

An unpublished theorem of Solovay, revisited

Ali Enayat* Vladimir Kanovei†

February 20, 2020

Abstract

A definable pair of disjoint non-OD sets of reals (hence, indiscernible sets) exists in the Sacks and E_0 -large generic extensions of the constructible universe L .

1	Introduction	1
2	Outline of the proof	3
3	Canonization results used in the proof	3
4	Corralling maps, Sacks case	5
5	Corralling maps, E_0 -large case	7
6	Increasing system of equivalence relations	8
7	Proof of the main theorem	9
8	Final remarks	11
9	History of this result	12
	References	13

1 Introduction

Let a *twin partition* be any partition of a given set U into two nonempty cells A and B . We refer to U as *the universe of discourse*, and each of A and B as a twin. Assume that some robust notion of definability D is chosen in advance, e.g., D might be ordinal definability OD, or D might be Δ_1^1 definability, or something similar. In this context, a twin partition $U = A \cup B$ can be called D -definable in one of two senses:

strongly D -definable, i.e., each of the twins A and B is D -definable;

weakly D -definable, meaning that the partition $\{A, B\}$ of U , considered as an unordered pair, is D -definable.

Strong D -definability clearly implies weak D -definability. The “twin problem” for a given notion of definability D is whether the converse holds. The twin problem obviously has a positive answer provided the domain of discourse U

*University of Gothenburg, Gothenburg, Sweden, ali.enayat@gu.se.

†IITP RAS, kanovei@googlegmail.com.

contains at least one D -definable element x , then one cell of the partition consists of those x' that share the same cell of the partition as x , and the other cell is just the complementary set. This provides a trivial positive solution for the twin problem when $U = \omega$, or when U is the class of ordinals, and generally when U admits a D -definable well-ordering. Now let's focus on the case when U is the set of real numbers.

The twin problem admits a positive solution in the case of Δ_1^1 definability. Indeed it follows from Theorem 3.1 below that if a Δ_1^1 equivalence relation E on a Δ_1^1 set U of reals has precisely two (or even countably many) equivalence classes then each E -class is itself a Δ_1^1 set. The problem also admits a positive solution in the case of Δ_2^1 definability because every non-empty Σ_2^1 set of reals contains a Δ_2^1 element (see, e.g., 4E.5 in Moschovakis [13]). But slightly above of Δ_2^1 there is a significant obstacle, as indicated by the following theorem.

Theorem 1.1 (the Sacks part originally by Solovay¹). *Let $a \in 2^\omega$ be either Sacks generic or E_0 -large generic² over \mathbf{L} . Then it is true in $\mathbf{L}[a]$ that there is a Σ_2^1 equivalence relation Q on 2^ω with exactly three equivalence classes, one of which is equal to $2^\omega \cap \mathbf{L}$, while two others are non-OD sets whose union is equal to the Π_2^1 set $2^\omega \setminus \mathbf{L}$.*

Under the assumptions of this theorem, we have we have a *weakly definable*, but not *strongly definable*, partition of the Π_2^1 set $U = 2^\omega \setminus \mathbf{L}$ into two equivalence classes of Q . Let A, B be those equivalence classes. As the relation Q is lightface Σ_2^1 , the unordered pair $\{A, B\}$ is an OD set, basically, a definable set, whose two elements (disjoint non-empty pointsets $A, B \subseteq 2^\omega \setminus \mathbf{L}$) are non-OD, hence, are OD-*indiscernible*.

Models of **ZF** or **ZFC** containing OD indiscernible pairs of (non-OD) disjoint sets of reals are well-known. Such is e.g. any Sacks \times Sacks extension $\mathbf{L}[a, b]$ of \mathbf{L} , where an OD pair of non-OD sets consists of the \mathbf{L} -degrees of the Sacks reals a, b , see [6] and also [2, 4]. Another model with an OD pair of *countable* disjoint non-OD sets is defined in [5]. Yet those examples fail to fulfill the property that the union of the two sets is equal to the whole domain of nonconstructible reals.

Generally, OD indiscernible pairs (not necessarily OD pairs) of disjoint sets of reals can be extracted from early works on Cohen forcing. In particular, if $\langle a, b \rangle$ is a Cohen-generic, over \mathbf{L} , pair of $a, b \in 2^\omega$, then the E_0 equivalence classes $[a]_{E_0}, [b]_{E_0}$ are OD indiscernible in $\mathbf{L}[a, b]$ (essentially by Feferman [3]) and so are the constructibility degrees $[a]_{\mathbf{L}} = \{x \in 2^\omega : \mathbf{L}[x] = \mathbf{L}[a]\}$ and $[b]_{\mathbf{L}}$ [4].

On the other hand, it is established in [8] that, in some models of **ZFC**, including the Sacks extension of the constructible universe \mathbf{L} , it is true that any

¹ See Section 9 on the history of the result

² That is, generic w.r.t. the forcing by perfect sets $P \subseteq 2^\omega$ such that the restricted relation $E_0 \upharpoonright P$ is not smooth, see below. Recall that the equivalence relation E_0 is defined on 2^ω so that $x E_0 y$ iff the set $\Delta(x, y) = \{k : x(k) \neq y(k)\}$ is finite.

countable OD (ordinal-definable) *set of reals* consists of OD elements. A similar result in much more general setting is known from [1, Thm 4.8] under a strong large cardinal hypothesis.

2 Outline of the proof

To prove Theorem 1.1, the required equivalence relation will be obtained as the union of an increasing transfinite sequence $\langle \mathbf{B}_\alpha \rangle_{\alpha < \omega_1}$ of **countable** Borel equivalence relations. The sequence is defined in \mathbf{L} , the ground universe. The following is a principal definition related to this construction.

Definition 2.1. A *double-bubble system*, DBS for brevity, is a pair of *countable* Borel equivalence relations $\langle \mathbf{B}, \mathbf{E} \rangle$ on 2^ω , such that each \mathbf{E} -class is the union of a pair of distinct \mathbf{B} -classes.

A DBS $\langle \mathbf{B}', \mathbf{E}' \rangle$ *extends* $\langle \mathbf{B}, \mathbf{E} \rangle$, in symbol $\langle \mathbf{B}, \mathbf{E} \rangle \preceq \langle \mathbf{B}', \mathbf{E}' \rangle$, if $\mathbf{B} \subseteq \mathbf{B}'$, $\mathbf{E} \subseteq \mathbf{E}'$, and for any $x, y \in 2^\omega$, if $x \mathbf{E} y$ but $x \not\mathbf{B} y$ then we still have $x \mathbf{B}' y$. \square

Thus the extension essentially means that the equivalence classes of the original equivalence relations are merged in countable bunches, but in such a way that the two \mathbf{B} -classes within the same \mathbf{E} -class are never merged. We are going to define a certain \preceq -increasing increasing sequence $\langle \langle \mathbf{B}_\alpha, \mathbf{E}_\alpha \rangle \rangle_{\alpha < \omega_1}$ of double-bubble systems $\langle \mathbf{B}_\alpha, \mathbf{E}_\alpha \rangle$ in \mathbf{L} , the ground universe, and $\mathbf{B} = \bigcup_\alpha \mathbf{B}_\alpha$ will be the equivalence relation required. This will take some effort.

Example 2.2. The most elementary example is $\mathbf{B} =$ the equality, and $x \mathbf{E} y$ iff $x(k) = y(k)$ for all $k \geq 1$; $\langle \mathbf{B}, \mathbf{E} \rangle$ is a DBS.

Another example consists of the equivalence relation \mathbf{E}_0 (see Footnote 2), and its subrelation $\mathbf{E}_0^{\text{even}}$, defined so that $x \mathbf{E}_0 y$ iff the set $\Delta(x, y)$ has finite even number of elements; $\langle \mathbf{E}_0^{\text{even}}, \mathbf{E}_0 \rangle$ is a DBS and obviously $\langle \mathbf{B}, \mathbf{E} \rangle \preceq \langle \mathbf{E}_0^{\text{even}}, \mathbf{E}_0 \rangle$. \square

3 Canonization results used in the proof

Here we present some well-known results of modern descriptive set theory involved in the proof of Theorem 1.1. We begin with the Silver Dichotomy theorem and a canonization corollary. See e.g. [14, Theorem 2.2] or [9, Section 10.1] for a proof of the “moreover” lightface version of Theorem 3.1.

Theorem 3.1 (Silver’s Dichotomy [15]). *Suppose that \mathbf{E} is a Π_1^1 equivalence relation on a Borel set $X \subseteq 2^\omega$. Then either \mathbf{E} has at most countably many equivalence classes, or there exists a perfect partial \mathbf{E} -transversal³.*

If moreover X is lightface Δ_1^1 and \mathbf{E} is lightface Π_1^1 then all equivalence classes are lightface Δ_1^1 in the “either” case. \square

³ A *partial transversal* is a set of pairwise inequivalent elements. A *full transversal* requires that in addition it has a non-empty intersection with any equivalence class in a given domain.

Corollary 3.2. *Suppose that E is a Π_1^1 equivalence relation on a Borel set $X \subseteq 2^\omega$. Then there is a perfect set $Y \subseteq X$ such that E coincides on Y with:*

- either (I) the total equivalence TOT making all reals equivalent;
- or (II) the equality, so that Y is a partial E -transversal.

If in addition E is countable⁴ then (I) is impossible.

Proof. In the “or” case of Theorem 3.1 we have (II). In the “either” case pick an uncountable equivalence class C and let $Y \subseteq C$ be any perfect set. \square

Corollary 3.3. *If $X \subseteq 2^\omega$ is a perfect set, and $f : X \rightarrow 2^\omega$ a Borel map, then there is a perfect set $Y \subseteq X$ such that $f \upharpoonright Y$ is a bijection or a constant.*

Proof. This is a well-known fact, of course, yet it immediately follows from Corollary 3.2. Indeed define a Borel equivalence relation E on X such that $x E y$ iff $f(x) = f(y)$. Apply Corollary 3.2. \square

Now we recall some definitions and results related to E_0 -large sets. A Borel set $X \subseteq 2^\omega$ is called E_0 -large if $E_0 \upharpoonright X$ is still a non-smooth⁵ equivalence relation. For instance 2^ω itself is E_0 -large, while any Borel partial E_0 -transversal is not. If $\mathbf{u} = \langle u_n^i \rangle_{n < \omega, i=0,1}$ is an array of strings $u_n^i \in 2^{<\omega}$, satisfying $\text{lh}(u_n^0) = \text{lh}(u_n^1) \geq 1$ and $u_n^0 \neq u_n^1$ for all n , then we call \mathbf{u} a E_0 -matrix, let

$$x_{\mathbf{u}}^a = u_0^{a(0)} \frown u_1^{a(1)} \frown u_2^{a(2)} \frown \dots \frown u_n^{a(n)} \frown \dots \in 2^\omega.$$

for any $a \in 2^\omega$, and define a canonical E_0 -large set $\mathcal{X}_{\mathbf{u}} = \{x_{\mathbf{u}}^a : a \in 2^\omega\}$. Each canonical E_0 -large set $\mathcal{X}_{\mathbf{u}}$ is perfect, and E_0 -large via the map $a \mapsto x_{\mathbf{u}}^a$. On the other hand, it is known (see e.g. [10, Section 7.1]) that each (Borel) E_0 -large set $X \subseteq 2^\omega$ contains a canonical E_0 -large subset $Y \subseteq X$.

If further $\mathbf{v} = \langle v_n^i \rangle_{n < \omega, i=0,1}$ is another E_0 -matrix, then we define a homeomorphism and E_0 -isomorphism $h_{\mathbf{u}\mathbf{v}} : \mathcal{X}_{\mathbf{u}} \xrightarrow{\text{onto}} \mathcal{X}_{\mathbf{v}}$ such that $h_{\mathbf{u}\mathbf{v}}(x_{\mathbf{u}}^a) = x_{\mathbf{v}}^a$ for all $a \in 2^\omega$. Maps of the form $h_{\mathbf{u}\mathbf{v}}$ will be called *canonical E_0 -large maps*.

Theorem 3.4 (Theorem 7.1 in [10], or else [12]). *Suppose that E is a Borel equivalence relation on 2^ω , and $X \subseteq 2^\omega$ is a E_0 -large set. Then there is a canonical E_0 -large set $Y \subseteq X$ such that E coincides on Y with:*

- either (I) the total equivalence relation TOT ;
- or (II) the relation E_0 ;

⁴ An equivalence relation is *countable* iff all its equivalence classes are at most countable.

⁵ Recall that an equivalence relation E on a Borel set X is *smooth* if there is a Borel map $f : X \rightarrow 2^\omega$ such that we have $x E y$ iff $f(x) = f(y)$ for all $x, y \in X$. The equivalence relation E_0 is non-smooth on 2^ω , meaning that such a Borel f does not exist. See Example 6.5 in [11].

– or (III) the equality.

In addition, if E is a countable equivalence relation then (I) is impossible, while if $E_0 \subseteq E$ then (III) is impossible. \square

Corollary 3.5. *If $X \subseteq 2^\omega$ is a Borel E_0 -large set, and $Z \subseteq X$ a Borel set, then there is a canonical E_0 -large set $Y \subseteq X$ such that $Y \subseteq Z$ or $Y \cap Z = \emptyset$.*

Proof. Define a Borel equivalence relation E on X such that $x F y$ iff $x, y \in Z$ or $x, y \in X \setminus Z$. Apply Theorem 3.4. As E has just two equivalence classes, only (I) is possible. \square

Corollary 3.6. *If $X \subseteq 2^\omega$ is a Borel E_0 -large set, and $f : X \rightarrow 2^\omega$ a Borel map, then there exists a canonical E_0 -large set $Y \subseteq X$ such that $f \upharpoonright Y$ is a bijection or a constant.*

Proof. Define a Borel equivalence relation E on X such that $x E y$ iff $f(x) = f(y)$. Apply Theorem 3.4. We have to prove that (II) is impossible. Suppose to the contrary that $E = E_0$ on a canonical E_0 -large set $Y \subseteq X$. In other words, we have $f(x) = f(y)$ iff $x E_0 y$ for all $x, y \in Y$. Thus f is a Borel reduction of $E_0 \upharpoonright Y$ to the equality, which contradicts to the assumption that Y is E_0 -large. \square

As a forcing notion, the set \mathbb{P}_{E_0} of all canonical E_0 -large (perfect) sets adjoins reals of minimal degree, preserves \aleph_1 , and has some other remarkable properties resembling the Sacks forcing, see e.g. [10, Section 7.1] and references thereof.

4 Corraling maps, Sacks case

Definition 4.1. Given a set $X \subseteq 2^\omega$ and a map $f : X \rightarrow 2^\omega$, a DBS $\langle B, E \rangle$:

- *corralls* f if $f(x) \in [x]_E$ for all $x \in X$;
- *positively corralls* f if $f(x) \in [x]_B$ for all $x \in X$;
- *negatively corralls* f if $f(x) \in [x]_E \setminus [x]_B$ for all $x \in X$. \square

Lemma 4.2. *Assume that $\langle B, E \rangle$ is a DBS, $X \subseteq 2^\omega$ is a perfect set, and $f : X \rightarrow 2^\omega$ is Borel and 1-1. There exist a perfect set $Y \subseteq X$ and a DBS $\langle B', E' \rangle$ which extends $\langle B, E \rangle$ and corralls $f \upharpoonright Y$.*

Proof. The sets $X' = \{x \in X : x E f(x)\}$ and $X'' = \{x \in X : x \not E f(x)\}$ are Borel, hence there is a perfect set X_0 with either $X_0 \subseteq X'$ or $X_0 \subseteq X''$. But if $X_0 \subseteq X'$ then $\langle B, E \rangle$ already corralls $f \upharpoonright X_0$, and we are done. Thus we assume that $X_0 \subseteq X''$, that is, $x \not E f(x)$ for all $x \in X_0$.

By Corollary 3.2, there is a perfect set $X_1 \subseteq X_0$ such that E, B coincide with the equality on X_1 . Define an equivalence relation \hat{E} on X_1 such that $x \hat{E} y$ iff $f(x) E f(y)$, and define \hat{B} similarly. Consider the \subseteq -minimal equivalence relation F defined on 2^ω such that $E \subseteq F$ and if $x, y \in 2^\omega$ and $f(x) E y$ then

$x \mathbf{F} y$. Thus $\widehat{\mathbf{E}}, \widehat{\mathbf{B}}, \mathbf{F}$ are countable Borel equivalence relations on X_1 . (The borelness of \mathbf{F} holds since all intended quantifiers in the definition of \mathbf{F} are over countable domains.) By Corollary 3.2, there is a perfect set $Y \subseteq X_1$ such that $\widehat{\mathbf{E}}, \widehat{\mathbf{B}}, \mathbf{F}$ coincide with the equality on Y , along with \mathbf{E}, \mathbf{B} . It follows, by the choice of X_0 , that if $x, y \in Y$ (whether equal or not) then $x \notin f(y)$.

We define the equivalence relations \mathbf{E}', \mathbf{B}' as follows.

If $x \in 2^\omega$ and the \mathbf{E} -class $[x]_{\mathbf{E}}$ does **not** intersect *the critical domain* $\Delta = Y \cup \{f(x) : x \in Y\}$, then put $[x]_{\mathbf{E}'} = [x]_{\mathbf{E}}$ and $[x]_{\mathbf{B}'} = [x]_{\mathbf{B}}$, so such a \mathbf{E} -class and its \mathbf{B} -subclasses are not changed. But within Δ some classes will be merged. Namely if $x \in Y$ then we have to merge $[x]_{\mathbf{E}}$ with $[f(x)]_{\mathbf{E}}$, hence put

$$[x]_{\mathbf{E}'} = [x]_{\mathbf{E}} \cup [f(x)]_{\mathbf{E}} \quad \text{and} \quad [x]_{\mathbf{B}'} = [x]_{\mathbf{B}} \cup [f(x)]_{\mathbf{B}},$$

and define the other \mathbf{B}' -class within $[x]_{\mathbf{E}'}$ as $[x]_{\mathbf{E}'} \setminus [x]_{\mathbf{B}'}$.

A routine verification shows that in either case the relations \mathbf{E}', \mathbf{B}' are Borel, and the pair $\langle \mathbf{B}', \mathbf{E}' \rangle$ is a DBS which extends $\langle \mathbf{B}, \mathbf{E} \rangle$ and positively corralls $f \upharpoonright Y$ (because we have $f(x) \in [x]_{\mathbf{B}'}$ for all $x \in Y$ simply by construction). \square

Lemma 4.3. *Let $\langle \mathbf{B}, \mathbf{E} \rangle$ be a DBS, and $R, X \subseteq 2^\omega$ be perfect sets. There exist: a perfect set $Y \subseteq X$, Borel 1-1 maps $f, g : Y \rightarrow R$, and a DBS $\langle \mathbf{B}', \mathbf{E}' \rangle$ which extends $\langle \mathbf{B}, \mathbf{E} \rangle$, positively corralls $f \upharpoonright Y$, and negatively corralls $g \upharpoonright Y$.*

Proof. By Corollary 3.2, there exist perfect partial \mathbf{E} -transversals $X_0 \subseteq X$ and $R_0 \subseteq R$. Let $R_0 = R_1 \cup R_2$ be a partition into two disjoint perfect sets. Then $[R_1]_{\mathbf{E}}$ and $[R_2]_{\mathbf{E}}$ are disjoint, hence there is a perfect set $Y \subseteq X_0$ such that $[Y]_{\mathbf{E}}$ does not intersect either $[R_1]_{\mathbf{E}}$ or $[R_2]_{\mathbf{E}}$. Let say $[Y]_{\mathbf{E}} \cap [R_1]_{\mathbf{E}} = \emptyset$.

Let $R_1 = R' \cup R''$ be a partition into two disjoint perfect sets. It follows by construction that (*) the Borel sets Y, R', R'' are pairwise disjoint and the union $\Delta = Y \cup R' \cup R''$ is a partial \mathbf{E} -transversal. Let $f : Y \rightarrow R'$ and $g : Y \rightarrow R''$ be arbitrary Borel 1-1 maps.

We define the equivalence relations \mathbf{E}', \mathbf{B}' as follows.

If $x \in 2^\omega$ and the \mathbf{E} -class $[x]_{\mathbf{E}}$ does **not** intersect *the critical domain* $\Delta = Y \cup Z' \cup Z''$, then put $[x]_{\mathbf{E}'} = [x]_{\mathbf{E}}$ and $[x]_{\mathbf{B}'} = [x]_{\mathbf{B}}$, so such a \mathbf{E} -class and its \mathbf{B} -subclasses are not changed. But within Δ some classes will be merged. Namely if $x \in Y$ then we have to merge $[x]_{\mathbf{E}}$ with $[f(x)]_{\mathbf{E}}$ and $[g(x)]_{\mathbf{E}}$, hence we put $[x]_{\mathbf{E}'} = [x]_{\mathbf{E}} \cup [f(x)]_{\mathbf{E}} \cup [g(x)]_{\mathbf{E}}$. We further define

$$[x]_{\mathbf{B}'} = [x]_{\mathbf{B}} \cup [f(x)]_{\mathbf{B}} \cup ([g(x)]_{\mathbf{E}} \setminus [g(x)]_{\mathbf{B}}),$$

and let $([x]_{\mathbf{E}} \setminus [x]_{\mathbf{B}}) \cup ([f(x)]_{\mathbf{E}} \setminus [f(x)]_{\mathbf{B}}) \cup [g(x)]_{\mathbf{B}}$ be the other \mathbf{B}' -class within $[x]_{\mathbf{E}'}$. A routine verification using (*) shows that the relations \mathbf{E}', \mathbf{B}' are Borel, and the pair $\langle \mathbf{B}', \mathbf{E}' \rangle$ is a DBS that extends $\langle \mathbf{B}, \mathbf{E} \rangle$, positively corralls $f \upharpoonright Y$, and negatively corralls $g \upharpoonright Y$. \square

5 Corralling maps, E_0 -large case

Here we prove two corralling lemmas similar to 4.2 and 4.3, yet with somewhat more complex proofs.

Lemma 5.1. *Assume that $\langle B, E \rangle$ is a DBS, $E_0 \subseteq E$, $X \subseteq 2^\omega$ is a canonical E_0 -large set, and $f : X \rightarrow 2^\omega$ is Borel and 1-1. There exist a canonical E_0 -large set $Y \subseteq X$ and a DBS $\langle B', E' \rangle$ which extends $\langle B, E \rangle$ and corralls $f \upharpoonright Y$.*

Proof. First of all, arguing as in the proof of Lemma 4.2 (but using Corollary 3.5), we get a canonical E_0 -large set $X_0 \subseteq X$ with $x \notin f(x)$ for all $x \in X_0$. By Theorem 3.4, there is a canonical E_0 -large perfect set $X_1 \subseteq X_0$ such that the relations E, B coincide with E_0 on X_1 . Define an equivalence relation \widehat{E} on X_1 such that $x \widehat{E} y$ iff $f(x) E f(y)$, and define \widehat{B} similarly. Consider the \subseteq -minimal equivalence relation F defined on 2^ω such that $E \subseteq F$ and if $x, y \in 2^\omega$ and $f(x) E y$ then $x F y$. Thus $\widehat{E}, \widehat{B}, F$ are countable Borel equivalence relations on X_1 . (The borelness of F holds since all intended quantifiers in the definition of F are over countable domains.) By Theorem 3.4, there is a canonical E_0 -large perfect set $Y \subseteq X_1$ such that each of these three equivalence relations is either of type (I) or of type (II) on Y . However, as each E -class contains two B -classes, \widehat{E} has to coincide with \widehat{B} on Y . Finally, as $E \subseteq F$, we have $F = E_0$ on Y . It follows by the choice of X_0 that if $x, y \in Y$ (whether equal or not) then $x \notin f(y)$.

To conclude, $E = B = F = E_0$ on Y , and also either $\widehat{E} = \widehat{B}$ is the equality on Y , or $\widehat{E} = \widehat{B} = E_0$ on Y . This leads to the following two cases.

In each case, we are going to define the equivalence relations E', B' required. If $x \in 2^\omega$ and the E -class $[x]_E$ does **not** intersect the critical domain $\Delta = Y \cup \{f(x) : x \in Y\}$, then put $[x]_{E'} = [x]_E$ and $[x]_{B'} = [x]_B$, so such a E -class and its B -subclasses are not changed. But within Δ some classes will be merged. In particular, we are going to merge $[x]_E$ with $[f(x)]_E$ for any $x \in Y$.

Case 1: $\widehat{E} = \widehat{B}$ is the equality on Y while $B = E = F = E_0$ on Y , thus if $x, y \in Y$ then first, $x \neq y$ implies $f(x) \notin f(y)$ and $f(x) \notin B f(y)$, and second, $[x]_E \cap Y = [x]_B \cap Y = [x]_{E_0} \cap Y$. If $x \in Y$ then put

$$[x]_{E'} = [x]_E \cup \bigcup_{y \in Y \cap [x]_{E_0}} [f(y)]_E \quad \text{and} \quad [x]_{B'} = [x]_B \cup \bigcup_{y \in Y \cap [x]_{E_0}} [f(y)]_B,$$

and define the other B' -class within $[x]_{E'}$ as $[x]_{E'} \setminus [x]_{B'}$.

Case 2: $E = B = \widehat{E} = \widehat{B} = F = E_0$ on Y , that is, if $x, y \in Y$ then

$$x E_0 y \iff x E y \iff x B y \iff f(x) E f(y) \iff f(x) B f(y).$$

Assume that $x \in Y$. Put $[x]_{E'} = [x]_E \cup [f(x)]_E = [y]_E \cup [f(y)]_E$ for any other $y \in Y \cap [x]_{E_0}$, and $[x]_{B'} = [x]_B \cup [f(x)]_B = [y]_B \cup [f(y)]_B$ for any other $y \in Y \cap [x]_{E_0}$. Define the other B' -class within $[x]_{E'}$ as $[x]_{E'} \setminus [x]_{B'}$.

A routine verification shows that in either case the relations E', B' are Borel, and the pair $\langle B', E' \rangle$ is a DBS which extends $\langle B, E \rangle$ and corralls $f \upharpoonright Y$ (because we have $f(x) \in [x]_{E'}$ for all $x \in Y$ simply by construction). \square

Lemma 5.2. *Let $\langle B, E \rangle$ be a DBS with $E_0 \subseteq E$, and $R, X \subseteq 2^\omega$ be canonical E_0 -large sets. There exist: a canonical E_0 -large set $Y \subseteq X$, canonical E_0 -large maps $f, g : Y \rightarrow R$, and a DBS $\langle B', E' \rangle$ that extends $\langle B, E \rangle$, positively corralls f , and negatively corralls g .*

Proof. By Theorem 3.4, we w.l.o.g. assume that E coincides with E_0 on R . By definition, $R = \mathbb{X}_r$ for a E_0 -matrix $r = \langle r_n^i \rangle_{n < \omega, i=0,1}$. Now let $\mathbf{p} = \langle p_n^i \rangle_{n < \omega, i=0,1}$, $\mathbf{q} = \langle q_n^i \rangle_{n < \omega, i=0,1}$, where $p_n^i = r_{2n}^0 \hat{\wedge} r_{2n+1}^i$, $q_n^i = r_{2n}^1 \hat{\wedge} r_{2n+1}^i$. Thus \mathbf{p}, \mathbf{q} are E_0 -matrices, and the sets $\mathbb{X}_p, \mathbb{X}_q$ satisfy $\mathbb{X}_p \cup \mathbb{X}_q \subseteq \mathbb{X}_r = R$ and $[\mathbb{X}_p]_{E_0} \cap [\mathbb{X}_q]_{E_0} = \emptyset$, hence, $[\mathbb{X}_p]_E \cap [\mathbb{X}_q]_E = \emptyset$ by the assumption above. It follows by Corollary 3.5 that there is a canonical E_0 -large set $X_0 \subseteq X$ satisfying $[X_0]_E \cap [\mathbb{X}_p]_E = \emptyset$ or $[X_0]_E \cap [\mathbb{X}_q]_E = \emptyset$. Let say $[X_0]_E \cap [\mathbb{X}_p]_E = \emptyset$. As just above, there exist E_0 -matrices $\mathbf{p}', \mathbf{p}''$ such that the canonical E_0 -large sets $R' = \mathbb{X}_{p'}$, $R'' = \mathbb{X}_{p''}$ satisfy $R' \cup R'' \subseteq \mathbb{X}_p$ and $[R']_E \cap [R'']_E = \emptyset$.

To conclude, we have canonical E_0 -large sets $X_0 \subseteq X$ and $R', R'' \subseteq R$ satisfying $[R']_E \cap [R'']_E = [X_0]_E \cap [R']_E = [X_0]_E \cap [R'']_E = \emptyset$. Theorem 3.4 yields a canonical E_0 -large set $Y = \mathbb{X}_u \subseteq X_0$ such that $E = B = E_0$ on Y . Consider the canonical E_0 -large maps $f = h_{\mathbf{u}p'} : Y \rightarrow R'$ and $g = h_{\mathbf{u}p''} : Y \rightarrow R''$.

We define the equivalence relations E', B' as follows.

If $x \in 2^\omega$ and the E -class $[x]_E$ does **not** intersect *the critical domain* $\Delta = Y \cup (f''Y) \cup (g''Y)$, then put $[x]_{E'} = [x]_E$ and $[x]_{B'} = [x]_B$, so such a E -class and its B -subclasses are not changed. But within Δ , if $x \in Y$ then we have to merge $[x]_E$ with $[f(x)]_E$ and $[g(x)]_E$, hence we put

$$[x]_{E'} = [x]_E \cup \bigcup_{y \in Y \cap [x]_{E_0}} [f(y)]_E \quad \text{and} \quad [x]_{B'} = [x]_B \cup \bigcup_{y \in Y \cap [x]_{E_0}} [f(y)]_B,$$

and define the other B' -class within $[x]_{E'}$ as $[x]_{E'} \setminus [x]_{B'}$. A routine verification shows that the relations E', B' are Borel, and the pair $\langle B', E' \rangle$ is a DBS that extends $\langle B, E \rangle$, positively corralls $f \upharpoonright Y$, and negatively corralls $g \upharpoonright Y$. \square

6 Increasing system of equivalence relations

Proposition 6.1 (in L). *There is an \preceq -increasing sequence of DBSs $\langle B_\alpha, E_\alpha \rangle$, $\alpha < \omega_1$, beginning with E_0 of Footnote 2 and $B_0 = E_0^{\text{even}}$ and such that*

- (i) *if $X \subseteq 2^\omega$ is perfect and $f : X \rightarrow 2^\omega$ Borel and 1-1, then there exist: a perfect $X' \subseteq X$ and an ordinal $\alpha < \omega_1$ such that $\langle B_\alpha, E_\alpha \rangle$ corralls $f \upharpoonright X'$;*
- (ii) *if $X, R \subseteq 2^\omega$ are perfect sets, then there exist: a perfect set $Y \subseteq X$, an ordinal $\alpha < \omega_1$, and Borel 1-1 maps $f, g : Y \rightarrow R$, such that $\langle B_\alpha, E_\alpha \rangle$ corralls f positively and corralls g negatively;*

- (iii) the sequence of pairs $\langle B_\alpha, E_\alpha \rangle$ is Δ_2^1 , in the sense that there exists a Δ_2^1 sequence of codes for Borel sets B_α and E_α .

Proof. An obvious inductive construction using lemmas 4.2, 4.3, that takes a Gödel-least code of all possible pairs fitting the given inductive step, with the obvious union at limit steps. \square

Proposition 6.2 (in \mathbf{L}). *There is an \preceq -increasing sequence of DBSs $\langle B_\alpha, E_\alpha \rangle$, $\alpha < \omega_1$, beginning with E_0 of Footnote 2 and $B_0 = E_0^{\text{even}}$ and such that*

- (i) if $X \subseteq 2^\omega$ is a Borel E_0 -large set and $f : X \rightarrow 2^\omega$ Borel and 1-1, then there exist: a canonical E_0 -large set $Y \subseteq X$ and $\alpha < \omega_1$ such that $\langle B_\alpha, E_\alpha \rangle$ corralls $f \upharpoonright Y$;
- (ii) if $X, R \subseteq 2^\omega$ are E_0 -large sets, then there exist: a canonical E_0 -large set $Y \subseteq X$, an ordinal $\alpha < \omega_1$, and canonical E_0 -large maps $f, g : Y \rightarrow R$, such that $\langle B_\alpha, E_\alpha \rangle$ corralls f positively and g negatively;
- (iii) the sequence of pairs $\langle B_\alpha, E_\alpha \rangle$ is Δ_2^1 , in the sense that there exists a Δ_2^1 sequence of codes for Borel sets B_α and E_α .

Proof. Similar. \square

7 Proof of the main theorem

Proof (Theorem 1.1, Sacks case). Fix, in \mathbf{L} , an \preceq -increasing sequence of DBSs $\langle B_\alpha, E_\alpha \rangle$, $\alpha < \omega_1$, satisfying conditions (i), (ii), (iii) of Proposition 6.1.

Arguing in a Sacks-generic extension $\mathbf{L}[a_0]$, we define a relation $B = \bigcup_{\alpha < \omega_1} B_\alpha$ on 2^ω ; thus $x B y$ iff $x B_\alpha y$ for some $\alpha < \omega_1$. (We identify Borel sets B_α and E_α , formally defined in \mathbf{L} , with their extensions, Borel sets in $\mathbf{L}[a_0]$ with the same codes.) Define a relation $E = \bigcup_{\alpha < \omega_1} E_\alpha$ on 2^ω similarly. Define the subdomain $U = 2^\omega \setminus \mathbf{L}$ of all new reals. Then $a_0 \in U$ and all reals in U have the same \mathbf{L} -degree by the minimality of Sacks reals, see e.g. [7, Theorem 15.34].

Lemma 7.1. *It is true in $\mathbf{L}[a_0]$ that*

- (i) E and B are equivalence relations and B is a subrelation of E ;
- (ii) B is lighface Σ_2^1 ;
- (iii) all reals $x, y \in U$ are E -equivalent;
- (iv) there are exactly two B -classes intersecting U — call them M, N ;
- (v) the sets M, N are not OD⁶, hence $M \cup N = U$.

⁶ Note that M, N are indiscernible in a stronger sense: if $R(M, N)$ holds for some OD relation R , then $R(N, M)$ holds. Indeed, otherwise M can be distinguished from N by the property: “ $R(\cdot, A)$ holds but $R(A, \cdot)$ fails, where A is the other element of the pair $\{M, N\}$ ”.

Proof. (i) To see that E is an equivalence relation, let $a, b, c \in W$ and suppose that $a E b$ and $a E c$. Then by definition we have $a E_\alpha b$ and $a E_\alpha c$ for some $\alpha < \omega_1$. However being an equivalence relation is absolute by Shoenfield's absoluteness theorem [7, Theorem 25.20]. Therefore $b B_\alpha c$ holds, as required.

(ii) holds by Theorem 6.1(iii).

(iii) Let $b \in U$; prove that $a_0 E b$. It is a known property of the Sacks forcing that there is a Borel 1-1 map $f : 2^\omega \rightarrow 2^\omega$ with a code in \mathbf{L} , such that $b = f(a_0)$.⁷ It follows then from Theorem 6.1(i) that there exists a perfect set $X \subseteq 2^\omega$, coded in \mathbf{L} and such that $a_0 \in X$ and E_α corralls $f \upharpoonright X$ for some α . In particular, $\langle a_0, b \rangle \in E_\alpha$, hence we have $a_0 E b$ as required.

(iv) Let $a, b, c \in U$; prove that two of these reals are B -equivalent. Note that $a E b E c$ by (iii), and hence there is an ordinal $\alpha < \omega_1$ such that $a E_\alpha b E_\alpha c$. However containing exactly two B_α -classes in each E_α -class is absolute. It follows that at least one pair among a, b, c is B_α -equivalent, as required.

(v) Suppose to the contrary that M is OD. Then M is Sacks-forced over \mathbf{L} , meaning that there is a perfect set $R \subseteq 2^\omega$, coded in \mathbf{L} and such that $R \cap U \subseteq M$ in $\mathbf{L}[a_0]$. By Proposition 6.1(ii), there exist: a perfect set $Y \subseteq 2^\omega$ coded in \mathbf{L} and containing a_0 , an ordinal $\alpha < \omega_1$, and Borel 1-1 maps $f, g : Y \rightarrow R$, also coded in \mathbf{L} and such that E_α corralls $f \upharpoonright Y$ positively and $g \upharpoonright Y$ negatively. In other words the reals $b = f(a_0)$ and $c = g(a_0)$ in $U \cap R$ satisfy $a_0 B_\alpha b$, $a_0 E_\alpha c$, but $\neg(a_0 B_\alpha c)$. It easily follows that $b \not B c$, which contradicts the fact that b, c belong to one and the same B -class. \square

To conclude, it is true in the Sacks extension $\mathbf{L}[a_0]$ that B is a Σ_2^1 equivalence relation on 2^ω , and the nonconstructible domain $U = 2^\omega \setminus \mathbf{L}$ (a Π_2^1 set) is equal to the union of two (non-empty) B -equivalence classes, which are non-OD sets. Now, to prove Theorem 1.1 (Sacks case), it suffices to define the required equivalence relation Q on 2^ω in $\mathbf{L}[a_0]$ as follows: $x Q y$ iff $x B y$ or just x, y both belong to \mathbf{L} . \square (Theorem 1.1, Sacks case)

Proof (Theorem 1.1, E_0 -large case). Rather similar to the proof of in the Sacks case above. **Arguing in a E_0 -large-generic extension $\mathbf{L}[a_0]$** , we define relations $B = \bigcup_{\alpha < \omega_1} B_\alpha$, $E = \bigcup_{\alpha < \omega_1} E_\alpha$ on 2^ω , and the subdomain $U = 2^\omega \setminus \mathbf{L}$; $a_0 \in U$.

Lemma 7.2. *It is true in $\mathbf{L}[a_0]$ that*

- (i) E and B are equivalence relations and B is a subrelation of E ;
- (ii) B is highface Σ_2^1 ;

⁷ Indeed, by the property of Borel reading of names, we have $b = f(a)$, where $f : 2^\omega \rightarrow 2^\omega$ is a Borel map with a code in \mathbf{L} . But any Borel $g : X \rightarrow 2^\omega$, defined on a perfect set $X \subseteq 2^\omega$, is 1-1 or a constant on a smaller perfect set, by Corollary 3.3. Thus there is a perfect set $Y \subseteq 2^\omega$, coded in \mathbf{L} , such that $a_0 \in Y$ and $f \upharpoonright Y$ is 1-1 or a constant. However if f is a constant, say $f(x) = z_0 \in 2^\omega$ for all $x \in Y$, then $f(a_0) = b = z_0 \in \mathbf{L}$, which contradicts to $b \notin \mathbf{L}$.

- (iii) all reals $x, y \in U$ are \mathbf{E} -equivalent;
- (iv) there are exactly two \mathbf{B} -classes intersecting U — call them M, N ;
- (v) the sets M, N are not OD, hence $M \cup N = U$.

Proof. The proof of claims (i), (ii), (iii), (iv) goes on similarly to Lemma 7.1, with some obvious changes *mutatis mutandis*, in particular, the reference to Corollary 3.3 has to be replaced by Corollary 3.6 in Footnote 7, the Proposition 6.1 by Proposition 6.2, and so on. But the last claim needs special attention because not all new reals in $\mathbf{L}[a_0]$ are \mathbf{E}_0 -large-generic unlike the Sacks case.

(v) First of all let's prove that each of the classes M, N of (iv) contains a real $b \in 2^\omega$ \mathbf{E}_0 -large-generic over \mathbf{L} . Indeed in view of (iv) it suffices to prove that (*) there are \mathbf{E}_0 -large-generic, but not \mathbf{B} -equivalent, reals $b, c \in \mathbf{L}[a_0] \cap 2^\omega$. Emulating the proof of Theorem 7.1(v), but using 6.2(ii) instead of 6.1(ii), we find a canonical \mathbf{E}_0 -large set $Y \subseteq 2^\omega$, coded in \mathbf{L} and containing a_0 , an ordinal $\alpha < \omega_1$, and canonical \mathbf{E}_0 -large maps $f, g : Y \rightarrow 2^\omega$, also coded in \mathbf{L} and such that \mathbf{E}_α corralls $f \upharpoonright Y$ positively and $g \upharpoonright Y$ negatively. We conclude that the reals $b = f(a_0)$ and $c = g(a_0)$ in U satisfy $a_0 \mathbf{B}_\alpha b$, $a_0 \mathbf{E}_\alpha c$, but $\neg (a_0 \mathbf{B}_\alpha c)$, so that $b \not\mathbf{B} c$. And finally, it is clear that b, c are \mathbf{E}_0 -large-generic along with a_0 . (Basically any image of a \mathbf{E}_0 -large-generic real $a \in 2^\omega$ via a canonical \mathbf{E}_0 -large map h , coded in \mathbf{L} , with $a \in \text{dom } h$, is \mathbf{E}_0 -large-generic by an easy argument.)

Now suppose to the contrary that M is OD. Let $\mu(\cdot)$ be an \in -formula, with ordinals as parameters, such that $M = \{x : \mu(x)\}$ in $\mathbf{L}[a_0]$. By (*), there is a real $b_0 \in M$ (in $\mathbf{L}[a_0]$), \mathbf{E}_0 -large-generic over \mathbf{L} . Then it is true in $\mathbf{L}[a_0] = \mathbf{L}[b_0]$ that $\mu(b_0)$ and any real x satisfying $\mu(x)$ also satisfies $x \mathbf{B} b_0$. This is \mathbf{E}_0 -large-forced over \mathbf{L} , meaning that there is a canonical \mathbf{E}_0 -large set $R \subseteq 2^\omega$, coded in \mathbf{L} and such that (1) $b_0 \in R$, (2) every real $b \in R \cap \mathbf{L}[x]$, \mathbf{E}_0 -large-generic over \mathbf{L} , satisfies $\mu(x)$ in $\mathbf{L}[b] = \mathbf{L}[b_0] = \mathbf{L}[a_0]$, and hence satisfies $b \mathbf{B} b_0$.

However, emulating the proof of Theorem 7.1(v) as above, we find a canonical \mathbf{E}_0 -large set $Y \subseteq 2^\omega$, coded in \mathbf{L} and containing b_0 , an ordinal $\alpha < \omega_1$, and canonical \mathbf{E}_0 -large maps $f, g : Y \rightarrow R$, also coded in \mathbf{L} and such that \mathbf{E}_α corralls $f \upharpoonright Y$ positively and $g \upharpoonright Y$ negatively. Then the reals $b = f(b_0)$ and $c = g(b_0)$ are \mathbf{E}_0 -large-generic over \mathbf{L} and satisfy $b_0 \mathbf{B}_\alpha b$ and $b_0 \mathbf{E}_\alpha c$ but $\neg (b_0 \mathbf{B}_\alpha c)$, hence $b \not\mathbf{B} c$, which contradicts (2) above. \square

\square (Theorem 1.1, \mathbf{E}_0 -large case)

8 Final remarks

Problem 8.1. It is interesting to figure out whether Theorem 1.1 holds in other extensions of \mathbf{L} by a single generic real, e.g. in extensions by a single Cohen-generic⁸, or a single Solovay-random, or a single Silver real. The random case is

⁸ Since adding a single Cohen real is equivalent to adding many Cohen reals, it is fairly easy to show that there are indiscernible sets of reals in Cohen extensions, e.g. $[a]_{\mathbf{L}}$ and $[b]_{\mathbf{L}}$

especially interesting as it is close to the Sacks case in some forcing details like the property of Borel reading of names of reals. One of the technical difficulties is to prove an analog of corraling lemmas in section 4 for perfect sets of positive measure. The merger of equivalence classes, rather transparent in the proof of lemmas 4.2, 4.3, becomes way more complex then. On the positive side, it turns out that Theorem 1.1 also holds for forcing by perfect non- σ -compact sets in ω^ω , to be published elsewhere.

Problem 8.2. In view of Theorem 1.1, one may ask whether there is a model in which every finite non-empty OD set contains an OD element but there are non-empty (infinite) OD sets containing no OD elements. Could the Solovay model [16] (where all projective sets are measurable) be such a model?

9 History of this result

The proof of Theorem 1.1 given above was manufactured by V. Kanovei in January 2020, after a short discussion at Mathoverflow⁹, on the basis of the following excerpt from an email message from R. M. Solovay to Ali Enayat, quoted here thanks to Solovay’s generous permission.

[Solovay to Enayat 25.10.2002:]

Here’s a freshly minted theorem.

Consider the Sacks extension of a model of $\mathbf{V} = \mathbf{L} + \mathbf{ZFC}$. Then LA does not hold.¹⁰

My proof is a bit involved. Here’s a high level - view.

By a transfinite construction of length \aleph_1 I construct a P -name E such that the following are forced:

- E is an equivalence relation on the set of non-constructible reals.
- E has precisely two equivalence classes.

for any Cohen-generic pair of reals $\langle a, b \rangle$, as shown in Theorem 3.1 of [2]. On the other hand, such indiscernibles hardly form an OD pair, or, equivalently, arise as equivalence classes of an OD equivalence relation E with only two equivalence classes.

⁹ <https://mathoverflow.net/questions/349243>

¹⁰ In the context of this exchange, LA is the Mycielski axiom, the axiom formulated by Mycielski, investigated in Enayat’s paper [2], in which it is referred to as the Leibniz-Mycielski axiom LM. LM states that given any pair of distinct sets a and b , there is some ordinal α , and some first order formula $\phi(x)$, such that \mathbf{V}_α contains a and b , and \mathbf{V}_α satisfies $\phi(a)$ but does not satisfy $\phi(b)$. The motivation for establishing Theorem 1.1 was the guess (privately communicated by Enayat to Solovay) that the consistency of $\mathbf{ZFC} + \text{LM} + “\mathbf{V} \neq \text{HOD}”$ can be shown by verifying that LM holds in the extension of the constructible universe by a Sacks real. The question of consistency of $\mathbf{ZFC} + \text{LM} + “\mathbf{V} \neq \text{HOD}”$ has proved to be more difficult than meets the eye, and remains open.

- In each perfect set with constructible code there are representatives of both equivalence classes.
- E is ordinal definable.

The two distinct but indiscernable members of the generic extension are the two equivalence classes of E .

The proof is a bit too involved to type in using a web-interface like yahoo. (Shades of Fermat’s margin!) The proof uses one standard but relatively deep fact from descriptive set theory. If B is an uncountable Borel set, then B contains a perfect subset.

– Bob

P.S. I don’t use much about \mathbf{L} . Just that it satisfies $\mathbf{V} = \text{OD}$ and is uniformly definable in any extension and that it satisfies CH.¹¹

[End]

The above proof of Theorem 1.1 in the Sacks case obviously more or less follows Solovay’s outline. In light of the key role of the Silver Dichotomy in the proof presented here, we don’t know to what degree it coincides with the original proof by Solovay in all important details.

Upon the completion of the proof, the co-authors contacted R. M. Solovay, with an invitation to join as a co-author of this note, but he unfortunately did not accept our invitation.

References

- [1] Andrés E. Caicedo and Richard Ketchersid. A trichotomy theorem in natural models of AD^+ . In *Set theory and its applications*, pages 227–258. *Contemporary Mathematics*, 533, 2011.
- [2] Ali Enayat. On the Leibniz – Mycielski axiom in set theory. *Fundam. Math.*, 181(3):215–231, 2004.
- [3] S. Feferman. Some applications of the notions of forcing and generic sets. *Fundam. Math.*, 56:325–345, 1965.
- [4] Gunter Fuchs, Victoria Gitman, and Joel David Hamkins. Ehrenfeucht’s lemma in set theory. *Notre Dame J. Formal Logic*, 59(3):355–370, 2018.
- [5] M. Golshani, V. Kanovei, and V. Lyubetsky. A Groszek – Laver pair of undistinguishable E_0 classes. *Mathematical Logic Quarterly*, 63(1–2):19–31, 2017.
- [6] M. Groszek and R. Laver. Finite groups of OD-conjugates. *Period. Math. Hung.*, 18:87–97, 1987.
- [7] Thomas Jech. *Set theory*. Springer-Verlag, Berlin-Heidelberg-New York, The third millennium revised and expanded edition, 2003.

¹¹ This is equally true for our proof.

- [8] V. Kanovei and V. Lyubetsky. Countable OD sets of reals belong to the ground model. *Arch. Math. Logic*, 57(3-4):285–298, 2018.
- [9] Vladimir Kanovei. *Borel equivalence relations. Structure and classification*. Providence, RI: American Mathematical Society (AMS), 2008.
- [10] Vladimir Kanovei, Marcin Sabok, and Jindřich Zapletal. *Canonical Ramsey theory on Polish spaces*. Cambridge: Cambridge University Press, 2013.
- [11] Alexander S. Kechris and Benjamin D. Miller. *Topics in orbit equivalence*, volume 1852 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.
- [12] Benjamin D. Miller. A classical proof of the Kanovei-Zapletal canonization. In *Set theory and its applications. Annual Boise extravaganza in set theory, Boise, ID, USA, 1995–2010*, pages 281–285. Providence, RI: AMS, 2011.
- [13] Yiannis N. Moschovakis. *Descriptive set theory*. Studies in Logic and the Foundations of Mathematics, Vol. 100. Amsterdam, New York, Oxford: North-Holland Publishing Company. XII, 637 p. Dfl. 150.00; \$ 73.25 , 1980.
- [14] Ramez L. Sami. A note on an effective Polish topology and Silver’s dichotomy theorem. *Proc. Am. Math. Soc.*, 147(9):4039–4044, 2019.
- [15] Jack H. Silver. Counting the number of equivalence classes of Borel and coanalytic equivalence relations. *Ann. Math. Logic*, 18(1):1–28, 1980.
- [16] R.M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. Math. (2)*, 92:1–56, 1970.