 6.2. Arguing in \mathfrak{M} , we define, by transfinite induction on ν , an array of countable \mathcal{Q} -names $\mathbf{t}_{\nu\gamma} \subseteq \mathcal{Q} \times \omega$, such that

- (1) if $\nu, \gamma < \omega_1$ then $\langle \mathbf{t}_{\mu\delta} \rangle_{\mu \leq \nu, \delta < \omega_1} \in \mathfrak{M}_\nu$, and $\langle \mathbf{t}_{\mu\delta} \rangle_{\mu \leq \nu, \delta < \gamma} \in \mathfrak{M}_{\nu\gamma}$, so that each particular $\mathbf{t}_{\nu\gamma}$ belongs to $\mathfrak{M}_{\nu, \gamma+1}$.

We also define *derived objects*, namely

- (2) reals $\mathbf{r}_{\nu\gamma} = \mathbf{t}_{\nu\gamma}[G^{\nu\gamma}] \subseteq \omega$, sets $\Sigma_{\nu\gamma} = \Sigma(\mathbf{r}_{\nu\gamma}, \mathbf{t}_{\nu\gamma}) \subseteq \mathcal{Q}$, forcing notions $\mathcal{Q}_{\nu\gamma} = \langle \mathcal{Q}; \leq_{\mathbf{t}_{\nu\gamma}} \rangle$, and
- (3) models $\mathfrak{N}_\nu = \mathbf{L}[\langle \mathbf{t}_{\mu\delta}, \mathbf{r}_{\mu\delta} \rangle_{\mu < \nu, \delta < \omega_1}]$, $\mathfrak{N}_{\nu\gamma} = \mathbf{L}[\langle \mathbf{t}_{\mu\delta}, \mathbf{r}_{\mu\delta} \rangle_{\mu < \nu, \delta < \gamma}]$ ($\gamma < \omega_1$);

which, by construction and the results of Section 3, satisfy the following:

- (4) $\mathbf{r}_{\nu\gamma} \in \mathfrak{M}_{\nu+1, \gamma+1}[G^{\nu\gamma}]$, $\mathcal{Q}_{\nu\gamma} = \langle \mathcal{Q}; \leq_{t_{\nu\gamma}} \rangle$ is a forcing notion in $\mathfrak{M}_{\nu, \gamma+1}$, and $\Sigma_{\nu\gamma} \subseteq \mathcal{Q}$ is a set $\mathcal{Q}_{\nu\gamma}$ -generic over $\mathfrak{M}_{\nu, \gamma+1}$ and over \mathfrak{M}_ν , and satisfying $\mathbf{L}[t_{\nu\gamma}, \Sigma_{\nu\gamma}] = \mathbf{L}[t_{\nu\gamma}, \mathbf{r}_{\nu\gamma}]$, by Corollary 4.4;
- (5) if $\nu, \gamma < \omega_1$ then the arrays $\langle \leq_{t_{\mu\delta}}, \mathcal{Q}_{\mu\delta} \rangle_{\mu \leq \nu, \delta < \omega_1}$, $\langle \Sigma_{\mu\delta}, \mathbf{r}_{\mu\delta} \rangle_{\mu < \nu, \delta < \omega_1}$ belong to \mathfrak{M}_ν , and the arrays $\langle \leq_{t_{\mu\delta}}, \mathcal{Q}_{\mu\delta} \rangle_{\mu \leq \nu, \delta < \gamma}$, $\langle \Sigma_{\mu\delta}, \mathbf{r}_{\mu\delta} \rangle_{\mu < \nu, \delta < \gamma}$ belong to $\mathfrak{M}_{\nu\gamma}$;
- (6) therefore $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$ and $\mathfrak{N}_{\nu\gamma} \subseteq \mathfrak{M}_{\nu\gamma}$, $\forall \gamma$.

Now the step. Suppose that $\nu < \omega_1$ and all sets $t_{\mu\delta} \subseteq \mathcal{Q} \times \omega$ ($\mu < \nu$, $\delta < \omega_1$) are defined, so that $\langle t_{\mu\delta} \rangle_{\mu < \nu, \delta < \omega_1} \in \mathfrak{M}_\nu$, and if $\gamma < \omega_1$ then $\langle t_{\mu\delta} \rangle_{\mu < \nu, \delta < \gamma} \in \mathfrak{M}_{\nu\gamma}$; this is slightly weaker than (1) since does not include ν itself. Then $\mathcal{Q}_{\mu\gamma}$, $\Sigma_{\mu\gamma}$, $\mathbf{r}_{\mu\gamma}$, \mathfrak{N}_ν , $\mathfrak{N}_{\nu\gamma}$ as in (2), (3) are defined as well. The goal is to define $t_{\nu\gamma}$, $\gamma < \omega_1$.

Note that $\mathfrak{M}_\nu = \mathbf{L}[\langle G^{\mu\gamma} \rangle_{\mu < \nu, \gamma < \omega_1}]$ is a \mathcal{Q}^{ω_1} -generic extension of \mathbf{L} , hence GCH is true in \mathfrak{M}_ν , and hence in $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$ as well. Therefore it holds in $\mathfrak{N}_{\nu\xi}$ that there exist only \aleph_1 -many countable sets $t \subseteq \mathcal{Q} \times \omega$; let $\langle t_{\nu\eta}^\xi \rangle_{\eta < \omega_1}$ be the Gödel-least (relative to $\langle t_{\mu\delta}, \mathbf{r}_{\mu\delta} \rangle_{\mu < \nu, \delta < \xi}$ as the parameter) enumeration of all such t in $\mathfrak{N}_{\nu\xi}$.

Let $\Omega = \{\gamma + 1 : \gamma < \omega_1\}$ (successor ordinals). Fix a bijection $\mathbf{b} : \Omega$ onto $\omega_1 \times \omega_1$, $\mathbf{b} \in \mathbf{L}$, satisfying $\mathbf{b}^{-1}(\xi, \eta) > \max\{\xi, \eta\}$ for all $\xi, \eta < \omega_1$. If $\gamma = \mathbf{b}^{-1}(\xi, \eta) \in \Omega$ then let $t_{\nu\gamma} = t_{\nu\eta}^\xi$. Put $t_{\nu\gamma} = \emptyset$ for all limit $\gamma < \omega_1$. The enumeration $\langle t_{\nu\gamma} \rangle_{\gamma < \omega_1}$ involves all at most countable sets $t \in \mathfrak{N}_\nu$, $t \subseteq \mathcal{Q} \times \omega$, the whole sequence $\langle t_{\nu\gamma} \rangle_{\gamma < \omega_1}$ belongs to \mathfrak{N}_ν , and if $\gamma < \omega_1$ then the subsequence $\langle t_{\nu\delta} \rangle_{\delta < \gamma}$ belongs to $\mathfrak{N}_{\nu\gamma}$.

This ends the inductive step.

After the inductive construction is accomplished, we let

$$\mathfrak{N} \upharpoonright_u = \mathbf{L}[\langle t_{\nu\gamma}, \mathbf{r}_{\nu\gamma} \rangle_{\langle \nu, \gamma \rangle \in u}] = \mathbf{L}[\langle t_{\nu\gamma}, \Sigma_{\nu\gamma} \rangle_{\langle \nu, \gamma \rangle \in u}], \quad \text{for all } u \subseteq \omega_1 \times \omega_1,$$

and then $\mathfrak{N} = \mathfrak{N} \upharpoonright_{\omega_1 \times \omega_1}$, and, equivalently to (3), $\mathfrak{N}_\nu = \mathfrak{N} \upharpoonright_{\nu \times \omega_1}$ and $\mathfrak{N}_{\nu\gamma} = \mathfrak{N} \upharpoonright_{\nu \times \gamma}$ for $\nu, \gamma < \omega_1$. We have by construction:

- (7) the whole sequence $\langle t_{\nu\gamma} \rangle_{\gamma < \omega_1}$ belongs to \mathfrak{N}_ν , and if $\gamma < \omega_1$ then the subsequence $\langle t_{\nu\delta} \rangle_{\delta < \gamma}$ belongs to $\mathfrak{N}_{\nu\gamma}$. \square

The next lemma explains further details.

Lemma 6.3. *Assume that $\nu < \omega_1$, $t \in \mathfrak{N}_\nu$, $t \subseteq \mathcal{Q} \times \omega$ is at most countable.*

Then there is an ordinal $\gamma < \omega_1$ such that $t = t_{\nu\gamma}$.

In this case, $\mathcal{Q}_t = \mathcal{Q}_{\nu\gamma}$, the set $\Sigma_{\nu\gamma} \in \mathfrak{N}_{\nu+1}$ is \mathcal{Q}_t -generic over \mathfrak{M}_ν , hence over the model $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$ as well, and the real $\mathbf{r}_{\nu\gamma} = t[G^{\nu\gamma}]$ satisfies $\mathbf{L}[t_{\nu\gamma}, \mathbf{r}_{\nu\gamma}] = \mathbf{L}[t_{\nu\gamma}, \Sigma_{\nu\gamma}]$ and belongs to $\mathfrak{N}_{\nu+1}$.

Proof. Recall that $G^{\nu\gamma}$ is \mathcal{Q} -generic over \mathfrak{M}_ν , hence over $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$ as well. It remains to use Theorem 3.6. \square

Remark 6.4. If $\nu < \omega_1$ then by construction the collection of all forcing notions $\mathcal{Q}_{\nu\gamma}$, $\gamma < \omega_1$, is equal to the Harrington fan $\text{HF}(\mathcal{Q})$ computed in \mathfrak{N}_ν . Thus \mathfrak{N} can be viewed as the ω_1 -long iterated $\text{HF}(\mathcal{Q})$ -generic extension of \mathbf{L} .

In particular, by 3.9(B), there is an index $\gamma < \omega_1$ such that $\mathcal{Q}_{\nu\gamma} = \mathcal{Q}$, hence, by Lemma 6.3, $\mathfrak{N}_{\nu+1}$ contains a set \mathcal{Q} -generic over \mathfrak{M}_ν and over $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$. Similarly by 3.9(C) there is an index $\gamma < \omega_1$ such that $\mathbf{t}_{\nu\gamma} = \mathbf{c} \in \mathbf{L}$ (see 3.9(C) on \mathbf{c}), and hence $\mathcal{Q}_{\nu\gamma} = \mathcal{Q}_{\mathbf{c}}$ adds a Cohen real $\mathbf{r}_{\nu\gamma} \subseteq \omega$ over \mathfrak{N}_ν . \square

7 Key lemmas

Lemma 7.1. *If $x \in 2^\omega \cap \mathfrak{N}$, then there is a set $G \in \mathfrak{N}$, \mathcal{Q} -generic over $\mathbf{L}[x]$, hence there are reals $g, a \in \mathfrak{N}$ such that the pair $\langle g, a \rangle$ is \mathcal{Q} -generic over $\mathbf{L}[x]$.*

Proof. By Lemma 6.1, x belongs to some $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$, $\nu < \omega_1$. By 6.4, the submodel $\mathfrak{N}_{\nu+1}$ contains a set \mathcal{Q} -generic set over \mathfrak{N}_ν , hence over $\mathbf{L}[x]$ as well. \square

Lemma 7.2. *Assume that $\nu, \gamma < \omega_1$, $u \in \mathbf{L}$, $u \not\subseteq \nu \times \gamma$, and $W = (\nu \times \gamma) \setminus u$. Then $\mathfrak{K} = (\mathfrak{N}|_u)[\langle G^{\mu\delta} \rangle_{\langle \mu, \delta \rangle \in W}]$ is a \mathcal{Q} -generic extension of $\mathfrak{N}|_u$, and $\mathfrak{N}_{\nu\gamma} \subseteq \mathfrak{K}$.*

Proof. Let $\mathcal{Q}(\mu, \delta) = \mathcal{Q}$, for all $\langle \mu, \delta \rangle \in W$. Note that \mathfrak{K} is a $\prod_{\langle \mu, \delta \rangle \in W} \mathcal{Q}(\mu, \delta)$ -generic extension of $\mathfrak{N}|_u$ by construction, hence essentially a \mathcal{Q}^ω -generic extension, yet \mathcal{Q}^ω is isomorphic to \mathcal{Q} as a forcing. To prove $\mathfrak{N}_{\nu\gamma} \subseteq \mathfrak{K}$, we check, by induction, that $\langle \mathbf{t}_{\kappa\delta}, \mathbf{r}_{\kappa\delta} \rangle_{\delta < \gamma} \in \mathfrak{K}$ for all $\kappa < \nu$. The induction hypothesis is $\kappa < \nu$ and $\mathfrak{N}_{\kappa\gamma} \subseteq \mathfrak{K}$, and the goal is to “effectively” prove that then $\langle \mathbf{t}_{\kappa\delta}, \mathbf{r}_{\kappa\delta} \rangle_{\delta < \gamma} \in \mathfrak{K}$. We first remind that $\langle \mathbf{t}_{\kappa\delta} \rangle_{\delta < \gamma} \in \mathfrak{N}_{\kappa\gamma}$ by (7) of Definition 6.2. Now, for any particular $\delta < \gamma$, if $\langle \kappa, \delta \rangle \in u$ then $\mathbf{r}_{\kappa\delta}$ belongs to $\mathfrak{N}|_u$, hence, to \mathfrak{K} as well, while if $\langle \kappa, \delta \rangle \in W$ then $G^{\kappa\delta}$ belongs to \mathfrak{K} , hence $\mathbf{r}_{\kappa\delta} = \mathbf{t}_{\kappa\delta}[G^{\kappa\delta}] \in \mathfrak{K}$, as required. \square

Definition 7.3 (autonomous sets). A set $x \in \mathfrak{N}$, $x \subseteq \mathbf{L}$, is *autonomous* if there is a countable set $u \in \mathbf{L}$, $u \subseteq \omega_1 \times \omega_1$ such that $\mathbf{L}[x] = \mathfrak{N}|_u$. \square

Lemma 7.4 (in \mathfrak{N}). *If z is an autonomous real and $t \in \mathbf{L}[z]$, $t \subseteq \mathcal{Q} \times \mathbf{L}[z]$, then there is a real b such that $\langle z, b \rangle$ is autonomous and $\mathbf{L}[z, b] = \mathbf{L}[z][\Sigma]$, where $b = t[\Sigma]$ and $\Sigma \subseteq \mathcal{Q}$, Σ is \mathcal{Q}_t -generic over $\mathbf{L}[z]$.*

Proof. Let a countable $u \in \mathbf{L}$, $u \subseteq \omega_1 \times \omega_1$ witness that z is autonomous. Then $u \subseteq \nu \times \omega_1$ for some $\nu < \omega_1$, and $t \in \mathfrak{N}_\nu$. By Lemma 6.3, $t = \mathbf{t}_{\nu\gamma}$ for some γ , and then $b = \mathbf{r}_{\nu\gamma}$ is as required. To see that $\langle z, b \rangle$ is autonomous note that $\mathbf{L}[z, b] = \mathfrak{N}|_v$, where $v = u \cup \{\langle \nu, \gamma \rangle\}$. \square

Lemma 7.5 (in \mathfrak{N}). *Let $x \in 2^\omega$ and $\varphi(\cdot)$ be a Σ_3^1 formula. Then*

- (i) *if \mathcal{Q} forces $\varphi(\dot{x})$ over $\mathbf{L}[x]$ then $\varphi(x)$ is true (in \mathfrak{N});³*
- (ii) *if x is autonomous and $\varphi(x)$ is true (in \mathfrak{N}) then \mathcal{Q} forces $\varphi(\dot{x})$ over $\mathbf{L}[x]$.*

Proof. (i) holds by Lemma 7.1, since the truth of Σ_3^1 formulas passes to bigger models by Shoenfield. To prove (ii), let $\mathbf{L}[x] = \mathfrak{N} \upharpoonright_u = \mathbf{L}[\langle t_{\nu\gamma}, r_{\nu\gamma} \rangle_{\langle \nu, \gamma \rangle \in u}]$, where $u \in \mathbf{L}$, $u \subseteq \omega_1 \times \omega_1$ is countable. Let $\varphi(\cdot)$ be $\exists z \psi(z, \cdot)$, ψ being Π_2^1 . Assume that $\varphi(x)$ is true in \mathfrak{N} . There is a real $z \in \mathfrak{N}$ such that $\psi(z, x)$ is true in \mathfrak{N} . There is an ordinal $\mu < \omega_1$, such that $u \not\subseteq \mu \times \mu$ and $z \in \mathfrak{N}_{\mu\mu} = \mathbf{L}[\langle t_{\nu\gamma}, r_{\nu\gamma} \rangle_{\nu, \gamma < \mu}]$. Then $z \in \mathfrak{K} = (\mathfrak{N} \upharpoonright_u)[\langle G^{\mu\delta} \rangle_{\langle \mu, \delta \rangle \in W}]$ by Lemma 7.2, where $W = (\mu \times \mu) \setminus u$. And \mathfrak{K} is a \mathcal{Q} -generic extension of $\mathfrak{N} \upharpoonright_u = \mathbf{L}[x]$ still by Lemma 7.2.

On the other hand, $\psi(z, x)$ is true in \mathfrak{K} by Shoenfield, hence $\varphi(x)$ is true in \mathfrak{K} as well. It follows that a condition in \mathcal{Q} forces $\varphi(\dot{x})$ over $\mathbf{L}[x]$. We conclude by Lemma 2.7 that \mathcal{Q} forces $\varphi(\dot{x})$ over $\mathbf{L}[x]$. \square

Recall that HC = hereditarily countable sets (in \mathfrak{N}).

Lemma 7.6. *If φ is a Σ_n formula containing \mathcal{Q} -names in HC (names of sets in $\text{HC}[\underline{G}]$), and if $p \in \mathcal{Q}$, then $p \Vdash_{\mathcal{Q}}^{\text{HC}} \varphi$ is a Σ_n^{HC} assertion about p, φ .*

Proof (sketch). If φ is a Σ_2^1 formula then, by the Mostowski absoluteness, $p \Vdash_{\mathcal{Q}}^{\text{HC}} \varphi$ iff $p \Vdash_{\mathcal{Q}}^M \varphi$ over some countable transitive model M of a sufficient fragment of **ZFC**, which is a Σ_1^{HC} relation. But, Σ_1^{HC} relations are the same as Σ_2^1 . This covers the case $n = 1$ of the lemma.

Step $\Sigma_n \rightarrow \Pi_n$. Suppose that φ is a Σ_n formula. Then $p \Vdash_{\mathcal{Q}}^{\text{HC}} \neg \varphi$ iff $\forall q (q \leq p \implies \neg q \Vdash_{\mathcal{Q}}^{\text{HC}} \varphi)$. This leads to a Π_n formula.

Step $\Pi_n \rightarrow \Sigma_{n+1}$. Let $\varphi(x)$ be a Π_n formula. Then $p \Vdash_{\mathcal{Q}}^{\text{HC}} \exists x \varphi(x)$ iff there is a \mathcal{Q} -name $t \in \text{HC}$ such that $p \Vdash_{\mathcal{Q}}^{\text{HC}} \varphi(t)$. This leads to a Σ_{n+1} formula. \square

Lemma 7.7. *There is a recursive correspondence $\varphi \mapsto \varphi^*$ between Σ_3^1 formulas (and hence between Π_3^1 formulas as well) such that for all reals $b \in \mathfrak{N}$, \mathcal{Q} forces $\varphi(\dot{b})$ over $\mathbf{L}[b]$ if and only if $\mathbf{L}[b] \models \varphi^*(b)$.*

Proof. Let $\sigma(z, t, p)$ be a Π_1 formula provided by Lemma 5.2.

Given φ a Σ_3^1 formula, define $\varphi^*(b)$ iff:

$$\exists t \in \mathbf{L} \exists p (\mathbf{L}[b] \models \sigma(b, t, p) \text{ and } p \Vdash_{\mathcal{Q}} \varphi(\dot{b}) \text{ over } \mathbf{L}[b]).$$
⁴

³ In this lemma, $\dot{x} = \{1\} \times x \in \mathbf{L}[x]$ is a \mathcal{Q} -name of x itself.

⁴ $\dot{b} = \{1\} \times b \in \mathbf{L}[b]$ is the canonical \mathcal{Q} -name of a real $b \subseteq \omega$, where 1 is the largest condition in \mathcal{Q} .

Prove that φ^* is as required. Since Σ_n^1 formulas correspond to the Σ_{n-1} definability in HC, φ^* is a Σ_2^{HC} formula by Lemma 7.6, hence essentially a Σ_3^1 formula.

Now suppose that b is a real in \mathfrak{N} and \mathcal{Q} forces $\varphi(\dot{b})$ over $\mathbf{L}[b]$. We have to prove that $\varphi^*(b)$ holds in $\mathbf{L}[b]$. Note that $b \in \mathfrak{N} \subseteq \mathfrak{M}$. Hence $b \in \mathfrak{M}_{\nu\nu}$ for some $\nu < \omega_1$. But $\mathfrak{M}_{\nu\nu}$ is a $\mathcal{Q}^{\nu \times \nu}$ -generic extension, hence, a \mathcal{Q} -generic extension of \mathbf{L} . Let say $\mathfrak{M}_{\nu\nu} = \mathbf{L}[H]$, where $H \subseteq \mathcal{Q}$, $H \in \mathfrak{M}$ is \mathcal{Q} -generic over \mathbf{L} ; H need not be in \mathfrak{N} . Thus $b = t[H]$, where $t \in \mathbf{L}$, $t \subseteq \mathcal{Q} \times \omega$. Let $\Sigma = \Sigma(b, t)$; then $\Sigma \subseteq \mathcal{Q}$ and $\mathbf{L}[\Sigma] = \mathbf{L}[b]$. Consider any $p \in \Sigma$; thus $\sigma(b, t, p)$ holds in \mathbf{L} . Under our assumptions, $p \Vdash_{\mathcal{Q}} \varphi(\dot{b})$ over $\mathbf{L}[b]$, hence we have $\varphi^*(b)$ in $\mathbf{L}[b]$.

To prove the converse, assume that $\varphi^*(b)$ holds in $\mathbf{L}[b]$, and this is witnessed by $t \in \mathbf{L}$ and p . In particular $\sigma(b, t, p)$ holds, thus $p \in \Sigma = \Sigma(b, t) \subseteq \mathcal{Q}$. Moreover, $p \Vdash_{\mathcal{Q}} \varphi(\dot{b})$ over $\mathbf{L}[b]$. It follows that if $G \subseteq \mathcal{Q}$ is generic over $\mathbf{L}[b]$ and $p \in G$ then $\varphi(b)$ is true in $\mathbf{L}[b][G]$. Thus \mathcal{Q} forces $\varphi(\dot{b})$ over $\mathbf{L}[b]$ by Lemma 2.7. \square

8 Reduction fails

In the remainder, we are going to prove that \mathfrak{N} is a model for Theorem 1.1. The following is the first part of the proof.

Theorem 8.1. *In \mathfrak{N} , there is a pair of Σ_3^1 sets of reals, not reducible to a pair of boldface Σ_3^1 sets.*

Proof. ⁵ **Arguing in \mathfrak{N}** , consider the Σ_3^1 set $A = \{g : \exists a T(g, a)\}$, and let $U \subseteq 2^\omega$ be a Σ_3^1 set, universal in the sense that $\{e < \omega : e \hat{\ } z \in U\} \notin \Pi_3^1(z)$ for every $z \in \omega^\omega$, where $e \hat{\ } z$ adds $e \in \omega$ as the leftmost term to $z \in \omega^\omega$.

Consider the Σ_3^1 sets $\omega^\omega \times A$, $U \times \omega^\omega$.

Suppose to the contrary that, in \mathfrak{N} , there are Σ_3^1 sets $A', U' \subseteq 2^\omega$, such that

$$A' \subseteq \omega^\omega \times A, U' \subseteq U \times \omega^\omega, A' \cap U' = \emptyset, A' \cup U' = (\omega^\omega \times A) \cup (U \times \omega^\omega).$$

Let z be an autonomous real such that A', U' are $\Sigma_3^1(z)$.

Lemma 8.2 (in \mathfrak{N}). *Assume that $d \in 2^\omega \setminus U$. Then $\{y : \langle d, y \rangle \in A'\} = A$ and there is $y \in 2^\omega$ such that $\langle d, y \rangle \in A'$ and y is Cohen generic over $\mathbf{L}[z, d]$.*

Proof. If $\langle d, y \rangle \in A'$ then $y \in A$ by construction. Conversely assume that $y \in A$. Then $\langle d, y \rangle \in \omega^\omega \times A$, but $\langle d, y \rangle \notin U \times \omega^\omega$ (as $d \notin U$). Therefore $\langle d, y \rangle \in A'$ as required. Prove the second claim. By Lemma 7.1, there is a pair $\langle g, a \rangle$ in \mathfrak{N} , \mathcal{Q} -generic over $\mathbf{L}[z, d]$. Thus $g \in 2^\omega$ is Cohen-generic over $\mathbf{L}[z, d]$ while $a \subseteq \omega$ satisfies $T(g, a)$, hence $g \in A$. Thus we are done with $y = g$. \square (Lemma 8.2)

⁵ Harrington relates the idea of the proof to Sami.

Consider the sets $K = \{e \hat{\ } z : e < \omega \wedge e \hat{\ } z \notin U\} \notin \Sigma_3^1(z)$ and

$$K' = \{e \hat{\ } z : \exists y (e < \omega \wedge \langle \langle z, e \rangle, y \rangle \in A' \wedge y \text{ is } C\text{-generic over } \mathbf{L}[z])\}.$$

Clearly $K \notin \Sigma_3^1(z)$ by the choice of U , while K' is $\Sigma_3^1(z)$, and we have $K \subseteq K'$ by Lemma 8.2. Thus $K \subsetneq K'$. We conclude that there is an integer e such that $e \hat{\ } z \in U$ and $e \hat{\ } z \in K'$, so that

$$\exists y (\langle e \hat{\ } z, y \rangle \in A' \wedge y \text{ is Cohen-generic over } \mathbf{L}[z]).$$

Fix such a number $e < \omega$.

Let $A'' = \{y : \langle e \hat{\ } z, y \rangle \in A'\}$; $A'' \subseteq A$. We claim that A'' is $\Delta_3^1(z)$. Indeed $y \notin A'' \iff \langle e \hat{\ } z, y \rangle \notin A'$. But $e \hat{\ } z \in U$, hence $\langle e \hat{\ } z, y \rangle \in U \times \omega^\omega$. Therefore $\langle e \hat{\ } z, y \rangle \notin A' \iff \langle e \hat{\ } z, y \rangle \in U'$. This yields the Π_3^1 definition for A'' .

Let φ be a Π_3^1 formula such that $y \in A'' \iff \varphi(z, y)$ in \mathfrak{N} . By the choice of e , there is a real $g \in A''$, Cohen-generic over $\mathbf{L}[z]$. So $\varphi(z, g)$ is true in \mathfrak{N} . It follows by Lemma 7.5 that \mathcal{Q} forces $\varphi(\dot{z}, \dot{g})$ over $\mathbf{L}[z, g]$.

So by Lemma 7.7 we have $\mathbf{L}[z, g] \models \varphi^*(z, g)$. By the genericity of g , there is a Cohen condition $\bar{g} \in \mathbb{C}$, $\bar{g} \subseteq g$ such that $\bar{g} \Vdash \text{“ } \mathbf{L}[z, \dot{g}] \models \varphi^*(z, \dot{g}) \text{”}$.

Recall that z is autonomous. Let this be witnessed by a countable $u \in \mathbf{L}$, $u \subseteq \nu \times \vartheta$, where $\nu, \vartheta < \omega_1$; thus $\mathbf{L}[z] = \mathfrak{N} \upharpoonright_u = \mathbf{L}[\langle t_{\mu\delta}, r_{\mu\delta} \rangle_{\langle \mu, \delta \rangle \in u}] \subseteq \mathfrak{M}_{\nu\vartheta} \subseteq \mathfrak{M}_\nu$. By 6.4, there is an ordinal $\gamma < \omega_1$ such that $t_{\nu\gamma} = \mathbf{c}$ and $\mathcal{Q}_{\nu\gamma}$ adds a Cohen real over \mathfrak{M}_ν , so $r_{\nu\gamma} \in 2^\omega$ is a Cohen real over \mathfrak{M}_ν . Changing appropriately a finite number of values $r_{\nu\gamma}(k)$, we get another real $g' \in 2^\omega$, Cohen-generic over \mathfrak{M}_ν , and satisfying $\bar{g} \subseteq g'$ and still $(\mathfrak{N} \upharpoonright_u)[g'] = (\mathfrak{N} \upharpoonright_u)[r_{\nu\gamma}]$, or in other words, $\mathbf{L}[z, g'] = (\mathfrak{N} \upharpoonright_u)[g'] = \mathfrak{N} \upharpoonright_v$, where $v = u \cup \{\langle \nu, \gamma \rangle\}$. Thus $\langle z, g' \rangle$ is autonomous.

To conclude, $\langle z, g' \rangle$ is autonomous and $\mathbf{L}[z, g'] \models \varphi^*(z, g')$ by the choice of g' and of $\bar{g} \in \mathbb{C}$. It follows by Lemma 7.7 that $\varphi(z, g')$ is true in some/every \mathcal{Q} -generic extension of $\mathbf{L}[z, g']$. Therefore $\varphi(z, g')$ is true in \mathfrak{N} by Shoenfield.

Thus $g' \in A''$ by the choice of φ , hence $g' \in A$ and $\exists a T(g', a)$ holds in \mathfrak{N} . Thus \mathcal{Q} forces $\exists a T(g', a)$ over $\mathbf{L}[z, g']$ by Lemma 7.5(ii), as $\langle z, g' \rangle$ is autonomous.

Lemma 8.3. \mathcal{Q} forces $\exists a T(g', a)$ over $\mathbf{L}[g']$.

Proof. It suffices to get a \mathcal{Q} -generic extension of $\mathbf{L}[g']$ in which $\exists a T(g', a)$ holds. Consider a set G' , \mathcal{Q} -generic over $\mathbf{L}[z, g']$. Then $\exists a T(g', a)$ holds in $\mathbf{L}[z, g', G']$ by the above. Recall that z belongs to a \mathcal{Q} -generic extension $\mathfrak{M}_{\nu\vartheta} \subseteq \mathfrak{M}_\nu$ of \mathbf{L} . Therefore $\mathbf{L}[z]$ is a \mathcal{Q}_t -generic extension of \mathbf{L} by Lemma 3.7, where $t \subseteq \mathcal{Q} \times \omega$ is countable. In other words, $\mathbf{L}[z] = \mathbf{L}[\Sigma]$, where $\Sigma \subseteq \mathcal{Q}$ is \mathcal{Q}_t -generic over \mathbf{L} .

On the other hand, g' is Cohen-generic over $\mathbf{L}[z]$ while G' is \mathcal{Q} -generic over $\mathbf{L}[z, g']$. It follows, by the product forcing theorem, that G' is \mathcal{Q} -generic over $\mathbf{L}[g']$ and $\mathbf{L}[z, g', G'] = \mathbf{L}[g', G'][\Sigma]$ is a \mathcal{Q}_t -generic extension of $\mathbf{L}[g', G']$.

Let $G \subseteq \Sigma$ be Σ -generic over $\mathbf{L}[g', G'][\Sigma]$. Then $\mathbf{L}[g', G', \Sigma, G]$ is a \mathcal{Q} -generic extension of $\mathbf{L}[g', G']$ by Theorem 3.6(iii), hence a \mathcal{Q} -generic extension of $\mathbf{L}[g']$ as well because $\mathcal{Q} \times \mathcal{Q}$ is order-isomorphic to \mathcal{Q} . Finally $\exists a T(g', a)$ holds in $\mathbf{L}[g', G', \Sigma, G]$ by Shoenfield as it holds in $\mathbf{L}[z, g', G']$, a smaller model. \square (Lemma)

But this contradicts Lemma 2.6.

\square (Theorem 8.1)

9 Separation holds

The next theorem is the final part of the proof of Theorem 1.1.

Theorem 9.1. *In \mathfrak{N} , any pair of disjoint Π_3^1 sets is separable by a Δ_3^1 set.*

Proof. *We argue in \mathfrak{N} .* Let $\varphi_0(\cdot, \cdot), \varphi_1(\cdot, \cdot)$ be Π_3^1 formulas and z be an autonomous real parameter, such that the Π_3^1 sets $A_i = \{x : \varphi_i(z, x)\}$, $i = 0, 1$, are disjoint. Let $\varphi_i^*(z, x)$ be the Π_3^1 -formulas provided by Lemma 7.7. Then the sets $B_i = \{x : \mathbf{L}[z, x] \models \varphi_i^*(z, x)\}$, $i = 0, 1$, are Π_3^1 , and $A_i \subseteq B_i$ by Lemma 7.5.

We claim that $B_0 \cap B_1 = \emptyset$. Assume to the contrary that $b \in B_0 \cap B_1$. The goal of the following argument is to get another real $b' \in B_0 \cap B_1$, with the extra property that $\langle z, b' \rangle$ is autonomous.

As z is autonomous, Lemma 7.2 implies that b belongs to a \mathcal{Q} -generic extension $\mathbf{L}[z][G]$ of $\mathbf{L}[z]$. Therefore, by Lemma 3.7, there is a countable set $t \subseteq \mathcal{Q} \times \omega$, $t \in \mathbf{L}[z]$, such that $b = t[G]$ and $\mathbf{L}[z, b]$ is a \mathcal{Q}_t -generic extension of $\mathbf{L}[z]$. By Lemma 2.7(ii), \mathcal{Q} forces “ $\varphi_0^*(\dot{z}, t[G])$ and $\varphi_1^*(\dot{z}, t[G])$ hold in $\mathbf{L}[z, t[G]]$ ” over $\mathbf{L}[z]$, therefore \mathcal{Q}_t forces “ $\varphi_0^*(\dot{z}, t[G]) \wedge \varphi_1^*(\dot{z}, t[G])$ ” over $\mathbf{L}[z]$. On the other hand, by Lemma 7.4, there is a real b' such that $\langle z, b' \rangle$ is autonomous and $\mathbf{L}[z, b'] = \mathbf{L}[z][\Sigma]$, where $b' = t[\Sigma]$ and $\Sigma \subseteq \mathcal{Q}$ is \mathcal{Q}_t -generic over $\mathbf{L}[z]$. Thus $\varphi_0^*(z, b')$ and $\varphi_1^*(z, b')$ hold in $\mathbf{L}[z, b']$, hence $b' \in B_0 \cap B_1$, and $\langle z, b' \rangle$ is autonomous.

But then $b' \in A_0 \cap A_1$ by Lemmas 7.7 and 7.5, contradiction.

Thus we have $B_0 \cap B_1 = \emptyset$. Then we separate B_0 from B_1 by a $\Delta_3^1(p)$ set of reals by a standard argument. Indeed let \leq_{zx}^G be the canonical “good” Gödel wellordering of the reals in $\mathbf{L}[z, x]$. Let $\exists y \vartheta_0(z, x, y)$ be the canonical transformation of $\neg \varphi_1^*(z, x)$ to Σ_3^1 -form, and $\exists y \vartheta_1(z, x, y)$ be the canonical transformation of $\neg \varphi_0^*(z, x)$ to Σ_3^1 -form, so that ϑ_i are Π_2^1 -formulas and

$$B_i \subseteq C_i = \{x : \mathbf{L}[x, z] \models \exists y \vartheta_i(z, x, y)\}, \quad i = 0, 1.$$

Here C_i is the complement to B_{1-i} , thus $C_0 \cup C_1 =$ all reals. If $x \in C_i$ then let $y_i(x)$ be the \leq_{zx}^G -least real y satisfying $\mathbf{L}[x, z] \models \vartheta_i(z, x, y)$. The sets

$$\begin{aligned} D_0 &= \{x \in C_0 : x \notin C_1 \vee y_0(x) \leq_{zx}^G y_1(x)\} \\ D_1 &= \{x \in C_1 : x \notin C_0 \vee y_1(x) <_{zx}^G y_0(x)\} \end{aligned}$$

then satisfy $A_i \subseteq B_i \subseteq D_i \subseteq C_i$ and $D_0 \cup D_1 =$ all reals. On the other hand, a standard argument (as in the proof of Σ_n^1 -reduction in \mathbf{L}) shows that both D_i are Σ_3^1 sets. It follows that D_0 is a Δ_3^1 set separating A_0 from A_1 . \square

10 Comments and questions

We may note the following substantial inventions in Harrington’s proof.

- The *localization property* in \mathfrak{N} , that is, the reduction of the truth of a formula $\varphi(x)$ in the final model \mathfrak{N} , first, to the truth in \mathcal{Q} -generic extensions of $\mathbf{L}[x]$ by Lemma 7.5, and second, to the truth in $\mathbf{L}[x]$ itself by Lemma 7.7. This is quite similar to the “important lemma” of Solovay [21, page 18], but achieved in a much less friendly generic model.
- The *Harrington fan* construction of 3.9 (see also Remark 6.4) which allows to inhibit (by Lemma 7.4) the fact that Lemma 7.5(ii) holds only for autonomous reals.
- Unlike the Separation counterexamples in specific models in [5, Part B] or say [9], the Reduction counterexample as in Theorem 8.1 (which Harrington grants to Sami) is not something explicitly designed by the intended definability structure of the generic model considered.

Harrington ends [5, Part C] with the following remark:

We believe that this result [= Theorem 1.1] can be generalized by replacing 3 by any integer $n \geq 3$. We also believe that that this result can be improved so as to obtain a model of **ZFC** in which both $\text{Sep}(\Pi_3^1, \Delta_3^1)$ and $\text{Sep}(\Sigma_3^1, \Delta_3^1)$ hold. At the moment though these beliefs are just expressions of faith (or is it hope?).

The second part of this “expressions of faith or hope” was partially materialized in [5, Part C], where, for an arbitrary $n \geq 3$, a model of **ZFC** is presented, in which $\text{Sep}(\Pi_n^1, \Delta_n^1)$ and $\text{Sep}(\Sigma_n^1, \Delta_n^1)$ (note the lightface classes!) both hold for sets of integers. (The proof is given for $n = 3$ only.) The rest presumably remains as open as it was in 1970s. Regarding the very recent results, a model in which Uniformization (and hence Reduction) holds for Π_3^1 was discussed in [3].

11 Reduction holds in some simple models

Here we sketch **the proof of Claim (I) of Theorem 1.2**. Proofs of the other two claims will appear elsewhere.

Let the set universe \mathbf{V} be an extension of \mathbf{L} by a transfinite sequence of Cohen-generic reals. The following is a known property of the Cohen forcing $\mathbb{C} = 2^{<\omega}$ and Cohen extensions.

Lemma 11.1. *If $x, y \in \mathbf{V}$ are reals then either $y \in \mathbf{L}[x]$ or $\mathbf{L}[x, y]$ is a \mathbb{C} -generic extension of $\mathbf{L}[x]$. In particular, y belongs to a \mathbb{C} -generic extension of $\mathbf{L}[x]$. \square*

Case $n = 3$. We claim that if $\varphi(x)$ is a Σ_3^1 or Π_3^1 formula with x as the only real parameter, then

$$\varphi(x) \text{ holds in } \mathbf{V} \quad \text{iff} \quad \mathbf{L}[x] \models \underbrace{\Lambda \text{ } \mathbb{C}\text{-forces } \varphi(\dot{x})}_{\varphi^*(x)} \text{ over the universe.} \quad (1)$$

Here $\Lambda \in \mathbb{C}$ (the empty sequence) is the weakest Cohen condition, and $\dot{x} = \{\Lambda\} \times x$ is the canonical Cohen name for a set x in the ground model.

To prove (1) for a Σ_3^1 formula $\varphi(x) := \exists x_1 \psi(x, y)$, ψ being Π_2^1 , assume that $\mathbf{V} \models \varphi(x)$, hence there is a real $y \in \mathbf{V}$ satisfying $\psi(x, y)$. But y belongs to a \mathbb{C} -generic extension of $\mathbf{L}[x][g]$ of $\mathbf{L}[x]$ by Lemma 11.1. Then $\exists y \psi(x, y)$ is true in $\mathbf{L}[x, g]$ by Shoenfield, and hence $\exists y \psi(\dot{x}, y)$ is \mathbb{C} -forced by Λ over $\mathbf{L}[x]$ (by the homogeneity of \mathbb{C}), that is, $\mathbf{L}[x] \models \varphi^*(x)$. Conversely let $\mathbf{L}[x] \models \varphi^*(x)$. Let $g \in \mathbf{V}$ be a \mathbb{C} -generic real over $\mathbf{L}[x]$. Then $\mathbf{L}[x, g] \models \varphi(x)$, hence $\mathbf{V} \models \varphi(x)$ by Shoenfield.

To check (1) for a Π_3^1 formula $\Phi(x) := \neg \varphi(x)$, φ being Σ_3^1 , assume first that $\mathbf{V} \models \Phi(x)$. Then $\mathbf{V} \not\models \varphi(x)$, hence by (1) $\varphi(\dot{x})$ is not \mathbb{C} -forced by Λ over $\mathbf{L}[x]$, thus by the homogeneity $\Phi(\dot{x})$ is forced, that is, $\mathbf{L}[x] \models \Phi^*(\dot{x})$. Conversely if $\mathbf{L}[x] \models \Phi^*(\dot{x})$, then definitely $\mathbf{L}[x] \not\models \varphi^*(\dot{x})$, thus $\mathbf{V} \not\models \varphi(x)$, and hence $\mathbf{V} \models \Phi(x)$.

Pretty similar to the proof of Lemma 7.6, $\varphi^*(\cdot)$ is a formula of type Σ_3^1 , resp., Π_3^1 formula provided φ itself is of this type. The next lemma will be used below.

Lemma 11.2. *Let $\varphi(x)$ be a Σ_3^1 or Π_3^1 formula. Let g be a real \mathbb{C} -generic over $\mathbf{L}[x]$, and $\mathbf{L}[x, g] \models \varphi^*(x)$. Then $\mathbf{L}[x] \models \varphi^*(x)$.*

Proof. By the homogeneity, it suffices to get a \mathbb{C} -generic real h over $\mathbf{L}[x]$, such that $\mathbf{L}[x, h] \models \varphi(x)$. Let g' be \mathbb{C} -generic over $\mathbf{L}[x, g]$. Then $\mathbf{L}[x, g, g'] \models \varphi(x)$, so it remains to make use of $h = \langle g, g' \rangle$. \square

Now, consider Σ_3^1 sets $A_0 = \{x : \varphi_0(x, z)\}$ and $A_1 = \{x : \varphi_1(x, z)\}$ in \mathbf{V} , where φ_i are Σ_3^1 formulas. Then $A_i = \{x : \mathbf{L}[x, z] \models \varphi_i^*(x, z)\}$ by the above, where $\varphi_i^*(x, z) := \exists y \Phi_i(x, z, y)$ are Σ_3^1 formulas by the claim, so Φ_i are Π_2^1 formulas.

If $x \in A_i$ then let $y_i(x, z)$ be the \leq_{xz}^G -least real y satisfying $\mathbf{L}[x, z] \models \Phi_i(x, z, y)$. The sets

$$\begin{aligned} B_0 &= \{x \in A_0 : x \notin A_1 \vee y_0(x, z) \leq_{xz}^G y_1(x, z)\} \\ B_1 &= \{x \in A_1 : x \notin A_0 \vee y_1(x, z) <_{xz}^G y_0(x, z)\} \end{aligned}$$

then satisfy $B_i \subseteq A_i$ and $B_0 \cap B_1 = \emptyset$, and belong to Σ_3^1 .

Case $n \geq 4$. Let say $n = 4$ exactly; it will be clear how to treat the general case. Suppose that $\varphi(x) := \exists y \psi(x, y)$ is a Σ_4^1 formula, ψ being Π_3^1 . Then a Π_3^1 formula $\psi^*(x, y)$ has been defined as above, such that $\mathbf{V} \models \psi(x, y)$ iff $\mathbf{L}[x, y] \models \psi^*(x, y)$, for all reals $x, y \in \mathbf{V}$. We claim that then

$$\mathbf{V} \models \varphi(x) \quad \text{iff} \quad \mathbf{L}[x] \models \underbrace{\Lambda \text{ C-forces } \text{“}\exists y \psi^*(\dot{x}, y)\text{”}}_{\varphi^*(x)} \quad (2)$$

Indeed assume that $\mathbf{V} \models \varphi(x)$, hence there is a real $y \in \mathbf{V}$ satisfying $\psi(x, y)$. It follows that $\mathbf{L}[x, y] \models \psi^*(x, y)$. But y belongs to a \mathbb{C} -generic extension of $\mathbf{L}[x]$ by Lemma 11.1. Therefore, $\exists y \psi^*(\dot{x}, y)$ is \mathbb{C} -forced by Λ over $\mathbf{L}[x]$ (by the homogeneity of \mathbb{C}), that is, $\mathbf{L}[x] \models \varphi^*(x)$.

To prove the converse, let $\mathbf{L}[x] \models \varphi^*(x)$. Let $g \in \mathbf{V}$ be a Cohen-generic real over $\mathbf{L}[x]$. Then we have $\mathbf{L}[x, g] \models \exists y \psi^*(x, y)$. Let this be witnessed by a real $y \in \mathbf{L}[x, g]$, thus $\mathbf{L}[x, g] \models \psi^*(x, y)$. However, by Lemma 11.1, either $\mathbf{L}[x, g] = \mathbf{L}[x, y]$ — and then $\mathbf{L}[x, y] \models \psi^*(x, y)$ and hence $\mathbf{V} \models \psi(x, y)$ and $\mathbf{V} \models \varphi(x)$, or $\mathbf{L}[x, g]$ is a \mathbb{C} -generic extension of $\mathbf{L}[x, y]$ — and then we still have $\mathbf{L}[x, y] \models \psi^*(x, y)$ by Lemma 11.2, and then $\mathbf{V} \models \varphi(x)$ as just above.

The proof of Reduction for a pair of Σ_4^1 -sets A_0, A_1 in \mathbf{V} goes on, on the base of (2), exactly as in the case $n = 3$ above.

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 - (B) Addendum. Models where Separation principles fail. May 74.
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 - (D) The constructible reals can be (almost) anything. Part II. May 75.
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