Definable Hamel bases and $\text{AC}_\omega(\mathbb{R})$

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Abstract

There is a model of $\text{ZF}$ with a $\Delta^1_3$ definable Hamel bases in which $\text{AC}_\omega(\mathbb{R})$ fails.

Answering a question from [9, p. 433] it was shown in [1] that there is a Hamel basis in the Cohen–Halpern–Lévy model. In this paper we show that in a variant of this model, there is a projective, in fact $\Delta^1_3$, Hamel basis.

Throughout this paper, by a Hamel basis we always mean a basis for $\mathbb{R}$, construed as a vector space over $\mathbb{Q}$. We denote by $E$ the Vitali equivalence relation, $x E y$ iff $x - y \in \mathbb{Q}$ for $x, y \in \mathbb{R}$. We also write $[x]_E = \{y : y E x\}$ for the $E$–equivalence class of $x$. A transversal for the set of all $E$–equivalence classes picks exactly one member from each $[x]_E$. The range of any such transversal is also called a Vitali set.

A set $\Lambda \subset \mathbb{R}$ is a Luzin set iff $\Lambda$ is uncountable but $\Lambda \cap M$ is at most countable for every meager set $M \subset \mathbb{R}$. A set $S \subset \mathbb{R}$ is a Sierpiński set iff $S$ is uncountable but $S \cap N$ is at most countable for every null set $N \subset \mathbb{R}$ (“null” in the sense of Lebesgue measure). A set $B \subset \mathbb{R}$ is a Bernstein set iff $B \cap P \neq \emptyset \neq P \setminus B$ for every perfect set $P \subset \mathbb{R}$. A Burstin basis is a Hamel basis which is also a Bernstein set. It is easy to see that $B \subset \mathbb{R}$ is a Burstin basis iff $B$ is a Hamel basis and $B \cap P \neq \emptyset$ for every perfect $P \subset \mathbb{R}$.

By $\text{AC}_\omega(\mathbb{R})$ we mean the statement that for all sequences $(A_n : n < \omega)$ such that $\emptyset \neq A_n \subset \mathbb{R}$ for all $n < \omega$ there is some choice function $f : \omega \to \mathbb{R}$, i.e., $f(n) \in A_n$ for all $n < \omega$.

D. Pincus and K. Prikry study the Cohen-Halpern-Lévy model $H$ in [9]. The model $H$ is obtained by adding a countable set of Cohen reals (say over $L$) without adding their enumeration; $H$ does not satisfy $\text{AC}_\omega(\mathbb{R})$. It is shown in [9] that there is a Luzin set in $H$, so that in $\text{ZF}$, the existence of a Luzin set does not even imply $\text{AC}_\omega(\mathbb{R})$. [1, Theorems 1.7 and 2.1] show that in $H$ there is a Bernstein set as well as a Hamel basis. As in $\text{ZF}$ the existence of a Hamel basis implies the existence of a Vitali set, the latter also reproves Feferman’s result (see [9]) according to which there is a Vitali set in $H$.

Therefore, in $\text{ZF}$ the conjunction of the following statements (1), (3), and (5) (which in $\text{ZF}$ implies (4)) does not yield $\text{AC}_\omega(\mathbb{R})$.

(1) There is a Luzin set.
(2) There is a Sierpiński set.
(3) There is a Bernstein set.
(4) There is a Vitali set.
(5) There is a Hamel basis.
(6) There is a Burstin basis.

(2) is false in $H$, see [1, Lemma 1.6]. We aim to prove that in $\text{ZF}$, the conjunction of all of these statements does not imply $\text{AC}_\omega(\mathbb{R})$, even if the respective sets are required to be projective. What we have at this point is:

**Theorem 0.1** There is a model of $\text{ZF}$ plus $\neg \text{AC}_\omega(\mathbb{R})$ in which the following hold true.

(a) There is a $\Delta^1_2$ Luzin set.
(b) There is a $\Delta^1_2$ Sierpiński set.
(c) There is a $\Delta^1_2$ Bernstein set.
(d) There is a $\Delta^1_2$ Hamel basis.

## 1 Jensen's perfect set forcing, revisited.

In what follows, we shall mostly think of reals as elements of the Cantor space $^{\omega 2}$. We shall need a variant of the Cohen-Halpern-Levy model. In order to construct our model, we need to introduce a variant of Jensen’s variant of Sacks forcing, see [6] (see also [7, Definition 6.1]), which we shall call $\mathbb{P}$. The reason why we can’t work with Jensen’s forcing directly is that it does not seem to have the Sacks property (see e.g. [2, Definition 2.15]).

By way of notation, if $Q$ is a forcing and $N > 0$ is any ordinal, then $Q(N)$ denotes the finite support product of $N$ copies of $Q$, ordered component-wise. In this paper, we shall only consider $Q(N)$ for $N \leq \omega$. If $\alpha$ is a limit ordinal, then $\prec_{J_\alpha}$ denotes the canonical well-ordering of $J_\alpha$, see [10, Definition 5.14 and p. 79], and $\prec_L = \bigcup \{\prec_{J_\alpha}: \alpha \text{ is a limit ordinal}\}$.

Let us work in $L$ until further notice. Let us first define $(\alpha_\xi, \beta_\xi: \xi < \omega_1)$ as follows: $\alpha_\xi$ is the least $\alpha > \sup(\{\beta_\xi: \xi < \xi\})$ such that $J_\alpha \models \text{ZFC}$, and $\beta_\xi$ is the least $\beta > \alpha_\xi$ such that $J_\beta(\mathbb{P}) = \omega$ (see [10, Definition 11.22]). $\text{AC}_\omega(\mathbb{R})$ is equivalent with $\mathbb{P}(\omega) \cap J_{\beta + \omega} \not\subseteq J_\beta$.

We shall also make use of a sequence $(f_\xi: \xi < \omega_1)$ which is defined as follows. Let $(f_\xi: \xi < \omega)$ be defined by the following trivial recursion: $f_\xi$ be the $\prec_L$-least $f$ such that $f \in (\omega J_{\omega_1} \cap J_{\omega_1}) \setminus \{f_\zeta: \zeta < \xi\}$. Then if $\pi$ denotes the Gödel pairing function, see [10, p. 35], we let $f_{\pi((\xi_1, \xi_2))} = f_{\xi_1}$. We will then have that $f_\xi \in J_{\alpha_\xi}$ for all $\xi$, and for each $f \in (\omega J_{\omega_1} \cap J_{\omega_1})$ the set of $\xi$ such that $f = f_\xi$ is cofinal in $\omega_1$.

Let us then define $(\mathbb{P}_\xi, Q_\xi: \xi \leq \omega_1)$. Each $\mathbb{P}_\xi$ will consist of perfect trees $T \subseteq \omega_2$ such that if $T \in \mathbb{P}_\xi$ and $s \in T$, then $T_s = \{t \in T: t \subseteq s \uplus s \subseteq t\} \in \mathbb{P}_\xi$ as well. Each $\mathbb{P}_\xi$ will be construed as a p.o. by stipulating $T \leq T'$ ("is stronger than" $T'$) iff $T \subseteq T'$. We will have that $\mathbb{P}_\xi \in J_{\alpha_\xi}$ and $\mathbb{P}_\xi \subseteq \mathbb{P}_\xi$ whenever $\xi \leq \xi \leq \omega_1$.

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1 The reader unfamiliar with the $J$-hierarchy may read $L_\alpha$ instead of $J_\alpha$.
2 Here, $\text{ZFC}^-$ denotes $\text{ZFC}$ without the power set axiom. Every $J_\alpha$ satisfies the strong form of $\text{AC}$ according to which every set is the surjective image of some ordinal.
3 We denote by $x \subset y$ the fact that $x$ is a (not necessarily proper) subset of $y$. 

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2
To start with, let \( \mathbb{P}_0 \) be the set of all basic clopen sets \( U_s = \{ t \in \omega \omega : t \subset s \lor s \subset t \} \), where \( s \in \omega \). If \( \lambda \leq \omega_1 \) is a limit ordinal, then \( \mathbb{P}_\lambda = \bigcup \{ \mathbb{P}_\xi : \xi < \lambda \} \).

Now fix \( \xi < \omega_1 \), and suppose that \( \mathbb{P}_\xi \) has already been defined. We shall define \( \mathbb{Q}_\xi \) and \( \mathbb{P}_{\xi+1} \).

Let \( g_\xi \in \omega J_{\alpha_\xi} \) be the following \( \omega \)-sequence. If there is some \( N < \omega \) such that \( f_\xi \)

is an \( \omega \)-sequence of subsets of \( \mathbb{P}_\xi(N) \), each of which is predense in \( \mathbb{P}_\xi(N) \), then for each \( n < \omega \) let \( g_\xi(n) \) be the open dense set

\[
\{(T_1, \ldots, T_N) \in \mathbb{P}_\xi(N) : \exists (T'_1, \ldots, T'_N) \in f_\xi(n) \ (T_1, \ldots, T_N) \leq (T'_1, \ldots, T'_N)\},
\]

and write \( N_\xi = N \). Otherwise we just set \( g_\xi(n) = \mathbb{P}_\xi(1) \) for each \( n < \omega \), and write \( N_\xi = 1 \). Let \( d_\xi \) be the \( < \beta_\xi + \omega \)-least \( d \in \omega \times \omega (\mathcal{P}(\mathbb{P}_\xi) \cap J_{\alpha_\xi}) \cap J_{\beta_\xi + \omega} \) such that

(i) for each \( (n, N) \in \omega \times \omega \), \( d(n, N) \) is an open dense subset of \( \mathbb{P}_\xi(N) \) which exists in \( J_{\beta_\xi} \),

(ii) for each \( N < \omega \) and each open dense subset \( D \) of \( \mathbb{P}_\xi(N) \) which exists in \( J_{\beta_\xi} \) there is some \( n < \omega \) with \( d(n, N) \subset D \),

(iii) \( d(n, N_\xi) \subset g_\xi(n) \) for each \( n < \omega \), and

(iv) \( d(n + 1, N) \subset d(n, N) \) for each \( (n, N) \in \omega \times \omega \).

Let us now look at the collection of all systems \( (T^m_s : m < \omega, s \in \omega \omega) \) with the following properties.

(a) \( T^m_s \in \mathbb{P}_\xi \) for all \( m, s \),

(b) for each \( T \in \mathbb{P}_\xi \) there are infinitely many \( m < \omega \) with \( T^m_s = T \),

(c) \( T^m_s \leq T^m_t \) for all \( m, t \supset s \),

(d) \( \text{stem}(T^m_{s-0}) \) and \( \text{stem}(T^m_{s-1}) \) are incompatible elements of \( T^m_s \) for all \( m, s \),

(e) if \( (m, s) \neq (m', s') \), where \( m, m' \in \omega \), \( \ln(s) = \ln(s') = n + 1 \) for some \( n \), then \( \text{stem}(T^m_s) \) and \( \text{stem}(T^m_{s'}) \) are incompatible, and

(f) for all \( N \leq n < \omega \) and all pairwise different \( (m_1, s_1), \ldots, (m_N, s_N) \) with \( m_1, \ldots, m_N < n \) and \( s_1, \ldots, s_N \in \omega + 1 \),

\[
(T^m_{s_1}, \ldots, T^m_{s_N}) \in d_\xi(n, N).
\]

It is easy to work in \( J_{\beta_\xi + \omega} \) and construct initial segments \( (T^m_s : m < \omega, s \in \omega \omega, \ln(s) \leq n) \) of such a system by induction on \( n < \omega \). Notice that (f) formulates a constraint only for \( m_1, \ldots, m_N < \ln(s_1) - 1 = \ldots = \ln(s_N) - 1 \), and writing \( n = \ln(s) - 1 \), there are \( \sum_{N=1}^{n} (n-2^{n+1} N)! \) (i.e., finitely many) such constraints.

We let \( (T^m_{s_\xi} : m < \omega, s \in \omega \omega) \) be the \( < \beta_\xi + \omega \)-least such system \( (T^m_s : m < \omega, s \in \omega \omega) \). For every \( m < \omega \), \( s \in \omega \omega \), we let

\[
A^m_{s, \xi} = \bigcap_{n \geq \ln(s)} \bigcup_{t \supset s, \ln(t) = n} T^m_t = \{ \text{stem}(T^m_t) \upharpoonright k : t \supset s, k < \omega \}.
\]
Notice that (e) implies that

\[ A_{s, s'}^\eta \cap A_{s, s'}^{\eta'} \text{ is finite, unless } m = m' \text{ and } s \subset s' \text{ or } s' \subset s. \]

(1) will imply that \( A_{s, s'}^\eta \) and \( A_{s, s'}^{\eta'} \) will be incompatible in every \( \mathbb{P}_\eta, \eta > \xi \), unless \( m = m' \) and \( s \subset s' \) or \( s' \subset s \).

We set \( Q_\xi = \{ A_{s, s'}^m : m < \omega, s \in \omega \} \). Finally, we set \( \mathbb{P}_{\xi+1} = \mathbb{P}_\xi \cup Q_\xi \).

**Lemma 1.1** Let \( N < \omega, \xi < \omega_1 \).

\[ D = \{(T_1, \ldots, T_N) \in Q_\xi(N) : \text{stem}(T_i) \perp \text{stem}(T_j) \text{ for } i \neq j\} \]

is dense in \( \mathbb{P}_{\xi+1}(N) \).

**Proof.** Let \( (T_1, \ldots, T_N) \in \mathbb{P}_{\xi+1}(N) \). For \( i \in \{1, \ldots, N\} \) such that \( T_i \in \mathbb{P}_\xi \) pick some \( m_i < \omega \) such that \( T_i = T_{m_i}^\eta \). And write \( s_i = \emptyset \). This is possible by (b).

If \( i \in \{1, \ldots, N\} \) is such that \( T_i \in Q_\xi \), then say \( T_i = A_{m_i, \xi}^\eta \). Now pick \( n > \max\{m_1, \ldots, m_N\} \) and \( t_1 \supset t_2 \supset \ldots \supset t_N \supset s \) such that \( \text{lh}(t_1) = \ldots = \text{lh}(t_N) = n + 1 \) and the \( (m_i, t_i) \) are pairwise different.

Then by (e) the finite sequences \( \text{stem}(T_{m_i}^{\eta}) \) are pairwise incompatible, so that by \( A_{m_i, \xi}^\eta \leq T_{m_i}^{\eta} \), the \( A_{m_i, \xi}^\eta \) are pairwise incompatible. But then \( (A_{m_1, \xi}^\eta, \ldots, A_{m_N, \xi}^\eta) \in D \) and \( (A_{m_1, \xi}^\eta, \ldots, A_{m_N, \xi}^\eta) \leq (T_1, \ldots, T_N) \). \( \square \)

**Lemma 1.2** (Sealing) Let \( N < \omega, \xi < \omega_1 \). If \( D \in J_{\beta, \xi} \) is predense in \( \mathbb{P}_\xi(N) \), then \( D \) is predense in all \( \mathbb{P}_\eta(N), \eta \geq \xi, \eta \leq \omega_1 \).

**Proof** by induction on \( \eta \). The cases \( \eta = \xi \) and \( \eta \) being a limit ordinal are trivial.

Suppose \( \eta \geq \xi, \eta < \omega_1 \), and \( D \) is predense in \( \mathbb{P}_\eta(N) \). Write \( D' = \{(T_1, \ldots, T_N) \in \mathbb{P}_\eta(N) : \exists (T'_1, \ldots, T'_N) \in D(T_1, \ldots, T_N) \leq (T'_1, \ldots, T'_N)\} \). As \( \beta \leq \beta_\eta, \beta_\eta \geq \beta \), \( D' \in J_{\beta, \xi} \) and by (ii) and (iv) there is some \( n_0 < \omega \) with \( d_\eta(n, N) \subset D' \) for every \( n > n_0 \).

To show that \( D' \) (and hence \( D \)) is predense in \( \mathbb{P}_{\eta+1}(N) \), by Lemma 1.1 it suffices to show that for all \( (T_1, \ldots, T_N) \in Q_\eta(N) \) there is some \( (T'_1, \ldots, T'_N) \in Q_\eta(N) \), \( (T'_1, \ldots, T'_N) \leq (T_1, \ldots, T_N) \), and \( (T'_1, \ldots, T'_N) \) is below some element of \( D' \).

So let \( (A_{s_1, \eta}^m, \ldots, A_{s_N, \eta}^m) \in Q_\eta(N) \) be arbitrary. Let

\[ n > \max\{\{n_0, N - 1, m_1, \ldots, m_N, \text{lh}(s_1), \ldots, \text{lh}(s_N)\}\}, \]

and let \( t_1 \supset s_1, \ldots, t_N \supset s_N \) be such that \( \text{lh}(t_1) = \ldots = \text{lh}(t_N) = n + 1 \). By increasing \( n \) further if necessary, we may certainly assume that \( t_1, \ldots, t_N \) are picked in such a way that \( (m_1, t_1), \ldots, (m_N, t_N) \) are pairwise different. Then

\[ (T_{m_1}^{n_1}, \ldots, T_{m_N}^{n_1}) \in d_\eta(n, N) \subset D' \]

by (f). But

\[ (A_{m_1, \eta}^1, \ldots, A_{m_N, \eta}^1) \leq (T_{m_1}^{n_1}, \ldots, T_{m_N}^{n_1}), \]

and also

\[ (A_{m_1, \eta}^1, \ldots, A_{m_N, \eta}^1) \leq (A_{s_1, \eta}^0, \ldots, A_{s_N, \eta}^0), \]

which means that \( (A_{s_1, \eta}^0, \ldots, A_{s_N, \eta}^0) \) is compatible with an element of \( D' \). \( \square \)

\textsuperscript{4}Here, \( \text{stem}(T_i) \perp \text{stem}(T_j) \) means that the stem of \( T_i \) is incompatible with the stem of \( T_j \).
Corollary 1.3 Let $N < \omega$, $\xi < \omega_1$.

\[(T_1, \ldots, T_N) \in \mathbb{Q}_\xi(N): \text{ stem}(T_i) \perp \text{ stem}(T_j) \text{ for } i \neq j\]

is predense in $\mathbb{P}(N)$.

Lemma 1.4 Let $N < \omega$. $\mathbb{P}(N)$ has the c.c.c.

Proof. Let $A \subset \mathbb{P}(N)$ be a maximal antichain, $A \in L$. Let $j: J_\beta \rightarrow J_{\omega_1}$ be elementary and such that $\beta < \omega_1$ and $\{\mathbb{P}, A\} \subseteq \text{ran}(j)$. Write $\xi = \text{crit}(j)$. We have that $j^{-1}(\mathbb{P}(N)) = \mathbb{P}(N) \cap J_\xi = \mathbb{P}_\xi(N)$ and $j^{-1}(A) = A \cap J_\xi = A \cap \mathbb{P}_\xi(N) \in J_\beta$ is a maximal antichain in $\mathbb{P}_\xi(N)$. Moreover, $\beta_\xi > \beta$, so that by Lemma 1.3 $A \cap \mathbb{P}_\xi(N)$ is predense in $\mathbb{P}(N)$. This means that $A = A \cap \mathbb{P}_\xi$ is countable. \(\square\)

Lemma 1.5 Let $N < \omega$. $(c_1, \ldots, c_N) \in N^{(\omega, 2)}$ is $\mathbb{P}(N)$-generic over $L$ iff for all $\xi < \omega_1$ there is an injection $t: \{1, \ldots, N\} \rightarrow \mathbb{Q}_\xi$ such that for all $i \in \{1, \ldots, N\}$, $c_i \in [t(i)]$.

Proof. “$\Longrightarrow$”: This readily follows from Corollary 1.3.

“$\Longleftarrow$”: Let $A \subset \mathbb{P}(N)$ be a maximal antichain, $A \in L$. By Lemma 1.4, we may certainly pick some $\xi < \omega_1$ with $A \subset \mathbb{P}_\xi(N)$ and $A \in J_\alpha$. Say $n_0$ is such that $d_\xi'(n, N) \subset \{(T_1, \ldots, T_N) \in \mathbb{P}_\xi: \exists (T'_1, \ldots, T'_N) \in A (T_1, \ldots, T_N) \subseteq (T'_1, \ldots, T'_N)\}$ for all $n \geq n_0$. By our hypothesis, we may pick pairwise different $(m_1, s_1), \ldots, (m_N, s_N)$ with $\text{lh}(s_1) = \ldots = \text{lh}(s_N) = n + 1$ for some $n \geq n_0$ and $c_i \in [T_{s_i}^m, \xi]$ for all $i \in \{1, \ldots, N\}$. But then $(T_{s_1}^{m_1}, \ldots, T_{s_N}^{m_N})$ is below an element of $A$, which means that the generic filter given by $(c_1, \ldots, c_N)$ meets $A$. \(\square\)

Corollary 1.6 Let $N < \omega$, and let $(c_1, \ldots, c_N) \in N^{(\omega, 2)}$ be $\mathbb{P}(N)$-generic over $L$. If $x \in L[(c_1, \ldots, c_N)]$ is $\mathbb{P}$-generic over $L$, then $x \in \{c_1, \ldots, c_N\}$.

Proof. If $x \in L[(c_1, \ldots, c_N)]$ is $\mathbb{P}$-generic over $L$, then $(c_1, \ldots, c_N, x) \in N^{(\omega, 2)}$ is $\mathbb{P}(N + 1)$-generic over $L$, hence $x \notin L[(c_1, \ldots, c_N)]$. Contradiction! \(\square\)

Corollary 1.7 Let $N < \omega$, and let $(c_1, \ldots, c_N) \in N^{(\omega, 2)}$ be $\mathbb{P}(N)$-generic over $L$. Then inside $L[(c_1, \ldots, c_N)]$, $\{c_1, \ldots, c_N\}$ is a (lightface) $\Pi^1_2$ set.

Proof. Let $\varphi(x)$ express that for all $\xi < \omega_1$ there is some $T \in \mathbb{Q}_\xi$ such that $x \in [T]$. The formula $\varphi(x)$ may be written in a $\Pi^1_2$ fashion, and it defines $\{c_1, \ldots, c_N\}$ inside $L[(c_1, \ldots, c_N)]$. \(\square\)

Lemma 1.8 (Sacks property) Let $N < \omega$, and let $g$ be $\mathbb{P}(N)$-generic over $L$. For each $f: \omega \rightarrow \omega$, $f \in L[a]$, there is some $g \in L$ with domain $\omega$ such that for each $n < \omega$, $f(n) = g(n)$ and $^5\text{Card}(g(n)) \leq (n + 1) \cdot 2^n + 1$.

Proof. Let $\tau \in L^{\mathbb{P}(N)}$, $\tau^g = f$. Let $(A_n: n < \omega) \in L$ be such that for each $n$, $A_n$ is a maximal antichain of $\bar{T} \in \mathbb{P}(N)$ such that $\exists m < \omega \bar{T} \models \tau(n) = \bar{m}$. We may pick some $\xi < \omega_1$ such that $\bigcup \{A_n: n < \omega\} \subseteq \mathbb{P}_\xi(N)$ and $(A_n: n < \omega) = f_\xi$.

^5In what follows, the only thing that will matter is that the bound on $\text{Card}(g(n))$ only depends on $n$ and not on the particular $g$.  

5
By Lemma 1.5, there are pairwise different \((m_1, s_1), \ldots, (m_N, s_N)\) such that
\[
(A_{s_1, \xi}, \ldots, A_{s_N, \xi}) \in g.
\]

Let
\[
n > \max(\{N - 1, m_1, \ldots, m_N, \text{lh}(s_1), \ldots, \text{lh}(s_N)\}).
\]

If \(t_1 \supset s_1, \ldots, t_N \supset t_N\) are such that \(\text{lh}(t_1) = \ldots = \text{lh}(t_N) = n + 1\), then \((T_{t_1, \xi}, \ldots, T_{t_N, \xi}) \in d_\xi(n, N) \subseteq A_n\), so that also
\[
\exists m < \omega \,(T_{t_1, \xi}^{m_1}, \ldots, T_{t_N, \xi}^{m_N}) \models \tau(\bar{n}) = \bar{m}.
\]

Therefore, if we let
\[
g(n) = \{m < \omega : \exists t_1 \supset s_1, \ldots, t_N \supset t_N \,(\text{lh}(t_1) = \ldots = \text{lh}(t_N) = n + 1 \land 
(T_{t_1, \xi}^{m_1}, \ldots, T_{t_N, \xi}^{m_N}) \models \tau(\bar{n}) = \bar{m})\},
\]
then \((A_{s_1, \xi}^{m_1}, \ldots, A_{s_N, \xi}^{m_N}) \models \tau(\bar{n}) \in (g(n))^{\uparrow}\), hence \(f(n) \in g(n)\), and \(\text{Card}(g(n)) = N \cdot 2^{n+1} \leq (n + 1) \cdot 2^{n+1}\) for all but finitely many \(n\). □

2 The variant of the Cohen–Helpern-Lévy model.

Let us force with \(\mathbb{P}(\omega)\) over \(L\), and let \(g\) be a generic filter. Let \(c_n\), \(n < \omega\), denote the Jensen reals which \(g\) adds. Let us write \(A = \{c_n : n < \omega\}\) for the set of those Jensen reals. The model
\[
H = H(L) = \text{HOD}_{A \cup \{A\}}^{L[g]}
\]
of all sets which inside \(L[g]\) are hereditarily definable from parameters in \(\text{OR} \cup A \cup \{A\}\) is the variant of the Cohen–Halpern–Lévy model (over \(L\)) which we shall work with. For the case of Jensen’s original forcing this model was first considered in [4].

For any finite \(a \subseteq A\), we write \(L[a]\) for the model constructed from the finitely many reals in \(a\).

Lemma 2.1 Inside \(H\), \(A\) is a (lightface) \(\Pi_1^1\) set.

Proof. Let \(\varphi(\cdot)\) be the \(\Pi_1^1\) formula from the proof of Lemma 1.7. If \(H \models \varphi(x), x \in L[a], a \in [A]^{<\omega}\), then \(L[a] \models \varphi(x)\) by Shoenfield, so \(x \in a \subseteq A\). On the other hand, if \(e \in A\), then \(L[e] \models \varphi(e)\) and hence \(H \models \varphi(e)\) again by Shoenfield. □

Fixing some Gödelization of formulae (or some enumeration of all the rud functions, resp.) at the outset, each \(L[a]\), \(a \in [A]^{<\omega}\), comes with a unique canonical global well-ordering \(<_a\) of \(L[a]\) by which we mean the one which is induced by the natural order of the elements of \(a\) and the fixed Gödelization device in the usual fashion. The assignment \(a \mapsto <_a, a \in [A]^{<\omega}\), is hence in \(H\).\(^6\) This is a crucial fact.

Let us fix a bijection
\[
(2) \quad e : \omega \rightarrow \omega \times \omega,
\]
and let us write \( ((n)_0, (n)_1) = e(n). \)

We shall also make use the following. Cf. [1, Lemma 1.2].

\(^6\)More precisely, the ternary relation consisting of all \((a, x, y)\) such that \(x <_a y\) is definable over \(H\).

6
Lemma 2.2  (1) Let \( a \in [A]^{<\omega} \) and \( X \subseteq L[a], X \in H, \) say \( X \in HOD_{b\cup(A)}^{L[g]} \), where \( b \supseteq a, b \in [A]^{<\omega}. \) Then \( X \in L[b], \)

(2) There is no well–ordering of the reals in \( H. \)

(3) \( A \) has no countable subset in \( H. \)

(4) \([A]^{<\omega}\) has no countable subset in \( H. \)

Proof sketch. (1) Every permutation \( \pi : \omega \to \omega \) induces an automorphism \( e_{\pi} \) of \( P(\omega) \) by sending \( p \) to \( q, \) where \( q(\pi(n)) = p(n) \) for all \( n < \omega. \) It is clear that no \( e_{\pi} \) moves the canonical name for \( A, \) call it \( \hat{A}. \) Let us also write \( \hat{c}_n \) for the canonical name for \( c_n, n < \omega. \) Now if \( a, b \) as in the statement of (1), say \( b = \{ c_{n_1}, \ldots, c_{n_k} \}, \) if \( p, q \in P(\omega), \) if \( \pi \upharpoonright \{ n_1, \ldots, n_k \} = \text{id}, \) \( p \upharpoonright \{ n_1, \ldots, n_k \} \) is compatible with \( q \upharpoonright \{ n_1, \ldots, n_k \}, \) and \( \text{supp}(\pi(p)) \cap \text{supp}(q) \subseteq \{ n_1, \ldots, n_k \}, \) if \( x \in L, \) if \( \alpha_1, \ldots, \alpha_m \) are ordinals, and if \( \varphi \) is a formula, then

\[
\begin{align*}
\text{ran}(\hat{x}) & \in L[b] \\
\text{ran}(\hat{x}) & \in L[b],
\end{align*}
\]

and \( \pi(p) \) is compatible with \( q, \) so that the statement \( \varphi(\hat{x}, \alpha_1, \ldots, \alpha_m, \hat{c}_{n_1}, \ldots, \hat{c}_{n_k}, \hat{A}) \) will be decided by conditions \( p \in P(\omega) \) with \( \text{supp}(p) \subseteq \{ n_1, \ldots, n_k \}. \) But every set in \( L[b] \) is coded by a set of ordinals, so if \( X \) is as in (1), this shows that \( X \in L[b]. \)

(2) Every real is a subset of \( L. \) Hence by (1), if \( L[g] \) had a well–ordering of the reals in \( HOD_{a\cup(A)}^{L[g]}, \) some \( a \in [A]^{<\omega}, \) then every real of \( H \) would be in \( L[a], \) which is nonsense.

(3) Assume that \( f : \omega \to A \) is injective, \( f \in H. \) Let \( x \in \omega \omega \) be defined by \( x(n) = f((n)_0)(x)_1, \) so that \( x \in H. \) By (1), \( x \in L[a] \) for some \( a \in [A]^{<\omega}. \) But then \( \text{ran}(f) \subseteq L[a], \) which is nonsense, as there is some \( n < \omega \) such that \( c_n \in \text{ran}(f) \setminus a. \)

(4) This readily follows from (3). \( \square \) (Lemma 2.2)

Let us recall another standard fact.

(3) If \( a, b \in [A]^{<\omega}, \) then \( L[a] \cap L[b] = L[a \cap b]. \)

To see this, let us assume without loss of generality that \( a \setminus b \neq \emptyset \neq b \setminus a, \) and say \( a \setminus b = \{ c_n : n \in I \} \) and \( b \setminus a = \{ c_n : n \in J \}, \) where \( I \) and \( J \) are non–empty disjoint finite subsets of \( \omega. \) Then \( a \setminus b \) and \( b \setminus a \) are mutually \( P(I) \)– and \( P(J) \)–generic over \( L[a \cap b]. \) But then \( L[a] \cap L[b] = L[a \cap b][a \setminus b] \cap L[a \cap b][b \setminus a] = L[a \cap b], \) cf. [10, Problem 6.12].

For any \( a \in [A]^{<\omega}, \) we write \( R_a = R \cap L[a] \) and \( R^+_a = R_a \setminus \{ R_b : b \subseteq a \}. \) (\( R^+_a : a \in [A]^{<\omega} \)) is a partition of \( R. \) By Lemma 2.2 (1),

(4) \[
R \cap H = \bigcup\{ R^+_a : a \in [A]^{<\omega} \},
\]

and \( R_a \cap R_b = R_{a \cap b} \) by (3), so that

(5) \[
R^+_a \cap R^+_b = \emptyset \text{ for } a, b \in [A]^{<\omega}, a \neq b.
\]

For \( x \in R, \) we shall also write \( a(x) \) for the unique \( a \in [A]^{<\omega} \) such that \( x \in R^+_a, \) and we shall write \#(x) = \text{Card}(a(x)).
Adrian Mathias showed that in the original Cohen–Halpern–Lévy model there is an definable function which assigns to each \( x \) an ordering \( <_x \) such that \(<_x \) is a well-ordering iff \( x \) can be well-ordered, cf. [8, p. 182]. The following is a special simple case of this, adapted to the current model \( H \).

**Lemma 2.3** (A. Mathias) In \( H \), the union of countably many countable sets of reals is countable.

**Proof.** Let us work inside \( H \). Let \( (A_n: n < \omega) \) be such that for each \( n < \omega \), \( A_n \subset \mathbb{R} \) and there exists some surjection \( f: \omega \to A_n \). For each such pair \( n \), \( f \) let \( y_n.f \in \omega^\omega \) be such that \( y_n.f(m) = f((m)_0)((m)_1) \). If \( a \in [A]^{<\omega} \) and \( y_n.f \in \mathbb{R}_n \), then \( A_n \in L[a] \). By (3), for each \( n \) there is a unique \( a_n \in [A]^{<\omega} \) such that \( A_n \in L[a_n] \) and \( b \supset a_n \) for each \( b \in [A]^{<\omega} \) such that \( A_n \in L[b] \). Notice that \( A_n \) is also countable in \( L[a_n] \).

Using the function \( n \mapsto a_n \), an easy recursion yields a surjection \( g: \omega \to \bigcup\{a_n: n < \omega\} \): first enumerate the finitely many elements of \( a_0 \) according to their natural order, then enumerate the finitely many elements of \( a_1 \) according to their natural order, etc. As \( A \) has no countable subset, \( \bigcup\{a_n: n < \omega\} \) must be finite, say \( a = \bigcup\{a_n: n < \omega\} \in [A]^{<\omega} \). But then \( \{A_n: n < \omega\} \subset L[a] \). (We don’t claim \( \{A_n: n < \omega\} \in L[a] \).)

For each \( n < \omega \), we may now let \( f_n \) the \(<_a\)-least surjection \( f: \omega \to A_n \). Then \( f(n) = f(n)_0((n)_1) \) for \( n < \omega \) defines a surjection from \( \omega \) onto \( \bigcup\{A_n: n < \omega\} \), as desired.

The following is not true in the original Cohen–Halpern–Lévy model. Its proof exploits the Sacks property, Lemma 1.8.

**Lemma 2.4** (1) Let \( M \subset H \) be a null set in \( H \). There is then a \( G_\delta \) null set \( M' \) with \( M' \supset M \) whose code is in \( L \).

(2) Let \( M \subset H \) be a meager set in \( H \). There is then an \( F_\sigma \) meager set \( M' \) with \( M' \supset M \) whose code is in \( L \).

**Proof.** (1) Let \( M \subset H \) be a null set in \( H \).

Let us work in \( H \). Let \( (\epsilon_n: n < \omega) \) be any sequence of positive reals. Let \( \bigcup_{s \in X} U_s \supset H \), where \( X \subset \omega^{<\omega} \) and \( \mu(\bigcup\{U_s: s \in X\}) \leq \epsilon_0 \).\(^7\) Let \( \epsilon: \omega \to X \) be onto. Let \( (k_n: n < \omega) \) be defined by: \( k_n = \) the smallest \( k \) (strictly bigger than \( k_{n-1} \) if \( n > 0 \)) such that \( \mu(\bigcup\{U_s: s \in \epsilon^n \omega \setminus k\}) \leq \epsilon_n \). Write \( k_{-1} = 0 \). We then have that \( \mu(\bigcup\{U_s: s \in \epsilon^n[k_{n-1}, k_n]\}) \leq \epsilon_n \) for every \( n < \omega \).

Now fix \( \epsilon > 0 \). Let

\[
\epsilon_n = \frac{\epsilon}{n \cdot 2^{n+2}},
\]

and let \( (k_n: n < \omega) \) and \( \epsilon: \omega \to \omega^{<\omega} \) be such that \( \bigcup_{s \in X} U_s \supset H \) and \( \mu(\bigcup\{U_s: s \in \epsilon^n[k_{n-1}, k_n]\}) \leq \epsilon_n \) for every \( n < \omega \). We may now apply Lemma 1.8 inside \( L[a] \) for some \( a \in [A]^{<\omega} \) such that \( \{e, (k_n: n < \omega)\} \subset L[a] \) and find a function \( g \in L \) with domain \( \omega \) such that for each \( n < \omega \), \( g(n) \) is a finite union \( U_n \) of basic open sets such that \( \{U_s: s \in \epsilon^n[k_{n-1}, k_n]\} \subset U_n \) and \( \mu(U_n) \leq \frac{1}{2^{n+2}} \). But then \( O = \bigcup\{O_n: n < \omega\} \supset M \) is open, \( O \) is coded in \( L \) (i.e., there is \( Y \in L, Y \subset \omega^{<2} \), with \( O = \bigcup\{U_s: s \in Y\} \), and \( \mu(O) \leq \epsilon \).

---

\(^7\)Here, \( \mu \) denotes Lebesgue measure.
We may hence for every $n < \omega$ let $\mathcal{O}_n$ be an open set with $\mathcal{O}_n \supset M$, $\mu(\mathcal{O}_n) \leq \frac{1}{2^n+1}$, and whose code in $L$ is $<_L$-least among all the codes giving such a set. Then $\bigcap\{\mathcal{O}_n : n < \omega\}$ is a $G_3$ null set with code in $L$ and which covers $M$.

(2) Let $M \in H$ be a meager set in $H$, say $M = \bigcup\{N_n : n < \omega\}$, where each $N_n$ is nowhere dense.

Let us again work in $H$. It is easy to verify that a set $P \subset \omega^2$ is nowhere dense iff there is some $z \in \omega^2$ and some strictly increasing $(k_n : n < \omega)$ such that for all $n < \omega$,

$$\{x \in \omega^2 : x \upharpoonright [k_n, k_{n+1}) = z \upharpoonright [k_n, k_{n+1})\} \cap P = \emptyset.$$

Look at $f : \omega \to \omega$, where $f(m) = k_{n+1}$ for the least $n$ with $m \leq k_n$. We may first apply Lemma 1.8 inside $L[a]$ for some $a \in [A]^{<\omega}$ such that $f \in L[a]$ and get a function $g : \omega \to \omega$, $g \in L$, such that $g(m) \geq f(m)$ for all $m < \omega$. Write $\ell_0 = 0$ and $\ell_{n+1} = g(\ell_n)$, so that for each $n$ there is some $n' \in \ell_{n+1}$.

Define $e : \omega \to \omega$ by $e(n) = \sum_{q=0}^{n}(q+1)2^{q+1}$. We may now apply Lemma 1.8 inside $L[a]$ for some $a \in [A]^{<\omega}$ such that $f \in L[a]$ and get some $n \mapsto (z^n_i : i \leq (n+1) \cdot 2^{n+1})$ inside $L$ such that for all $n$, $i$, $z^n_i : e(n) \to 2$, and for all $n$ there is some $i$ with $z \upharpoonright e(n) = z^n_i$. From this we get some $z' : \omega \to \omega$, $z' \in L$, such that for all $n$ there is some $n'$ with $z' \upharpoonright \ell_{n', \ell_{n'+1}} = z \upharpoonright \ell_{n', \ell_{n'+1}}$. But then, writing

$$D = \{x \in \omega^2 : \exists n x \upharpoonright [\ell_{e(n)}, \ell_{e(n+1)}) = z' \upharpoonright [\ell_{e(n)}, \ell_{e(n+1)})\},$$

$D \in L$, and $D$ is open and dense.

We may hence for every $n < \omega$ let $\mathcal{O}_n$ be an open dense set with $\mathcal{O}_n \cap N_n = \emptyset$, whose code in $L$ is $<_L$-least among all the codes giving such a set. Then $\bigcup\{\omega^2 \setminus \mathcal{O}_n : n < \omega\}$ is an $F_\sigma$ meager set with code in $L$ and which covers $M$. $\square$

**Corollary 2.5** In $H$, there is a $\Delta^1_2$ Sierpiński set as well as a $\Delta^1_2$ Luzin set.

**Proof.** There is a $\Delta^1_2$ Luzin set in $L$. By Lemma 2.4 (2), any such set is still a Luzin set in $H$. The same is true with “Luzin” replaced by “Sierpiński” and Lemma 2.4 (2) replaced by Lemma 2.4 (1). $\square$

**Lemma 2.6** In $H$, there is a $\Delta^1_3$ Bernstein set.

**Proof.** In this proof, let us think of reals as elements of the Cantor space $\omega^2$.

Let us work in $H$.

We let

$$B = \{x \in \mathbb{R} : \exists \text{ even } n \left(2^n < \#(x) \leq 2^{n+1}\right)\} \quad \text{and} \quad B' = \{x \in \mathbb{R} : \exists \text{ odd } n \left(2^n < \#(x) \leq 2^{n+1}\right)\}.$$

Obviously, $B \cap B' = \emptyset$.

Let $P \subset \mathbb{R}$ be perfect. We aim to see that $P \cap B \neq \emptyset \neq P \cap B'$.
Say \( P = \{ T \in \omega^2 : \forall n \ x \ | \ n \in T \} \), where \( T \subseteq \omega^2 \) is a perfect tree. Modulo some fixed natural bijection \( \omega^2 \leftrightarrow \omega \), we may identify \( T \) with a real. By (4), we may pick some \( a \in [A]^{<\omega} \) such that \( T \subseteq L[a] \). Say \( \text{Card}(a) < 2^n \), where \( n \) is even.

Let \( b \in [A]^{2^{n+1}} \), \( b \supset a \), and let \( x \in \mathbb{R}^+ \). In particular, \( \#(x) = 2^{n+1} \). It is easy to work in \( L[b] \) and construct some \( z \in [T] \) such that \( x \leq_T z \oplus T \), \(^8 \) e.g., arrange that if \( z \nmid m \) is the \( k \)th splitting node of \( T \) along \( z \), where \( k \leq m < \omega \), then \( z(m) = 0 \) if \( x(k) = 0 \) and \( z(m) = 1 \) if \( x(k) = 1 \).

If we had \( \#(z) \leq 2^n \), then \( \#(z \oplus T) \leq \#(z) + \#(T) < 2^n + 2^n = 2^{n+1} \), so that \( \#(z) = 2^{n+1} \) by \( x \leq_T z \oplus T \). Contradiction! Hence \( \#(z) > 2^n \). By \( z \in L[b] \), \( \#(z) \leq 2^{n+1} \). Therefore, \( z \in P \cap B \).

The same argument shows that \( P \cap B' \neq \emptyset \). \( B \) (and also \( B' \)) is thus a Bernstein set.

We have that \( x \in B \) iff

\[
\exists a \in [A]^{<\omega} \exists n \exists J_\alpha[a] \quad (x \in J_\alpha[a] \land 2^n < \text{Card}(a) \leq 2^{n+1} \land \forall b \subseteq a \forall J_\beta[b] x \notin J_\beta[b]),
\]

which is true iff

\[
\forall a \in [A]^{<\omega} \forall J_\alpha[a] (x \in J_\alpha[a] \rightarrow \exists a' \subset a \exists n \exists J_\alpha'[a']) (x \in J_\alpha'[a'] \land 2^n < \text{Card}(a) \leq 2^{n+1} \land \forall b \subseteq a' \forall J_\beta[b] x \notin J_\beta[b]).
\]

By Lemma 2.1, this shows that \( B \) is \( \Delta_3^1 \). \( \square \)

Recall that for any \( a \in [A]^{<\omega} \), we write \( R_a = \mathbb{R} \cap L[a] \). Let us now also write \( R_{<a} = \text{span}([ \mathbb{R}_a \setminus b \subseteq a ]) \), and \( R^*_a = R_a \setminus R_{<a} \). In particular, \( R_{<\emptyset} = \{0\} \) by our above convention that \( \text{span}(\emptyset) = \{0\} \), and \( R^*_a = (\mathbb{R} \cap L) \setminus \{0\} \).

The proof of Claim 2.8 below will show that

\[
(9) \quad \mathbb{R} \cap H = \text{span}\left( \bigcup \{ R^*_a : a \in [A]^{<\omega} \} \right).
\]

Also, we have that \( R^*_a \subset \mathbb{R}^+_a \), so that by (5),

\[
(10) \quad R^*_a \cap R^*_b = \emptyset \quad \text{for } a, b \in [A]^{<\omega}, a \neq b.
\]

**Lemma 2.7** In \( H \), there is a \( \Delta_3^1 \) Hamel basis.

**Proof.** We call \( X \subset R^*_a \) linearly independent over \( R_{<a} \) iff whenever

\[
\sum_{n=1}^{m} q_n \cdot x_n \in R_{<a},
\]

where \( m \in \mathbb{N}, m \geq 1 \), and \( q_n \in \mathbb{Q} \) and \( x_n \in X \) for all \( n, 1 \leq n \leq m \), then \( q_1 = \ldots = q_m = 0 \). In other words, \( X \subset R^*_a \) is linearly independent over \( R_{<a} \) iff

\[
\text{span}(X) \cap R_{<a} = \{0\}.
\]

\( ^8 \)Here, \( (x \oplus y)(2n) = x(n) \) and \( (x \oplus y)(2n + 1) = y(n), n < \omega. \)
We call \( X \subset \mathbb{R}_a^* \) maximal linearly independent over \( \mathbb{R}_{<a} \) iff \( X \) is linearly independent over \( \mathbb{R}_{<a} \) and no \( Y \supseteq X, Y \subset \mathbb{R}_a^* \) is still linearly independent over \( \mathbb{R}_{<a} \). In particular, \( X \subset \mathbb{R}_a^* = (\mathbb{R} \cap L) \setminus \{0\} \) is linearly independent over \( \mathbb{R}_{<\emptyset} = \{0\} \) iff \( X \) is a Hamel basis for \( \mathbb{R} \cap L \).

For any \( a \in [A]^{<\omega} \), we let \( b_a = \{x_i^a : i < \theta^a\} \), some \( \theta^a \leq \omega_1 \), be the unique set such that

(i) for each \( i < \theta^a \), \( x_i^a \) is the \( <_a \)-least \( x \in \mathbb{R}_a^* \) such that \( \{x_j^a : j < i\} \cup \{x\} \) is linearly independent over \( \mathbb{R}_{<a} \), and

(ii) \( b_a \) is maximal linearly independent over \( \mathbb{R}_{<a} \).

By the above crucial fact, the function \( a \mapsto b_a \) is well–defined and exists inside \( H \). In particular,

\[
B = \bigcup \{b_a : a \in [A]^{<\omega}\}
\]

is an element of \( H \).

We claim that \( B \) is a Hamel basis for the reals of \( H \), which will be established by Claims 2.8 and 2.9.

**Claim 2.8** \( \mathbb{R} \cap H \subset \text{span}(B) \).

**Proof** of Claim 2.8. Assume not, and let \( n < \omega \) be the least size of some \( a \in [A]^{<\omega} \) such that \( \mathbb{R}_a^* \setminus \text{span}(B) \neq \emptyset \). Pick \( x \in \mathbb{R}_a^* \setminus \text{span}(B) \neq \emptyset \), where \( \text{Card}(a) = n \).

We must have \( n > 0 \), as \( b_0 \) is a Hamel basis for the reals of \( L \). Then, by the maximality of \( b_a \), while \( b_a \) is linearly independent over \( \mathbb{R}_{<a} \), \( b_a \cup \{x\} \) cannot be linearly independent over \( \mathbb{R}_{<a} \). This means that there are \( q \in \mathbb{Q}, q \neq 0, m \in \mathbb{N}, m \geq 1, \) and \( q_n \in \mathbb{Q} \setminus \{0\} \) and \( x_n \in b_a \) for all \( n, 1 \leq n \leq m \), such that

\[
z = q \cdot x + \sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a}.
\]

By the definition of \( \mathbb{R}_{<a} \) and the minimality of \( n \), \( z \in \text{span}(\bigcup \{b_c : c \subseteq a\}) \), which then clearly implies that \( x \in \text{span}(\bigcup \{b_c : c \subseteq a\}) \subset \text{span}(B) \).

This is a contradiction! \( \square \) (Claim 2.8)

**Claim 2.9** \( B \) is linearly independent.

**Proof** of Claim 2.9. Assume not. This means that there are \( 1 \leq k < \omega, a_i \in [A]^{<\omega} \) pairwise different, \( m_i \in \mathbb{N}, m_i \geq 1 \) for \( 1 \leq i \leq k \), and \( q_n^i \in \mathbb{Q} \setminus \{0\} \) and \( x_n^i \in b_{a_i} \) for all \( i \) and \( n \) with \( 1 \leq i \leq k \) and \( 1 \leq n \leq m_i \) such that

\[
\sum_{n=1}^{m_i} q_n^i \cdot x_n^i + \ldots + \sum_{n=1}^{m_k} q_n^k \cdot x_n^k = 0.
\]

By the properties of \( b_{a_i} \), \( \sum_{n=1}^{m_i} q_n^i \cdot x_n^i \in \mathbb{R}_{a_i}^* \), so that (11) buys us that there are \( z_i \in \mathbb{R}_{a_i}^*, z_i \neq 0, 1 \leq i \leq k, \) such that

\[
z_1 + \ldots + z_k = 0.
\]
There must be some $i$ such that there is no $j$ with $a_j \supseteq a_i$, which implies that $a_j \cap a_i \subsetneq a_i$ for all $j \neq i$. Let us assume without loss of generality that $a_j \cap a_i \subsetneq a_i$ for all $j, 1 < j \leq k$.

Let $a_1 = \{c_\ell : \ell \in I\}$, where $I \in [\omega]^{<\omega}$, and let $a_j \cap a_1 = \{c_\ell : \ell \in I_j\}$, where $I_j \subsetneq I$, for $1 < j \leq l$.

In what follows, a nice name $\tau$ for a real is a name of the form

$$\tau = \bigcup_{n,m < \omega} \{(n,m)^\gamma\} \times A_{n,m},$$

where each $A_{n,m}$ is a maximal antichain of conditions of the forcing in question deciding that $\tau(n) = \bar{m}$.

We have that $z_1$ is $\mathbb{P}(I)$-generic over $L$, so that we may pick a nice name $\tau_1 \in L^{\mathbb{P}(I)}$ for $z_1$ with $(\tau_1)^{\mathbb{P}(I)} = z_1$. Similarly, for $1 < j \leq k$, $z_j$ is $\mathbb{P}(I_j)$-generic over $L[g \restriction (\omega \setminus I)]$, so that we may pick a nice name $\tau_j \in L[g \restriction (\omega \setminus I)]^{\mathbb{P}(I_j)}$ for $z_j$ with $(\tau_j)^{\mathbb{P}(I_j)} = z_j$. We may construe each $\tau_j, 1 < j \leq k$, as a name in $L[g \restriction (\omega \setminus I)]^{\mathbb{P}(I)}$ by replacing each $p: I_j \to \mathbb{P}$ in an antichain as in (13) by $p': I \to \mathbb{P}$, where $p'(\ell) = p(\ell)$ for $\ell \in I_j$ and $p'(\ell) = \emptyset$ otherwise. Let $p \in g \restriction I$ be such that

$$p \restriction_{L[g \restriction (\omega \setminus I)]} \tau_1 + \tau_2 + \ldots + \tau_k = 0.$$

We now have that inside $L[g \restriction (\omega \setminus I)]$, there are nice $\mathbb{P}(I)$-names $\tau'_j, 1 < j \leq k$ (namey, $\tau'_j, 1 < j \leq k$), such that still inside $L[g \restriction (\omega \setminus I)]$

(1) $p \restriction_{L[g \restriction (\omega \setminus I)]} \tau_1 + \tau'_2 + \ldots + \tau'_k = 0,$ and

(2) for all $j, 1 < j \leq k$ and for all $p$ in one of the antichains of the nice name $\tau'_j$, $\supp(p) \subseteq I_j$.

By Lemma 1.4, the nice names $\tau_1, \tau'_2, \ldots, \tau'_k$ may be coded by reals, and both (1) and (2) are arithmetic in such real codes for $\tau_1, \tau'_2, \ldots, \tau'_k$, so that by $\tau_1 \in L^{\mathbb{P}(I)}$ and $\Sigma^1_1$-absoluteness between $L$ and $L[g \restriction (\omega \setminus I)]$ there are inside $L$ nice $\mathbb{P}(I)$-names $\tau''_j, 1 < j \leq k$, such that in $L$, (1) and (2) hold true. But then, writing $z'_1 = (\tau''_j)^{\mathbb{P}(I)}$, we have by (2) that $z'_1 \in \mathbb{R}_{I_j}$ for $1 < j \leq k$, and $z_1 + z'_2 + \ldots + z'_k = 0$ by (1). But then $z_1 \in \mathbb{R}^k \cap \mathbb{R}_{<k}$, which is absurd. \(\square\) (Claim 2.9)

We now have that $x \in B$ iff

$$\exists a \in [A]^{<\omega} \exists J_0[a] \exists (x_i : i \leq \theta) \in J_0[a] \exists X \subset \theta + 1 \text{ (the } x_i \text{ enumerate the first } \theta + 1 \text{ reals in } J_0[a] \text{ acc. to } <_a \land \theta \in X \land x = x_\theta \land \forall i \in \theta \setminus X \exists J_\beta[a] J_\beta[a] \models \{x_j : j \in X \cap i\} \cup \{x_i\} \text{ is not linearly independent over } \mathbb{R}_{<a} \land \forall i \in X \forall J_\beta[a] J_\beta[a] \models \{x_j : j \in X \cap i\} \cup \{x_i\} \text{ is linearly independent over } \mathbb{R}_{<a}),$$

which is true iff

$$\forall a \in [A]^{<\omega} \forall J_0[a] \forall (x_i : i \leq \theta) \in J_0[a] \forall X \subset \theta + 1 \text{ (the } x_i \text{ enumerate the first } \theta + 1 \text{ reals in } J_0[a] \text{ acc. to } <_a \land x = x_\theta \land \forall i \in (\theta + 1) \setminus X \exists J_\beta[a] J_\beta[a] \models \{x_j : j \in X \cap i\} \cup \{x_i\} \text{ is not linearly independent over } \mathbb{R}_{<a} \land \forall i \in X \forall J_\beta[a] J_\beta[a] \models \{x_j : j \in X \cap i\} \cup \{x_i\} \text{ is linearly independent over } \mathbb{R}_{<a} \rightarrow \theta \in X).$$

By Lemma 2.1, this shows that $B$ is $\Delta^3_3$. \(\square\)
References


