METRIC COMPLETIONS, THE HEINE–BOREL PROPERTY, AND APPROACHABILITY

VLADIMIR KANOVEI, MIKHAIL G. KATZ, AND TAHL NOWIK

Abstract. We show that the metric universal cover of a plane with a puncture yields an example of a nonstandard hull properly containing the metric completion of a metric space. As mentioned by do Carmo, a nonextendible Riemannian manifold can be noncomplete, but in the broader category of metric spaces it becomes extendible. We give a short proof of a characterisation of the Heine–Borel property of the metric completion of a metric space M in terms of the absence of inapproachable finite points in ${}^*\!M$.

Keywords: galaxy; halo; metric completion; nonstandard hull; universal cover; Heine–Borel property

1. Introduction

A p-adic power series example of the phenomenon of inapproachability in a nonstandard hull of a metric space M appears in Goldblatt [1, p. 252]. Recall that a point $x \in {}^*\!M$ is approachable if for each $\varepsilon \in \mathbb{R}^+$ there is some (standard) $x_\varepsilon \in M$ such that ${}^*\!d(x, x_\varepsilon) < \varepsilon$ (op. cit., p. 236). Otherwise x is called inapproachable.

A nonstandard hull of a metric space M can in general contain points that need to be discarded (namely, the inapproachable ones) in order to form the metric completion of M. We provide a more geometric example of such a phenomenon stemming from differential geometry. The example is the metric universal cover of a plane with one puncture; see Definition 3.1.

Let ${}^*\mathbb{R}$ be a hyperreal field extending $\mathbb{R}$. Denote by ${}^\flat\mathbb{R} \subseteq {}^*\mathbb{R}$ the subring consisting of finite hyperreals. The ring ${}^\flat\mathbb{R}$ is the domain of the standard part function $\mathrm{st}\colon {}^\flat\mathbb{R} \to \mathbb{R}$. Here $\mathrm{st}(x)$ for $x \in {}^\flat\mathbb{R}$ is the real number corresponding to the Dedekind cut on $\mathbb{R}$ defined by x, via the embedding $\mathbb{R} \hookrightarrow {}^\flat\mathbb{R}$.

Let ${}^*\mathbb{Q} \subseteq {}^*\mathbb{R}$ be the subfield consisting of hyperrational numbers. Let $F \subseteq {}^*\mathbb{Q}$ be the ring of finite hyperrationals, so that $F = {}^*\mathbb{Q} \cap {}^\flat\mathbb{R}$. Let $I \subseteq F$ be the ideal of hyperrational infinitesimals. If $x \in {}^*\mathbb{Q}$ then its *halo* is the (co)set $\mathrm{hal}(x) = x + I \subseteq {}^*\mathbb{Q}$. The following result is well known; see e.g., [2].

Theorem 1.1. *The ideal $I \subseteq F$ is maximal, and the quotient field $\widehat{\mathbb{Q}} = F/I$ is naturally isomorphic to $\mathbb{R}$, so that we have a short exact sequence $0 \to I \to F \to \mathbb{R} \to 0$.*

Proof. A typical element of $\widehat{\mathbb{Q}}$ is a halo, namely $\mathrm{hal}(x) \subseteq {}^*\mathbb{Q}$, where each $x \in F$ can be viewed as an element of the larger ring ${}^\flat\mathbb{R} \subseteq {}^*\mathbb{R}$. Then the map

$$\phi(\mathrm{hal}(x)) = \mathrm{st}(x)$$

is the required map $\phi\colon \widehat{\mathbb{Q}} \to \mathbb{R}$. To show surjectivity of ϕ, note that over $\mathbb{R}$ we have

$$(\forall \epsilon \in \mathbb{R}^+)(\forall y \in \mathbb{R})(\exists q \in \mathbb{Q})\ [|y - q| < \epsilon]. \tag{1.1}$$

Recall that Robinson's transfer principle (see [3]) asserts that every first-order formula, e.g., (1.1), has a hyperreal counterpart obtained by starring the ranges of the bound variables. We apply the upward transfer principle to (1.1) to obtain

$$(\forall \epsilon \in {}^*\mathbb{R}^+)(\forall y \in {}^*\mathbb{R})(\exists q \in {}^*\mathbb{Q})\ \big[|y - q| < \epsilon\big]. \tag{1.2}$$

Now choose an infinitesimal $\epsilon > 0$. Then formula (1.2) implies that for each real number y there is a hyperrational q with $y \approx q$, where $\approx$ is the relation of infinite proximity (i.e., $y - q$ is infinitesimal). Therefore we obtain $\phi(\mathrm{hal}(q)) = y$, as required. □

A framework for differential geometry via infinitesimal displacements was developed in [4]. An application to small oscillations of the pendulum appeared in [5]. The reference [1] is a good introductory exposition of Robinson's techniques, where the reader can find the definitions and properties of the notions exploited in this text. Additional material on Robinson's framework can be found in [6]. The historical significance of Robinson's framework for infinitesimal analysis in relation to the work of Fermat, Gregory, Leibniz, Euler, and Cauchy has been analyzed respectively in [7], [8], [9], [10], [11], and elsewhere. The approach is not without its critics; see e.g., [12].

2. Ihull construction

In Section 1 we described a construction of $\mathbb{R}$ starting from ${}^*\mathbb{Q}$. More generally, one has the following construction. In the literature this construction is often referred to as the *nonstandard hull construction*, which we will refer to as the *ihull construction* ("i" for *infinitesimal*) for short. The general construction takes place in the context of an arbitrary metric space M.

Given a metric space (M, d), we consider its natural extension ${}^*M$. The distance function d extends to a hyperreal-valued function ${}^*d$ on ${}^*M$

as usual. The halo of $x \in {}^*M$ is defined to be the set of points in ${}^*M$ at infinitesimal distance from x.

Let $\approx$ be the relation of infinite proximity in ${}^*M$. Denote by $F \subseteq {}^*M$ the set of points of ${}^*M$ at finite distance from any point of M (i.e., the galaxy of any element in M). The quotient

$$F/\approx$$

is called the *ihull* of M and denoted $\widehat{M}$. In this terminology, Theorem 1.1 asserts that the ihull of $\mathbb{Q}$ is naturally isomorphic to $\mathbb{R}$. Thus, ihulls provide a natural way of obtaining completions; see Morgan ([13], 2016) for a general framework for completions. We will exploit the following notation for halos.

Definition 2.1. We let $\overset{\circ}{x}$ be the halo of $x \in {}^*M$.

In general the ihull $\widehat{M}$ of a metric space M consists of halos $\overset{\circ}{x}$, where $x \in F$, with distance d on $\widehat{M}$ defined to be

$$d\left(\overset{\circ}{x}, \overset{\circ}{y}\right) = \mathrm{st}({}^*d(x, y)). \tag{2.1}$$

Note that that M may be viewed as a subset of $\widehat{M}$. Hence it is meaningful to speak of the closure of M in $\widehat{M}$. We will denote such closure $cl(M)$ to distinguish it from the abstract notion of the metric completion $\overline{M}$ of M (see Section 4). The closure $cl(M)$ is indeed the completion in the sense that it is complete and M is dense in it. The metric completion $\overline{M}$ of M is the approachable part of $\widehat{M}$, and coincides with the closure $cl(M)$ of M in $\widehat{M}$; see [1, Chapter 18] for a detailed discussion.

3. Universal cover of plane with a puncture

The ihull $\widehat{M}$ may in general be larger than the metric completion $\overline{M}$ of M. An example of such a phenomenon was given in [1, p. 252] in terms of p-adic series. We provide a more geometric example of such a phenomenon stemming from differential geometry.

We start with the standard flat metric $dx^2 + dy^2$ in the (x, y)-plane, which can be written in polar coordinates (r, θ) as $dr^2 + r^2 d\theta^2$ where θ is the usual polar angle in $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 3.1. Let M be the metric universal cover of $\mathbb{R}^2 \setminus \{0\}$ (the plane minus the origin), coordinatized by (r, ζ) where $r > 0$ and ζ is an arbitrary real number.

In formulas, M can be given by the coordinate chart $r > 0$, $\zeta \in \mathbb{R}$, equipped with the metric

$$dr^2 + r^2 d\zeta^2. \tag{3.1}$$

Formula (3.1) provides a description of the metric universal cover of the flat metric on $\mathbb{R}^2 \setminus \{0\}$, for which the covering map $M \to \mathbb{R}^2 \setminus \{0\}$ sending $(r, \zeta) \mapsto (r, \theta)$ induces a local Riemannian isometry, where θ corresponds to the coset $\zeta + 2\pi\mathbb{Z}$. Recall that a number is called *appreciable* when it is finite but not infinitesimal.

Theorem 3.2. *Points of the form $\overbrace{(r,\zeta)}$ for appreciable r and infinite ζ are in the ihull $\widehat{M}$ but are not approachable from M.*

Proof. The distance function d of M extends to the ihull $(\widehat{M}, d)$ as in (2.1). Here points of $\widehat{M}$ are halos in the finite part of ${}^*M$. Notice that in $\widehat{M}$ the origin has been "restored" and can be represented in coordinates (r, ζ) by a point $(\epsilon, 0)$ in ${}^*M$ where $\epsilon > 0$ is infinitesimal.

Consider a point $(1, \zeta) \in {}^*M$ where ζ is infinite. Let us show that the point $(1, \zeta)$ is at a finite distance ${}^*d$ from the point $(\epsilon, 0)$; namely the standard part of the distance is 1. Indeed, the triangle inequality applied to the sequence of points $(1, \zeta)$, $(\frac{1}{\zeta^2}, \zeta)$, $(\frac{1}{\zeta^2}, 0)$, $(\epsilon, 0)$ yields the bound

$$d^*\big((1,\zeta),(\epsilon,0)\big) \le (1 - \tfrac{1}{\zeta^2}) + \tfrac{1}{\zeta^2}\zeta + |\tfrac{1}{\zeta^2} - \epsilon| \approx 1.$$

Therefore

$$d\Big(\overbrace{(1,\zeta)}, \overbrace{(\epsilon,0)}\Big) \le 1$$

by (2.1). Hence $\overbrace{(1,\zeta)} \in \widehat{M}$.

On the other hand, let us show that the point $\overbrace{(1,\zeta)} \in \widehat{M}$ is not approachable from M. Consider the rectangle K defined by the image in $\widehat{M}$ of

$${}^*[\tfrac{1}{2}, 2] \times {}^*[\zeta - 1, \zeta + 1] \subseteq {}^*M.$$

The metric d of $\widehat{M}$ restricted to the rectangle K dominates the product metric $dr^2 + \frac{1}{4} d\zeta^2$ by (3.1). The boundary of K separates the interior of K from the complement of K. Thus, to reach the finite part one must first traverse the boundary. Therefore K includes the metric ball of radius $\frac{1}{2}$ centered at $\overbrace{(1,\zeta)}$. This ball contains no standard points. Hence the point $\overbrace{(1,\zeta)} \in \widehat{M}$ is not approachable. $\square$

Note that what is responsible for the inapprochability is the fact that the closure $\overline{M} \subseteq \widehat{M}$ does not have the Heine–Borel property (and is not even locally compact). Namely, the boundary of the metric unit ball in $\overline{M}$ centered at the origin $\overgroup{(\epsilon, 0)}$ is a line.

Do Carmo ([14], 1992, p. 152) views the universal cover M of the plane with a puncture as an example of a Riemannian manifold that is *nonextendible* but not complete. Indeed, M is nonextendible in the category of Riemannian manifolds, but M is extendible in the category of metric spaces, in such a way that near the "extended" origin $\overgroup{(\epsilon, 0)} \in \overline{M} \subseteq \widehat{M}$, the Heine–Borel property is violated. In Section 4 we show that such a result holds more generally.

4. An approachable criterion for the Heine–Borel property

For the sake of completeness we provide a short proof of a relation between approachability and the Heine–Borel property for metric spaces. For related results in the context of uniform spaces see Henson–Moore [15], [16] (but note that they use a different notion of "finiteness" for a point $x \in {}^*M$). For a study of the relation between the Heine–Borel property and local compactness, see [17].

We show that the Heine–Borel property for the completion of a metric space has a characterisation in terms of the absence of finite inapproachable points; see Theorem 4.3. The following result appears in Luxemburg [18, Theorem 3.14.1, p. 78] and Hurd–Loeb [19, Proposition 3.14]; cf. Davis [2, Theorems 5.19 and 5.20, p. 93].

Proposition 4.1. *Let M be a metric space. Then the following two properties are equivalent:*

(1) *every approachable point in ${}^*M$ is nearstandard;*
(2) *M is complete.*

Definition 4.2. A metric space M is *Heine–Borel* (HB) if every closed and bounded subset of M is compact.

We fix a point $p \in M$. Let $n \in \mathbb{N}$. Let $B_n = \{x \in M : d(x,p) \leq n\}$. Clearly M is HB if and only if the sets B_n are all compact. Let $\overline{M}$ be the completion of M. Let $\overline{B}_n = \{x \in \overline{M} : d(x,p) \leq n\}$, for the same fixed $p \in M$. By transfer, we have ${}^*B_n = \{x \in {}^*M : d(x,p) \leq n\}$, and similarly ${}^*\overline{B}_n = \{x \in {}^*\overline{M} : d(x,p) \leq n\}$.

Clearly, an HB metric space is complete (given a Cauchy sequence, find a convergent subsequence in the closure of its set of points).

Theorem 4.3. *Let M be a metric space. The following three properties are equivalent:*

(1) *Every finite point in* ${}^*M$ *is approachable;*
(2) *the completion* $\overline{M}$ *is Heine–Borel;*
(3) $\overline{M} = \widehat{M}$ *(the metric completion is already all of the ihull).*

Proof. Assume $\overline{M}$ is HB. Let $a \in {}^*M$ be finite. Then we have $a \in {}^*B_n \subseteq {}^*\overline{B}_n$ for some $n \in \mathbb{N}$. Since $\overline{M}$ is assumed to be HB, the ball $\overline{B}_n$ is compact. Hence there is a point $x \in \overline{B}_n$ with $x \approx a$. Now let $\epsilon > 0$ be standard. Since M is dense in $\overline{M}$ there is a point $y \in M$ such that $d(y, x) < \epsilon$, and therefore $d(y, a) < \epsilon$.

Conversely, assume every finite point in ${}^*M$ is approachable. As a first step we show that every finite point in ${}^*\overline{M}$ is approachable. Let $a \in {}^*\overline{B}_n$ and fix a standard $\epsilon > 0$. Since M is dense in $\overline{M}$, the following holds for our fixed n and ϵ:

$$(\forall x \in \overline{B}_n)(\exists y \in B_{n+1})[d(x, y) < \epsilon].$$

By transfer we obtain

$$(\forall x \in {}^*\overline{B}_n)(\exists y \in {}^*B_{n+1})[d(x, y) < \epsilon]. \tag{4.1}$$

Applying (4.1) with $x = a$, we obtain a point $b \in {}^*B_{n+1}$ with $d(a, b) < \epsilon$. Every finite point in ${}^*M$ is approachable by assumption. Therefore there is a point $x \in M \subseteq \overline{M}$ with $d(x, b) < \epsilon$. Thus $d(x, a) < 2\epsilon$, showing that every finite point in ${}^*\overline{M}$ is approachable.

We now prove that $\overline{M}$ is HB by showing that each $\overline{B}_n$ is compact. Let $a \in {}^*\overline{B}_n$. We need to find a point $x \in \overline{B}_n$ with $x \approx a$. We have shown above that a is approachable. By Proposition 4.1, a is nearstandard, i.e., there is a point $x \in \overline{M}$ with $x \approx a$. Since $d(a, p) \leq n$, and x, p are both standard, we also have $d(x, p) \leq n$, i.e., $x \in \overline{B}_n$. □

Combining Proposition 4.1 and Theorem 4.3, we obtain the following corollary, which also appears in Goldbring [20, Proposition 9.23].

Corollary 4.4. *Let* M *be a metric space. The following two properties are equivalent:*

(1) *Every finite point in* ${}^*M$ *is nearstandard;*
(2) M *is Heine–Borel.*

Acknowledgments

We are grateful to the referees for helpful comments on an earlier version of the manuscript. V. Kanovei was partially supported by RFBR grant no 18-29-13037.

References

[1] Robert Goldblatt, *Lectures on the hyperreals. An introduction to nonstandard analysis*, Graduate Texts in Mathematics **188**, Springer-Verlag, New York, 1998.

[2] Martin Davis, *Applied nonstandard analysis*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York–London–Sydney, 1977. Reprinted by Dover, NY, 2005. See `http://store.doverpublications.com/0486442292.html`

[3] Abraham Robinson, *Non-standard analysis*, North-Holland Publishing, Amsterdam, 1966.

[4] Tahl Nowik, Mikhail Katz, *Differential geometry via infinitesimal displacements*, Journal of Logic and Analysis **7** (2015), no. 5, 1–44. See `http://www.logicandanalysis.org/index.php/jla/article/view/237/106` and `http://arxiv.org/abs/1405.0984`

[5] Vladimir Kanovei, Karin Katz, Mikhail Katz, Tahl Nowik, *Small oscillations of the pendulum, Euler's method, and adequality*. Quantum Studies: Mathematics and Foundations **3** (2016), no. 3, 231–236. See `http://dx.doi.org/10.1007/s40509-016-0074-x` and `https://arxiv.org/abs/1604.06663`

[6] Peter Fletcher, Karel Hrbacek, Vladimir Kanovei, Mikhail Katz, Claude Lobry, Sam Sanders, *Approaches to analysis with infinitesimals following Robinson, Nelson, and others*, Real Analysis Exchange **42** (2017), no. 2, 193–252. See `https://arxiv.org/abs/1703.00425` and `http://msupress.org/journals/issue/?id=50-21D-61F`

[7] Jacques Bair, Mikhail Katz, David Sherry, *Fermat's dilemma: Why did he keep mum on infinitesimals? and the European theological context*, Foundations of Science **23** (2018), no. 3, 559–595. See `http://dx.doi.org/10.1007/s10699-017-9542-y` and `https://arxiv.org/abs/1801.00427`

[8] Tiziana Bascelli, Piotr Błaszczyk, Vladimir Kanovei, Karin Katz, Mikhail Katz, Semen Kutateladze, Tahl Nowik, David Schaps, David Sherry, *Gregory's sixth operation*, In *The Best Writing on Mathematics 2019*, 195–207. Edited by Mircea Pitici. Princeton University Press, Princeton, NJ, 2019. See `https://books.google.co.il/books?id=RcmXDwAAQBAJ`

[9] Jacques Bair, Piotr Błaszczyk, Robert Ely, Peter Heinig, Mikhail Katz, *Leibniz's well-founded fictions and their interpretations*, Mat. Stud. **49** (2018), no. 2, 186–224. See `http://dx.doi.org/10.15330/ms.49.2.186-224` and `https://arxiv.org/abs/1812.00226`

[10] Jacques Bair, Piotr Błaszczyk, Robert Ely, Valérie Henry, Vladimir Kanovei, Karin Katz, Mikhail Katz, Semen Kutateladze, Thomas McGaffey, Patrick Reeder, David Schaps, David Sherry, Steve Shnider, *Interpreting the infinitesimal mathematics of Leibniz and Euler*, Journal for General Philosophy of Science **48** (2017), no. 2, 195–238. See `http://dx.doi.org/10.1007/s10838-016-9334-z` and `https://arxiv.org/abs/1605.00455`

[11] Jacques Bair, Piotr Błaszczyk, Robert Ely, Valérie Henry, Vladimir Kanovei, Karin Katz, Mikhail Katz, Taras Kudryk, Semen Kutateladze, Thomas McGaffey, Thomas Mormann, David Schaps, David Sherry, *Cauchy, infinitesimals and ghosts of departed* quantifiers, Mat. Stud. **47** (2017), no. 2, 115–144. See `http://dx.doi.org/10.15330/ms.47.2.115-144` and `https://arxiv.org/abs/1712.00226`

[12] Mikhail Katz, Eric Leichtnam, *Commuting and noncommuting infinitesimals*, American Mathematical Monthly **120** (2013), no. 7, 631–641. See `http://dx.doi.org/10.4169/amer.math.monthly.120.07.631` and `https://arxiv.org/abs/1304.0583`

[13] John C. Morgan, II, *Completion from an abstract perspective*, Real Anal. Exchange **41** (2016), no. 2, 233–261.

[14] Manfredo do Carmo, *Riemannian geometry*, Translated from the second Portuguese edition by Francis Flaherty, Mathematics: Theory & Applications, Birkhäuser, Boston, MA, 1992.

[15] C. Ward Henson, Lawrence C. Moore, Jr., *Invariance of the nonstandard hulls of locally convex spaces*, Duke Math. J. **40** (1973), 193–205.

[16] C. Ward Henson, Lawrence C. Moore, Jr., *Invariance of the nonstandard hulls of a uniform space*, Victoria Symposium on Nonstandard Analysis (Univ. Victoria, Victoria, B.C., 1972), pp. 85–98, Lecture Notes in Math. **369**, Springer, Berlin, 1974.

[17] Robert Williamson, Ludvik Janos, *Constructing metrics with the Heine-Borel property*, Proc. Amer. Math. Soc. **100** (1987), no. 3, 567–573.

[18] Wilhelmus Luxemburg, *A general theory of monads*, In *Applications of Model Theory to Algebra, Analysis, and Probability* (Internat. Sympos., Pasadena, Calif., 1967), pp. 18–86, Holt, Rinehart and Winston, New York, 1969.

[19] Albert Hurd, Peter Loeb, *An introduction to nonstandard real analysis*, Pure and Applied Mathematics, **118**. Academic Press, Orlando, FL, 1985.

[20] Isaac Goldbring, *Lecture notes on Nonstandard Analysis* (UCLA Summer School in Logic, 2009 & 2012), Preprint, 2014. See `https://www.math.uci.edu/~isaac/NSA%20notes.pdf`

V. Kanovei, IPPI RAS, Moscow, Russia
E-mail address: `kanovei@googlemail.com`

M. Katz, Department of Mathematics, Bar Ilan University, Ramat Gan 5290002 Israel
E-mail address: `katzmik@macs.biu.ac.il`

T. Nowik, Department of Mathematics, Bar Ilan University, Ramat Gan 5290002 Israel
E-mail address: `tahl@math.biu.ac.il`