On the significance of parameters and the projective level in the Choice and Collection axioms*

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Abstract

We make use of generalized iterations of a version of the Jensen forcing to define a cardinal-preserving generic model of **ZF** for any $\mathbb{n} \ge 1$ and each of the following four Choice hypotheses:

(1) $\mathbf{DC}(\mathbf{\Pi}^1_{\mathbb{n}}) \wedge \neg\mathbf{AC}_\omega(\mathit{\Pi}^1_{\mathbb{n}+1})$;

(2) $\mathbf{AC}_\omega(\mathrm{OD}) \wedge \mathbf{DC}(\mathit{\Pi}^1_{\mathbb{n}+1}) \wedge \neg\mathbf{AC}_\omega(\mathbf{\Pi}^1_{\mathbb{n}+1})$;

(3) $\mathbf{AC}_\omega \wedge \mathbf{DC}(\mathbf{\Pi}^1_{\mathbb{n}}) \wedge \neg\mathbf{DC}(\mathit{\Pi}^1_{\mathbb{n}+1})$;

(4) $\mathbf{AC}_\omega \wedge \mathbf{DC}(\mathit{\Pi}^1_{\mathbb{n}+1}) \wedge \neg\mathbf{DC}(\mathbf{\Pi}^1_{\mathbb{n}+1})$.

Thus if **ZF** is consistent and $\mathbb{n} \ge 1$ then each of these four conjunctions (1)–(4) is consistent with **ZF**.

As for the second main result, let $\mathbf{PA}^0_2$ be the 2nd-order Peano arithmetic without the Comprehension schema **CA**. For any $\mathbb{n} \ge 1$, we define a cardinal-preserving generic model of **ZF**, and a set $M \subseteq \mathscr{P}(\omega)$ in this model, such that $\langle \omega ; M \rangle$ satisfies

(5) $\mathbf{PA}^0_2 + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty) + \mathbf{CA}(\mathbf{\Sigma}^1_{\mathbb{n}+1}) + \neg\mathbf{CA}(\mathbf{\Sigma}^1_{\mathbb{n}+2})$.

Thus $\mathbf{CA}(\mathbf{\Sigma}^1_{\mathbb{n}+1})$ does not imply $\mathbf{CA}(\mathbf{\Sigma}^1_{\mathbb{n}+2})$ in $\mathbf{PA}^0_2$ even in the presence of the full parameter-free Choice $\mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty)$.

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I Introduction and preliminaries

1 The main results

This paper is devoted to interrelations between different restricted forms of the axioms of countable *independent choice* $\mathbf{AC}_\omega$ and *dependent choice* $\mathbf{DC}$. These forms will be distinguished by limiting the set defining the structure of the sequence of choices by one or another effective (*lightface*) or classical (*boldface*) projective class, resp. $\mathit{\Sigma}(\mathit{\Pi})^1_n$, $\boldsymbol{\Sigma}(\boldsymbol{\Pi})^1_n$. The following theorem, our first main result, shows that all three factors play a role in determining the strength of these forms of the countable axiom of choice, namely, the variant of the axiom ($\mathbf{AC}_\omega$ or $\mathbf{DC}$), the index n, as well as the assumption (boldface classes) or exclusion (lightface classes) parameters in the definitions of choice sets. Note that OD = *ordinal-definable* sets in (2).

Theorem 1.1 (1st main theorem)**.** *Assume that* $\mathbb{n} \ge 1$*. Then there exist cardinal-preserving generic extensions* $\mathbf{V}_1$*,* $\mathbf{V}_2$*,* $\mathbf{V}_3$*,* $\mathbf{V}_4$ *of* $\mathbf{L}$*, in each of which* $\mathbf{ZF}$ *holds and the corresponding conjunction from the next list holds*:

(1) $\mathbf{DC}(\boldsymbol{\Pi}^1_{\mathbb{n}}) \wedge \neg\mathbf{AC}_\omega(\mathit{\Pi}^1_{\mathbb{n}+1})$;

(2) $\mathbf{AC}_\omega(\mathrm{OD}) \wedge \mathbf{DC}(\mathit{\Pi}^1_{\mathbb{n}+1}) \wedge \neg\mathbf{AC}_\omega(\boldsymbol{\Pi}^1_{\mathbb{n}+1})$;

(3) $\mathbf{AC}_\omega \wedge \mathbf{DC}(\boldsymbol{\Pi}^1_{\mathbb{n}}) \wedge \neg\mathbf{DC}(\mathit{\Pi}^1_{\mathbb{n}+1})$;

(4) $\mathbf{AC}_\omega \wedge \mathbf{DC}(\mathit{\Pi}^1_{\mathbb{n}+1}) \wedge \neg\mathbf{DC}(\boldsymbol{\Pi}^1_{\mathbb{n}+1})$.

Thus if $\mathbf{ZF}$ *is consistent then each conjunction is compatible with* $\mathbf{ZF}$.

The content of Theorem 1.1 is graphically presented in figures 1, 2, 3. The figures and the theorem will be commented upon in Sections 2 and 3.

Our second main result is related to the Comprehension schema in 2nd order Peano arithmetic $\mathbf{PA}_2$. For the sake of brevity, let $\mathbf{PA}_2^0$ be the 2nd order Peano arithmetic entirely without Comprehension, and let $\mathbf{CA}(K)$ be the Comprehension schema $\exists\, x\, \forall\, k\, (k \in x \iff \varphi(k))$, limited to a given collection K of formulas φ. Thus $\mathbf{CA}(\boldsymbol{\Sigma}^1_n)$, resp. $\mathbf{CA}(\mathit{\Sigma}^1_n)$ is the Comprehension schema for Σ^1_n formulas *with*, resp. *without* parameters, and *parameters* are formally free variables other than k above.

Theorem 1.2 (2nd main theorem)**.** *Assume that* $\mathbb{n} \ge 1$*. Then there is a cardinal-preserving generic extension of* $\mathbf{L}$*, and a set* $M \subseteq \mathscr{P}(\omega)$ *in this extension, such that* $\mathbf{L} \cap \mathscr{P}(\omega) \subseteq M$ *and* $\langle \omega\,; M\rangle$ *models the theory* $\mathbf{PA}_2^0 + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty) + \mathbf{CA}(\boldsymbol{\Sigma}^1_{\mathbb{n}+1}) + \neg\mathbf{CA}(\boldsymbol{\Sigma}^1_{\mathbb{n}+2})$*. Thus* $\mathbf{CA}(\boldsymbol{\Sigma}^1_{\mathbb{n}+1})$ *does not imply* $\mathbf{CA}(\boldsymbol{\Sigma}^1_{\mathbb{n}+2})$ *even in the presence of the full parameter-free* $\mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty)$.

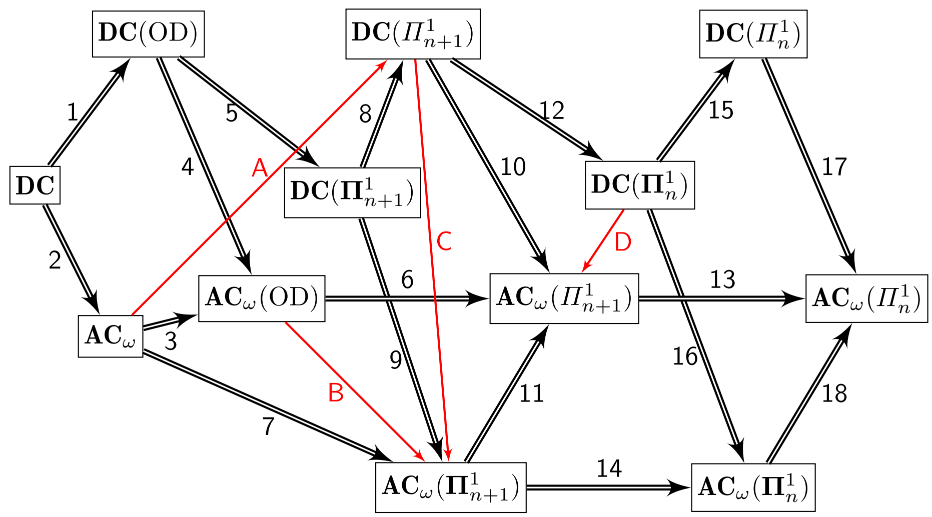

Figure 1: Provable $\Longrightarrow$ and unprovable $\rightarrow$ implications in **ZF**

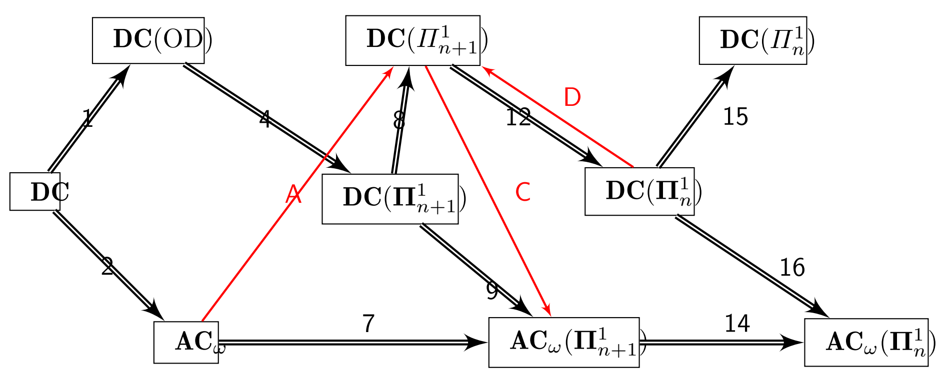

Figure 2: Provable $\Longrightarrow$ and unprovable $\rightarrow$ implications in **ZF** + **AC**$_\omega$(OD)

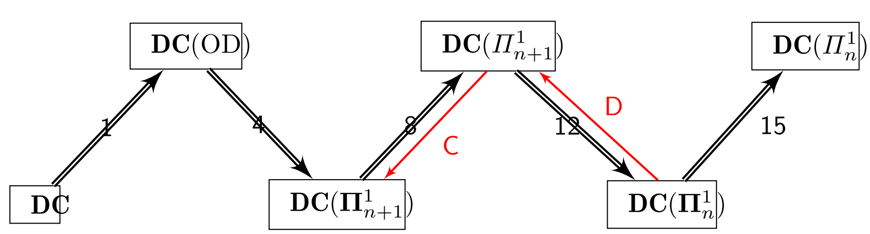

Figure 3: Provable $\Longrightarrow$ and unprovable $\rightarrow$ implications in **ZF** + **AC**$_\omega$

2 Comments on figures

All *unprovable* implications on the figures are such in virtue of Theorem 1.1.

All *provable* implications are rather self-evident, except for $\mathbf{DC}(K) \Longrightarrow \mathbf{AC}_\omega(K)$ for different classes K (arrows 2, 9, 10, 16, 17) – which are well-known anyway, and the implication $\mathbf{DC}(\mathit{\Pi}^1_{n+1}) \Longrightarrow \mathbf{DC}(\mathbf{\Pi}^1_n)$ (arrow 12) proved by Lemma 2.2(v) below.

We consider the *Baire space* $\mathscr{N} = \omega^\omega$, whose points are called *reals* in modern set theory, as well as product spaces of the form $\omega^m \times \mathscr{N}^k$, ω being discrete as usual. Sets in these spaces are called *pointsets*. See [50] on lightface and boldface *projective hierarchies* of pointsets.

The next definition presents the versions of $\mathbf{AC}_\omega$ and $\mathbf{DC}$ used here.

Definition 2.1. Let K be any poinclass (a collection of pointsets). The following *axioms*, or *principles* are introduced:

$\mathbf{AC}_\omega(K)$: if $P \subseteq \omega \times \omega^\omega$, $P \in K$, and $\mathtt{dom}\, P = \omega$ then there is a map $x : \omega \to \omega^\omega$ such that $\forall\, k\, P(k, x(k))$.

$\mathbf{DC}^-(K)$: if $P \subseteq (\omega^\omega)^2$, $P \in K$, and $\mathtt{dom}\, P = \omega^\omega$, then there is a map $x : \omega \to \omega^\omega$ such that $\forall\, k\, P(x(k), x(k{+}1))$.

$\mathbf{DC}(K)$: if $P \subseteq (\omega^\omega)^2$, $P \in K$, and $\mathtt{dom}\, P = \omega^\omega$, then, **for any** $a \in \omega^\omega$, there is $x : \omega \to \omega^\omega$ such that $\forall\, k\, P(x(k), x(k{+}1))$ and $x(0) = a$.

$\mathbf{DC}^*(K)$: if $P \subseteq (\omega^\omega)^2$, $P \in K$, $\mathtt{ran}\, P \subseteq \mathtt{dom}\, P$, then, for any $a \in \mathtt{dom}\, P$ there is $x : \omega \to \omega^\omega$ such that $\forall\, k\, P(x(k), x(k{+}1))$ and $x(0) = a$.

Simply $\mathbf{AC}_\omega$, $\mathbf{DC}$, $\mathbf{DC}^-$, $\mathbf{DC}^*$ mean $\mathbf{AC}_\omega$(all sets), $\mathbf{DC}$(all sets), *etc.* □

This definition can be used, for instance, for descriptive-set-theoretic, DST for brevity, pointclasses K of the form $\mathit{\Sigma}^1_n$ (lightface), $\mathbf{\Sigma}^1_n$ (boldface), $\mathit{\Sigma}^1_\infty = \bigcup_n \mathit{\Sigma}^1_n$, same for Π, Δ — and then the corresponding axiom will be called *a DST form of countable* $\mathbf{AC}$. Non-descriptive forms are obtained *e.g.* in cases $K = \mathrm{OD}$ (all ordinal-definable pointsets), or $K = \mathrm{ROD}$ (all real-ordinal-definable pointsets), or $K =$ all pointsets of any kind.

The axiom of (countable) dependent choices is known in slightly different versions, in particular $\mathbf{DC}, \mathbf{DC}^-, \mathbf{DC}^*$ as above. Clearly the equivalence $\mathbf{DC} \Longleftrightarrow \mathbf{DC}^- \Longleftrightarrow \mathbf{DC}^*$ holds in $\mathbf{ZF}$; this is why $\mathbf{DC}^-$, the minimal form so to speak, is usually considered (and denoted by just $\mathbf{DC}$) in modern set theory. However $\mathbf{DC}(K)$ as we define it turns out to be more convenient in the case of DST classes K, in particular, because, as far as we know, claim (v) of Lemma 2.2 takes the form $\mathbf{DC}^-(\mathit{\Pi}^1_{n+2}) \Longrightarrow \mathbf{DC}^-(\mathbf{\Pi}^1_n)$ via an argument by Guzicki [20]. This leaves the interrelations between $\mathbf{DC}^-(\mathit{\Pi}^1_{n+1})$ and $\mathbf{DC}^-(\mathbf{\Pi}^1_n)$ to be an open problem. This is why we prefer to consider **DC** rather than $\mathbf{DC}^-$ (in the notation of Definition 2.1) in this paper. The form **DC** was considered, by the way, in earlier papers [5, 20, 43].

The next lemma proves some elementary connections. In particular, claim (iii) implies that there is no need whatsoever to consider Σ-limited forms of the choice principles as they can be substituted by Π-forms.

Lemma 2.2. (i) $\mathbf{DC}^*(K) \Longrightarrow \mathbf{DC}(K) \Longrightarrow \mathbf{DC}^-(K)$ *for any* K ;

(ii) *if* K *is any boldface or lightface projective class, or the class* OD*, or the class of all sets, then*: $\mathbf{DC}^-(K) \Longrightarrow \mathbf{AC}_\omega(K)$;

(iii) $\mathbf{AC}_\omega(\mathit{\Pi}^1_n) \Longleftrightarrow \mathbf{AC}_\omega(\mathit{\Sigma}^1_{n+1})$, $\mathbf{AC}_\omega(\mathbf{\Pi}^1_n) \Longleftrightarrow \mathbf{AC}_\omega(\mathbf{\Sigma}^1_{n+1})$, *and the same holds for* **DC** *and* $\mathbf{DC}^*$;

(iv) $\mathbf{DC}(\mathit{\Pi}^1_n) \Longleftrightarrow \mathbf{DC}(\mathit{\Sigma}^1_{n+1}) \Longleftrightarrow \mathbf{DC}^*(\mathit{\Pi}^1_n) \Longleftrightarrow \mathbf{DC}^*(\mathit{\Sigma}^1_{n+1})$,
$\mathbf{DC}(\mathbf{\Pi}^1_n) \Longleftrightarrow \mathbf{DC}(\mathbf{\Sigma}^1_{n+1}) \Longleftrightarrow \mathbf{DC}^*(\mathbf{\Pi}^1_n) \Longleftrightarrow \mathbf{DC}^*(\mathbf{\Sigma}^1_{n+1})$;

(v) $\mathbf{DC}(\mathit{\Pi}^1_{n+1}) \Longrightarrow \mathbf{DC}(\mathbf{\Pi}^1_n)$, *and hence* $\mathbf{DC}(\mathit{\Sigma}^1_\infty) \Longleftrightarrow \mathbf{DC}(\mathbf{\Sigma}^1_\infty)$;

(vi) $\mathbf{DC}(\mathrm{OD}) \Longleftrightarrow \mathbf{DC}(\mathrm{ROD})$.

(vii) $\mathbf{DC}(\mathbf{\Pi}^1_1)$ *holds in* **ZF** *and in* $\mathbf{PA}_2$ *by the* $\mathbf{\Pi}^1_1$*-uniformization theorem.*

Proof. (i) is trivial. (ii),(vii) are standard facts, see *e.g.* [5] or [20].

(iii) As an example, to prove the lightface-**DC** claim in (iii) (also a rather known fact as a whole), assume that $a \in \omega^\omega$, and $P \subseteq (\omega^\omega)^2$ is a $\mathit{\Sigma}^1_{n+1}$ set with $\mathtt{dom}\, P = \omega^\omega$. Then $P(x,y) \Longleftrightarrow \exists\, z\, Q(x,y,z)$, where $Q \subseteq (\omega^\omega)^3$ is $\mathit{\Pi}^1_n$. It remains to apply $\mathbf{DC}(\mathit{\Pi}^1_n)$ to any $a' \in \omega^\omega$ with $(a')_0 = a$ and the $\mathit{\Pi}^1_n$ set $P' = \{\langle x,y\rangle \in (\omega^\omega)^2 : Q((x)_0,(y)_0,(y)_1)\}$. (Recall that if $x \in \omega^\omega$ then $(x)_k \in \omega^\omega$ is defined by $(x)_k(j) = x(2^k(2j+1)-1,\ \forall\, j.)$

(iv) The scheme of the proof is

$$\mathbf{DC}(\mathit{\Pi}^1_n) \Longrightarrow \mathbf{DC}(\mathit{\Sigma}^1_{n+1}) \Longrightarrow \mathbf{DC}^*(\mathit{\Pi}^1_n) \Longrightarrow \mathbf{DC}^*(\mathit{\Sigma}^1_{n+1}) \Longrightarrow \mathbf{DC}(\mathit{\Pi}^1_n).$$

Here the 1st and 3rd implications follow from (iii), so it remains to establish the 2nd one. Assume that $P \subseteq (\omega^\omega)^2$ is a $\mathit{\Pi}^1_n$ set with $\mathtt{ran}\, P \subseteq \mathtt{dom}\, P$, and $a \in \mathtt{dom}\, P$. It suffices to apply $\mathbf{DC}(\mathit{\Pi}^1_n)$ to the $\mathit{\Delta}^1_{n+1}$ set

$$Q = \{\langle x,y\rangle \in (\omega^\omega)^2 : P((x)_0,(x)_1) \Longrightarrow \big[P((y)_0,(y)_1) \wedge (y)_0 = (x)_1\big]\}$$

and any $a' \in \omega^\omega$ with $(a')_0 = a$ and $P((a')_0,(a')_1)$.

(v) is a bit trickier. Assume that $a \in \omega^\omega$, and $P \subseteq (\omega^\omega)^2$ is a $\mathbf{\Pi}^1_n$ set with $\mathtt{dom}\, P = \omega^\omega$. Then $P(x,y) \iff S(x,y,p)$, where $S \subseteq (\omega^\omega)^3$ is lightface $\mathit{\Pi}^1_n$, and $p \in \omega^\omega$. It remains to apply $\mathbf{DC}(\mathit{\Pi}^1_{n+1})$ to the $\mathit{\Pi}^1_{n+1}$ set

$$Q = \{\langle x, y\rangle : (y)_1 = (x)_1 \wedge \big[\exists\, z\, S((x)_0, z, (x)_1) \implies S((x)_0, (y)_0, (x)_1)\big]\}$$

and any $a' \in \omega^\omega$ with $(a')_0 = a$ and $(a')_1 = p$.

(vi) is similar to (v). □

3 Further comments on Theorem 1.1

It is quite clear that $\mathbf{AC} \implies \mathbf{DC} \implies \mathbf{AC}_\omega$. Studies in the early years of modern set theory by Gödel, Cohen, Levy, Jensen, demonstrated that neither implication is reversible in $\mathbf{ZF}$, $\mathbf{AC}$ is consistent with $\mathbf{ZF}$, but $\mathbf{AC}_\omega$ is independent of $\mathbf{ZF}$ and $\mathbf{DC}$ is independent of $\mathbf{ZF} + \mathbf{AC}_\omega$ (Jensen [26]).

Furthermore Levy [44] demonstrated that the generic collapse of cardinals below $\aleph_\omega$ (called the Levy collapse, see Solovay [58]) results in a generic extension of $\mathbf{L}$ in which $\mathbf{AC}_\omega(\mathit{\Pi}^1_2)$ fails, which is the strongest possible failure since $\mathbf{AC}_\omega(\mathbf{\Sigma}^1_2)$ is a theorem of $\mathbf{ZF}$.

Using rather similar arguments, Guzicki [20] proved that the Levy-style generic collapse below $\aleph_{\omega_1}$ results in a generic extension of $\mathbf{L}$ in which $\mathbf{AC}_\omega(\mathbf{\Pi}^1_2)$ fails, but $\mathbf{AC}_\omega(\mathrm{OD})$ holds, so that $\mathbf{AC}_\omega(\mathrm{OD})$ (for ordinal-definable sets) does not imply $\mathbf{AC}_\omega(\mathbf{\Pi}^1_2)$, let alone the full $\mathbf{AC}_\omega$. This can be compared with an opposite result for the *dependent choice* axiom $\mathbf{DC}(\mathbf{\Sigma}^1_\infty)$, which is equivalent to the parameter-free form $\mathbf{DC}(\mathit{\Sigma}^1_\infty)$ by Lemma 2.2.

Recent research has shown that similar consistency results can be obtained via non-collapse forcing, and in some cases using the consistency of 2nd order Peano arithmetic $\mathbf{PA}_2$ as the blanket assumption (see Section 71).

Enayat [8] used the finite-support infinite product of Jensen's minimal-$\mathit{\Delta}^1_3$-real forcing [25] to define a non-collapse permutation model of $\mathbf{ZF}$ with an infinite Dedekind-finite $\mathit{\Pi}^1_2$ set of reals, which easily yields the refutation of $\mathbf{AC}_\omega(\mathit{\Pi}^1_2)$. Friedman *e.a.* [13] used another generalization of Jensen's forcing to get a non-collapse model of $\mathbf{ZF} + \mathbf{AC}_\omega$ in which $\mathbf{DC}(\mathit{\Pi}^1_2)$ fails. (This result by a different method was also announced by Simpson [56], but in fact never published, see notes in [13, p. 4] and [22, p. 5].) Our own studies [42, 39] provided a Sacks-iterated, cardinal-preserving model of $\mathbf{ZF} + \mathbf{AC}_\omega(\mathrm{OD})$ in which $\mathbf{AC}_\omega(\mathbf{\Pi}^1_3)$ fails, and another such a model of $\mathbf{ZF}$ in which $\mathbf{AC}_\omega(\mathbf{\Sigma}^1_3)$ fails — which is admiddedly not the expected optimal failure of $\mathbf{AC}_\omega(\mathbf{\Pi}^1_2)$, resp., $\mathbf{AC}_\omega(\mathbf{\Sigma}^1_2)$ in those cases.

Some results related to parameter-free versions of the Separation and Replacement axiom schemata in $\mathbf{ZFC}$ also are known from [7, 45, 51].

Our Theorem 1.1 substantially strengthens the abovementioned results and maintains further clarification of the role of the projective level and parameters in the descriptive-theoretic axioms $\mathbf{AC}_\omega(K)$ and $\mathbf{DC}(K)$. Some parts of the theorem were published, in Russian, in a technical report [29].

4 Comments on Theorem 1.2

Following [5, 43, 57] we define the second order Peano arithmetic $\mathbf{PA}_2$ as a theory in the language $\mathcal{L}(\mathbf{PA}_2)$ with two sorts of variables – for natural numbers and for sets of them. We use j, k, m, n for variables over ω and x, y, z for variables over $\mathscr{P}(\omega)$, reserving capital letters for subsets of $\mathscr{P}(\omega)$ and other sets. The axioms are as follows in (1), (2), (3), (4):

(1) **Peano's axioms** for numbers.

(2) The **Induction** schema: $\Phi(0) \wedge \forall k\, (\Phi(k) \Longrightarrow \Phi(k+1)) \Longrightarrow \forall k\, \Phi(k)$, for every formula $\Phi(k)$ in $\mathcal{L}(\mathbf{PA}_2)$, and in $\Phi(k)$ we allow parameters, *i.e.*, free variables other than k. (We do not formulate Induction as one sentence here because the Comprehension schema **CA** will not be always assumed in full generality by default.)

(3) **Extensionality** for sets of natural numbers.

(4) The **Comprehension** schema **CA**: $\exists x\, \forall k\, (k \in x \Longleftrightarrow \Phi(k))$, for every formula Φ in which x does not occur, and in Φ we allow parameters.

$\mathbf{PA}_2$ is also known as A_2^- (see *e.g.* an early survey [5]), as Z_2 (see *e.g.* Simpson [57] and Friedman [9]), az Z_2^- (in [53] or elsewhere). The schema of Choice (see below) is not included in $\mathbf{PA}_2$ in this paper.

Let $\mathbf{PA}_2^0$ to be the (1)+(2)+(3) subtheory of $\mathbf{PA}_2$ (no Comprehension).

The principles $\mathbf{AC}_\omega$ and $\mathbf{DC}$ as in Definition 2.1 can be naturally reformulated as axiom schemata in the context of $\mathbf{PA}_2$.

Definition 4.1. Let K be a type of formulas of $\mathcal{L}(\mathbf{PA}_2)$, *e.g.* $\varSigma^1_n$ (lightface, real parameters not allowed), $\mathbf{\Sigma}^1_n$ (boldface, real parameters allowed), $\varSigma^1_\infty = \bigcup_n \varSigma^1_n$, same for Π. The next axiom schemata in $\mathcal{L}(\mathbf{PA}_2)$ are considered:

$\mathbf{AC}_\omega(K)$: $\forall k\, \exists x\, \Phi(k, x) \Longrightarrow \exists x\, \forall k\, \Phi(k, (x)_k)$, for every formula Φ in K, where as usual $(x)_k = \{j : 2^k(2j+1) - 1 \in x\}$.

$\mathbf{DC}(K)$: $\forall x\, \exists y\, \Phi(x, y) \Longrightarrow \forall x\, \exists z\, \forall k\, \big((z)_0 = x \wedge \Phi((z)_k, (z)_{k+1})\big)$, for any formula Φ in K.

$\mathbf{CA}(K)$: $\exists x\, \forall k\, \big(k \in x \Longleftrightarrow \Phi(k, (x)_k)\big)$, for any formula Φ in K.

Thus $\mathbf{CA}(\mathbf{\Sigma}^1_\infty)$ is the full Comprehension **CA** whereas $\mathbf{CA}(\varSigma^1_\infty)$ is the parameter-free Comprehension, *etc.* □

Discussing the structure and deductive properties of $\mathbf{PA}_2$, Kreisel [43, § III, page 366] wrote that the selection of subsystems "is a central problem". In particular, Kreisel notes, that

> [...] if one is convinced of the significance of something like a given axiom schema, it is natural to study details, such as the effect of parameters.

Recall that *parameters* in this context are free variables in axiom schemata that are not explicitly specified, in $\mathbf{PA}_2$, **ZFC**, and other similar theories. Thus the most obvious way to study "the effect of parameters" is to compare the strength of a given axiom schema with its parameter-free subschema, *e.g.* $\mathbf{CA}(\mathbf{\Sigma}^1_n)$ *vs.* $\mathbf{CA}(\varSigma^1_n)$. Working in this direction, it is established in our recent papers [42, 39] that 1) there is a cardinal-preserving generic extension of $\mathbf{L}$, and a set $M \subseteq \mathscr{P}(\omega)$ in this extension, such that $\mathscr{P}(\omega) \cap \mathbf{L} \subseteq M$ and M is a model of $\mathbf{PA}^0_2 + \mathbf{CA}(\varSigma^1_\infty) + \mathbf{CA}(\mathbf{\Sigma}^1_2) + \neg\,\mathbf{CA}(\mathbf{\Sigma}^1_4)$, and 2) if $\mathbf{PA}_2$ is consistent then $\mathbf{PA}^0_2 + \mathbf{CA}(\varSigma^1_\infty) + \mathbf{CA}(\mathbf{\Sigma}^1_2)$ does not prove $\mathbf{CA}(\mathbf{\Sigma}^1_4)$.

5 Brief review of the forcing notions involved

The models we built to prove Theorems 1.1 and 1.2 have their own interesting history. It starts with forcing by perfect sets, or *Sacks forcing* [52] which produces generic reals of minimal degree. Further studies discovered and studied countable-support *iterations* of Sacks forcing [6, 47, 19], and then *generalized iterations* [18], that is, iterations along any partial order I in the ground model M. In this case, a generic I-array $\boldsymbol{v} : I \to \mathscr{D} = 2^\omega$ is added, so that each $\boldsymbol{v}(i)$ is Sacks-generic over the model $M[\boldsymbol{v}\restriction\{j : j < i\}]$, and it happens that the structure of I as a poset in M is reflected in the structure of M-degrees of reals in the extension $M[\boldsymbol{v}]$. This connection can be used in coding by degrees of constructibility, see *e.g.* [46, p. 143].

As another application of generalized Sacks forcing iterations, in combination with the technique of "symmetric" generic extensions, cardinal-preserving generic models have been constructed with analytically definable violations of certain forms of the axiom of choice in the domain of reals.

Example 5.1. Taking $\mathbf{L}$ as the ground model and $\boldsymbol{I} = {\omega_1}^{<\omega} \smallsetminus \Lambda$ in $\mathbf{L}$ (all non-$\varnothing$ tuples of countable ordinals) leads to an $\boldsymbol{I}$-iterated Sacks generic array $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ of reals as above. Let Ω consist of all countable well-founded (*i.e.*, no infinite paths) initial segments $\xi \subseteq \boldsymbol{I}$ in $\mathbf{L}$. Then the symmetric subclass $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}]) \subseteq \mathbf{L}[\boldsymbol{v}]$ (Definition 23.3), generated by the set $\{\mathbf{x}\restriction\eta : \eta \in \Omega\}$, is a cardinal-preserving model of $\mathbf{ZF} + \mathbf{AC}_\omega$ in which $\mathbf{DC}$ fails (Jensen [26]), and more precisely $\mathbf{DC}(\mathit{\Pi}^1_3)$ fails. Some other constructions within $\mathbf{L}[\boldsymbol{v}]$ lead to some other models, *e.g.* of $\mathbf{ZF} + \neg\mathbf{AC}_\omega(\mathit{\Pi}^1_3)$, $\mathbf{ZF} + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty) + \neg\mathbf{AC}_\omega(\mathbf{\Pi}^1_3)$, $\mathbf{PA}^0_2 + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty) + \mathbf{CA}(\mathbf{\Sigma}^1_2) + \neg\mathbf{CA}(\mathbf{\Sigma}^1_4)$, [39, 42]. □

Admittedly, counter-examples obtained this way are one projective level higher than can be expected from the known positive results. For instance, instead of $\mathbf{ZF} + \mathbf{AC}_\omega + \neg\mathbf{DC}(\mathit{\Pi}^1_3)$ in the first counter-example one may want to get a model for $\mathbf{ZF} + \mathbf{AC}_\omega + \neg\mathbf{DC}(\mathit{\Pi}^1_3)$, since $\mathbf{DC}(\mathit{\Sigma}^1_2)$ is provable. This goal was achieved with the help of Jensen $\mathit{\Pi}^1_2$-real singleton forcing.

Example 5.2. Jensen forcing is a proper subset $J \in \mathbf{L}$ of the Sacks forcing, that is, it consists of (some) perfect sets of reals (or corresponding perfect trees). It satisfies CCC, and forces generic $\mathit{\Pi}^1_2$-real singletons [25]. In fact Jensen forcing is not a unique forcing notion in virtue of its definition, as *e.g.* Sacks forcing, but rather a family of similar forcing notions obtained by the same ω_1-long inductive construction in $\mathbf{L}$ based on the diamond $\diamondsuit_{\omega_1}$. □

Example 5.3. Countable-support iterated Jensen forcing of length ω_2 was defined and studied by Abraham [1, 2]. □

Example 5.4. Enayat [8] used a finite-support infinite product of Jensen forcing to get a permutation model of $\mathbf{ZF}$ with an infinite Dedekind-finite $\mathit{\Pi}^1_2$ set of reals, which implies the refutation of $\mathbf{AC}_\omega(\mathit{\Pi}^1_2)$. □

Example 5.5. By [33], it is forced by the finite-support product of ω copies of Jensen forcing that the set of basic Jensen-generic reals is a countable $\mathit{\Pi}^1_2$ set containing no OD real. □

Example 5.6. A somewhat modified forcing notion, say $\mathbb{J}'$, rather similar to Jensen forcing J, is used in [17]. Instead of a single generic real by $\mathbb{J}$, it adjoins a E_0-*equivalence class* of $\mathbb{J}$-generic reals. (Recall that reals $a, b \in 2^\omega$ are E_0-equivalent if $a(n) = b(n)$ for all but finite n. See some generalizations in [31, 34].) It turns out that this $\mathbb{J}'$-generic E_0-class is a (countable) $\mathit{\Pi}^1_2$ set containing no OD elements in the extension. □

Example 5.7. Capitalizing on Examples 5.5 and 5.1, a generalized finite-support $\boldsymbol{I}$-iteration of Jensen forcing is defined and studied in [13], to prove (among other results) that $\mathbf{ZF} + \mathbf{AC}_\omega + \neg\mathbf{DC}(\mathit{\Pi}^1_2)$ holds in a model similar to $\mathfrak{N}$ of Example 5.1. Some other constructions within $\boldsymbol{I}$-iterated Jensen extensions of $\mathbf{L}$ lead to some other cardinal-preserving models, *e.g.* of

$\mathbf{ZF} + \neg\mathbf{AC}_\omega(\mathit{\Pi}^1_2)$,

$\mathbf{ZF} + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty) + \neg\mathbf{AC}_\omega(\mathbf{\Pi}^1_2)$,
$\mathbf{PA}^0_2 + \mathbf{CA}(\mathit{\Sigma}^1_\infty) + \mathbf{CA}(\mathbf{\Sigma}^1_2) + \neg\mathbf{CA}(\mathbf{\Sigma}^1_3)$ (see [15] on the latter),
which suitably strengthen the results of Example 5.1. □

Another fundamental direction in these studies was discovered by Harrington [21]. This is the construction of generic models in which one or another effect is achieved at a given level n of the projective hierarchy, but not at previous levels. The results of Theorems 1.1 and 1.2 belong to this type, of course.

Example 5.8. Following Harrington's ideas, we defined a deneric extension $\mathbf{L}[a]$ in [41], by a real a that is $\mathit{\Delta}^1_{n+1}$ in $\mathbf{L}[a]$ for a given $n \geq 2$, and such that any $\mathit{\Sigma}^1_n$ real in $\mathbf{L}[a]$ is constructible. (Note that the Jensen forcing gives the result for $n = 2$ because of the Shoenfiend absoluteness.)

The technique of [41] involves a Harrington-style modification of the original construction of Jensen forcing in $\mathbf{L}$ in the form $J = \bigcup_{\alpha<\omega_1} J_\alpha$, each J_α being a countable set of perfect trees in $2^{<\omega}$. The modification essentially requires the sequence of J_αs to be "$\mathit{\Delta}^1_n$-generic" itself in the sense that it meets every $\mathit{\Delta}^1_n$ set dense in the "supertree" of all possible countable beginnings of the construction. (Harrington carried this out in [21] w.r.t. a rather similar almost-disjoint forcing of [24].)

Let $J(n)$ be the resulting forcing. Then still 1) there is a single $J(n)$-generic real a in $\mathbf{L}[a]$, 2) being $J(n)$-generic turns out to be a $\mathit{\Pi}^1_n$ formula, so $\{a\}$ is $\mathit{\Pi}^1_n$ and a itself is $\mathit{\Delta}^1_{n+1}$, and 3) the $\mathit{\Delta}^1_n$-genericity of the construction obscures things enough for all $\mathit{\Sigma}^1_n$ reals in $\mathbf{L}[a]$ being constructible. □

As a first approximation, the proof of our main results can be seen as using suitable submodels of the generalized $\boldsymbol{I}$-iteration (as in Examples 5.7 and 5.1) of a Harrington-style "$\mathit{\Delta}^1_{\mathfrak{n}}$-generic" version $J(\mathfrak{n})$ of Jensen forcing.

In fact, the proof will unfold somewhat differently, in particular, the CCC property, instrumental in [13, 15, 59], will not be pursued, but the ideas outlined in Examples 5.1, 5.7, 5.8 will be included.

6 The structure of the paper

The implementation of the plan outlined in Example 5.8 is organized as follows. It turns out that the usual approach to iterations of Jensen or similar forcing based on perfect trees, as in [13], leads to significant technical difficulties, which we have not been able to completely overcome, especially with regard to Harrington's idea of "generic" forcing constructions. This is why we have to turn to a purely geometric method of working with such iterations, developed in [29, 30, 32]. It presents iterated forcing conditions as *iterated perfect sets* in spaces $\mathscr{D}^\xi$, where $\mathscr{D} = 2^\omega$ is the Cantor spase and $\xi \subseteq \boldsymbol{I}$ is a countable initial segment in $\boldsymbol{I}$. These sets are introduced and studied in Chapters II and III, with the *fusion construction* in the latter.

Any set $\mathscr{X} \in \mathbf{L}$ of iterated perfect sets, satisfying some natural conditions, can be viewed as a forcing notion that adjoins a generic $\boldsymbol{I}$-array of reals. Such forcing notions $\mathscr{X}$, called *normal forcings*, corresponding $\mathscr{X}$-generic arrays $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$, extensions $\mathbf{L}[\boldsymbol{v}]$, their symmetric subextensions, and associates forcing relations, are studied in Chapter IV, and the consequences of the fusion property of $\mathscr{X}$ are also analyzed.

Chapter V introduces those symmetric submodels of generic extensions $\mathbf{L}[\boldsymbol{v}]$ that are specifically involved in the proof of Theorem 1.1. Two key conditions for a forcing $\mathscr{X}$ are formulated that guarantee, along with the fusion property, that these symmetric submodels give the desired result. The first one is *the definability property*, which claims that the binary relation

x, y are reals, and $x = \boldsymbol{v}(\boldsymbol{i}), y = \boldsymbol{v}(\boldsymbol{j})$ for some ***even*** tuples $\boldsymbol{i} \subset \boldsymbol{j}$ in $\boldsymbol{I}$

is $\Pi^1_{\mathbb{n}+1}$, for a given $\mathbb{n}$, in any suitable submodel of any $\mathscr{X}$-generic extension $\mathbf{L}[\boldsymbol{v}]$. The 2nd one, *the odd expansion property*, requires that if $\xi \in \mathbf{L}$ is an initial segment, $\varphi(x)$ a $\Pi^1_{\mathbb{n}}$ formula with reals in $\mathbf{L}[\boldsymbol{v}\restriction\xi]$ as parameters, and $\mathbf{L}[\boldsymbol{v}] \models \exists\, x\, \varphi(x)$ then such an x exists in $\mathbf{L}[\boldsymbol{v}\restriction\tau]$ for some $\tau \in \mathbf{L}$ (still an initial segment) such that $\tau \smallsetminus \xi$ consists only of odd tuples. (A tuple in $\boldsymbol{I}$ is odd if its last term is an odd ordinal.)

The goal of Chapter VI is to replace the the $\mathbb{n}$-odd expansion property with a more convenient $\mathbb{n}$-completeness property for $\mathscr{X}$. For this purpose, we introduce an auxiliary forcing relation $X \,\texttt{forc}\, \varphi$ in $\mathbf{L}$, where X is an iterated perfect set and φ is a formula of a certain extension of the language of 2nd order Peano arithmetic $\mathbf{PA}_2$. Then, a normal forcing is $\mathbb{n}$-complete, if for any closed Σ^1_n formula φ of the extended language, the set of all $X \in \mathscr{X}$ satisfying $X \,\texttt{forc}\, \varphi$ or $X \,\texttt{forc}\, \neg\varphi$, is dense in $\mathscr{X}$. This is how Harrington's idea of "generic" forcing notions (Example 5.8) is realized in our proof.

Note that $\texttt{forc}$ is not connected with any $\mathscr{X}$, but if $\mathscr{X}$ is $\mathbb{n}$-complete then $\texttt{forc}$ coincides with the usual $\mathscr{X}$-forcing relation up to Σ^1_{n+1} formulas. This allows to show that $\mathbb{n}$-completeness implies $\mathbb{n}$-odd expansion. Hence overall the whole task related to Theorem 1.1 is reduced to the following:

(∗) for a given $\mathbb{n} \geq 1$, find a normal forcing $\mathscr{X}$ in $\mathbf{L}$, satisfying the fusion, $\mathbb{n}$-definability, and $\mathbb{n}$-completeness properties.

The construction of such a forcing $\mathscr{X}$ is accomplished in Chapters VII–X. This is a difficult task. We define the forcing $\mathscr{X}$ required as a sort of limit of an ω_1-sequence of countable collections of iterated perfect sets, called *rudiments*. Rudiments, and sequences of rudiments increasing in the sense of a *refinement* relation $\sqsubseteq$, are studied in Chapter VII.

We introduce some properties of an $\sqsubseteq$-increasing ω_1-sequence of rudiments in Chapter VIII, which imply that the associated limit forcing $\mathscr{X}$ satisfies $(*)$ above. The properties are summed up in the notion of 1-5-$\mathbb{n}$ extension, such that $(*)$ above is reduced to the following:

(†) for a given $\mathbb{n} \geq 1$, construct an $\sqsubseteq$-increasing $\Sigma_{\mathbb{n}}^{\mathbf{HC}}$-definable ω_1-sequence of rudiments in $\mathbf{L}$, such that each term is a 1-5-$\mathbb{n}$ extension of the subsequence of all previous terms.

We prove the existence of 1-5-$\mathbb{n}$ extensions in Chapter IX, and then finish (†) and the proof of Theorem 1.1 in Chapter X by the construction of a sequence required by taking the $\leqslant_{\mathbf{L}}$-least possible 1-5-$\mathbb{n}$ extension at each step of the construction.

Chapter XI presents the proof of Theorem 1.2. We use a symmetric submodel of a $\mathscr{X}$-generic extension $\mathbf{L}[\boldsymbol{v}]$ of $\mathbf{L}$, for the same forcing $\mathscr{X}$.

The paper ends with a usual conclusion-style material in Chapter XII. In particular, we'll touch on the evaluation of those proof theoretic tools used in the arguments. We show in Section 71 why the formal consistency of second order Peano arithmetic $\mathbf{PA}_2$ suffices to prove the natural consistency corollaries of Theorems 1.1, 1.2 w.r.t. $\mathbf{PA}_2$ or $\mathbf{PA}_2^0$. This is a crucial advantage comparably to some earlier results, like *e.g.* the abovementioned results by Levy [44] and Guzicki [20] which definitely cannot be obtained on the base of the consistency of $\mathbf{PA}_2$.

7 Definability, constructibility, diamond prerequisites

Recall that $\mathbf{HC} = H\omega_1 = \{x : \mathrm{TC}\,(x) \text{ is at most countable}\}$, the set of all *hereditarily countable* sets. The $\in$-definability in $\mathbf{HC}$ is connected with the descriptive set theoretic definability by the following classical result:

Proposition 7.1 (see *e.g.* 25.25 in Jech [23]). *If* $n \geq 1$ *and* $X \subseteq \omega^\omega$ *then*

$$X \in \Sigma_{n+1}^1 \iff X \in \Sigma_n^{\mathbf{HC}} \quad \textit{and} \quad X \in \Pi_{n+1}^1 \iff X \in \Pi_n^{\mathbf{HC}},$$

and $X \in \Sigma_{n+1}^1(p) \iff X \in \Sigma_n^{\mathbf{HC}}(p)$ *for any parameter* $p \in \omega^\omega$, *etc.* □

Assume $\mathbf{V} = \mathbf{L}$ in the remainder of this section.

It is known that $\mathbf{HC} = \mathbf{L}_{\omega_1}$ provided $\mathbf{V} = \mathbf{L}$. Let $\leqslant_{\mathbf{L}}$ be the Gödel well-ordering of $\mathbf{L}$. If $\alpha < \omega_1$ then we let $\mathbb{c}_\alpha$ be the αth member of $\mathbf{HC} = \mathbf{L}_{\omega_1}$ in the sense of $\leqslant_{\mathbf{L}}$, and $\mathbf{HC}_{<\alpha} = \{\mathbb{c}_\gamma : \gamma < \alpha\}$. The following is well-known.

Proposition 7.2 ($\mathbf{V} = \mathbf{L}$). *The relation* $\leqslant_{\mathbf{L}} \restriction \mathbf{HC}$ *has length* ω_1, *therefore* $\mathbf{HC} = \{\mathbb{c}_\alpha : \alpha < \omega_1\}$ *and* $\mathbf{HC}_{<\alpha} \in \mathbf{HC}$ *for all* $\alpha < \omega_1$. *In addition*:

(i) $\leqslant_{\mathbf{L}} \restriction \mathbf{HC}$ *is a* $\Delta^{\mathbf{HC}}_1$ *relation, the set* $\{\mathbf{HC}_{<\alpha} : \alpha < \omega_1\}$ *is* $\Delta^{\mathbf{HC}}_1$, *too*;

(ii) *the maps* $\alpha \mapsto \mathbb{c}_\alpha$ *and* $\alpha \mapsto \mathbf{HC}_{<\alpha}$ *are* $\Delta^{\mathbf{HC}}_1$ *as well*;

(iii) $\leqslant_{\mathbf{L}} \restriction \mathbf{HC}$ *is* ***good****, in the sense that if* $p \in \mathbf{HC}$, $n \ge 1$, *and* $P(\cdot,\cdot,\cdot)$ *is a ternary* $\Delta^{\mathbf{HC}}_n(p)$ *relation on* $\mathbf{HC}$, *then so are the binary relations* $\exists\, x \leqslant_{\mathbf{L}} y\, P(x,y,z)$ *and* $\forall\, x \leqslant_{\mathbf{L}} y\, P(x,y,z)$. □

The *diamond principle* $\diamondsuit_{\omega_1}$ is true in $\mathbf{L}$ by [23, Thm 13.21], hence there is a $\Delta^{\mathbf{HC}}_1$ sequence of sets $\boldsymbol{S}_\alpha \subseteq \alpha$, $\alpha < \omega_1$, such that

(A) if $X \subseteq \mathbf{HC}$ then the set $\{\alpha < \omega_1 : \boldsymbol{S}_\alpha = X \cap \alpha\}$ is stationary in ω_1.

The $\Delta^{\mathbf{HC}}_1$-definability property is achieved by taking the $\leqslant_{\mathbf{L}}$-least possible $\boldsymbol{S}_\alpha$ at each step α in the standard construction of $\boldsymbol{S}_\alpha$, as *e.g.* in [23]. Define

$$\overline{\boldsymbol{S}}_\alpha = \{\mathbb{c}_\gamma : \gamma \in S_\alpha\} \text{ for } \alpha < \omega_1, \quad \text{hence} \quad \overline{\boldsymbol{S}}_\alpha \subseteq \mathbf{HC}_{<\alpha} := \{\mathbb{c}_\gamma : \gamma < \alpha\}.$$

We get the following as an easy corollary of (A) and Proposition 7.2.

Proposition 7.3 ($\mathbf{V} = \mathbf{L}$). *The map* $\alpha \mapsto \overline{\boldsymbol{S}}_\alpha$ *is* $\Delta^{\mathbf{HC}}_1$.
If $\overline{\boldsymbol{S}} \subseteq \mathbf{HC}$ *then the set* $\{\alpha < \omega_1 : \overline{\boldsymbol{S}}_\alpha = \overline{\boldsymbol{S}} \cap \mathbf{HC}_{<\alpha}\}$ *is stationary.* □

II Iterated perfect sets

The proof of our main results involves the engine of generalized product-iterated Sacks forcing developed in [30, 32] on the basis of earlier papers [6, 18, 19] and others. We consider the constructible universe $\mathbf{L}$ as the ground model for any forcing in the remainder.

8 Spaces and projections

Arguing in L in this section, we define, in $\mathbf{L}$, the set $\boldsymbol{I} = \omega_1^{<\omega} \smallsetminus \{\Lambda\} \in \mathbf{L}$ of all non-empty tuples $\boldsymbol{i} = \langle \gamma_0, \dots, \gamma_n \rangle$, $n < \omega$, of ordinals $\gamma_k < \omega_1$, partially ordered by *the strict extension* $\subset$ of tuples. $\boldsymbol{I}$ is a tree without the minimal node Λ (the empty tuple), which we exclude. We put

$$\begin{array}{rclcl} \boldsymbol{I}[{<}2] & = & 2^{<\omega} \smallsetminus \{\Lambda\} & = & \{\boldsymbol{i} \in \boldsymbol{I} : \mathtt{ran}\, \boldsymbol{i} \subseteq \{0,1\}\}, \\ \boldsymbol{I}[{<}\omega] & = & \omega^{<\omega} \smallsetminus \{\Lambda\} & = & \{\boldsymbol{i} \in \boldsymbol{I} : \mathtt{ran}\, \boldsymbol{i} \subseteq \omega\}, \end{array}$$

and generally $\boldsymbol{I}[{<}\alpha] = \alpha^{<\omega} \smallsetminus \{\Lambda\} = \{\boldsymbol{i} \in \boldsymbol{I} : \mathtt{ran}\, \boldsymbol{i} \subseteq \alpha\}$, so $\boldsymbol{I}[{<}\omega_1] = \boldsymbol{I}$.

If $\boldsymbol{i} \in \boldsymbol{I}$ then $\mathtt{lh}(\boldsymbol{i})$ is the length of $\boldsymbol{i}$; $\mathtt{lh}(\boldsymbol{i}) \geq 1$ since Λ is excluded.

Our plan is to define a generic extension $\mathbf{L}[\mathbf{a}]$ of $\mathbf{L}$ by an array $\mathbf{a} = \langle \mathbf{a}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}}$ of reals $\mathbf{a}_{\boldsymbol{i}} \subseteq \omega$, in which the structure of iterated genericity of $\mathbf{a}_{\boldsymbol{i}}$ will be determined by this set $\boldsymbol{I}$.

Let $\boldsymbol{\Xi}$ be the set of all at most countable initial segments (in the sense of $\subset$) $\zeta \subseteq \boldsymbol{I}$. If $\zeta \in \boldsymbol{\Xi}$ then IS_ζ is the set of all initial segments of ζ.

Greek letters ξ, η, ζ, ϑ, τ will denote sets in $\boldsymbol{\Xi}$.

Characters $\boldsymbol{i}$, $\boldsymbol{j}$ are used to denote *elements* of $\boldsymbol{I}$.

For any $\boldsymbol{i} \in \zeta \in \boldsymbol{\Xi}$, we consider initial segments $[{\subset}\boldsymbol{i}] = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{j} \subset \boldsymbol{i}\}$, $[{\subseteq}\boldsymbol{i}] = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{j} \subseteq \boldsymbol{i}\}$, $\zeta[\boldsymbol{i}{\not\subseteq}] = \{\boldsymbol{j} \in \zeta : \boldsymbol{i} \not\subseteq \boldsymbol{j}\}$, $\zeta[\boldsymbol{i}{\not\subset}] = \{\boldsymbol{j} \in \zeta : \boldsymbol{i} \not\subset \boldsymbol{j}\}$. Clearly $[{\subset}\boldsymbol{i}] \subsetneqq [{\subseteq}\boldsymbol{i}] \subseteq \zeta$ and $\zeta[\boldsymbol{i}{\not\subseteq}] \subsetneqq \zeta[\boldsymbol{i}{\not\subset}] \subseteq \zeta$.

Let $\mathscr{D} = 2^\omega \subseteq \omega^\omega$ be the *Cantor space*. For any set ξ, $\mathscr{D}^\xi$ is the product of ξ-many copies of $\mathscr{D}$ with the product topology. Then every $\mathscr{D}^\xi$ is a compact space, homeomorphic to $\mathscr{D}$ itself unless $\xi = \varnothing$.

Definition 8.1 (projections)**.** Assume that $\eta \subseteq \xi$ belong to $\boldsymbol{\Xi}$.

If $x \in \mathscr{D}^\xi$ then let $x{\downarrow}\eta = x \restriction \eta \in \mathscr{D}^\eta$ denote the usual restriction. If $X \subseteq \mathscr{D}^\xi$ then let $X{\downarrow}\eta = \{x{\downarrow}\eta : x \in X\}$. Moreover if $\mathscr{X}$ consists of sets $X \subseteq \mathscr{D}^\xi$ for different supersets ξ of η then let $\mathscr{X}{\downarrow}\eta = \{X{\downarrow}\eta : X \in \mathscr{X}\}$.

If $Y \subseteq \mathscr{D}^\eta$ then let $Y{\uparrow}\xi = \{x \in \mathscr{D}^\xi : x{\downarrow}\eta \in Y\}$ (*lifting*).

We define $X{\downarrow}_{\subseteq \boldsymbol{i}} = X{\downarrow}[{\subseteq}\boldsymbol{i}]$, $X{\uparrow}^{\subseteq \boldsymbol{i}} = X{\uparrow}[{\subseteq}\boldsymbol{i}]$, and similarly $X{\downarrow}_{\subset \boldsymbol{i}}$, $X{\uparrow}^{\subset \boldsymbol{i}}$, $x{\downarrow}_{\subseteq \boldsymbol{i}}$ *etc.* for points x, and $\mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}}$ *etc.* for collections $\mathscr{X}$ of sets.

Finally, we let $X{\Downarrow}\boldsymbol{i} = \{x(\boldsymbol{i}) : x \in X\}$. (Note a different arrow.) □

9 Iterated perfect sets and projection–keeping

We argue in L in this section. To describe the key idea, recall that the Sacks forcing consists of perfect subsets of $\mathscr{D}$, that is, sets of the form $H"\mathscr{D} = \{H(a) : a \in \mathscr{D}\}$, where $H : \mathscr{D} \xrightarrow{\text{onto}} X$ is a homeomorphism.

To get a product Sacks model, with two factors (the case of a two-element unordered set as the length of iteration), we have to consider sets $X \subseteq \mathscr{D}^2$ of the form $X = H"\mathscr{D}^2$ where H is any homeomorphism defined on $\mathscr{D}^2$ so that it splits in obvious way into a pair of one-dimentional homeomorphisms.

To get an iterated Sacks model, with two stages of iteration (the case of a two-element ordered set as the length of iteration), we make use of sets $X \subseteq \mathscr{D}^2$ of the form $X = H"\mathscr{D}^2$, where H is any homeomorphism defined on $\mathscr{D}^2$ such that if $H(a_1, a_2) = \langle x_1, x_2 \rangle$ and $H(a'_1, a'_2) = \langle x'_1, x'_2 \rangle$ then $a_1 = a'_1 \iff x_1 = x'_1$.

The combined product/iteration case results in the following definition.

Definition 9.1 ([30, 32]). For any $\zeta \in \boldsymbol{\Xi}$, $\mathbf{IPS}_\zeta$ (iterated perfect sets of dimension ζ) is the collection of all sets $X \subseteq \mathscr{D}^\zeta$ such that there is a homeomorphism $H : \mathscr{D}^\zeta \xrightarrow{\text{onto}} X$ satisfying

$$x_0{\downarrow}\xi = x_1{\downarrow}\xi \iff H(x_0){\downarrow}\xi = H(x_1){\downarrow}\xi$$

for all $x_0, x_1 \in \mathtt{dom}\, H$ and $\xi \in \boldsymbol{\Xi}$, $\xi \subseteq \zeta$. Homeomorphisms H satisfying this requirement will be called *projection–keeping*, PKH for brevity. In other words, sets in $\mathbf{IPS}_\zeta$ are images of $\mathscr{D}^\zeta$ via PKHs.

We put $\mathbf{IPS} = \bigcup_{\xi \in \boldsymbol{\Xi}} \mathbf{IPS}_\xi$. Sets in **IPS** are called *iterated perfect sets*, IPS in brief. If $X \in \mathbf{IPS}_\xi$ then let $\|X\| = \xi$ (*dimension* of X).

We let $\mathbf{IPS}_{\subset \boldsymbol{i}} = \mathbf{IPS}_{[\subset \boldsymbol{i}]}$, $\mathbf{IPS}_{\subseteq \boldsymbol{i}} = \mathbf{IPS}_{[\subseteq \boldsymbol{i}]}$ for the sake of brevity. □

Remark 9.2. Suppose that $\zeta \in \boldsymbol{\Xi}$ in **L**. The set $\mathbf{IPS}_\zeta$, defined in **L**, can be considered as a forcing notion. It is established in [32, Thm 1 and Subsection 6.1] that $\mathbf{IPS}_\zeta$ adjoins a generic array $\boldsymbol{v} \in \mathscr{D}^\zeta$ of reals $\boldsymbol{v}(\boldsymbol{i}) \in \mathscr{D}$, $\boldsymbol{i} \in \zeta$, such that each each $\boldsymbol{v}(\boldsymbol{i})$ is Sacks-generic over $\mathbf{L}[\boldsymbol{v}{\downarrow}_{\subset \boldsymbol{i}}]$. Thus $\mathbf{IPS}_\zeta$ works as a generalized ζ-iteration of the Sacks (perfect set) forcing. This is why we call sets in **IPS** *iterated perfect sets*. □

Remark 9.3. The empty set $\varnothing \in \boldsymbol{\Xi}$, $\mathscr{D}^\varnothing = \{\varnothing\}$, $\mathbb{1} = \{\varnothing\} \in \mathbf{IPS}_\varnothing$. □

Lemma 9.4 (Lemma 7 in [32]). *If H is a PKH defined on $X \in \mathbf{IPS}_\zeta$ then the image $H"X = \{H(x) : x \in X\}$ belongs to $\mathbf{IPS}_\zeta$.*

Proof. The superposition of two PKHs is a PKH. □

Lemma 9.5. *If* $X \in \mathbf{IPS}_\zeta$, $\eta \in \mathrm{IS}_\zeta$, $\boldsymbol{i} \in \zeta \smallsetminus \eta$, *then there exist points* $x, y \in X$ *with* $x{\downarrow}\eta = y{\downarrow}\eta$ *but* $x(\boldsymbol{i}) \neq y(\boldsymbol{i})$.

Proof. There is a PKH $H : \mathscr{D}^\zeta \xrightarrow{\text{onto}} X$. Assume w.l.o.g. that $\eta = \zeta[\boldsymbol{i}\not\subseteq]$ (otherwise consider $\eta' = \zeta[\boldsymbol{i}\not\subseteq]$). Obviously there are points $x', y' \in \mathscr{D}^\zeta$ with $x'{\downarrow}\eta = y'{\downarrow}\eta$ but $x'(\boldsymbol{i}) \neq y'(\boldsymbol{i})$, hence $x'{\downarrow}_{\subseteq \boldsymbol{i}} \neq y'{\downarrow}_{\subseteq \boldsymbol{i}}$. Their H-values $x = H(x'), y = H(y')$ then satisfy $x{\downarrow}\eta = y{\downarrow}\eta$ but $x{\downarrow}_{\subseteq \boldsymbol{i}} \neq y{\downarrow}_{\subseteq \boldsymbol{i}}$. Yet $[\subset \boldsymbol{i}] \subseteq \eta$, so that $x{\downarrow}_{\subset \boldsymbol{i}} = y{\downarrow}_{\subset \boldsymbol{i}}$. And this implies $x(\boldsymbol{i}) \neq y(\boldsymbol{i})$. □

10 Some basic properties of iterated perfect sets

We argue in L in this section. Here follows a collection of some results related to iterated perfect sets, partially taken from [30, 32].

Lemma 10.1 (Proposition 4 in [32]). *Let* $\zeta \in \boldsymbol{\Xi}$. *Every set* $X \in \mathbf{IPS}_\zeta$ *is closed and satisfies the following properties*:

(i) *if* $\boldsymbol{i} \in \zeta$ *and* $z \in X{\downarrow}_{\subset \boldsymbol{i}}$ *then* $\mathbf{D}_{Xz}(\boldsymbol{i}) = \{x(\boldsymbol{i}) : x \in X \wedge x{\downarrow}_{\subset \boldsymbol{i}} = z\}$ *is a perfect set in* $\mathscr{D}$,

(ii) *if* $\xi \in \boldsymbol{\Xi}$, $\xi \subseteq \zeta$, *and a set* $X' \subseteq X$ *is open in* X *(in the relative topology) then the projection* $X'{\downarrow}\xi$ *is open in* $X{\downarrow}\xi$ *— in other words, the projection from* X *to* $X{\downarrow}\xi$ *is an open map,*

(iii) *if* $\xi, \eta \in \mathrm{IS}_\zeta$, $x \in X{\downarrow}\xi$, $y \in X{\downarrow}\eta$, *and* $x{\downarrow}(\xi \cap \eta) = y{\downarrow}(\xi \cap \eta)$, *then* $x \cup y \in X{\downarrow}(\xi \cup \eta)$.

Proof (sketch). Clearly $\mathscr{D}^\zeta$ satisfies (i), (ii), (iii), and one easily shows that projection–keeping homeomorphisms preserve the requirements. □

Lemma 10.2 (routine from (iii)). *Suppose that* $\xi, \eta, \vartheta \in \boldsymbol{\Xi}$, $\vartheta \cup \eta \subseteq \xi$, *and* $X \in \mathbf{IPS}_\xi$. *Then* $X{\downarrow}(\eta \cup \vartheta) = (X{\downarrow}\vartheta{\uparrow}(\eta \cup \vartheta)) \cup (X{\downarrow}\eta{\uparrow}(\eta \cup \vartheta))$. □

Lemma 10.3 (Lemma 5 in [32]). *Suppose that* $\xi, \eta, \vartheta \in \boldsymbol{\Xi}$, $\vartheta \cup \eta \subseteq \xi$, $W \in \mathbf{IPS}_\xi$, $C \subseteq W{\restriction}\eta$ *is any set, and* $U = W \cap (C{\uparrow}\xi)$. *Then*

(i) $U{\downarrow}\vartheta = (W{\downarrow}\vartheta) \cap (C{\downarrow}(\vartheta \cap \eta){\uparrow}\vartheta)$;

(ii) *if* $\vartheta = [\subseteq \boldsymbol{i}]$, $\boldsymbol{i} \in \xi$, *then* $U{\downarrow}_{\subseteq \boldsymbol{i}} = (W{\downarrow}_{\subseteq \boldsymbol{i}}) \cap (C{\downarrow}\sigma{\uparrow}^{\subseteq \boldsymbol{i}})$, *where* $\sigma = \eta \cap [\subseteq \boldsymbol{i}]$, *in particular, if* $\boldsymbol{i} \in \eta$ *then* $U{\downarrow}_{\subseteq \boldsymbol{i}} = C{\downarrow}\sigma{\uparrow}^{\subseteq \boldsymbol{i}}$. □

Lemma 10.4 (Lemma 6 in [32]). *If* $\xi \subseteq \zeta$ *belong to* $\boldsymbol{\Xi}$, *and* $X \in \mathbf{IPS}_\zeta$, *then* $X{\downarrow}\xi \in \mathbf{IPS}_\xi$. □

Lemma 10.5 (Lemma 9 in [32]). *Suppose that* $\zeta \in \mathbf{\Xi}$, $\eta \in \mathrm{IS}_\zeta$, $X \in \mathbf{IPS}_\zeta$, $Y \in \mathbf{IPS}_\eta$, *and* $Y \subseteq X{\downarrow}\eta$. *Then* $Z = X \cap (Y{\uparrow}\zeta)$ *belongs to* $\mathbf{IPS}_\zeta$.

In particular $Y{\uparrow}\zeta \in \mathbf{IPS}_\zeta$ (lifting), *since obviously* $\mathscr{D}^\zeta \in \mathbf{IPS}_\zeta$. □

Lemma 10.6 (Lemma 9 in [30]). *If* $\eta \subseteq \xi$ *belong to* $\mathbf{\Xi}$, $X, Y \in \mathbf{IPS}_\xi$, *and* $X{\restriction}\eta = Y{\restriction}\eta$, *then there is a PKH* $H : X \xrightarrow{\text{onto}} Y$ *such that* $H(x){\downarrow}\eta = x{\downarrow}\eta$ *for all* $x \in X$. □

Lemma 10.7. *Suppose that* $\eta \subseteq \xi$ *belong to* $\mathbf{\Xi}$, $X \in \mathbf{IPS}_\xi$, $Y = X{\downarrow}\eta \in \mathbf{IPS}_\eta$, *and* $H : \mathscr{D}^\eta \xrightarrow{\text{onto}} Y$ *is a PKH. Then there is a PKH* $K : \mathscr{D}^\xi \xrightarrow{\text{onto}} X$ *such that* $K(x){\downarrow}\eta = H(x{\downarrow}\eta)$ *for all* $x \in \mathscr{D}^\xi$.

Proof. The set $Y' = Y{\uparrow}\xi$ belongs to $\mathbf{IPS}_\xi$ by Lemma 10.5. Therefore by Lemma 10.6 there is a PKH $J : Y' \xrightarrow{\text{onto}} X$ such that $J(x){\downarrow}\eta = x{\downarrow}\eta$ for all $x \in Y'$. Yet by the choice of H, the map $H' : \mathscr{D}^\xi \to Y'$ defined by $H'(x){\downarrow}\eta = H(x{\downarrow}\eta)$ and $H'(x){\downarrow}(\xi \smallsetminus \eta) = x{\downarrow}(\xi \smallsetminus \eta)$ for all $x \in \mathscr{D}^\xi$, is a PKH $\mathscr{D}^\xi \xrightarrow{\text{onto}} Y'$. Thus the superposition $K(x) = J(H'(x))$ is a PKH $\mathscr{D}^\xi \xrightarrow{\text{onto}} X$, and if $x \in \mathscr{D}^\xi$ then $K(x){\downarrow}\eta = J(H'(x)){\downarrow}\eta = H'(x){\downarrow}\eta = H(x{\downarrow}\eta)$. □

Corollary 10.8. *Let* $\xi, \eta \in \mathbf{\Xi}$, $\vartheta = \xi \cup \eta$, $X \in \mathbf{IPS}_\xi$, $Y \in \mathbf{IPS}_\eta$, $X{\downarrow}(\xi \cap \eta) = Y{\downarrow}(\xi \cap \eta)$. *Then* $Z = (X{\uparrow}\vartheta) \cap (Y{\uparrow}\vartheta) \in \mathbf{IPS}_\vartheta$, $Z{\downarrow}\xi = X$, $Z{\downarrow}\eta = Y$.

Proof. The set $X' = X{\uparrow}\vartheta$ belongs to $\mathbf{IPS}_\vartheta$ by Lemma 10.5. In addition, $X'{\downarrow}\eta = X{\downarrow}(\xi \cap \eta){\uparrow}\eta$ by Lemma 10.3 (with $C = X$, $W = \mathscr{D}^\vartheta$). Then $Y \subseteq X'{\downarrow}\eta$, because $Y{\downarrow}(\xi \cap \eta) = X{\downarrow}(\xi \cap \eta)$. We conclude that $X' \cap (Y{\uparrow}\vartheta) \in \mathbf{IPS}_\vartheta$ by Lemma 10.5. Finally, $X' \cap (Y{\uparrow}\vartheta) = Z$ by construction.

To check that say $Z{\downarrow}\xi = X$, let $x \in X$. There is $y \in Y$ with $x{\downarrow}(\xi \cap \eta) = y{\downarrow}(\xi \cap \eta)$. Then $z = x \cup y \in Z$ by construction, and $z{\downarrow}\xi = x$. □

11 Clopen subsets

We argue in L in this section.

The next lemma highlights the Sacks-iterated character of sets in $\mathbf{IPS}_\xi$ in case $\xi = [\subseteq \boldsymbol{i}]$. Let a *perfect tree* be any (nonempty) tree $T \subseteq 2^{<\omega}$ with no endpoints, such that $B(T) = \{t \in T : t^\frown 0 \in T \wedge t^\frown 1 \in T\}$, the set of all splitting points, is cofinal in T.

Let $\mathbf{PT} = \{T \subseteq 2^{<\omega} : T \text{ is a perfect tree}\}$, a closed set in $\mathscr{P}(2^{<\omega})$.

If $T \in \mathbf{PT}$ then $[T] = \{x \in 2^\omega : \forall k\, (x{\restriction}k \in T)\}$, a perfect set.

Conversely, $\mathbf{tree}(X) = \{s \in 2^{<\omega} : [s] \cap X \neq \varnothing\} \in \mathbf{PT}$ for any perfect set $X \subseteq 2^\omega$, where $[s] = \{x \in 2^\omega : s \subset x\}$ for $s \in 2^{<\omega}$.

Lemma 11.1 (Lemma 11 in [32])**.** *Assume that $\boldsymbol{i} \in \boldsymbol{I}$, $Y \in \mathbf{IPS}_{\subset \boldsymbol{i}}$, $\mathcal{T}$ continuously maps Y into $\mathscr{P}(2^{<\omega})$ so that $\mathcal{T}(y) \in \mathbf{PT}$ for all $y \in Y$. Then $X = \{x \in \mathscr{D}^{[\subseteq \boldsymbol{i}]} : x{\downarrow}_{\subset \boldsymbol{i}} \in Y \wedge x(i) \in [\mathcal{T}(x{\downarrow}_{\subset \boldsymbol{i}})]\} \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$.* $\square$

The following is a converse to Lemma 11.1.

Recall that perfect sets $\mathbf{D}_{Xy}(\boldsymbol{i})$ are defined by Lemma 10.1(i).

Lemma 11.2. *Let $\boldsymbol{i} \in \boldsymbol{I}$, $X \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$, $Y = X{\downarrow}_{\subset \boldsymbol{i}} \in \mathbf{IPS}_{\subset \boldsymbol{i}}$, and if $y \in Y$ then $\mathcal{T}_X(y) = \mathbf{tree}(\mathbf{D}_{Xy}(\boldsymbol{i}))$. Then $\mathcal{T}_X$ continuously maps Y into $\mathbf{PT}$.*

Proof. Let $s \in 2^{<\omega}$. Then the set $Y_s = \{y \in Y : s \in \mathcal{T}_X(y)\}$ satisfies $Y_s = X_s{\downarrow}_{\subset \boldsymbol{i}}$, where $X_s = \{x \in X : s \subset x(\boldsymbol{i})\}$. It follows that Y_s is clopen in Y by Lemma 10.1(ii). By similar reasons, the set $Y'_s = \{y \in Y : s \notin \mathcal{T}_X(y)\}$ is clopen in Y as well. $\square$

We continue with assorted results on clopen subsets of sets in $\mathbf{IPS}$. The next lemma fails for $\mathbf{IPS}_\xi$ in case $\xi \in \boldsymbol{\Xi}$ is not linearly ordered by $\subseteq$.

Lemma 11.3. *Let $\boldsymbol{i} \in \boldsymbol{I}$, $X \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$. Then every set $\varnothing \neq Y \subseteq X$, clopen in X, belongs to $\mathbf{IPS}_{\subseteq \boldsymbol{i}}$ as well.*

Proof. We argue by induction on $\mathtt{lh}(\boldsymbol{i})$. If $\mathtt{lh}(\boldsymbol{i}) = 1$ then $[\subseteq \boldsymbol{i}] = \{\boldsymbol{i}\}$, and hence $\mathbf{IPS}_{\subset \boldsymbol{i}}$ is essentially the family of all perfect sets $P \subseteq \mathscr{D}$. Thus we can refer to the fact that a clopen subset of a perfect set is perfect, too.

Now suppose that $\mathtt{lh}(\boldsymbol{i}) = \ell \geq 2$, and let $\boldsymbol{j} = \boldsymbol{i} \restriction (\ell - 1)$. By Lemma 9.4, it suffices to consider the case $X = \mathscr{D}^{[\subseteq \boldsymbol{i}]}$, so that let $Y \subseteq \mathscr{D}^{[\subseteq \boldsymbol{i}]}$ be clopen. By a simple topological argument, Y has the form $Y = \bigcup_{k<n}(U_k \times P_k)$, where all $U_k \subseteq \mathscr{D}^{[\subset \boldsymbol{i}]}$ are clopen and pairwise disjoint, and $P_k \subseteq \mathscr{D}$ are clopen, so that there are perfect trees T_k satisfying $P_k = [T_k]$.

On the other hand, the set $Y' = Y{\downarrow}_{\subset \boldsymbol{i}} = \bigcup_{k<n} U_k$ belongs to $\mathbf{IPS}_{\subset \boldsymbol{i}} = \mathbf{IPS}_{\subseteq \boldsymbol{j}}$ by the inductive hypothesis, and the map $\mathcal{T}(y) = T_k$ in case $y \in U_k$ is continuous. It remains to apply Lemma 11.1. $\square$

Lemma 11.4. *If $\eta \subseteq \zeta$ belong to $\boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\zeta$, and $U \subseteq X$ is clopen in X then $U{\downarrow}\eta$ is clopen in $X{\downarrow}\eta$.*

Proof. By Lemma 9.4, it suffices to prove the result for $X = \mathscr{D}^\zeta$, in which case the result is obvious. $\square$

Lemma 11.5. *If $\zeta \in \boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\zeta$, $U \subseteq X$ is open in X, and $x_0 \in U$, then there is a set $X' \in \mathbf{IPS}_\zeta$, $X' \subseteq U$, clopen in X and containing x_0.*

Proof. By Lemma 9.4, it suffices to prove the result for $X = \mathscr{D}^\zeta$. Note that if $x_0 \in X' \subseteq \mathscr{D}^\zeta$ and X' is open in $\mathscr{D}^\zeta$ then there exists a basic clopen set $C \subseteq X'$ containing x_0. (*Basic clopen sets* are those of the form

$$C = \{x \in \mathscr{D}^\zeta : u_1 \subset x(\boldsymbol{i}_1) \wedge \ldots \wedge u_m \subset x(\boldsymbol{i}_m)\},$$

where $m \in \omega$, $\boldsymbol{i}_1, ..., \boldsymbol{i}_m \in \zeta$ are pairwise different, and $u_1, ..., u_m \in 2^{<\omega}$.) One easily proves that every set C of this type actually belongs to $\mathbf{IPS}_\zeta$. $\square$

Lemma 11.6. *Suppose that tuples $\boldsymbol{j} \subset \boldsymbol{i}$ belong to $\boldsymbol{I}$, $X \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$, $Y \in \mathbf{IPS}_{\subseteq \boldsymbol{j}}$, $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{j}}$, and $Z = X \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}})$. Let $\varnothing \neq Z' \subseteq Z$ be clopen in Z. Then there exist sets $X' \subseteq X$ and $Y' \subseteq Y$, clopen in resp. X, Y, such that $Y' \subseteq X'{\downarrow}_{\subseteq \boldsymbol{j}}$, and $Z' = X' \cap (Y'{\uparrow}^{\subseteq \boldsymbol{i}})$.*

Under the conditions of the lemma, note that $Z \in \mathbf{IPS}_\xi$ by Lemma 10.5, whereas $X', Z' \in \mathbf{IPS}_\xi$, $Y' \in \mathbf{IPS}_\eta$ by Lemma 11.3.

Proof. By the compactness, there is a set $C \subseteq X$, clopen in X, such that $Z' = Z \cap C$. Put $X' = C$. To define Y', note that $C' = C{\downarrow}_{\subseteq \boldsymbol{j}}$ is clopen in $X{\downarrow}_{\subseteq \boldsymbol{j}}$ by Lemma 10.1(ii). Therefore $Y' = Y \cap C'$ is clopen in Y. $\square$

Lemma 11.7. *If $X, Y \in \mathbf{IPS}_\zeta$, $\eta \subseteq \zeta$ belong to $\boldsymbol{\Xi}$, $\boldsymbol{i} \in \zeta \smallsetminus \eta$, and $X{\downarrow}\eta = Y{\downarrow}\eta$, then there exists $k < \omega$ and sets $X', Y' \in \mathbf{IPS}_\zeta$, $X' \subseteq X$, $Y' \subseteq Y$, clopen in resp. X, Y and such that $X'{\downarrow}\eta = Y'{\downarrow}\eta$, and $x(\boldsymbol{i})(k) = 0$ but $y(\boldsymbol{i})(k) = 1$ for all $x \in X'$ and $y \in Y'$, or vice versa.*

Proof. By Lemma 9.5, there are points $x_0 \in X$, $y_0 \in Y$ with $x_0{\downarrow}\eta = y_0{\downarrow}\eta$ but, for some k, $x_0(\boldsymbol{i})(k) = 0$ while $y_0(\boldsymbol{i})(k) = 1$ (or vice versa). By Lemma 11.5, there is a set $A \in \mathbf{IPS}_\zeta$, $x_0 \in A \subseteq X$, clopen in X, and such that $x(\boldsymbol{i})(k) = 0$ for all $x \in A$. Then $A{\downarrow}\eta$ is clopen in $X{\downarrow}\eta$ by Lemma 11.4.

Note that $x_0{\downarrow}\eta \in A{\downarrow}\eta$ by construction, therefore $y_0{\downarrow}\eta \in A{\downarrow}\eta$ as well.

Furthermore, $B = \{y \in Y : y{\downarrow}\eta \in A{\downarrow}\eta\}$ is clopen in Y, and $y_0 \in B$. Still by Lemma 11.5, there is a set $Y' \in \mathbf{IPS}_\zeta$, $y_0 \in Y' \subseteq B$, clopen in Y, and such that $y(\boldsymbol{i})(k) = 1$ for all $y \in Y'$.

It remains to define $X' = ((Y'{\downarrow}\eta){\uparrow}\zeta) \cap A$ and apply Lemma 11.4 to check that X' is clopen in X, and Lemma 10.5 to check that $X' \in \mathbf{IPS}_\zeta$. $\square$

Corollary 11.8. *If $X \in \mathbf{IPS}_\zeta$, and $\boldsymbol{i} \neq \boldsymbol{j}$ belong to ζ, then there exists $Z \in \mathbf{IPS}_\zeta$, $Z \subseteq X$, clopen in X, and such that $(Z{\Downarrow}\boldsymbol{i}) \cap (Z{\Downarrow}\boldsymbol{j}) = \varnothing$.*

Proof. Let say $\boldsymbol{j} \not\subseteq \boldsymbol{i}$, so that $\boldsymbol{i} \notin \eta = \zeta[\boldsymbol{j}\not\subseteq]$. Lemma 11.7 (with $X = Y$) yields relatively clopen sets $X', Y' \subseteq X$ in $\mathbf{IPS}_\zeta$ with $X' {\restriction} \eta = Y' {\restriction} \eta$, and $k < \omega$, such that $x(\boldsymbol{i})(k) = 0$ for all $x \in X'$ and $x(\boldsymbol{i})(k) = 1$ for all $x \in Y'$.

Now note that $U = X'\restriction\eta = Y'\restriction\eta \in \mathbf{IPS}_\eta$ by Lemma 10.4, and U is clopen in $X\restriction\eta$ by Lemma 11.4. Lemma 11.5 implies that there is a relatively clopen $V \subseteq U$, $V \in \mathbf{IPS}_\eta$, such that either (0) $u(\boldsymbol{j})(k) = 0$ for all $u \in V$ or (1) $u(\boldsymbol{j})(k) = 1$ for all $u \in U$. Let say (1) hold. Then the set $Z = X' \cap (V{\uparrow}\zeta) \subseteq X$ belongs to $\mathbf{IPS}_\zeta$ by Lemma 10.5, is clopen in X, and if $x \in Z$ then $x(\boldsymbol{j})(k) = 1$ but $x(\boldsymbol{i})(k) = 0$ by construction, as required. □

We leave the proof of the following generalization of 11.7/11.8 to the reader; it is rather routine and similar to the above.

Lemma 11.9. *Let $X, Y \in \mathbf{IPS}_\zeta$, $\eta \subseteq \zeta$ belong to $\boldsymbol{\Xi}$, $X{\downarrow}\eta = Y{\downarrow}\eta$, $\boldsymbol{i}, \boldsymbol{j} \in \zeta$, and either $\boldsymbol{i} \neq \boldsymbol{j}$ or $\boldsymbol{i} = \boldsymbol{j} \notin \eta$. Then there is $k < \omega$ and sets $X', Y' \in \mathbf{IPS}_\zeta$, $X' \subseteq X$, $Y' \subseteq Y$, clopen in resp. X, Y, and such that still $X'{\downarrow}\eta = Y'{\downarrow}\eta$, and $x(\boldsymbol{i})(k) = 0$ but $y(\boldsymbol{j})(k) = 1$ for all $x \in X'$, $y \in Y'$, or vice versa.* □

12 Assembling sets from projections

We still argue in L in this section.

Lemma 12.1. *Assume that $\xi_0, \xi_1, \xi_2, \dots \in \boldsymbol{\Xi}$, $\vartheta = \bigcup_n \xi_n$, and $X \in \mathbf{IPS}_\vartheta$. Then $X = \bigcap_n (X{\downarrow}\xi_n{\uparrow}\vartheta)$. In particular, $X = \bigcap_{\boldsymbol{i}\in\vartheta}(X{\downarrow}_{\subseteq\boldsymbol{i}}{\uparrow}\vartheta)$.*

Proof. The relation $X \subseteq X' = \bigcap_n (X{\downarrow}\xi_n){\uparrow}\vartheta$ is obvious. To prove $X' \subseteq X$, consider the following cases.

Case 1: simply $\vartheta = \xi_0 \cup \xi_1$. Apply Lemma 10.2.

Case 2: $\vartheta = \xi_0 \cup \xi_1 \cup \ldots \cup \xi_n$. Argue by induction using Case 1.

Case 3: general case. By the result for Case 2, we can w.l.o.g. assume that $\xi_n \subseteq \xi_{n+1}$ for all n. Then apply the compactness. □

It follows by Lemma 12.1 that each set $X \in \mathbf{IPS}_\vartheta$ is fully determined by the coherent system of its projections $X{\downarrow}_{\subseteq\boldsymbol{i}} = X{\downarrow}[\subseteq\boldsymbol{i}] \in \mathbf{IPS}_{\subseteq\boldsymbol{i}}$, where $\boldsymbol{i} \in \vartheta$ and $[\subseteq\boldsymbol{i}] = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{j} \subseteq \boldsymbol{i}\}$. The next lemma shows that conversely any coherent system of iterated perfect sets results in a set in $\mathbf{IPS}_\vartheta$.

Lemma 12.2. *Let $\xi_0, \xi_1, \xi_2, \dots \in \boldsymbol{\Xi}$, $\vartheta = \bigcup_n \xi_n$, and sets $X_n \in \mathbf{IPS}_{\xi_n}$ satisfy the* coherence condition

(∗) $X_n{\downarrow}(\xi_k \cap \xi_n) = X_k{\downarrow}(\xi_k \cap \xi_n)$ *for all k, n.*

Then $X = \bigcap_n (X_n{\uparrow}\vartheta)$ belongs to $\mathbf{IPS}_\vartheta$, and $X{\downarrow}\xi_n = X_n$, $\forall\, n$.

In particular, if $\xi_0, \xi_1, \xi_2, \dots$ are pairwise disjoint, then (∗) holds by default, hence $X = \bigcap_n (X_n{\uparrow}\vartheta)$ belongs to $\mathbf{IPS}_\vartheta$ and $X{\downarrow}\xi_n = X_n$, $\forall\, n$.

Proof. By Corollary 10.8, we w.l.o.g. assume that $\xi_0 \subseteq \xi_1 \subseteq \xi_2 \subseteq \dots$. Lemma 10.7 yields a sequence of PKHs $H_n : \mathscr{D}^{\xi_n} \xrightarrow{\text{onto}} X_n$ s.t. $H_{n+1}(x){\downarrow}\xi_n = H_n(x{\downarrow}\xi_n)$ for all n and $x \in \mathscr{D}^{\xi_{n+1}}$. This allows us to define a PKH $H : \mathscr{D}^{\vartheta} \xrightarrow{\text{onto}} X$ by simply $H(x){\downarrow}\xi_n = H_n(x{\downarrow}\xi_n)$ for all n and $x \in \mathscr{D}^{\vartheta}$. □

The lemma leads to another representation of iterated perfect sets. Let $\vartheta \in \boldsymbol{\Xi}$. If $X \subseteq \mathscr{D}^{\vartheta}$ then the system of projections $X{\downarrow}_{\subseteq \boldsymbol{i}}$, $\boldsymbol{i} \in \vartheta$, will be called *the projection tree* of X. Generally, a *projection tree* is any system of sets $X_{\boldsymbol{i}}$, $\boldsymbol{i} \in \vartheta$, satisfying the *coherence condition* in the form

(†) $X_{\boldsymbol{i}} \subseteq \mathscr{D}^{[\subseteq \boldsymbol{i}]}$, and if $\boldsymbol{i} \subset \boldsymbol{j}$ belong to ϑ then $X_{\boldsymbol{i}} = X_{\boldsymbol{j}}{\downarrow}_{\subseteq \boldsymbol{i}}$.

Corollary 12.3 (of Lemma 12.2)**.** *Let $\vartheta \in \boldsymbol{\Xi}$. If $X \in \mathbf{IPS}_{\vartheta}$ then the system of sets $X{\downarrow}_{\subseteq \boldsymbol{i}}$, $\boldsymbol{i} \in \vartheta$, satisfies (†), and $X = \bigcap_{\boldsymbol{i} \in \vartheta}(X{\downarrow}_{\subseteq \boldsymbol{i}}{\uparrow}\vartheta)$.*

Conversely, if sets $X_{\boldsymbol{i}} \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ satisfy (†) (i.e., form a coherent projection tree)*, then $X = \bigcap_{\boldsymbol{i} \in \vartheta}(X_{\boldsymbol{i}}{\uparrow}\vartheta) \in \mathbf{IPS}_{\vartheta}$ and $X{\downarrow}_{\subseteq \boldsymbol{i}} = X_{\boldsymbol{i}}$ for all $\boldsymbol{i}$.* □

Thus sets in $\mathbf{IPS}_{\vartheta}$ are in natural 1-1 correspondence with coherent projection trees of sets $X_{\boldsymbol{i}} \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$.

13 Permutations

Let **Perm** be the group of all bijections $\pi : \boldsymbol{I} \xrightarrow{\text{onto}} \boldsymbol{I}$, $\pi \in \mathbf{L}$, $\subset$-*invariant* in the sense that $\boldsymbol{i} \subset \boldsymbol{j} \iff \pi(\boldsymbol{i}) \subset \pi(\boldsymbol{j})$ for all $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}$. Thus $\mathbf{Perm} \in \mathbf{L}$. Bijections $\pi \in \mathbf{Perm}$ will be called *permutations*. Any $\pi \in \mathbf{Perm}$ is *length-preserving*, so that $\mathtt{lh}(\boldsymbol{i}) = \mathtt{lh}(\pi(\boldsymbol{i}))$ for all $\boldsymbol{i} \in \xi$,

The superposition $\circ$ is the group operation: $(\pi \circ \rho)(\boldsymbol{i}) = \pi(\rho(\boldsymbol{i}))$.

To define an important subgroup of **Perm**, recall that every ordinal α can be represented in the form $\alpha = \lambda + m$, where $\lambda \in \mathbf{Ord}$ is a limit ordinal and $m < \omega$; then α is called *odd*, resp., *even*, if the number n is odd, resp., even. A tuple $\boldsymbol{i} = \langle \alpha_0, \dots, \alpha_k \rangle \in \boldsymbol{I}$ is *odd*, resp., *even*, if such is the last term α_k. If $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}$ then $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$ will mean that $\mathtt{lh}(\boldsymbol{i}) = \mathtt{lh}(\boldsymbol{j})$ and if $k < \mathtt{lh}(\boldsymbol{i})$ then the ordinals $\boldsymbol{i}(k)$ and $\boldsymbol{j}(k)$ have the same parity.

Let $\boldsymbol{\Pi}$ be the subgroup of all permutations $\pi \in \mathbf{Perm}$, such that $\boldsymbol{i} \approx_{\mathtt{par}} \pi(\boldsymbol{i})$ for every $\boldsymbol{i} \in \boldsymbol{I}$, that is, *parity-preserving* permutations.

Example 13.1. Suppose that $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}$, $\mathtt{lh}(\boldsymbol{i}) = \mathtt{lh}(\boldsymbol{j})$. Define a permutation $\pi = \pi_{\boldsymbol{ij}} \in \mathbf{Perm}$ satisfying $\pi(\boldsymbol{i}) = \boldsymbol{j}$ as follows. Let $\boldsymbol{k} \in \boldsymbol{I}$.

If $\boldsymbol{k}(0) \notin \{\boldsymbol{i}(0), \boldsymbol{j}(0)\}$ then put $\pi(\boldsymbol{k}) = \boldsymbol{k}$.

If $\boldsymbol{k}(0) = \boldsymbol{i}(0)$ then there is a largest number $1 \le m \le \mathtt{lh}(\boldsymbol{i}) = \mathtt{lh}(\boldsymbol{j})$ such that $\boldsymbol{k}{\restriction}m = \boldsymbol{i}{\restriction}m$. Then $\boldsymbol{k} = (\boldsymbol{i}{\restriction}m)^\frown\boldsymbol{k}'$ (concatenation of tuples) for some tuple $\boldsymbol{k}' \in \boldsymbol{I} \cup \{\Lambda\}$. Put $\pi(\boldsymbol{k}) = (\boldsymbol{j}{\restriction}m)^\frown\boldsymbol{k}'$.

Similarly, if $\boldsymbol{k}(0) = \boldsymbol{j}(0)$ then there is a largest number $1 \le m \le \mathtt{lh}(\boldsymbol{i}) = \mathtt{lh}(\boldsymbol{j})$ such that $\boldsymbol{k}\restriction m = \boldsymbol{j}\restriction m$. Then accordingly $\boldsymbol{k} = (\boldsymbol{j}\restriction m)^\frown \boldsymbol{k}'$ for some $\boldsymbol{k}' \in \boldsymbol{I} \cup \{\Lambda\}$. Put $\pi(\boldsymbol{k}) = (\boldsymbol{i}\restriction m)^\frown \boldsymbol{k}'$.

Easily $\pi \in \mathbf{Perm}$, $\pi^{-1} = \pi$, $\pi(\boldsymbol{i}) = \boldsymbol{j}$, and if $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$ then $\pi \in \boldsymbol{\Pi}$. □

Actions. Any permutation $\pi \in \mathbf{Perm}$ induces a transformation left-acting on several types of objects as follows.

- If $\xi \in \boldsymbol{\Xi}$, or generally $\xi \subseteq \boldsymbol{I}$, then $\pi{\bullet}\xi := \pi``\xi = \{\pi(\boldsymbol{i}) : \boldsymbol{i} \in \xi\}$.
- If $\xi \subseteq \boldsymbol{I}$ and $x \in \mathscr{D}^\xi$ then $\pi{\bullet}x \in \mathscr{D}^{\pi\bullet\xi}$ is defined by $(\pi{\bullet}x)(\pi(\boldsymbol{i})) = x(\boldsymbol{i})$ for all $\boldsymbol{i} \in \xi$. That is, formally $\pi{\bullet}x = x \circ \pi^{-1}$, the superposition.
- If $\xi \subseteq \boldsymbol{I}$ and $X \subseteq \mathscr{D}^\xi$ then $\pi{\bullet}X := \{\pi{\bullet}x : x \in X\} \subseteq \mathscr{D}^{\pi\bullet\xi}$.
- If $G \subseteq \mathbf{IPS}$ then $\pi{\bullet}G := \{\pi{\bullet}X : X \in G\}$.

Lemma 13.2. *Let $\pi, \rho \in \mathbf{Perm}$, $\eta \in \boldsymbol{\Xi}$, and $v \in \mathscr{D}^{\boldsymbol{I}}$. Then*

(i) $\pi{\bullet}(\rho{\bullet}v) = (\pi\circ\rho){\bullet}v$ — *the group action property*;

(ii) $(\pi{\bullet}v){\downarrow}(\pi{\bullet}\eta) = \pi{\bullet}(v{\downarrow}\eta)$, *equivalently,* $(\pi{\bullet}v){\downarrow}\eta = \pi{\bullet}(v{\downarrow}(\pi^{-1}{\bullet}\eta))$.

Proof. $\pi{\bullet}(\rho{\bullet}v) = (v\circ\rho^{-1})\circ\pi^{-1} = v\circ(\pi\circ\rho)^{-1} = (\pi\circ\rho){\bullet}v$. □

Thus in general $\pi{\bullet}(v{\downarrow}\eta) = (\pi{\bullet}v){\downarrow}(\pi{\bullet}\eta)$ is not equal to $(\pi{\bullet}v){\downarrow}\eta$!

Lemma 13.3. *If $\pi \in \mathbf{Perm}$ and $X \in \mathbf{IPS}_\xi$ then $\pi{\bullet}X \in \mathbf{IPS}_{\pi\bullet\xi}$.*
Moreover π is an $\subseteq$-preserving automorphism of $\mathbf{IPS}$. □

III Splitting/fusion construction

We argue in L in this chapter.

We'll make use of a construction of sets in $\mathbf{IPS}_\zeta$ as $X = \bigcap_{m\in\omega}\bigcup_{u\in 2^m} X_u$ where all X_u belong to $\mathbf{IPS}_\zeta$ and 2^m = all 0,1-tuples of length m. This chapter presents the technique, originally developed in [30, 32], with some changes, and outlines some applications as well.

14 Vertical splitting

Given $\boldsymbol{i} \in \zeta \in \boldsymbol{\Xi}$, and a set $X \in \mathbf{IPS}_\zeta$, we are going to divide X into a disjoint union $U \cup V$ of sets in $\mathbf{IPS}_\zeta$ such that $U{\downarrow}\zeta[\boldsymbol{i}\not\subseteq] = V{\downarrow}\zeta[\boldsymbol{i}\not\subseteq] = X{\downarrow}\zeta[\boldsymbol{i}\not\subseteq]$, and in the same time, if $y \in X{\downarrow}_{\subset \boldsymbol{i}}$ then the cross-sections $\mathbf{D}_{Uy}(\boldsymbol{i})$, $\mathbf{D}_{Vy}(\boldsymbol{i})$ have strictly smaller size than $\mathbf{D}_{Xy}(\boldsymbol{i}) = \{x(\boldsymbol{i}) : x \in X \wedge x{\downarrow}_{\subset \boldsymbol{i}} = y\}$.

Still assuming that $\boldsymbol{i} \in \zeta \in \boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\zeta$, and $y \in X{\downarrow}_{\subset \boldsymbol{i}}$, recall that $P = \mathbf{D}_{Xy}(\boldsymbol{i})$ is a perfect set in $\mathscr{D} = 2^\omega$ by Lemma 10.1(i). It follows that there is a unique tuple $u = \mathbf{u}_{Xy}(\boldsymbol{i}) \in 2^{<\omega}$ of length $m = \mathtt{lh}(u) = \mathbf{m}_{Xy}(\boldsymbol{i}) \in 2^{<\omega}$, such that $u \subset p$ for all $p \in P = \mathbf{D}_{Xy}(\boldsymbol{i})$, and in the same there exist $p_0, p_1 \in P$ with $p_0(m) = 0$ and $p_1(m) = 1$. We let, for $e = 0, 1$,

$$X_{\to \boldsymbol{i},e} = \{x \in X : x(\boldsymbol{i})(\mathbf{m}_{Xy}(\boldsymbol{i})) = e\}.$$

Lemma 14.1. *Let $\boldsymbol{i} \in \zeta \in \boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\zeta$, $X_e = X_{\to \boldsymbol{i},e}$, $e = 0, 1$. Then*

(i) *the sets X_e belong to $\mathbf{IPS}_\zeta$ and are clopen in X, $X = X_0 \cup X_1$, $X_0{\downarrow}_{\subseteq \boldsymbol{i}} \cap X_1{\downarrow}_{\subseteq \boldsymbol{i}} = \varnothing$, $X_0{\downarrow}\zeta[\boldsymbol{i}\not\subseteq] = X_1{\downarrow}\zeta[\boldsymbol{i}\not\subseteq] = X{\downarrow}\zeta[\boldsymbol{i}\not\subseteq]$;*

(ii) *if $y \in X{\downarrow}_{\subset \boldsymbol{i}}$ then $\mathbf{m}_{X_0,y}(\boldsymbol{i}) > \mathbf{m}_{Xy}(\boldsymbol{i})$, $\mathbf{m}_{X_1,y}(\boldsymbol{i}) > \mathbf{m}_{Xy}(\boldsymbol{i})$ strictly;*

(iii) *if $\tau \in \boldsymbol{\Xi}$, $\boldsymbol{i} \in \tau \subseteq \zeta$, $Z = X{\downarrow}\tau$, $Z_e = Z_{\to \boldsymbol{i},e}$, then $Z_e = X_e{\downarrow}\tau$ and $X_e = X \cap (Z_e{\uparrow}\zeta)$.*

Proof. Claims (ii), (iii) hold by construction.

Claim (i) **Case 1:** $\zeta = [\subseteq \boldsymbol{i}]$ (in other words, $\boldsymbol{i}$ is the largest tuple in ζ). By Lemma 10.1(ii), if $u \in 2^{<\omega}$ then $S_u = \{y \in X{\downarrow}_{\subset \boldsymbol{i}} : \exists p \in \mathbf{D}_{Xy}(\boldsymbol{i})(u \subset p)\}$ is a set relatively clopen in $Y = X{\downarrow}_{\subset \boldsymbol{i}}$. Therefore

$$Y_u = (S_{u^\frown 0} \cap S_{u^\frown 1}) \smallsetminus \textstyle\bigcup_{v\in 2^m, v\neq u} S_v, \text{ where } m = \mathtt{lh}(u)$$

is clopen in Y as well. Therefore, by the compactness of the spaces considered, the set $A = \{u \in 2^{<\omega} : Y_u \neq \varnothing\}$ is finite. It follows that, for $e = 0, 1$,

$$X_e := X_{\to \boldsymbol{i},e} = \textstyle\bigcup_{u\in A}\{x \in X : x{\downarrow}_{\subset \boldsymbol{i}} \in Y_u \wedge u^\frown e \subset x(\boldsymbol{i})\}$$

is clopen in X, hence $X_e \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ by Lemma 11.3. The rest of claims is obvious in Case 1.

(i) **Case 2:** any ζ. Let $Z = X{\downarrow}_{\subseteq \boldsymbol{i}}$, $Z_e = Z_{\to \boldsymbol{i},e}$. Then $X_e = X \cap (Z_e {\uparrow} \zeta)$ by (iii). Apply the results of Case 1 for Z, and then Lemma 10.5. □

15 Splitting systems

First of all let us specify requirements which imply an appropriate behaviour of a system of sets $X_u \in \mathbf{IPS}_\zeta, u \in 2^m$, with respect to projections. We need to determine, for any pair of tuples $u, v \in 2^m$ ($m < \omega$), the largest initial segment $\xi = \zeta[u,v]$ of ζ such that the projections $X_u {\restriction} \xi$ and $X_v {\restriction} \xi$ have to be equal, to maintain the construction in proper way.

Assume that $\zeta \in \boldsymbol{\Xi}$ and $\phi : \omega \to \boldsymbol{I}$ is any map, not necessarily $\phi : \omega \to \zeta$. We define, for any pair of tuples $u, v \in 2^m$, $m < \omega$, an initial segment

$$\left.\begin{array}{rcl} \zeta_\phi[u,v] & = & \bigcap_{l<m,\ u(l) \neq v(l)} \zeta[\phi(l) \not\subseteq] \quad = \\ & = & \{\boldsymbol{j} \in \zeta : \neg\, \exists\, l < m \left(u(l) \neq v(l) \wedge \phi(l) \subseteq \boldsymbol{j}\right)\} \end{array}\right\} \in \mathrm{IS}_\zeta\,.$$

Definition 15.1. Let still $\zeta \in \boldsymbol{\Xi}$ and $\phi : \omega \to \boldsymbol{I}$. A *$\phi$-split system* (rather $(\phi{\restriction} m)$-split as the notion depends only on $\phi{\restriction} m$) in $\mathbf{IPS}_\zeta$, of *height* m, is a family $\langle X_u \rangle_{u \in 2^m}$ of sets $X_u \in \mathbf{IPS}_\zeta$ satisfying, for all $u, v \in 2^m$:

- S1: $X_u {\downarrow} \zeta_\phi[u,v] = X_v {\downarrow} \zeta_\phi[u,v]$ (*projection-coherence*), *and*
- S2: if $k < m$, $\sigma \in \boldsymbol{\Xi}$, $\sigma \subseteq \zeta$, but $\sigma \not\subseteq \zeta_\phi[u,v]$ then $(X_u {\downarrow} \eta) \cap (X_v {\downarrow} \eta) = \varnothing$.

If in addition the following strengthening of S2 holds, then $\langle X_u \rangle_{u \in 2^m}$ will be a *strong* ϕ-split system:

- S3: if $k < m$, $\boldsymbol{i} = \phi(k) \in \zeta \smallsetminus \zeta_\phi[u,v]$, then $(X_u {\Downarrow} \boldsymbol{i}) \cap (X_v {\Downarrow} \boldsymbol{i}) = \varnothing$ — and then $X_u {\downarrow}_{\subseteq \boldsymbol{j}} \cap X_v {\downarrow}_{\subseteq \boldsymbol{j}} = \varnothing$ for all $\boldsymbol{j} \in \zeta \smallsetminus \zeta_\phi[u,v]$.

We proceed with a few related definitions.

(A) A system $\langle X'_u \rangle_{u \in 2^m}$ *narrows* $\langle X_u \rangle_{u \in 2^m}$ if $X'_u \subseteq X_u$ for all u, and a *clopenly* narrows, if in addition each X'_u is clopen in X_u.

(B) A system $\langle X_{u'} \rangle_{u' \in 2^{m+1}}$ is an *expansion* of $\langle X_u \rangle_{u \in 2^m}$ iff we have $X_{u^\frown e} \subseteq (X_u)_{\to \boldsymbol{i},e}$ for all $u \in 2^m$ and $e = 0, 1$, where $\boldsymbol{i} = \phi(m)$, and a *clopen* expansion, if in addition each $X_{u^\frown e}$ is clopen in X_u.

(C) A system $\langle Y_u \rangle_{u \in 2^m}$ of sets $Y_u \in \mathbf{IPS}_\vartheta$, where $\zeta \subseteq \vartheta \in \boldsymbol{\Xi}$, is a *lifting* of $\langle X_u \rangle_{u \in 2^m}$, iff $Y_u {\downarrow} \zeta \subseteq X_u$ for all $u \in 2^m$, and a *clopen* lifting, if in addition each $Y_u {\downarrow} \zeta$ is clopen in X_u. □

A set $\zeta \in \Xi$, and $\phi : \omega \to \boldsymbol{I}$, remain fixed in the following lemmas.

Lemma 15.2. *Let $\langle X_u \rangle_{u \in 2^m}$ be a system in* **IPS**$_\zeta$ *satisfying* S1 *and* S2*, and $u, v \in 2^m$. Then either $X_u = X_v$ or $X_u \cap X_v = \varnothing$.*

Proof. If $\zeta_\phi[u,v] = \zeta$ then $X_u = X_v$ by S1. If $\boldsymbol{i} \in \zeta \smallsetminus \zeta_\phi[u,v]$ then $(X_u{\downarrow}_{\subseteq \boldsymbol{i}}) \cap (X_v{\downarrow}_{\subseteq \boldsymbol{i}}) = \varnothing$ by S2, and hence $X_u \cap X_v = \varnothing$. □

The next lemma proves that any split system admits a narrowing that honors a shrink of one of its sets to a given smaller set in **IPS**.

Lemma 15.3. *Let $\langle X_u \rangle_{u \in 2^m}$ be a system in* **IPS**$_\zeta$ *satisfying* S1*, $u_0 \in 2^m$, $X \in$* **IPS**$_\zeta$*, $X \subseteq X_{u_0}$. Then the sets $Y_u = X_u \cap (X{\downarrow}\zeta_\phi[u,u_0]{\uparrow}\zeta)$, $u \in 2^m$, belong to* **IPS**$_\zeta$*, and the system $\langle Y_u \rangle_{u \in 2^m}$ narrows $\langle X_u \rangle_{u \in 2^m}$ and satisfies* S1 *and $Y_{u_0} = X$* (since $\zeta_\phi[u_0,u_0] = \zeta$).

If the given set X is clopen in X_{u_0}, then each Y_u is clopen in X_u.

Proof. The sets Y_u belong to **IPS**$_\zeta$ by Lemma 10.5, because each $X{\downarrow}\zeta_\phi[u,u_0]$ belongs to **IPS**$_{\zeta_\phi[u,u_0]}$ by Lemma 10.4 (since $X{\downarrow}\zeta_\phi[u,u_0] \subseteq X_u{\downarrow}\zeta_\phi[u,u_0]$). The clopenness claim follows from Lemma 11.4.

That $\langle Y_u \rangle_{u \in 2^m}$ satisfies S1 see the proof of Lemma 12 in [32]. □

There is a remarkable strenthening of the lemma.

Corollary 15.4. *Under the assumptions of Lemma 15.3, if in addition $u_1 \in 2^m$, $Y \in$* **IPS**$_\zeta$*, $Y \subseteq X_{u_1}$, $Y{\downarrow}\zeta_\phi[u_0,u_1] = X{\downarrow}\zeta_\phi[u_0,u_1]$, then the sets*

$$Z_u = X_u \cap (X{\downarrow}\zeta_\phi[u,u_0]{\uparrow}\zeta) \cap (Y{\downarrow}\zeta_\phi[u,u_1]{\uparrow}\zeta),\ u \in 2^m,$$

belong to **IPS**$_\zeta$*, and the system $\langle Z_u \rangle_{u \in 2^m}$ narrows $\langle X_u \rangle_{u \in 2^m}$ and satisfies* S1 *and $Z_{u_0} = X$, $Z_{u_1} = Y$.*

If X, Y are clopen in resp. X_{u_0}, X_{u_1}, then each Z_u is clopen in X_u.

Proof. The sets $Y_u = X_u \cap (X{\downarrow}\zeta_\phi[u,u_0]{\uparrow}\zeta) \in$ **IPS**$_\zeta$ form a S1-system $\langle Y_u \rangle_{u \in 2^m}$, which narrows $\langle X_u \rangle_{u \in 2^m}$, with $Y_{u_0} = X$, by Lemma 15.3.

Note that $Y \subseteq Y_{u_1}$. (Indeed $Y_{u_1} = X_{u_1} \cap (X{\downarrow}\eta{\uparrow}\zeta)$ by construction, but $Y \subseteq X_{u_1}$ and $Y{\downarrow}\eta = X{\downarrow}\eta$.) It remains to apply Lemma 15.3 yet again, because $Z_u = Y_u \cap (Y{\downarrow}\zeta_\phi[u,u_1]{\uparrow}\zeta)$ by construction. □

Lemma 15.5. *Let $\langle X_u \rangle_{u \in 2^m}$ be a system in* **IPS**$_\zeta$ *satisfying* S1*. There is a system $\langle Y_u \rangle_{u \in 2^m}$ in* **IPS**$_\zeta$*, which still satisfies* S1*, clopenly narrows $\langle X_u \rangle_{u \in 2^m}$, and satisfies* S3 *as well.*

Proof. Pick any pair of $u_0, v_0 \in 2^m$, and let $\eta = \zeta_\phi[u_0, v_0]$, so that $X_{u_0}{\downarrow}\eta = X_{v_0}{\downarrow}\eta$ by S1. Let $\boldsymbol{i} = \phi(k) \in \zeta \smallsetminus \eta$, $k < m$. By Lemma 11.7, there exist sets $U, V \in \mathbf{IPS}_\zeta$, $U \subseteq X_{u_0}$, $V \subseteq X_{v_0}$, clopen in resp. Y_{u_0}, Y_{v_0} and such that still $U{\downarrow}\eta = V{\downarrow}\eta$, but $U{\Downarrow}\boldsymbol{i} \cap V{\Downarrow}\boldsymbol{i} = \varnothing$. By Corollary 15.4, there is a system $\langle X'_u \rangle_{u \in 2^m}$ of sets $X'_u \in \mathbf{IPS}_\zeta$, $X'_u \subseteq X_u$, clopen in X_u, which satisfies S1 and $X'_{u_0} = U$, $X'_{u_0} = V$, so that $X'_{u_0}{\Downarrow}\boldsymbol{i} \cap X'_{v_0}{\Downarrow}\boldsymbol{i} = \varnothing$.

Thus we have succeeded to clopenly narrow $\langle X_u \rangle_{u \in 2^m}$ to a system $\langle X'_u \rangle_{u \in 2^m}$ still satisfying S1, and also satisfying S3 for a given triple of $u_0, v_0 \in 2^m$ and $\boldsymbol{i} = \phi(k) \in \zeta \smallsetminus \zeta_\phi[s_0, t_0]$, $k < m$. It remains to iterate this narrowing construction for all such triples. ☐

The next two lemmas provide expansions and liftings.

Lemma 15.6. *Any split system $\langle X_u \rangle_{u \in 2^m}$ in $\mathbf{IPS}_\zeta$ admits a clopen expansion by the split system $\langle Y_s \rangle_{s \in 2^{m+1}}$, where $Y_{u^\frown e} = (X_u)_{\to \boldsymbol{i}, e}$, $\boldsymbol{i} = \phi(m)$.*

Proof. In view of Lemma 14.1(i), it suffices to establish S1 for the new system. Let $s = u^\frown e$, $t = v^\frown \varepsilon$ be tuples in 2^{m+1}, $\boldsymbol{i} = \phi(m)$, $\eta = \zeta_\phi[u, v]$, $\sigma = \zeta[\boldsymbol{i} \not\subseteq]$, $\xi = \zeta_\phi[s, t]$. The goal is to prove (*) $X_s{\downarrow}\xi = Y_s{\downarrow}\xi$.

Case 1: $\xi \subseteq \sigma$. Then $X_s{\downarrow}\xi = X_s{\downarrow}\sigma{\downarrow}\xi = X_u{\downarrow}\xi = X_u{\downarrow}\eta{\downarrow}\xi$ (here Lemma 14.1 is used for the middle equality), and accordingly $X_t{\downarrow}\xi = X_v{\downarrow}\eta{\downarrow}\xi$. Yet $X_u{\downarrow}\eta = X_v{\downarrow}\eta$ by S1 for $\langle X_u \rangle_{u \in 2^m}$. This yields (*).

Case 2: $\xi \not\subseteq \sigma$. This means $\boldsymbol{i} \in \eta$, $e = \varepsilon$, and $\xi = \eta$. Then $X_s{\downarrow}\eta = (X_u{\downarrow}\eta)_{\to \boldsymbol{i}, e} = (X_v{\downarrow}\eta)_{\to \boldsymbol{i}, e} = X_t{\downarrow}\eta$ (by Lemma 14.1(iii) and S1 for the given system), which implies (*) yet again since $\xi = \eta$. ☐

Lemma 15.7. *Assume that $\zeta \subseteq \vartheta$ belong to $\boldsymbol{\Xi}$, $\langle X_u \rangle_{u \in 2^m}$ is a ϕ-split system in $\mathbf{IPS}_\zeta$, and $Y_u = X_u{\uparrow}\vartheta$ for all $u \in 2^m$. Then $\langle Y_u \rangle_{u \in 2^m}$ is a ϕ-split system in $\mathbf{IPS}_\vartheta$.*

Proof. To prove S1 for $\langle Y_u \rangle_{u \in 2^m}$, let $u, v \in 2^m$. It can be the case that $\zeta_\phi[u, v] \subsetneqq \vartheta_\phi[u, v]$, but definitely $\zeta_\phi[u, v] = \zeta \cap \vartheta_\phi[u, v]$ holds. Therefore

$$Y_u{\downarrow}\vartheta_\phi[u, v] = X_u{\downarrow}\zeta_\phi[u, v]{\uparrow}\vartheta_\phi[u, v], \quad Y_v{\downarrow}\vartheta_\phi[u, v] = X_v{\downarrow}\zeta_\phi[u, v]{\uparrow}\vartheta_\phi[u, v].$$

by Lemma 10.3 (with $W = \mathscr{D}^\vartheta$). However $X_u{\downarrow}\zeta_\phi[u, v] = X_v{\downarrow}\zeta_\phi[u, v]$. ☐

16 Fusion sequences

We argue in L in this section.

Given $\zeta \in \boldsymbol{\Xi}$, a map $\phi : \omega \to \boldsymbol{I}$ is *ζ-admissible*, if the preimage $\phi^{-1}(\boldsymbol{i}) = \{k : \phi(k) = \boldsymbol{i}\}$ of every $\boldsymbol{i} \in \zeta$ is infinite, and in addition if $\boldsymbol{j} \subset \boldsymbol{i} = \phi(k)$ then $\boldsymbol{j} = \phi(\ell)$ for some $\ell < k$. Yet we do *not* assume that $\phi(k) \in \zeta$, $\forall\, k$.

Definition 16.1. Suppose that $\zeta \in \mathbf{\Xi}$, and $\phi : \omega \xrightarrow{\text{onto}} \zeta$ is ζ-admissible.

An indexed family of sets $X_u \in \mathbf{IPS}_\zeta$, $u \in 2^{<\omega}$, is a *ϕ-fusion sequence* in $\mathbf{IPS}_\zeta$ if, for every $m \in \omega$, the subfamily $\langle X_u \rangle_{u \in 2^m}$ is a ϕ-split system, expanded by $\langle X_u \rangle_{u \in 2^{m+1}}$ in the sense of Definition 15.1(B). □

Theorem 16.2. *Under the assumption of Definition 16.1, let $\langle X_u \rangle_{u \in 2^{<\omega}}$ be a ϕ-fusion sequence in $\mathbf{IPS}_\zeta$. Then $X = \bigcap_{m \in \omega} \bigcup_{u \in 2^m} X_u$ belongs to $\mathbf{IPS}_\zeta$.*

Proof. To begin with, prove that (*) if $a \in 2^\omega$ then the intersection $F(a) = \bigcap_m \bigcup X_{a \restriction m}$ is a singleton. Indeed if $\boldsymbol{i} \in \zeta$ and $m < \omega$ then let $\kappa_m^{\boldsymbol{i}}$ be equal to the number of all $k < m$ such that $\phi(k) = \boldsymbol{i}$. Thus if $u \in 2^m$ and $y \in (X_u){\downarrow}_{\subset \boldsymbol{i}}$ then we have $\mathbf{m}_{X_u, \boldsymbol{i}}(y) \geq \kappa_m^{\boldsymbol{i}}$ by construction. Now, as $\kappa_m^{\boldsymbol{i}} \to \infty$ with $m \to \infty$, the set $\mathbf{D}_{F(a), \boldsymbol{i}}(y)$ is a singleton for any $y \in F(p){\downarrow}_{\subset \boldsymbol{i}}$. This implies (*) because ζ is well-founded. Thus $F(a) = \{f(a)\}$, where $f : 2^\omega \to \mathscr{D}^\zeta$ is continuous, still by compactness.

Having (*) established, we can then follow the proof of Theorem 14 in [32]. (Note that (*) was established in [32] under different suppositions, because the well-foundednes of ζ was not assumed there.) Namely we let $D = \mathscr{D}^\zeta$, and then define D_u by induction on $u \in 2^{<\omega}$ so that $D_{u^\frown e} = (D_u)_{\to \boldsymbol{i}, e}$, where $\boldsymbol{i} = \phi(m)$ and $m = \mathtt{lh}(u)$. Then $\langle D_u \rangle_{u \in 2^{<\omega}}$ is a ϕ-fusion sequence in $\mathbf{IPS}_\zeta$ by Lemma 15.6.

Thus similarly to (*) there is a continuous map $d : 2^\omega \to \mathscr{D}^\zeta$ such that for any $a \in 2^\omega$, $\bigcap_m D_{a \restriction m} = \{d(p)\}$. Moreover, by the equality $X = X_0 \cup X_1$ of Lemma 14.1, we have $\mathtt{ran}\, d = \mathscr{D}^\zeta$, hence $d^{-1} : \mathscr{D}^\zeta \xrightarrow{\text{onto}} 2^\omega$ is continuous.

If $a, b \in 2^\omega$ then define $\zeta_\phi[a, b] = \bigcap_{m < \omega} \zeta_\phi[a \restriction m, b \restriction m]$. (Note that $\zeta_\phi[a, b] = \zeta$ iff $a = b$.) We conclude from S1 and S2 that

$$(\dagger) \begin{cases} \left.\begin{array}{rcl} x_a \restriction \zeta_\phi[a,b] & = & x_b \restriction \zeta_\phi[a,b] \quad \text{and} \\ d_a \restriction \zeta_\phi[a,b] & = & d_b \restriction \zeta_\phi[a,b] \end{array}\right\} & \text{for all } \ a,\, b \in 2^\omega \\ x_a{\downarrow}_{\leqslant \boldsymbol{i}} \neq x_b{\downarrow}_{\leqslant \boldsymbol{i}} \ \text{ and } \ d_a{\downarrow}_{\leqslant \boldsymbol{i}} \neq d_b{\downarrow}_{\leqslant \boldsymbol{i}} & \text{whenever } \ \boldsymbol{i} \notin \zeta_\phi[a,b] \end{cases}$$

This allows to define a homeomorphism $H : D = \mathscr{D}^\zeta$ onto X by $H(d(a)) = f(a)$ for all $a \in 2^\omega$. We claim that H is projection-keeping — which implies $X \in \mathbf{IPS}_\zeta$. Indeed let $\xi \in \mathbf{\Xi}$, $\xi \subseteq \zeta$, and, say, $d(a), d(b) \in \mathscr{D}^\zeta$, $d(a){\downarrow}\xi = d(b){\downarrow}\xi$. Then we have $\xi \subseteq \zeta_\phi[a, b]$ by the second part of (†), hence $f(a){\downarrow}\xi = f(b){\downarrow}\xi$ holds by the first part of (†), as required. □

The classical theorem, that any uncountable Borel or $\mathbf{\Sigma}^1_1$ set includes a perfect subset, does not generalize for $\mathbf{IPS}_\zeta$: if $\mathtt{card}\, \zeta \geqslant 2$ then easily there is an uncountable closed $W \subseteq \mathscr{D}^\zeta$ which does not include a subset in $\mathbf{IPS}_\zeta$. However the following weaker claim (Corollary 16 in [32]) survives.

Corollary 16.3. *Assume that $X \in \mathbf{IPS}_\zeta$, and a set $A \subseteq X$ has the relative Baire property in X but is not relative meager in X. Then there exists a set $Y \in \mathbf{IPS}_\zeta$, $Y \subseteq A$.*

Proof. It suffices to prove the result in case $X = \mathscr{D}^\zeta$. As A has the BP but not meager, there is a basic clopen set $\varnothing \neq B \subseteq X$ (see the proof of Lemma 11.5) such that $A \cap B$ is comeager in B, so that there are relatively open dense sets $D_n \subseteq B$ satisfying $\bigcap_n D_n \subseteq A \cap B$. Now Lemmas 15.3 and 15.6 yield a fusion sequence $\langle X_u \rangle_{u \in 2^{<\omega}}$ in $\mathbf{IPS}_\zeta$, such that $X_\Lambda \subseteq X$, and each X_u is clopen and satisfies $X_u \subseteq D_m$ for all $m \in \omega$ and $u \in 2^m$. The set $Y = \bigcap_{m \in \omega} \bigcup_{u \in 2^m} X_u$ is as required. □

17 Uniform shrinking

Say that a set $X \in \mathbf{IPS}_\zeta$ is *uniform*, if for any pair of tuples $\boldsymbol{i} \subset \boldsymbol{j}$ in ζ and any $x, y \in X$, we have $x(\boldsymbol{j}) = y(\boldsymbol{j}) \Longrightarrow x(\boldsymbol{i}) = y(\boldsymbol{i})$. As the first application of the splitting/fusion technique, we prove a lemma on uniform shrinking.

Lemma 17.1 (in **L**). *If $\zeta \in \boldsymbol{\Xi}$ and $X \in \mathbf{IPS}_\zeta$ then there is a uniform set $Y \in \mathbf{IPS}_\zeta$, $Y \subseteq X$.*

Proof. Let $\phi : \omega \xrightarrow{\text{onto}} \zeta$ be ζ-admissible. Lemmas 15.3, 15.6, 15.5 yield a fusion sequence $\langle X_u \rangle_{u \in 2^{<\omega}}$ in $\mathbf{IPS}_\zeta$, such that $X_\Lambda \subseteq X$ and the layer $\langle X_u \rangle_{u \in 2^m}$ satisfies S3 of Definition 15.1 for all m. Then $Y = \bigcap_n \bigcup_{u \in 2^n} X_u \in \mathbf{IPS}_\zeta$ by Theorem 16.2, and $Y \subseteq X$. We claim that Y is uniform.

Indeed suppose that $\boldsymbol{i} \subset \boldsymbol{j}$ belong to ζ, and x, y in Y satisfy $x(\boldsymbol{i}) \neq y(\boldsymbol{i})$, say $x(\boldsymbol{i})(k) = 0$ but $y(\boldsymbol{i})(k) = 1$ for some $k < \omega$. Then $x \neq y$, hence there exists m and some $u \neq v$ in 2^m such that $x \in X_u$, $y \in X_v$. We can take m big enough for $x'(\boldsymbol{i})(k) = 0$ for all $x' \in X_u$ but $y'(\boldsymbol{i})(k) = 1$ for all $y' \in X_v$. Thus $(X_u {\Downarrow} \boldsymbol{i}) \cap (X_v {\Downarrow} \boldsymbol{i}) = \varnothing$.

Now consider the initial segment $\eta = \zeta_\phi[u, v] \subseteq \zeta$. Then $X_u {\downarrow} \eta = X_v {\downarrow} \eta$ by S1 of Definition 15.1. It follows that $\boldsymbol{i} \notin \eta$ since $(X_u {\Downarrow} \boldsymbol{i}) \cap (X_v {\Downarrow} \boldsymbol{i}) = \varnothing$. Therefore $\boldsymbol{j} \notin \eta$ either. But then $(X_u {\Downarrow} \boldsymbol{j}) \cap (X_v {\Downarrow} \boldsymbol{j}) = \varnothing$ by S3 of Definition 15.1. We conclude that $x(\boldsymbol{j}) \neq y(\boldsymbol{j})$, as required. □

18 Axis/avoidance shrinking

We continue to argue in L. Here we set up some notions related to continuous maps $F : \mathscr{D}^\xi \to \omega^\omega$, $\xi \in \boldsymbol{\Xi}$. Let

$$\begin{aligned} \mathbf{CF}_\xi &= \{F : \mathscr{D}^\xi \to \omega^\omega : F \text{ is continuous}\}, \\ \mathbf{CF}^*_\xi &= \{F : \mathscr{D}^\xi \to \mathscr{D} : F \text{ is continuous}\} \subseteq \mathbf{CF}_\xi, \end{aligned}$$

and $\mathbf{CF} = \bigcup_{\xi \in \boldsymbol{\Xi}} \mathbf{CF}_\xi$, $\mathbf{CF}^* = \bigcup_{\xi \in \boldsymbol{\Xi}} \mathbf{CF}^*_\xi$. Let $\|f\| = \xi$ in case $f \in \mathbf{CF}_\xi$.

Definition 18.1 (in $\mathbf{L}$)**.** Assume that $\sigma \subseteq \tau$ belong to $\boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\tau$, $\boldsymbol{i} \in \tau$, $S \subseteq \mathscr{D}$, and $F \in \mathbf{CF}^*_\sigma$.

If $F(x{\downarrow}\sigma) = x(\boldsymbol{i})$ for all $x \in X$, say that F is *an $\boldsymbol{i}$-axis map on X*.

If $F(x{\downarrow}\sigma) \notin S$ for all $x \in X$, then say that F *avoids* S on X. □

We prove several lemmas here, related to axis maps and avoidance, which culminate in a dichotomy theorem (Theorem 19.1).

Lemma 18.2. *If $\boldsymbol{i} \in \tau \in \boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\tau$, and $F \in \mathbf{CF}^*_\tau$ is not an $\boldsymbol{i}$-axis map on X, then there is $Y \in \mathbf{IPS}_\tau$, $Y \subseteq X$, such that F avoids $Y{\Downarrow}\boldsymbol{i}$ on Y.*

Proof. We have $F(x_0) \neq x(\boldsymbol{i})$ for some $x_0 \in X$, say $F(x_0)(k) = 0$ and $x_0(\boldsymbol{i})(k) = 1$ for some k, so $X' = \{x \in X : F(x)(k) = 0 \wedge x(\boldsymbol{i})(k) = 1\} \neq \varnothing$. But X' is open in X. Take any $Y \in \mathbf{IPS}_\tau, Y \subseteq X'$ by Lemma 11.5. □

Lemma 18.3. *If $\eta \subseteq \tau$ and ξ belong to $\boldsymbol{\Xi}$, $\boldsymbol{i} \in \tau \smallsetminus \eta$, $X \in \mathbf{IPS}_\xi$, $Y \in \mathbf{IPS}_\tau$, and $F \in \mathbf{CF}^*_\xi$, then there exist relatively clopen sets $X' \subseteq X$ and $Y' \subseteq Y$ in resp. $\mathbf{IPS}_\xi$, $\mathbf{IPS}_\tau$, such that $Y'{\downarrow}\eta = Y{\downarrow}\eta$ and F avoids $Y'{\Downarrow}\boldsymbol{i}$ on X'.*

Proof. Pick any $x_0 \in X$. Let $p_0 = F(x_0)$, $Q_m = \{p \in \mathscr{D} : p{\restriction}m = p_0{\restriction}m\}$,

$$U_m = \{u \in Y{\downarrow}\eta : \exists\, y \in Y\ (y{\downarrow}\eta = u \wedge y(\boldsymbol{i}) \notin Q_m)\}$$

for all $m < \omega$. Then $U_m \subseteq U_{m+1}$, $\forall\, m$. Further, Lemma 11.4 implies that each set U_m is clopen in $Y{\downarrow}\eta \in \mathbf{IPS}_\eta$. Moreover, we have $Y{\downarrow}\eta = \bigcup_m U_m$. (Because if $u \in Y{\downarrow}\eta$ then $\{y(\boldsymbol{i}) : y \in Y \wedge y{\downarrow}\eta = u\}$ is a perfect set.) It follows by the compactness of $\mathscr{D}^\eta$ that $Y{\restriction}\eta = U_m$ for some m.

Now let $Y' = \{y \in Y : y(\boldsymbol{i}) \notin Q_m\}$. The set $S = \{x \in X : F(x) \in Q_m\}$ is clopen in X, and $p_0 \in S$, hence there exists a relatively clopen $X' \in \mathbf{IPS}_\xi, X' \subseteq S$. We claim that X', Y' are as required. Indeed $Y'{\downarrow}\eta = Y{\downarrow}\eta$ holds by the choice of m, whereas F avoids $Y'{\Downarrow}\boldsymbol{i}$ on X' by construction. It remains to show that $Y' \in \mathbf{IPS}_\tau$ and that Y' is relatively clopen in Y.

Note that $Y' = Y \cap (V{\uparrow}\tau)$, where $V = \{v \in Y{\downarrow}_{\subseteq \boldsymbol{i}} : v(\boldsymbol{i}) \notin Q_m\}$ is clopen in $Y{\downarrow}\eta$ by Lemma 11.4. Lemma 11.3 implies that $V \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$. Then $Y' \in \mathbf{IPS}_\tau$ by Lemma 10.5, as required. □

Corollary 18.4. *If $\eta \subseteq \tau$ belong to $\boldsymbol{\Xi}$, $X, Y \in \mathbf{IPS}_\tau$, $X{\downarrow}\eta = Y{\downarrow}\eta$, $F \in \mathbf{CF}^*_\tau$, $\boldsymbol{i} \in \tau$, and either $\boldsymbol{i} \notin \eta$ or F is not an $\boldsymbol{i}$-axis map on X, then there exist relatively clopen sets $X' \subseteq X$ and $Y' \subseteq Y$ in $\mathbf{IPS}_\tau$, such that $X'{\downarrow}\eta = Y'{\downarrow}\eta$ and F avoids $Y'{\Downarrow}\boldsymbol{i}$ on X'.*

Proof. Suppose that $\boldsymbol{i} \notin \eta$. Then by Lemma 18.3 there exist relatively clopen sets $X' \subseteq X$ and $Y'' \subseteq Y$ in $\mathbf{IPS}_\tau$, such that $Y'' \!\downarrow\! \eta = Y \!\downarrow\! \eta$ and F avoids $Y'' \Downarrow \boldsymbol{i}$ on X'. Take $Y' = Y'' \cap (X' \!\downarrow\! \eta \!\uparrow\! \tau)$, and we are done.

Now suppose that $\boldsymbol{i} \in \eta$ and F is not an $\boldsymbol{i}$-axis map on X. Lemma 18.2 yields a relatively clopen $X' \in \mathbf{IPS}_\tau$, $X' \subseteq X$, such that F avoids $X' \Downarrow \boldsymbol{i}$ on X'. Take $Y' = Y \cap (X' \!\downarrow\! \eta \!\uparrow\! \tau)$, and we are done. □

Corollary 18.5. *If $\eta \subseteq \tau$ belong to $\boldsymbol{\Xi}$, $X, Y \in \mathbf{IPS}_\tau$, $X \!\downarrow\! \eta = Y \!\downarrow\! \eta$, $F \in \mathbf{CF}^*_\tau$, $\boldsymbol{i} \in \tau \smallsetminus \eta$, then there exist relatively clopen sets $X' \subseteq X$ and $Y' \subseteq Y$ in $\mathbf{IPS}_\tau$, such that $X' \!\downarrow\! \eta = Y' \!\downarrow\! \eta$ and $(Y' \Downarrow \boldsymbol{i}) \cap (X' \Downarrow \boldsymbol{i}) = \varnothing$.*

Proof. Use Corollary 18.4 for $F(x) = x(\boldsymbol{i})$. □

19 Axis/avoidance dichotomy theorem

And now the main result goes, a dichotomy theorem.

Theorem 19.1. *If $\tau \in \boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\tau$, and $F \in \mathbf{CF}_\tau$ then there is a set $Y \in \mathbf{IPS}_\tau$, $Y \subseteq X$, such that one of the two following claims holds:*

(i) *F avoids $Y \Downarrow \boldsymbol{i}$ on Y for all $\boldsymbol{i} \in \tau$;*

(ii) *there is $\boldsymbol{j} \in \tau$ such that F is a $\boldsymbol{j}$-axis map on Y and F avoids $Y \Downarrow \boldsymbol{i}$ on Y for all $\boldsymbol{i} \in \tau$, $\boldsymbol{i} \neq \boldsymbol{j}$.*

Proof. To begin with, prove that if $U \in \mathbf{IPS}_\tau$ and $\boldsymbol{i} \neq \boldsymbol{j}$ belong to τ then

(1) F cannot be both $\boldsymbol{i}$-axis map on U and $\boldsymbol{j}$-axis map on U.

Indeed suppose otherwise. Let say $\boldsymbol{i} \not\subseteq \boldsymbol{j}$, so that $\boldsymbol{i} \notin \eta = [\subseteq \boldsymbol{j}]$. Corollary 18.5 with $X = Y = U$ (note that $\boldsymbol{j} \in \eta$) yields sets $X', Y' \in \mathbf{IPS}_\tau$ such that $X' \cup Y' \subseteq U$, $X' \Downarrow \boldsymbol{j} = Y' \Downarrow \boldsymbol{j}$, but $(X' \Downarrow \boldsymbol{i}) \cap (Y' \Downarrow \boldsymbol{i}) = \varnothing$. Thus $X' \Downarrow \boldsymbol{i} \neq X' \Downarrow \boldsymbol{j}$ or $Y' \Downarrow \boldsymbol{i} \neq Y' \Downarrow \boldsymbol{j}$, both cases leading to a contradiction with the contrary assumption. This ends the proof of (1).

Coming back to the theorem, we have two cases.

Case 1: there exist $\boldsymbol{j} \in \tau$ and $Z \in \mathbf{IPS}_\tau$, $Z \subseteq X$, such that F is a $\boldsymbol{j}$-axis map on Z. Let $\delta = \tau \smallsetminus \{\boldsymbol{j}\}$ in this case.

Case 2: not case 1. Let $\delta = \tau$ and $Z = X$ in this case.

It follows from (1) that in both cases

(2) if $U \in \mathbf{IPS}_\tau$, $U \subseteq Z$, $\boldsymbol{i} \in \delta$, then F is **not** an $\boldsymbol{i}$-axis map on U.

Now fix any τ-admissible map $\phi : \omega \xrightarrow{\text{onto}} \tau$. The next claim is a consequence of (2) and Corollary 18.4, by means of Corollary 15.4 applied consecutively enough many times:

(3) If $\boldsymbol{i} \in \delta$ and $m < \omega$ then any ϕ-split system $\langle X_u \rangle_{u \in 2^m}$ of sets $X_u \subseteq Z$ in $\mathbf{IPS}_\tau$ admits a narrowing $\langle X'_u \rangle_{u \in 2^m}$ such that if $u, v \in 2^m$ then F avoids $X'_v \Downarrow \boldsymbol{i}$ on X'_u, and hence F avoids $X'_m \Downarrow \boldsymbol{i}$ on $X'_m = \bigcup_{u \in 2^m} X'_u$.

With this "narrowing" result, Lemmas 15.3 and 15.6 yield a fusion sequence $\langle X_u \rangle_{u \in 2^{<\omega}}$ in $\mathbf{IPS}_\tau$, such that $X_\Lambda \subseteq Z$, and, for each m, F avoids $X_m \Downarrow \boldsymbol{i}$ on $X_m = \bigcup_{u \in 2^m} X_u$, where $\boldsymbol{i} = \phi(m) \in \delta$. Then $Y = \bigcap_n \bigcup_{u \in 2^n} X_u \in \mathbf{IPS}_\tau$, $Y \subseteq Z \subseteq X$, and F avoids $Y \Downarrow \boldsymbol{i}$ on Y for all $\boldsymbol{i} \in \delta$, as required. □

20 Avoidable sets

Assume that $U \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$, $\boldsymbol{i} \in \boldsymbol{I}$. Say that a set $S \subseteq \mathscr{D} = 2^\omega$ is *U-avoidable on $\boldsymbol{i}$* if there exists a relatively clopen set $V \subseteq U$ satisfying $V \downarrow_{\subset \boldsymbol{i}} = U \downarrow_{\subset \boldsymbol{i}}$ and $S \cap (V \Downarrow \boldsymbol{i}) = \varnothing$. Thus avoidability in this sense means that not U itself but a certain clopen subset of U with the same projection avoids S.

Theorem 20.1. *Suppose that $\xi \in \boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\xi$, $F \in \mathbf{CF}^*_\xi$, and $\mathcal{U} \subseteq \bigcup_{\boldsymbol{i} \in \boldsymbol{I}} \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ is countable. Then there is a set $Y \in \mathbf{IPS}_\xi$, $Y \subseteq X$, such that the image $S = F"Y$ is U-avoidable on $\boldsymbol{i}$ for all $\boldsymbol{i} \in \boldsymbol{I}$ and $U \in \mathbf{IPS}_{\subseteq \boldsymbol{i}} \cap \mathcal{U}$.*

Proof. Lemma 18.3 ($\tau = [\subseteq \boldsymbol{i}]$, $\eta = [\subset \boldsymbol{i}]$) implies:

(1) if $Z \in \mathbf{IPS}_\xi$, $\boldsymbol{i} \in \boldsymbol{I}$, $U \in \mathbf{IPS}_{\subseteq \boldsymbol{i}} \cap \mathcal{U}$, then there is a relatively clopen set $Z' \subseteq Z$, $Z' \in \mathbf{IPS}_\xi$, such that $F"Z'$ is U-avoidable on $\boldsymbol{i}$.

Now fix any ξ-admissible map $\phi : \omega \xrightarrow{\text{onto}} \xi$. The next claim is a consequence of (1) and Corollary 18.4, by means of Corollary 15.4 applied consecutively enough many times:

(2) If $\boldsymbol{i} \in \boldsymbol{I}$, $U \in \mathbf{IPS}_{\subseteq \boldsymbol{i}} \cap \mathcal{U}$, and $m < \omega$, then any ϕ-split system $\langle X_u \rangle_{u \in 2^m}$ of sets $X_u \in \mathbf{IPS}_\xi$ admits a narrowing $\langle X'_u \rangle_{u \in 2^m}$ in $\mathbf{IPS}_\xi$ such that $F"X'_m$ is U-avoidable on $\boldsymbol{i}$, where $X'_m = \bigcup_{u \in 2^m} X'_u$.

Using this result and the countability of $\mathcal{U}$, Lemmas 15.3 and 15.6 yield a fusion sequence $\langle X_u \rangle_{u \in 2^{<\omega}}$ in $\mathbf{IPS}_\xi$, such that $X_\Lambda \subseteq X$, and, for each $\boldsymbol{i} \in \boldsymbol{I}$ and $U \in \mathbf{IPS}_{\subseteq \boldsymbol{i}} \cap \mathcal{U}$ there is a number m, such that $F"X_m$ is U-avoidable on $\boldsymbol{i}$, where $X_m = \bigcup_{u \in 2^m} X_u$. Then $Y = \bigcap_m \bigcup_{u \in 2^m} X_u \in \mathbf{IPS}_\xi$, $Y \subseteq X$, and $F"Y$ is U-avoidable for all $\boldsymbol{i} \in \boldsymbol{I}$ and $U \in \mathbf{IPS}_{\subseteq \boldsymbol{i}} \cap \mathcal{U}$. □

Remark 20.2. The theorem will be applied only in cases when the given set $\mathcal{U} \subseteq \bigcup_{\boldsymbol{i} \in \boldsymbol{I}} \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ satisfies the property that if $\varnothing \neq V \subseteq U \in \mathcal{U}$ is relatively clopen in U then $V \in \mathcal{U}$ as well. In this case, the condition of relative clopenness of V in the definition of being "U-avoidable on $\boldsymbol{i}$" can be replaced by just $V \in \mathcal{U}$, and then Theorem 20.1 still holds. □

IV Normal forcing notions

It will take considerable effort to actually define the forcing notion $\mathscr{X} \subseteq \mathbf{IPS}$ in the constructible universe $\mathbf{L}$ for the proof of Theorem 1.1. Yet we can gradually formulate some conditions on $\mathscr{X}$ that will bring a number of useful consequences related to the corresponding $\mathscr{X}$-generic extensions of $\mathbf{L}$, and which will be fulfilled in the final construction of $\mathscr{X}$.

21 Normal forcings

We argue in L in this section.

Any set $\mathscr{X} \subseteq \mathbf{IPS}$ can be viewed as a forcing notion, with the partial order $\downarrow\subseteq$ on $\mathbf{IPS}$ defined by: $X \downarrow\subseteq Y$ iff $\eta = \|Y\| \subseteq \|X\|$ and $X{\downarrow}\eta \subseteq Y$. But we have to somehow resrict the generality, to make sure that $\mathscr{X}$ adjoins $\boldsymbol{\Xi}$-arrays of reals (*i.e.*, points of $\mathscr{D}$), similarly to $\mathbf{IPS}$ itself. Recall that

$$\begin{aligned} \mathscr{X}{\downarrow}\eta &= \{X{\downarrow}\eta : X \in \mathscr{X} \wedge \eta \subseteq \|X\|\}, \\ \mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}} &= \mathscr{X}{\downarrow}\eta, \text{ where } \eta = [\subseteq \boldsymbol{i}] = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{j} \subseteq \boldsymbol{i}\}, \\ \|X\| &= \xi, \text{ in case } X \subseteq \mathscr{D}^{\xi}, \end{aligned}$$

by Section 8, for any $\mathscr{X} \subseteq \mathbf{IPS}$, and $\mathscr{D} = 2^{\omega}$, the Cantor space.

Say that $\mathscr{X} \subseteq \mathbf{IPS}$ is a *normal forcing*, $\mathscr{X} \in \mathbf{NF}$ for brevity, iff the following conditions 1°–6° hold:

1°. $\mathscr{X} \subseteq \mathbf{IPS}$, and if $\tau \in \boldsymbol{\Xi}$ then $\mathscr{D}^{\tau} \in \mathscr{X}$.

2°. If $\xi \subseteq \tau$ belong to $\boldsymbol{\Xi}$ and $X \in \mathscr{X} \cap \mathbf{IPS}_{\tau}$ then $X{\downarrow}\xi \in \mathscr{X}$, and hence $\mathscr{X}{\downarrow}\xi = \mathscr{X} \cap \mathbf{IPS}_{\xi}$. In particular the set $\mathbb{1} = \{\varnothing\} = X{\downarrow}\varnothing$ belongs to $\mathscr{X}{\downarrow}\varnothing$, and $\mathbb{1} \downarrow\subseteq X$ for any $X \in \mathscr{X}$.

3°. If $\xi \subseteq \tau$ belong to $\boldsymbol{\Xi}$, $X \in \mathscr{X}{\downarrow}\tau$, $Y \in \mathscr{X}{\downarrow}\xi$, and $Y \subseteq X{\downarrow}\eta$, then $X \cap (Y{\uparrow}\tau) \in \mathscr{X}{\downarrow}\tau$. In particular, if $Y \in \mathscr{X}{\downarrow}\xi$ then $Y{\uparrow}\tau \in \mathscr{X}{\downarrow}\tau$.

4°. If $\tau \in \boldsymbol{\Xi}$, $X \in \mathscr{X}{\downarrow}\tau$, $Y \in \mathbf{IPS}_{\tau}$, $Y \subseteq X$ is clopen in X, then $Y \in \mathscr{X}$.

5°. $\mathscr{X}$ is $\boldsymbol{\Pi}$-invariant: if $X \in \mathbf{IPS}$ then $X \in \mathscr{X} \iff \pi{\bullet}X \in \mathscr{X}$.

6°. If $\tau \in \boldsymbol{\Xi}$, $X \in \mathbf{IPS}_{\tau}$, and $X{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \tau$, then $X \in \mathscr{X}$.

Quite clearly $\mathbf{IPS}$ itself belongs to $\mathbf{NF}$: $\mathscr{D}^{\tau} \in \mathscr{X}$ in 1° holds via the identity PKH, 2° holds by Lemma 10.4, 3° holds by Lemma 10.5, 4° and 6° are obvious, 5° holds by Lemma 13.3, so that $\mathbf{IPS}$ is even $\mathbf{Perm}$-invariant. The next lemma provides some other similarities.

Lemma 21.1. *Let $\mathscr{X} \in$ **NF**. Under the assumptions of Lemma 15.3, Corollary 15.4, Lemma 15.5, Lemma 15.6, if all the given sets X_u, X, Y belong to $\mathscr{X}$, then the resulting sets Y_u, Z_u, Y_s belong to $\mathscr{X}$ as well.*

Under the assumptions of Lemma 11.5, if $X \in \mathscr{X}$ then $X' \in \mathscr{X}$, too.

Under the assumptions of Lemma 11.7, if $X, Y \in \mathscr{X}$ then $X', Y' \in \mathscr{X}$.

Under the assumptions of Lemma 12.2, if $X_k \in \mathscr{X}$, $\forall k$, then $X \in \mathscr{X}$.

Under the assumptions of Corollary 10.8, if $X, Y \in \mathscr{X}$ then $Z \in \mathscr{X}$.

Proof. Make use of 3° above w.r.t. Lemma 15.3 and Corollaries 15.4 and 10.8, of 4° above w.r.t. Lemmas 15.5, 15.6, 11.5, 11.7, and of 6° w.r.t. Lemma 12.2. □

Definition 21.2. If $\mathcal{P} \subseteq \mathbf{IPS}$ then let $\mathbf{NH}(\mathcal{P})$ (the *normal hull* of $\mathcal{P}$) be the least set $\mathscr{X} \in \mathbf{NF}$ with $\mathcal{P} \subseteq \mathscr{X}$. It is equal to the intersection of all sets $\mathscr{Y} \in \mathbf{NF}$ with $\mathcal{P} \subseteq \mathscr{Y}$. □

22 Kernels of normal forcings

We still argue in L.

Here we show that each normal forcing $\mathscr{X}$ is the normal hull of its smaller and simpler part called *the kernel*. If $\xi \subseteq \boldsymbol{I}$ then let a ξ-*kernel* be a system $\mathcal{K} = \langle \mathcal{K}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \xi}$ of sets $\mathcal{K}_{\boldsymbol{i}} \subseteq \mathbf{IPS}_{\subseteq \boldsymbol{i}}$, satisfying 1*–5* below.

1*. If tuples $\boldsymbol{j} \subset \boldsymbol{i}$ belong to ξ and $Y \in \mathbf{IPS}_{\subseteq \boldsymbol{j}}$ then $Y = X{\downarrow}_{\subseteq \boldsymbol{j}}$ for some $X \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$.

2*. If tuples $\boldsymbol{j} \subset \boldsymbol{i}$ belong to ξ and $X \in \mathcal{K}_{\boldsymbol{i}}$ then $X{\downarrow}_{\subseteq \boldsymbol{j}} \in \mathcal{K}_{\boldsymbol{j}}$.

3*. If tuples $\boldsymbol{j} \subset \boldsymbol{i}$ belong to ξ, $X \in \mathcal{K}_{\boldsymbol{i}}$, $Y \in \mathcal{K}_{\boldsymbol{j}}$, and $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{j}}$, then $Z = X \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}}) \in \mathcal{K}_{\boldsymbol{i}}$.

4*. If $\boldsymbol{i} \in \xi$, $X \in \mathcal{K}_{\boldsymbol{i}}$, $\varnothing \neq Y \subseteq X$ is clopen in X, then $Y \in \mathcal{K}_{\boldsymbol{i}}$.

5*. If tuples $\boldsymbol{j} \approx_{\mathtt{par}} \boldsymbol{i}$ belong to ξ and $X \in \mathcal{K}_i$ then $\pi_{\boldsymbol{ij}} \cdot X \in \mathcal{K}_{\boldsymbol{j}}$. (See Example 13.1 on $\pi_{\boldsymbol{ij}}$.)

Say that $\mathcal{K}$ is a *strong* ξ-*kernel*, if in addition the following 1* holds.

1*s. If $\boldsymbol{i} \in \xi$ then $\mathscr{D}^{[\subseteq \boldsymbol{i}]} \in \mathcal{K}_{\boldsymbol{i}} \subseteq \mathbf{IPS}_{\subseteq \boldsymbol{i}}$.

Lemma 22.1. *In the presence of 3*, condition 1*s implies 1*.*

Proof. As $X = \mathscr{D}^{[\subseteq \boldsymbol{i}]} \in \mathcal{K}_{\boldsymbol{i}}$ by 1*s, the set $Z = Y{\uparrow}^{\subseteq \boldsymbol{i}} = X \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}})$ belongs to $\mathcal{K}_{\boldsymbol{i}}$ by 3*, and obviously $Y = Z{\downarrow}_{\subseteq \boldsymbol{j}}$. □

Lemma 22.2. *Let* $\mathscr{X} \in \mathbf{NF}$. *Then* $\mathbf{Ker}(\mathscr{X}) = \langle \mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}}$ (the *kernel* of $\mathscr{X}$) *is a strong* $\boldsymbol{I}$*-kernel.*

Proof. Infer $1^{*\mathrm{s}}$ and 2^*–5^* from $1°$–$5°$ above. Apply Lemma 11.3 for 4^*. □

Conversely, every $\boldsymbol{I}$-kernel defines a normal forcing via $6°$.

Lemma 22.3. *Let* $\mathcal{K} = \langle \mathcal{K}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}}$ *be a strong* $\boldsymbol{I}$*-kernel. Then* $\mathscr{X} = \mathbf{NH}(\mathcal{K}) \in \mathbf{NF}$, $\mathcal{K} = \mathbf{Ker}(\mathscr{X})$ *— so that* $\mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}} = \mathcal{K}_{\boldsymbol{i}}$ *for all* $\boldsymbol{i} \in \boldsymbol{I}$, *and if* $\xi \in \boldsymbol{\Xi}$ *then* $\mathscr{X}{\downarrow}\xi$ *is equal to the set* $\mathscr{Y}_\xi = \{X \in \mathbf{IPS}_\xi : \forall\, \boldsymbol{i} \in \xi\, (X{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}})\}$.

Proof. We claim that the set $\mathscr{Y} = \bigcup_{\xi \in \boldsymbol{\Xi}} \mathscr{Y}_\xi$ belongs to $\mathbf{NF}$. As $6°$ of Section 21 obviously holds for $\mathscr{Y}$ by construction, we derive $1°$–$5°$ for $\mathscr{Y}$ from 1^* and 2^*–5^* for $\mathcal{K}$. Here $1°$,$2°$,$5°$ are entirely obvious.

Make use of Lemma 11.4 for $4°$. Now focus on $3°$. Thus assume that $\xi \subseteq \tau$ belong to $\boldsymbol{\Xi}$, $X \in \mathscr{Y}{\downarrow}\tau$, $Y \in \mathscr{Y}{\downarrow}\xi$, and $Y \subseteq X{\downarrow}\eta$; prove that $Z = X \cap (Y{\uparrow}\tau) \in \mathscr{Y}{\downarrow}\tau$. We have to check that $Z{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}$ for all $\boldsymbol{i} \in \tau$. If $\boldsymbol{i} \in \xi$ then $Z{\downarrow}_{\subseteq \boldsymbol{i}} = Y{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}$. If $\boldsymbol{i} \in \tau \smallsetminus \xi$ and $\eta = \xi \cap [\subseteq \boldsymbol{i}]$ then $Z{\downarrow}_{\subseteq \boldsymbol{i}} = X{\downarrow}_{\subseteq \boldsymbol{i}} \cap (Y{\downarrow}\eta){\uparrow}^{\subseteq \boldsymbol{i}}$ by Lemma 10.3, hence yet again $Z{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}$ by 3^*, as required. Thus $\mathscr{Y} \in \mathbf{IPS}$, and hence $\mathscr{X} \subseteq \mathscr{Y}$ by the minimality of $\mathscr{X}$.

Moreover $\mathscr{Y}{\downarrow}_{\subseteq \boldsymbol{i}} = \mathcal{K}_{\boldsymbol{i}}$ by construction. Therefore, as $\mathcal{K}_{\boldsymbol{i}} \subseteq \mathscr{X}$, we have $\mathscr{Y} \subseteq \mathscr{X}$ by $6°$ of Section 21 for $\mathscr{X}$. Thus $\mathscr{Y} = \mathscr{X}$ and we are done. □

We may note that in fact even dyadic $\boldsymbol{I}[{<}2]$-kernels suffice to produce normal forcings. Recall that $\boldsymbol{I}[{<}2] = 2^{<\omega} \smallsetminus \{\Lambda\}$, the set of all non-empty dyadic tuples. Obviously for any $\boldsymbol{i} \in \boldsymbol{I}$ there is a unique dyadic tuple $\underline{\boldsymbol{i}} \in \boldsymbol{I}[{<}2]$ satisfying $\boldsymbol{i} \approx_{\mathtt{par}} \underline{\boldsymbol{i}}$. Indeed put $\mathtt{lh}(\underline{\boldsymbol{i}}) = \mathtt{lh}(\boldsymbol{i})$ and

$$\text{for all } k < \mathtt{lh}(\underline{\boldsymbol{i}}) = \mathtt{lh}(\boldsymbol{i}), \quad \underline{\boldsymbol{i}}(k) = \begin{cases} 0 & \text{in case} \quad \boldsymbol{i}(k) \text{ is even} \\ 1 & \text{in case} \quad \boldsymbol{i}(k) \text{ is odd} \end{cases}. \qquad (*)$$

Lemma 22.4. *Assume that* $2 \le \alpha < \omega_1$ *and* $\mathcal{K} = \langle \mathcal{K}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}[<\alpha]}$ *is an* $\boldsymbol{I}[{<}\alpha]$*-kernel. Put* $\mathcal{K}^{\mathsf{ex}}_{\boldsymbol{i}} := \pi_{\boldsymbol{i},\underline{\boldsymbol{i}}} \boldsymbol{\cdot} \mathcal{K}_{\underline{\boldsymbol{i}}}$ *for all* $\boldsymbol{i} \in \boldsymbol{I}$. *Then* $\mathcal{K}^{\mathsf{ex}} = \langle \mathcal{K}^{\mathsf{ex}}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}}$ *is an* $\boldsymbol{I}$*-kernel,* $\mathcal{K}^{\mathsf{ex}}_{\boldsymbol{i}} = \mathcal{K}_{\boldsymbol{i}}$ *for all* $\boldsymbol{i} \in \boldsymbol{I}[{<}\alpha]$, *and if* $\mathcal{K}$ *is strong then so is* $\mathcal{K}^{\mathsf{ex}}$. □

Thus to define a normal forcing $\mathscr{X}$ it suffices to first define an auxiliary $\boldsymbol{I}[{<}2]$-kernel $\mathcal{K}$ and then let $\mathscr{X} = \mathbf{NH}(\mathcal{K}^{\mathsf{ex}})$ by Lemmas 22.4 and 22.3.

23 Generic arrays

According to the formulation of Theorem 1.2, we are going to establish our main results in this paper by means of suitable generic extensions of $\mathbf{L}$, the constructible universe, under the *consistent* assumption that $\omega_2^{\mathbf{L}} < \omega_1$ in the universe, intended to imply the existence of generic extensions. The forcing notions considered in this process will be normal forcings as in Section 21 defined in $\mathbf{L}$. As the notion of iterated perfect set and many related notions are definitely non-abcolute, we add the following warning.

Blanket agreement 23.1. The definition of $\mathbf{IPS}$ in Section 8 and all other relevant definitions in Sections 8–21, are assumed to be relativized to $\mathbf{L}$ by default, and we'll not bother to add the sign $^{\mathbf{L}}$ of relativization. In other words, $\boldsymbol{I}$ is $(\boldsymbol{I})^{\mathbf{L}}$, $\boldsymbol{\Xi}$ is $(\boldsymbol{\Xi})^{\mathbf{L}}$, $\mathbf{IPS} = (\mathbf{IPS})^{\mathbf{L}}$, $\boldsymbol{\Pi} = (\boldsymbol{\Pi})^{\mathbf{L}}$, $\mathbf{NF} = (\mathbf{NF})^{\mathbf{L}}$, *etc.*

In addition, $\omega_2^{\mathbf{L}} < \omega_1$ will be our blanket assumption in the unoverse. □

Under $\omega_2^{\mathbf{L}} < \omega_1$, if $\zeta \in \boldsymbol{\Xi}$ (*i.e.*, $\zeta \in \mathbf{L}$ and $\mathbf{L} \models \zeta \in \boldsymbol{\Xi}$) then every set $X \in \mathbf{IPS}_\zeta$ is a countable subset of $\mathscr{D}^\zeta$ in the universe. However it transforms to a perfect set in the universe by the closure operation: *the topological closure* $X^\#$ *of a set* $X \in \mathbf{IPS}_\zeta$ *is closed in* $\mathscr{D}^\zeta$ *in the universe.* (And in fact $X^\#$ satisfies the definition of $\mathbf{IPS}_\zeta$ in the universe.)

Let $\mathscr{X} \subseteq \mathbf{IPS}$, $\mathscr{X} \in \mathbf{L}$ be a normal forcing, that is, 1°–6° of Section 21 hold (in $\mathbf{L}$), and $\mathscr{X}$ is ordered by $\downarrow\subseteq$, meaning that

$$\textit{if } X \downarrow\subseteq Y \textit{ then } X \textit{ is a stronger condition.}$$

Let $G \subseteq \mathscr{X}$ be a filter $\mathscr{X}$-generic over $\mathbf{L}$. It easily follows from Lemma 21.1 w.r.t. Lemma 11.5, that there is a unique array $\boldsymbol{v} = \boldsymbol{v}[G] = \langle \boldsymbol{v_i} \rangle_{\boldsymbol{i} \in \boldsymbol{I}} \in \mathscr{D}^{\boldsymbol{I}}$, called *$\mathscr{X}$-generic array* (over $\mathbf{L}$), all terms $\boldsymbol{v_i} = \boldsymbol{v_i}[G] = \boldsymbol{v}(\boldsymbol{i})$ being reals (*i.e.*, elements of $\mathscr{D} = 2^\omega$), such that the equivalence

$$\boldsymbol{v}\downarrow\zeta \in X^\# \iff X \in G$$

holds for all $X \in \mathscr{X}$ and $\zeta = \|X\| \in \boldsymbol{\Xi}$. Then the model $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{v}[G]] = \mathbf{L}[\langle \boldsymbol{v_i}[G] \rangle_{\boldsymbol{i} \in \boldsymbol{I}}]$ is an *$\mathscr{X}$-generic extension of* $\mathbf{L}$. **Equivalently,** an array $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic iff the set $\mathscr{G}_{\boldsymbol{v}} \cap \mathscr{X}$ is $\mathscr{X}$-generic over $\mathbf{L}$, where

$$\mathscr{G}_{\boldsymbol{v}} = \{X \in \mathbf{IPS} : \boldsymbol{v}\downarrow\zeta \in X^\#, \text{ where } \zeta = \|X\|\} \subseteq \mathbf{IPS}$$

and $X^\#$ is the topological closure of $X \subseteq \mathscr{D}^\zeta$ in $\mathscr{D}^\zeta$ as above.

Lemma 23.2. *Assume that $\mathscr{X} \subseteq \mathbf{IPS}$, $\mathscr{X} \in \mathbf{L}$ is a normal forcing, and $\omega_2^{\mathbf{L}} < \omega_1$. If $X \in \mathscr{X}$ then there is an $\mathscr{X}$-generic (over $\mathbf{L}$) array $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ satisfying $\boldsymbol{v}\downarrow\xi \in X^\#$, where $\xi = \|X\|$. If $\boldsymbol{v}$ is such then*:

(i) *if $\mathscr{Y} \in \mathbf{L}, \mathscr{Y} \subseteq \mathscr{X}$ is pre-dense in $\mathscr{X}$, then $\mathscr{G}_{\boldsymbol{v}} \cap \mathscr{Y} \neq \varnothing$;*

(ii) *if $\tau \in \boldsymbol{\Xi}$, and a set $\mathscr{Y} \in \mathbf{L}, \mathscr{Y} \subseteq (\mathscr{X}\downarrow\tau)$ is pre-dense in $\mathscr{X}\downarrow\tau$, then $\mathscr{G}_{\boldsymbol{v}} \cap \mathscr{Y} \neq \varnothing$.*

Proof. (i) is obvious. To prove (ii), it suffices to show that the set

$$\mathscr{Y}' = \{X \in \mathscr{X} : \tau \subseteq \xi = \|X\| \wedge \exists\, Y \in \mathscr{Y} (X{\downarrow}\tau \subseteq Y)\}$$

is dense in $\mathscr{X}$. **Arguing in L**, assume that $Z_0 \in \mathscr{X}$, $\eta = \|Z_0\|$. Let $\xi = \eta \cup \tau$. Then $Z = Z_0{\uparrow}\xi \in \mathscr{X}{\downarrow}\xi$ and $Z_1 = Z{\downarrow}\tau \in \mathscr{X}{\downarrow}\tau$ by 3°, 2°. By the pre-density, Z_1 is compatible with some $Y \in \mathscr{Y}$, so that there exists $U \in \mathscr{X}{\downarrow}\tau$, $U \subseteq Y \cap Z_1$. Then $X = Z \cap (U{\uparrow}\xi) \in \mathscr{X}{\downarrow}\xi$ by 3°, and $X{\downarrow}\tau = U \subseteq Y$, therefore $X \in \mathscr{Y}'$. Moreover $X \subseteq Z$, hence $X \downarrow\subseteq Z_0 = Z{\uparrow}\eta$ by construction. This ends the proof of the density of $\mathscr{Y}'$. □

Definition 23.3 (symmetric subextensions)**.** Assume that $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ and $\Omega \subseteq \mathbf{\Xi}$. We put $\mathbf{W}_\Omega[\boldsymbol{v}] = \{\rho{\bullet}(\boldsymbol{v}{\downarrow}\eta) : \rho \in \mathbf{\Pi} \wedge \eta \in \Omega\}$.

We'll use symmetric subclasses $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$ of generic extensions $\mathbf{L}[\boldsymbol{v}]$, $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$, for suitable sets $\Omega \subseteq \mathbf{\Xi}$ in $\mathbf{L}$, as models for Theorem 1.1. By definition, $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$ is the least transitive subclass of $\mathbf{L}[\boldsymbol{v}]$ which contains the set $\mathbf{W}_\Omega[\boldsymbol{v}]$ and satisfies **ZF**. □

24 Forcing relation

Assume that $\mathscr{X} \in \mathbf{NF}$ is a normal forcing, *i.e.*, $\mathscr{X} \in \mathbf{L}$ and it holds in $\mathbf{L}$ that $\mathscr{X} \in \mathbf{NF}$, see Blanket assumption 23.1. To study $\mathscr{X}$-generic extensions of $\mathbf{L}$, we make use of a *forcing language* $\mathscr{L}$, containing the following proper $\mathbf{L}$-class $\mathbf{N}(\mathscr{L})$ of basic names:

- $\dot{x}$ for any $x \in \mathbf{L}$ — we'll typically *identify* $\dot{x}$ with x itself, as usual;
- $\underline{\sigma\boldsymbol{v}}$ for any $\sigma \in \mathbf{\Pi}$ — names of this form will be called *unlimited*;
- *derived* names $\underline{\sigma\boldsymbol{v}}{\downarrow}\eta$ for any $\sigma \in \mathbf{\Pi}$ and $\eta \in \mathbf{\Xi}$;
- in particular $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{v}}{\downarrow}\eta$ will be shorthands for resp. $\underline{\varepsilon\boldsymbol{v}}$ and $\underline{\varepsilon\boldsymbol{v}}{\downarrow}\eta$, where $\varepsilon \in \mathbf{\Pi}$ is the identity;
- $\underline{\mathbf{W}}_\Omega$ for any $\Omega \in \mathbf{L}$, $\Omega \subseteq \mathbf{\Xi}$.

All those names belong to $\mathbf{L}$ as $\mathbf{\Pi}, \mathbf{\Xi} \in \mathbf{L}$ by Blanket agreement 23.1.

The name $\underline{\boldsymbol{v}}$ will be involved as the canonical name for a generic array $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$. Accordingly each $\underline{\sigma\boldsymbol{v}}$ will work as a name for $\sigma{\bullet}\underline{\boldsymbol{v}}$, so in principle it is a derived name. Yet we'd like to have each $\underline{\sigma\boldsymbol{v}}$ as an independent name so to speak, in order to define an action of $\mathbf{\Pi}$ on basic names. Accordingly, each derived name $\underline{\sigma\boldsymbol{v}}{\downarrow}\eta$ will work as a name for $(\sigma{\bullet}\underline{\boldsymbol{v}}){\downarrow}\eta = \sigma{\bullet}(\underline{\boldsymbol{v}}{\downarrow}\eta')$, where $\eta' = \sigma^{-1}{\bullet}\eta$ (recall Lemma 13.2). Finally, $\underline{\mathbf{W}}_\Omega$ is a name for $\mathbf{W}_\Omega[\underline{\boldsymbol{v}}] = \{\rho{\bullet}(\underline{\boldsymbol{v}}{\downarrow}\eta) : \rho \in \mathbf{\Pi} \wedge \eta \in \Omega\}$.

An $\mathscr{L}$-formula is ***limited*** iff it contains unlimited names $\underline{\pi\boldsymbol{v}}$ only via derived names $\underline{\sigma\boldsymbol{v}}{\downarrow}\eta$, $\sigma \in \mathbf{\Pi}$ and $\eta \in \mathbf{\Xi}$.

Given $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ in the universe and an $\mathscr{L}$-formula φ, we define the *valuation* $\varphi[\boldsymbol{v}]$ by the substitution of the valuations resp.

$$\dot{x}[\boldsymbol{v}] = x, \quad (\underline{\sigma \boldsymbol{v}})[\boldsymbol{v}] = \sigma{\bullet}\boldsymbol{v}, \quad \mathbf{W}_\Omega[\boldsymbol{v}] = \{\rho{\bullet}(\boldsymbol{v}{\downarrow}\eta) : \rho \in \mathbf{\Pi} \wedge \eta \in \Omega\} \ ,$$

for any basic names resp. $\dot{x}$, $\underline{\pi\boldsymbol{v}}$, $\underline{\mathbf{W}}_\Omega$ in $\mathbf{N}(\mathscr{L})$ that occur in φ. All those sets belong to the extension $\mathbf{L}[\boldsymbol{v}] = \mathbf{L}[\mathscr{G}_{\boldsymbol{v}}]$, of course.

Definition 24.1 (forcing relation)**.** Let $\mathscr{X} \in \mathbf{NF}$ is a normal forcing, in particular, $\mathscr{X} \in \mathbf{L}$, and φ be a closed $\mathscr{L}$*-formula* (with names in $\mathbf{N}(\mathscr{L})$ as parameters). Let $X \in \mathscr{X}$, $\zeta = \|X\|$. We define $X \Vdash_{\mathscr{X}} \varphi$, iff $\varphi[\boldsymbol{v}]$ holds in $\mathbf{L}[\boldsymbol{v}]$ whenever $\boldsymbol{v}$ is an $\mathscr{X}$-generic array over $\mathbf{L}$, satisfying $\boldsymbol{v}{\downarrow}\zeta \in X^{\#}$. □

The next routine lemma contains an important claim; it involves one more definition. Suppose that $X \in \mathbf{IPS}$ and $\mathscr{Y} \subseteq \mathbf{IPS}$. We define

$X \subseteq^{\mathtt{fin}} \bigcup\mathscr{Y}$**,** iff there is a finite set $\mathscr{Y}' \subseteq \mathscr{Y}$ such that 1) $\|Y\| \subseteq \xi = \|X\|$ for all $Y \in \mathscr{Y}'$, and 2) $X \subseteq \bigcup_{Y\in\mathscr{Y}'}(Y{\uparrow}\xi)$.

$X \subseteq^{\mathtt{fd}} \bigcup\mathscr{Y}$**,** iff in addition 3) $(Y{\uparrow}\xi) \cap (Z{\uparrow}\xi) = \varnothing$ for all $Y \neq Z$ in $\mathscr{Y}'$.

Lemma 24.2. *Under the assumptions of Definition 24.1, if* $X \in \mathscr{X}$, $\mathscr{Y} \subseteq \mathscr{X}$, $X \subseteq^{\mathtt{fin}} \bigcup\mathscr{Y}$, *and* $Y \Vdash_{\mathscr{X}} \varphi$ *for all* $Y \in \mathscr{Y}$, *then* $X \Vdash_{\mathscr{X}} \varphi$.

Proof. To check that every $X \in \mathscr{X}$ satisfying $X \subseteq^{\mathtt{fin}} \bigcup\mathscr{Y}$ is compatible with some $Y \in \mathscr{Y}$ use 4° of Section 21, and Lemma 11.5. □

25 Forcing and permutations

Automorphisms of forcing notions have been widely used to define models with various effects related to the axiom of choice, basically since Cohen's times. Define the left action of permutations $\pi \in \mathbf{\Pi}$ on names, as follows:

$$\begin{aligned} \pi{\bullet}\dot{x} &= \dot{x}; \\ \pi{\bullet}\underline{\sigma\boldsymbol{v}} &= \underline{(\sigma\circ\pi^{-1})\boldsymbol{v}}, \quad \text{in particular,} \quad \pi{\bullet}\underline{\boldsymbol{v}} = \underline{(\pi^{-1})\boldsymbol{v}}; \\ \pi{\bullet}\underline{\mathbf{W}}_\Omega &= \underline{\mathbf{W}}_{\{\pi\xi\,:\,\xi\in\Omega\}}. \end{aligned}$$

The group action property holds, for instance:

$$\rho{\bullet}(\pi{\bullet}\underline{\sigma\boldsymbol{v}}) = \rho{\bullet}\underline{(\sigma\circ\pi^{-1})\boldsymbol{v}} = \underline{(\sigma\circ\pi^{-1}\circ\rho^{-1})\boldsymbol{v}} = \underline{(\sigma\circ(\rho\circ\pi)^{-1})\boldsymbol{v}} = (\rho\circ\pi){\bullet}\underline{\sigma\boldsymbol{v}}.$$

If $\pi \in \mathbf{\Pi}$ and φ is an $\mathscr{L}$-formula then we let $\pi\varphi$ be obtained by the substitution of $\pi{\bullet}\nu$ for any name ν in φ.

If $\eta \in \mathbf{\Xi}$, $\Omega \subseteq \mathbf{\Xi}$, $\mathscr{X} \subseteq \mathbf{IPS}$ then define the following subgroups of $\mathbf{\Pi}$:

$$\begin{aligned} \mathbf{\Pi}(\eta) &= \{\pi \in \mathbf{\Pi} : \forall \boldsymbol{i} \in \eta\,(\boldsymbol{i} = \pi(\boldsymbol{i}))\}, \\ \mathbf{Inv}(\Omega) &= \{\pi \in \mathbf{\Pi} : \forall \xi \in \mathbf{\Xi}\,(\xi \in \Omega \iff \pi{\bullet}\xi \in \Omega)\}. \end{aligned}$$

If φ is an $\mathscr{L}$-formula, then let

$$\begin{aligned} \mathbf{Inv}(\varphi) &= \bigcap\{\mathbf{Inv}(\Omega) : \Omega = \mathbf{\Xi} \vee \underline{\mathbf{W}_\Omega} \text{ occurs in } \varphi\}; \\ \|\varphi\| &= \bigcup\{\sigma^{-1}{\bullet}\eta : \underline{\sigma \boldsymbol{v}}{\downarrow}\eta \text{ occurs in } \varphi\}, \quad \text{thus } \|\varphi\| \in \mathbf{\Xi}. \end{aligned}$$

Lemma 25.1. *Let φ be an $\mathscr{L}$-formula and $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$. Then:*

(i) *if $\pi \in \mathbf{\Pi}$ then the formulas $\varphi[\boldsymbol{v}]$ and $(\pi\varphi)[\pi{\bullet}\boldsymbol{v}]$ coincide;*

(ii) *$\|\pi\varphi\| = \pi{\bullet}\|\varphi\|$, and if $\pi \in \mathbf{Inv}(\varphi)$ then any name $\underline{\mathbf{W}_\Omega}$ in φ does not change in $\pi\varphi$;*

(iii) *if $\pi \in \mathbf{\Pi}(\|\varphi\|) \cap \mathbf{Inv}(\varphi)$, and φ is a limited formula, then the formulas $\varphi[\boldsymbol{v}]$, $(\pi\varphi)[\boldsymbol{v}]$ coincide.*

Proof. (i) Let $\underline{\sigma \boldsymbol{v}}$ occur in φ. Then it changes to $\underline{(\sigma \circ \pi^{-1})\boldsymbol{v}}$ in $\pi\varphi$. It remains to note that by the group action property

$$(\sigma \circ \pi^{-1}){\bullet}(\pi{\bullet}\boldsymbol{v}) = (\sigma \circ \pi^{-1} \circ \pi){\bullet}\boldsymbol{v} = \sigma{\bullet}\boldsymbol{v}.$$

Further, any name $\underline{\mathbf{W}_\Omega}$ in φ changes to $\underline{\mathbf{W}_{\Omega'}}$, where $\Omega' = \{\pi{\bullet}\xi : \xi \in \Omega\}$. Using Lemma 13.2, we obtain:

$$\begin{aligned} \mathbf{W}_{\Omega'}[\pi{\bullet}\boldsymbol{v}] &= \{\rho{\bullet}((\pi{\bullet}\boldsymbol{v}){\downarrow}\eta_1) : \rho \in \mathbf{\Pi} \wedge \eta_1 \in \Omega'\} \\ = \{\rho{\bullet}((\pi{\bullet}\boldsymbol{v}){\downarrow}(\pi{\bullet}\eta)) &: \rho \in \mathbf{\Pi} \wedge \eta \in \Omega\} = \{\rho{\bullet}(\pi{\bullet}(\boldsymbol{v}{\downarrow}\eta)) : \rho \in \mathbf{\Pi} \wedge \eta \in \Omega\} \\ = \{(\rho\circ\pi){\bullet}(\boldsymbol{v}{\downarrow}\eta)) &: \rho \in \mathbf{\Pi} \wedge \eta \in \Omega\} = \{\rho_1{\bullet}(\boldsymbol{v}{\downarrow}\eta)) : \rho_1 \in \mathbf{\Pi} \wedge \eta \in \Omega\}, \end{aligned}$$

because $\{\rho\circ\pi : \rho \in \mathbf{\Pi}\} = \mathbf{\Pi}$.

(ii) If $\underline{\mathbf{W}_\Omega}$ is a name in φ then it changes to $\underline{\mathbf{W}_{\Omega'}}$ in $\pi\varphi$, where $\Omega' = \{\pi{\bullet}\eta : \eta \in \Omega\} = \Omega$ since $\pi \in \mathbf{\Gamma}(\Omega)$. This $\underline{\mathbf{W}_{\Omega'}}$ is identic to $\underline{\mathbf{W}_\Omega}$. Further,

$$\begin{aligned} \|\pi\varphi\| &= \bigcup\{{\sigma_1}^{-1}{\bullet}\eta : \underline{\sigma_1 \boldsymbol{v}}{\downarrow}\eta \text{ occurs in } \pi\varphi\} \\ &= \bigcup\{(\sigma\circ\pi^{-1})^{-1}{\bullet}\eta : \underline{\sigma \boldsymbol{v}}{\downarrow}\eta \text{ occurs in } \varphi\} \\ &= \bigcup\{\pi{\bullet}(\sigma^{-1}{\bullet}\eta) : \underline{\sigma \boldsymbol{v}}{\downarrow}\eta \text{ occurs in } \varphi\} \qquad = \quad \pi{\bullet}\|\varphi\|\,. \end{aligned}$$

(iii) If $\underline{\sigma \boldsymbol{v}}{\downarrow}\eta$ occurs in φ then it changes to $\underline{(\sigma\circ\pi^{-1})\boldsymbol{v}}{\downarrow}\eta$ in $\pi\varphi$. The $\boldsymbol{v}$-valuation of $\underline{(\sigma\circ\pi^{-1})\boldsymbol{v}}{\downarrow}\eta$ is equal (by Lemma 13.2) to

$$\sigma{\bullet}(\pi^{-1}{\bullet}\boldsymbol{v}){\downarrow}\eta = \sigma{\bullet}((\pi^{-1}{\bullet}\boldsymbol{v}){\downarrow}(\sigma^{-1}{\bullet}\eta)) = \sigma{\bullet}(\boldsymbol{v}{\downarrow}(\sigma^{-1}{\bullet}\eta)) = (\sigma{\bullet}\boldsymbol{v}){\downarrow}\eta,$$

since π and π^{-1} are the identities on $\sigma^{-1}{\bullet}\eta$ (because $\pi \in \mathbf{\Pi}(\|\varphi\|)$). But this is equal to the $\boldsymbol{v}$-valuation of the original derived name $\underline{\boldsymbol{\sigma v}}{\downarrow}\eta$ in φ.

If $\underline{\mathbf{W}}_\Omega$ is a name in φ then it does not change in $\pi\varphi$ by (ii). □

Theorem 25.2. *Assume that, in* $\mathbf{L}$*,* $\mathscr{X} \in \mathbf{NF}$ *is a normal forcing,* φ *is a closed* $\mathscr{L}$*-formula, and* $\pi \in \mathbf{\Pi}$*. Let* $X \in \mathscr{X}$*. Then* $X \Vdash_{\mathscr{X}} \varphi$ *iff* $\pi{\bullet}X \Vdash_{\mathscr{X}} \pi\varphi$*.*

Proof. As $\mathscr{X}, \pi \in \mathbf{L}$ (see Blanket agreement 23.1), an array $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic over $\mathbf{L}$ iff so is $\pi{\bullet}\boldsymbol{v}$. Now the result follows from Lemma 25.1(i). □

Corollary 25.3. *Under the assumptions of Theorem 25.2, suppose that* $\tau \in \mathbf{\Xi}$*,* φ *is a closed limited formula,* $\|\varphi\| \subseteq \tau$*,* $\pi \in \mathbf{\Pi}(\tau) \cap \mathbf{Inv}(\varphi)$*,* $X \in \mathscr{X}$*. Then* $X \Vdash_{\mathscr{X}} \varphi$ *iff* $\pi{\bullet}X \Vdash_{\mathscr{X}} \varphi$*.*

Proof. The result follows from Theorem 25.2 and Lemma 25.1(iii). □

Corollary 25.4. *Under the assumptions of Thm 25.2, let* $\tau \subsetneq \eta$ *belong to* $\mathbf{\Xi}$*,* $\varphi(x)$ *be a limited formula,* $\|\varphi\| \subseteq \tau$*,* $\pi \in \mathbf{\Pi}(\tau) \cap \mathbf{Inv}(\varphi)$*,* $X \in \mathscr{X}$*,* $\sigma = \pi``\eta$*. Then* $X \Vdash_{\mathscr{X}} (\exists\, x \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\eta])\, \varphi(x)$ *iff* $\pi{\bullet}X \Vdash_{\mathscr{X}} (\exists\, x \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\sigma])\, \varphi(x)$*.*

Proof. Assume that $X \Vdash_{\mathscr{X}} (\exists\, x \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\eta])\, \varphi(x)$. Then, by Theorem 25.2, $\pi{\bullet}X \Vdash_{\mathscr{X}} (\exists\, x \in \mathbf{L}[\underline{\pi^{-1}\boldsymbol{v}}{\downarrow}\eta])\, \pi\varphi(x)$. Yet if $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ then, by Lemma 13.2, $(\pi^{-1}{\bullet}\boldsymbol{v}){\downarrow}\eta = \pi^{-1}{\bullet}(\boldsymbol{v}{\downarrow}\sigma)$, hence obviously $\mathbf{L}[(\pi^{-1}{\bullet}\boldsymbol{v}){\downarrow}\eta] = \mathbf{L}[\boldsymbol{v}{\downarrow}\sigma]$. We conclude that $\pi{\bullet}X \Vdash_{\mathscr{X}} (\exists\, x \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\sigma])\, \pi\varphi(x)$. And finally, here we can replace $\pi\varphi(x)$ by $\varphi(x)$ by Lemma 25.1(iii). □

26 Isolation and the narrowing theorem

Suppose that $\eta \in \mathbf{\Xi}$. It often happens in similar cases that sentences relativized to $\mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$ are decided by forcing conditions X satisfying $\|X\| \subseteq \eta$. The following theorem belongs to this category.

Definition 26.1 (in $\mathbf{L}$). Assume that $\Gamma \subseteq \mathbf{\Pi}$ is a subgroup. Say that $\eta \in \mathbf{\Xi}$ is Γ-*isolated* if (*) for each $\xi \in \mathbf{\Xi}$ with $\eta \subseteq \xi$ there is a permutation $\pi \in \Gamma \cap \mathbf{\Pi}(\eta)$ satisfying $\xi \cap (\pi{\bullet}\xi) = \eta$. □

Lemma 26.2 (in $\mathbf{L}$). *Each* $\eta \in \mathbf{\Xi}$ *is* $\mathbf{\Pi}$*-isolated.*

Proof. Let $\eta \subseteq \xi \in \mathbf{\Xi}$; define $\pi \in \mathbf{\Pi}(\eta)$ with $\xi \cap (\pi{\bullet}\xi) = \eta$. Let $\lambda < \omega_1$ be a limit ordinal $>$ all ordinals $\boldsymbol{j}(k)$, where $\boldsymbol{j} \in \xi$ and $k < \mathtt{lh}(\boldsymbol{j})$.

Define, in $\mathbf{L}$, $B : \omega_1 \xrightarrow{\text{onto}} \omega_1$ by $B(\gamma) = B^{-1}(\gamma) = \lambda + \gamma$ for all $\gamma < \lambda$, and $B(\gamma) = \gamma$ for $\gamma \ge \lambda + \lambda$. If $\boldsymbol{i} \in \boldsymbol{I}$ then define $\beta(\boldsymbol{i}) = \boldsymbol{i}' \in \boldsymbol{I}$ so that $\mathtt{lh}(\boldsymbol{i}') = \mathtt{lh}(\boldsymbol{i})$ and $\boldsymbol{i}'(\ell) = B(\boldsymbol{i}(\ell))$ for all $\ell < \mathtt{lh}(\boldsymbol{i})$. Clearly $\beta \in \mathbf{\Pi}$.

Now let $\boldsymbol{i} \in \boldsymbol{I}$. There is a largest number $m_{\boldsymbol{i}} \le \mathtt{lh}(\boldsymbol{i})$ such that $\boldsymbol{i}\restriction m_{\boldsymbol{i}} \in \eta$. Then $\boldsymbol{i} = (\boldsymbol{i}\restriction m_{\boldsymbol{i}})^\frown \boldsymbol{k}$ for some $\boldsymbol{k} \in \boldsymbol{I} \cup \{\Lambda\}$. Put $\pi(\boldsymbol{i}) = (\boldsymbol{i}\restriction m_{\boldsymbol{i}})^\frown \beta(\boldsymbol{k})$. □

Theorem 26.3 (the narrowing theorem, in $\mathbf{L}$). *Assume that $X \in \mathscr{X} \in \mathbf{NF}$, φ is a closed limited $\mathscr{L}$-formula, $\eta \in \boldsymbol{\Xi}$ is $\mathbf{Inv}(\varphi)$-isolated, and $\|\varphi\| \subseteq \eta \subseteq \|X\|$. Then $X \Vdash_{\mathscr{X}} \varphi$ iff $X{\downarrow}\eta \Vdash_{\mathscr{X}} \varphi$.*

Proof. Suppose to the contrary that $X \Vdash_{\mathscr{X}} \varphi$ but $X{\downarrow}\eta \nVdash_{\mathscr{X}} \varphi$. There is a condition $U \in \mathscr{X}$ such that $U \downarrow\subseteq (X{\downarrow}\eta)$ and $U \Vdash_{\mathscr{X}} \neg\varphi$. Let $\xi = \|X\|$, $\tau = \|U\|$. By (*) of Definition 26.1, there is a permutation $\pi \in \mathbf{Inv}(\varphi) \cap \boldsymbol{\Pi}(\eta)$ satisfying $(\pi{\bullet}(\xi \cup \tau)) \cap (\xi \cup \tau) = \eta$, in particular, $(\pi{\bullet}\tau) \cap \xi = \eta$.

Let $Y = \pi{\bullet}U$ and $\zeta = \|Y\| = \pi{\bullet}\tau$. Then $Y \in \mathscr{X}$ (since $\pi \in \boldsymbol{\Gamma}(\mathscr{X})$, $\xi \cap \zeta = \eta$, and (most important!) $Y \Vdash_{\mathscr{X}} \neg\varphi$ by Corollary 25.3.

Furthermore, $Y{\downarrow}\eta = U{\downarrow}\eta$ (since $\pi \in \boldsymbol{\Pi}(\eta)$), in particular, $Y{\downarrow}\eta \subseteq X{\downarrow}\eta$. Therefore $X' = X \cap (Y{\downarrow}\eta{\uparrow}\xi) \in \mathscr{X}$, $X' \subseteq X$, $X'{\downarrow}\eta = Y{\downarrow}\eta$. Let $\vartheta = \xi \cup \zeta$. It follows by Lemma 21.1 w.r.t. Corollary 10.8 that the set $Z = (X'{\uparrow}\vartheta) \cap (Y{\uparrow}\vartheta)$ belongs to $\mathscr{X}$, and obviously $Z \downarrow\subseteq Y$ and $Z \downarrow\subseteq X' \subseteq X$. Thus X and Y are compatible in $\mathscr{X}$. But X, Y force contradictory sentences. □

Corollary 26.4. *Assume that $\mathscr{X} \in \mathbf{NF}$, $\boldsymbol{i} \in \boldsymbol{I} \smallsetminus \eta$, and $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic. Then $\boldsymbol{v}(\boldsymbol{i}) \notin \mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$.*

Proof. Suppose towards the contrary that $\boldsymbol{v}(\boldsymbol{i}) \notin \mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$. Then there is a parameter-free $\in$-formula $\varphi(\cdot,\cdot,\cdot)$, and a parameter $p \in \mathbf{L}$, such that,

$$\text{for all } k < \omega\text{:} \quad \boldsymbol{v}(\boldsymbol{i})(k) = 1 \text{ iff } \mathbf{L}[\boldsymbol{v}{\downarrow}\eta] \models \varphi(p, \boldsymbol{v}{\downarrow}\eta, k).$$

Then there exists such a condition $X \in \mathscr{X} \cap \mathscr{G}_{\boldsymbol{v}}$ that

$$X \Vdash_{\mathscr{X}} \forall\, k\, \big(\underline{\boldsymbol{v}}(\boldsymbol{i})(k) = 1 \iff \varphi(\dot{p}, \underline{\boldsymbol{v}}{\downarrow}\eta, k\big). \tag{1}$$

Let $\xi = \|X\|$. We may assume that $\eta \subseteq \xi$, as otherwise replace X by $X' = X{\uparrow}(\eta \cup \xi)$, which still belongs to $\mathscr{X}$ by 3° of Section 21. And we may assume that $\boldsymbol{i} \in \xi$ by the same reason. Lemma 11.7 implies that there exists $k < \omega$ and sets $Y, Z \in \mathbf{IPS}_\xi$, clopen in X and such that $Y{\downarrow}\eta = Z{\downarrow}\eta$ and $y(\boldsymbol{i})(k) = 1$ but $z(\boldsymbol{i})(k) = 0$ for all $y \in Y$ and $z \in Z$ (or vice versa). Then $Y, Z \in \mathscr{X}$ by 4° of Section 21, and $Y \Vdash_{\mathscr{X}} \underline{\boldsymbol{v}}(\boldsymbol{i})(k) = 1$ but $Z \Vdash_{\mathscr{X}} \underline{\boldsymbol{v}}(\boldsymbol{i})(k) = 0$.

It follows by (1) that $Y \Vdash_{\mathscr{X}} \varphi(\dot{p}, \underline{\boldsymbol{v}}{\downarrow}\eta, k\big)$, hence $Y{\downarrow}\eta \Vdash_{\mathscr{X}} \varphi(\dot{p}, \underline{\boldsymbol{v}}{\downarrow}\eta, k\big)$ by Theorem 26.3 (applicable by Lemma 26.2). We have $Z{\downarrow}\eta \Vdash_{\mathscr{X}} \neg\varphi(\dot{p}, \underline{\boldsymbol{v}}{\downarrow}\eta, k\big)$ by the same reasons. However $Y{\downarrow}\eta = Z{\downarrow}\eta$, which is a contradiction. □

Corollary 26.5. *Assume that $\mathscr{X} \in \mathbf{NF}$, φ is a closed limited $\mathscr{L}$-formula, $\eta \in \boldsymbol{\Xi}$ is $\mathbf{Inv}(\varphi)$-isolated, and $\|\varphi\| \subseteq \eta$, $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic, and $\mathbf{L}[\boldsymbol{v}] \models \varphi[\boldsymbol{v}]$. Then there is $X \in \mathscr{X}{\downarrow}\eta \cap \mathscr{G}_{\boldsymbol{v}}$ such that $X \Vdash_{\mathscr{X}} \varphi$.*

Proof. The set $\mathscr{D} = \{X \in \mathscr{X}{\downarrow}\eta : X \Vdash_{\mathscr{X}} \varphi$ or $X \Vdash_{\mathscr{X}} \neg\varphi\}$ is pre-dense in $\mathscr{X}$ by Theorem 26.3. □

Corollary 26.6. *Assume that $\mathscr{X} \in \mathbf{NF}$, $\varphi(\cdot)$ is a limited $\mathscr{L}$-formula, $\eta \in \boldsymbol{\Xi}$ is $\mathbf{Inv}(\varphi)$-isolated, and $\|\varphi\| \subseteq \eta$, $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic, and $A \in \mathbf{L}$. Then the set $S = \{a \in A : \mathbf{L}[\boldsymbol{v}] \models \varphi(a)\}$ belongs to $\mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$.*

Proof. We have $S = \{a \in A : \exists X \in \mathscr{X}{\downarrow}\eta \cap \mathscr{G}_{\boldsymbol{v}} \,(X \Vdash_{\mathscr{X}} \varphi(\mathring{a}))\}$. On the other hand, $\mathscr{X}{\downarrow}\eta \cap \mathscr{G}_{\boldsymbol{v}} = \{X \in \mathbf{IPS}_\eta : \boldsymbol{v}{\downarrow}\eta \in X^\#\} \in \mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$. □

Corollary 26.7. *Assume that $\mathscr{X} \in \mathbf{NF}$, $\Omega \in \mathbf{L}$, $\Omega \subseteq \boldsymbol{\Xi}$ is $\cup$-closed (under finite unions), all $\eta \in \Omega$ are $\mathbf{Inv}(\Omega)$-isolated, $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic, and $S \in \mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$, $S \subseteq \mathbf{L}$. Then $S \in \mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$ for some $\eta \in \Omega$.*

Proof. First of all, $S \subseteq A$ for some $A \in \mathbf{L}$. Then, as $S \in \mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$, we have $S = \{a \in A : \mathbf{L}[\boldsymbol{v}] \models \varphi(a)\}$, where φ contains only $x \in \mathbf{L}$, $\mathbf{W}_\Omega[\boldsymbol{v}]$, and some $\boldsymbol{v}{\downarrow}\eta$, $\eta \in \Omega$, as parameters. Then $S \in \mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$ by Corollary 26.6. □

Corollary 26.8. *Assume that $\mathscr{X}$, Ω are as in Corollary 26.7, $\psi(\cdot)$ is a limited $\mathscr{L}$-formula, $X \in \mathscr{X}$, $A \in \mathbf{L}$, $X \Vdash_{\mathscr{X}} \exists S \in \mathbf{L}(\underline{\mathbf{W}}_\Omega)\,(S \subseteq A \wedge \psi(x))$. Then there exists a condition $Y \in \mathscr{X}$, and $\eta \in \Omega$, such that $Y \downarrow\subseteq X$ and $Y \Vdash_{\mathscr{X}} \exists S \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\eta]\,(S \subseteq A \wedge \psi(x))$.*

Proof. By Lemma 23.2, there exists a $\mathscr{X}$-generic array $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ satisfying $X \in \mathscr{G}_{\boldsymbol{v}}$. There is $S \in \mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$ such that $\mathbf{L}[\boldsymbol{v}] \models \psi(S)$ and $S \subseteq A$. We have $S \in \mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$ for some $\eta \in \Omega$ by Corollary 26.7. Then some $Z \in \mathscr{G}_{\boldsymbol{v}} \cap \mathscr{X}$ satisfies $Z \Vdash_{\mathscr{X}} \exists S \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\eta]\,(S \subseteq A \wedge \psi(x))$. But Z and X are compatible in $\mathscr{X}$, so take any $Y \in \mathscr{X}$ with $Y \downarrow\subseteq X$ and $Y \downarrow\subseteq Z$. □

27 Fusion property

Arguing in L, say that a set $\mathscr{X} \in \mathbf{NF}$ has *the fusion property*, if for any sequence $\langle \mathscr{Y}_k \rangle_{k<\omega} \in \mathbf{L}$ of dense sets $\mathscr{Y}_k \subseteq \mathscr{X}$, the set

$$\mathscr{Y} = \{X \in \mathscr{X} : \forall\, k\, (X \subseteq^{\mathtt{fd}} \textstyle\bigcup \mathscr{Y}_k)\}$$

is dense in $\mathscr{X}$ too. (See before Lemma 24.2 on $\subseteq^{\mathtt{fd}}$.) The fusion property is another formalization of some features of the Sacks forcing. It differs from the more common Axiom A, but it fits more to applications in this paper. The following theorem presents several principal applications.

Theorem 27.1. *Assume that, in $\mathbf{L}$, $\mathscr{X} \in \mathbf{NF}$ has the fusion property, and $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic over $\mathbf{L}$. Then:*

(i) *if* $h \in \mathbf{L}[\boldsymbol{v}]$, $h : \omega \to \mathbf{L}$, *then there is a map* $H \in \mathbf{L}$ *such that* $\mathtt{dom}\, H = \omega$, *and, for each* $k < \omega$, $h(k) \in H(k)$ *and* $H(k)$ *is finite;*

(ii) *every* $\mathbf{L}$*-cardinal remains a cardinal in* $\mathbf{L}[\boldsymbol{v}]$;

(iii) *if* $x \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}]$ *then* $x \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$ *for some* $\xi \in \boldsymbol{\Xi}$;

(iv) *if* $\xi \in \boldsymbol{\Xi}$ *and* $a \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$ *then there is a continuous map* $F : \mathscr{D}^\xi \to \omega^\omega$ *such that* $a = F(\boldsymbol{v}{\downarrow}\xi)$, *and* F *is* coded in $\mathbf{L}$ *in the sense that the restriction* $F_{\mathbf{L}} = F \restriction (\mathbf{L} \cap \mathscr{D}^\xi)$ *belongs to* $\mathbf{L}$.

Note that if $F_{\mathbf{L}} = F \restriction (\mathbf{L} \cap \mathscr{D}^\xi) \in \mathbf{L}$ in (iv) then $\mathbf{L} \models$ "$F_{\mathbf{L}} : \mathscr{D}^\xi \to \omega^\omega$ is continuous" and $F = F_{\mathbf{L}}^{\#}$ (the topological closure of $F_{\mathbf{L}}$ in $\mathscr{D}^\xi \times \omega^\omega$).

Proof. (i) There is an $\in$-formula $\varphi(v, k, m)$, with ordinals as parameters, such that $h = \{\langle k, x\rangle \in \omega \times \mathbf{L} : \mathbf{L}[\boldsymbol{v}] \models \varphi(\boldsymbol{v}, k, m)\}$, and

(1) if $X \in \mathscr{X}$ then $X \Vdash_{\mathscr{X}} \big(\{\langle k, x\rangle : \varphi(\boldsymbol{v}, k, m)\}$ is a map $\omega \to \mathbf{L}\big)$.

Arguing in L, define the sets $\mathscr{Z}_m = \{X \in \mathscr{X} : \exists\, x\, (X \Vdash_{\mathscr{X}} \varphi(\underline{\boldsymbol{v}}, m.\dot{x})\}$. By (1), each $\mathscr{Z}_m$ is open dense in $\mathscr{X}$. Thus $\mathscr{Z} = \{X \in \mathscr{X} : \forall\, m\, (Z \subseteq^{\mathtt{fd}} \bigcup \mathscr{Z}_m)\}$ is dense as well by the fusion property. It follows that there exists $Z \in \mathscr{Z} \cap \mathscr{G}_{\boldsymbol{v}}$, so that for each m there exists a finite subset $\mathscr{Y}_m \subseteq \mathscr{Z}_m$ with $Z \subseteq \bigcup_{Y \in \mathscr{Y}_m} (Y {\uparrow} \zeta)$, where $\zeta = \dim Z$ and $\dim Y \subseteq \zeta$ for all $Y \in \mathscr{Y}_m$.

By definition and (1), for each $m < \omega$ and $Y \in \mathscr{Y}_m$ there is a unique set $x_{mY} \in \mathbf{L}$ satisfying $Y \Vdash_{\mathscr{X}} \varphi(\boldsymbol{v}, m, \dot{x}_{mY})$. Let $H(m) = \{x_{mY} : Y \in \mathscr{Y}_m\}$. Then H is as required by Lemma 24.2.

(ii) is a simple corollary of (i).

(iii) is a simple corollary of (ii).

(iv) As $x \in \mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$, there is an $\in$-formula $\varphi(v, k, m)$, with ordinals as parameters, such that $a = \{\langle k, m\rangle \in \omega \times \omega : \mathbf{L}[\boldsymbol{v}] \models \varphi(\boldsymbol{v}{\downarrow}\xi, k, m)\}$, and

(2) if $X \in \mathscr{X}$ then $X \Vdash_{\mathscr{X}} \forall\, k < \omega\, \exists !\, m < \omega\, \varphi(\underline{\boldsymbol{v}}{\downarrow}\xi, k, m)$.

Let $\Psi(\boldsymbol{v}{\downarrow}\xi)$ be the conclusion of (iv) after 'then'. Assume towards the contrary that (iv) fails, so that there exists $X_0 \in \mathscr{X} \cap G_{\boldsymbol{v}}$, $X_0 \Vdash_{\mathscr{X}} \neg\, \Psi(\underline{\boldsymbol{v}}{\downarrow}\xi)$. We may w.l.o.g. assume by Theorem 26.3 that $\|X_0\| = \xi$, *i.e.* $X_0 \in \mathscr{X}{\downarrow}\xi$.

Arguing in L, define the sets $\mathscr{Y}_{km} = \{X \in \mathscr{X}{\downarrow}\xi : X \Vdash_{\mathscr{X}} \varphi(\underline{\boldsymbol{v}} \restriction \xi, k, m)\}$. By (2) and Theorem 26.3, each set $\mathscr{Y}_k = \bigcup_m \mathscr{Y}_{km}$ is open dense in $\mathscr{X}{\downarrow}\xi$. Therefore $\mathscr{Y} = \{X \in \mathscr{X}{\downarrow}\xi : \forall\, k\, (X \subseteq^{\mathtt{fd}} \bigcup \mathscr{Y}_k)\}$ is dense as well by the fusion property. It follows that there exists $X \in \mathscr{Y}$, $X \subseteq X_0$.

Then for any $k < \omega$ there is a finite $\mathscr{Y}'_k \subseteq \mathscr{Y}_k$ satisfying $X \subseteq \bigcup \mathscr{Y}'_k$, and if $Y \neq Z$ belong to $\mathscr{Y}'_k$ then $Y \cap Z = \varnothing$. Then for each k we have a partition $\mathscr{Y}'_k = \bigcup_m \mathscr{Y}'_{km}$, where $\mathscr{Y}'_{km} = \mathscr{Y}_{km} \cap \mathscr{Y}'_k$. This enables us to define a continuous map $F_0 : X \to \omega^\omega$ such that if $a \in X$ then $F_0(x)(k) = m$ iff $x \in \bigcup \mathscr{Y}'_{km}$. Let $F : \mathscr{D}^\xi \to \omega^\omega$ be a continuous extension of F_0 from X to the whole $\mathscr{D}^\xi$, still defined in $\mathbf{L}$. Then we have $X \Vdash_{\mathscr{X}} \Psi(\underline{\boldsymbol{v}}{\downarrow}\xi)$ by routine arguments, contrary to the choice of $X \subseteq X_0$. $\square$

28 The case of the full forcing IPS

The next theorem shows that $\mathbf{IPS}$ itself has the fusion property. Our more elaborated forcing notions $\mathscr{X} \subseteq \mathbf{IPS}$, defined below, will have it, too.

Theorem 28.1 (in $\mathbf{L}$). **IPS** *has the fusion property.*

Proof. Beginning the proof, we w.l.o.g. assume that (*) each $\mathscr{Y}_k$ is open dense. *i.e.*, if $Y \in \mathscr{Y}_k$, $Z \in \mathbf{IPS}$, and $Z \mathbin{\downarrow\subseteq} Y$ then $Z \in \mathscr{Y}_k$ as well — for if not then replace $\mathscr{Y}_k$ with $\mathscr{Y}'_k = \{Y' \in \mathbf{IPS} : \exists\, Y \in \mathscr{Y}_k (Y' \mathbin{\downarrow\subseteq} Y)\}$.

Fix some $X_0 \in \mathbf{IPS}$ and let $\eta_0 = \|X_0\|$. Our plan is to define:

(1) a sequence $\eta_0 \subseteq \xi_0 \subseteq \xi_1 \subseteq \xi_2 \subseteq \dots$ of $\xi_k \in \mathbf{\Xi}$, and $\xi = \bigcup_k \xi_k$;

(2) a ξ-admissible map $\phi : \omega \xrightarrow{\text{onto}} \xi$, so that

   (a) if $\boldsymbol{i} \in \xi$ then the preimage $\phi^{-1}(\boldsymbol{i}) = \{k : \phi(k) = \boldsymbol{i}\}$ is infinite,

   (b) $\boldsymbol{i} \subset \boldsymbol{j} = \phi(k) \in \xi$ implies $\boldsymbol{i} = \phi(\ell)$ for some $\ell < k$,

   (c) and in addition we require that $\phi(k) \in \xi_{k+1}$, $\forall\, k$;

(3) a system $\langle X_s \rangle_{s \in 2^{<\omega}}$ of sets $X_s \in \mathbf{IPS}_{\xi_m}$ whenever $s \in 2^m$, such that $X_\Lambda \subseteq X_0$, and $\langle X_s \rangle_{s \in 2^m}$ is a ϕ-split system (Definition 15.1), $\forall\, m$;

(4) if $s \in 2^m$ and $e = 0, 1$ then a set $X_{s^\frown e} \subseteq X_s{\uparrow} xi_{m+1}$;

(5) finally, a set $X_s \in \mathscr{Y}_m$ for all $s \in 2^m$.

If this construction is accomplished then sets $Y_s = X_s{\uparrow}\xi \in \mathbf{IPS}_\xi$ form a ϕ-fusion sequence by Lemma 15.7, so that $Y = \bigcap_m \bigcup_{s \in 2^m} X_s \in \mathbf{IPS}_\xi$ by Theorem 16.2, and we obviously have $Y \mathbin{\downarrow\subseteq} X_0$, and $Y \subseteq^{\mathtt{fin}} \bigcup \mathscr{Y}_m$, $\forall\, m$.

To maintain the construction, we pick any $X_\Lambda \in \mathscr{Y}_0$, $X_\Lambda \mathbin{\downarrow\subseteq} X_0$, by the density, let $\xi_0 = \|X_\Lambda\|$, and let $\phi(0)$ be any 1-term tuple in ξ_0.

Now the step $m \to m+1$, so that we assume that ξ_m, $\phi{\restriction}m$, and all sets $X_s \in \mathbf{IPS}_{\xi_m}$, $s \in 2^m$, are defined such that (1)–(5) hold wherever applicable.

Stage 1. Pick any $s_0 \in 2^m$. By the density, there is a set $Y \in \mathscr{Y}_{m+1}$, $Y \mathbin{\downarrow\subseteq} X_{s_0}$. Let $\eta = \|Y\|$; $\xi_m \subseteq \eta$. Let $Y_s = X_s{\uparrow}\eta$, so that $\langle Y_s \rangle_{s \in 2^m}$ is still a ϕ-split system by Lemma 15.7, and $Y \subseteq Y_{s_0}$. Let $Y'_s = Y_s \cap (Y{\downarrow}\eta_\phi[s, s_0]{\uparrow}\eta)$ for all $s \in 2^m$. Then $\langle Y'_s \rangle_{s \in 2^m}$ is still a ϕ-split system in $\mathbf{IPS}_\eta$ by Lemma 15.3, $Y'_s \mathbin{\downarrow\subseteq} X_s$ for all $s \in 2^m$, and $Y'_{s_0} = Y' \in \mathscr{Y}_{m+1}$.

Stage 2. Iterating Stage 1 (with all $s_0 \in 2^m$ involved one by one), we get a set $\zeta \in \mathbf{\Xi}$ with $\xi_m \subseteq \zeta$ and a ϕ-split system $\langle Z_s \rangle_{s\in 2^m}$ of sets $Z_s \in \mathbf{IPS}_\zeta$, such that $Z_s \in \mathscr{Y}_{m+1}$ (here we refer to the open density assumption (*) above) and $Z_s \downarrow\subseteq X_s$ for all $s \in 2^m$. Let $\xi_{m+1} = \zeta$.

Stage 3. We pick $\phi(m) \in \xi_{m+1}$ such that condition (2)b is preserved.

Stage 4. By Lemma 15.6, there is a ϕ-split system $\langle X_u \rangle_{u\in 2^{m+1}}$ is $\mathbf{IPS}_{\xi_{m+1}}$ expanding $\langle Z_s \rangle_{s\in 2^m}$, *i.e.* $X_{s^\frown e} \subseteq Z_s$ for all $s^\frown e \in 2^{m+1}$.

As the sets ξ_m obtained in the course of the construction are countable, we can maintain Stage 3 at all inductive steps in such a way that condition (2)a holds. This ends the construction and the proof. □

29 Fusion property implies countable choice

The two theorems below in this section are major applications of the fusion property and Theorem 27.1.

Theorem 29.1. *Assume that* $\mathscr{X} \in \mathbf{NF}$ *has the fusion property, a set* $\Omega \subseteq \mathbf{\Xi}$, $\Omega \in \mathbf{L}$ *is* $\cup$*-closed* (under the finite $\cup$)*, each* $\eta \in \Omega$ *is* $\mathbf{Inv}(\Omega)$*-isolated,* $\tau_0 \in \Omega$*, and*

(∗) *if* $\langle \sigma_k \rangle_{k<\omega} \in \mathbf{L}$ *is a sequence of sets* $\sigma_k \in \Omega$*, and* $\sigma_k \cap \sigma_\ell = \tau_0$ *for all* $k \neq \ell$*, then* $\bigcup_k \sigma_k \in \Omega$.

Let $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *be* $\mathscr{X}$*-generic. Then* $\mathbf{AC}_\omega$ *holds in* $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$ *for all relations* $P \subseteq \omega \times \omega^\omega$ *of class* $\mathrm{OD}(\mathbf{W}_\Omega[\boldsymbol{v}], \boldsymbol{v}{\downarrow}\tau_0)$.

In particular if $\tau_0 = \varnothing$ *then* $\mathbf{AC}_\omega(\mathrm{OD})$ *holds in* $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$.

Proof. Fix a set $P \in \mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$, $P \subseteq \omega \times \omega^\omega$, $\mathrm{OD}(\mathbf{W}_\Omega[\boldsymbol{v}], \boldsymbol{v}{\downarrow}\tau_0)$ in $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$, with $\mathtt{dom}\, P = \omega$. There is an $\in$-formula $\varphi(\cdot,\cdot,k,x)$ satisfying

$$P = \{\langle k,x\rangle : \mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}]) \models \varphi(\mathbf{W}_\Omega[\boldsymbol{v}], \boldsymbol{v}{\downarrow}\tau_0, k, x)\}.$$

As $\mathtt{dom}\, P = \omega$, for any k there is a real $x_k \in \omega^\omega \cap \mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$ with $\langle k, x_k\rangle \in P$, and then, by Corollary 26.7, there is a set $\xi_k \in \Omega$ such that $x_k \in \mathbf{L}[\boldsymbol{v}\restriction\xi_k]$. In other words,

(1) $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}]) \models \exists\, x \in \mathbf{L}[\boldsymbol{v}{\downarrow}\xi_k]\, \varphi(\mathbf{W}_\Omega[\boldsymbol{v}], \boldsymbol{v}{\downarrow}\tau_0, k, x)$.

Here the enumerations $k \mapsto x_k, \xi_k$ are maintained in $\mathbf{L}[\boldsymbol{v}]$, not in $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$, of course. However Theorem 27.1(i) yields a map $H \in \mathbf{L}$ such that $\mathtt{dom}\, H = \omega$ and $\xi_k \in H(k)$ for all k. Let $\eta_k = \bigcup(\Omega \cap H(k))$; $\eta_k \in \Omega$ because Ω is $\cup$-closed. Now $k \mapsto \eta_k$ is a map in $\mathbf{L}$, and $\xi_k \subseteq \eta_k$, hence still $x_k \in \mathbf{L}[\boldsymbol{v}\restriction\xi_k]$. We can assume that $\tau_0 \subseteq \eta_k$, $\forall\, k$, of course. Now (1) implies

(2) $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}]) \models \exists\, x \in \mathbf{L}[\boldsymbol{v}{\downarrow}\eta_k]\, \varphi(\mathbf{W}_\Omega[\boldsymbol{v}], \boldsymbol{v}{\downarrow}\tau_0, k, x)$.

Coming back to the theorem, assume to the contrary that

$$\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}]) \models \neg\,\exists\, f\, \forall\, k\, \varphi(\mathbf{W}_\Omega[\boldsymbol{v}], \boldsymbol{v}{\downarrow}\tau_0, k, f(k)).$$

Putting it all together, we get a condition $X \in \mathscr{G}_{\boldsymbol{v}}$ which $\Vdash_{\mathscr{X}}$-forces this:

(A) $\mathbf{L}(\underline{\mathbf{W}}_\Omega) \models \neg\,\exists\, f\, \forall\, k\, \varphi(\underline{\mathbf{W}}_\Omega, \underline{\boldsymbol{v}}{\downarrow}\tau_0, k, f(k))$; and

(B) $\mathbf{L}(\underline{\mathbf{W}}_\Omega) \models \exists\, x \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\eta_k]\, \varphi(\underline{\mathbf{W}}_\Omega, \underline{\boldsymbol{v}}{\downarrow}\tau_0, k, x)$, for each $k < \omega$.

We can assume that $\eta = \bigcup_k \eta_k \subseteq \|X\|$. Then we get by Theorem 26.3:

(3) $X{\downarrow}\tau_0 \Vdash_{\mathscr{X}} \big(\mathbf{L}(\underline{\mathbf{W}}_\Omega) \models \neg\,\exists\, f\, \forall\, k < \omega\, \varphi(\underline{\mathbf{W}}_\Omega, \underline{\boldsymbol{v}}{\downarrow}\tau_0, k, f(k))\big)$; and

(4) $X{\downarrow}\eta_k \Vdash_{\mathscr{X}} \big(\mathbf{L}(\underline{\mathbf{W}}_\Omega) \models \exists\, x \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\eta_k]\, \varphi(\underline{\mathbf{W}}_\Omega, \underline{\boldsymbol{v}}{\downarrow}\tau_0, k, x)\big)$, $\forall\, k < \omega$.

This is because the formula $\big(...\big)$ in (A) satisfies $\boldsymbol{\Gamma}\big(...\big) = \Omega$ and $\|\big(...\big)\| = \tau_0 \in \Omega$, and similarly for (B) with $\|\big(...\big)\| = \eta_k \in \Omega$, and the isolation condition of the theorem is also used.

Arguing in $\mathbf{L}$ and using the $\mathbf{Inv}(\Omega)$-isolation of τ_0, we get a sequence of permutations $\pi_k \in \mathbf{Inv}(\Omega) \cap \boldsymbol{\Pi}(\tau_0)$ by induction, satisfying $\sigma_k \cap \sigma_j = \tau_0$ whenever $k \neq j$, where $\sigma_k = \pi_k{\bullet}\eta_k \in \Omega$. Let $Y_k = \pi_k{\bullet}(X{\downarrow}\eta_k)$. Then

(5) $Y_k \Vdash_{\mathscr{X}} \big(\mathbf{L}(\underline{\mathbf{W}}_\Omega) \models \exists\, x \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\sigma_k]\, \varphi(\underline{\mathbf{W}}_\Omega, \underline{\boldsymbol{v}}{\downarrow}\tau_0, k, x)\big)$, $\forall\, k < \omega$.

holds by (4) by Corollary 25.4. Note that $Y_k \in \mathscr{X}{\downarrow}\sigma_k$ by 5° in Section 21.

Note that $\sigma = \bigcup_k \sigma_k \in \Omega$ by $(*)$ of the theorem. The sets Y_k satisfy $Y_k{\downarrow}\tau_0 = X{\downarrow}\tau_0$, $\forall\, k$, since $\pi_k \in \boldsymbol{\Pi}(\tau_0)$. Thus $Y = \bigcap_k (Y_k{\uparrow}\sigma) \in \mathscr{X}{\downarrow}\sigma$ by Lemma 21.1 (w.r.t. Lemma 12.2). As obviously $Y{\downarrow} \subseteq Y_k$, (5) implies:

$$Y \Vdash_{\mathscr{X}} \big(\mathbf{L}(\underline{\mathbf{W}}_\Omega) \models \forall\, k\, \exists\, x \in \mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\sigma]\, \varphi(\underline{\mathbf{W}}_\Omega, \underline{\boldsymbol{v}}{\downarrow}\tau_0, k, x)\big),$$

and hence (because any Y forces that $\mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\sigma]$ is Gödel-wellordered)

(6) $Y \Vdash_{\mathscr{X}} \big(\mathbf{L}(\underline{\mathbf{W}}_\Omega) \models \exists\, f\, \forall\, k < \omega\, \varphi(\underline{\mathbf{W}}_\Omega, \underline{\boldsymbol{v}}{\downarrow}\tau_0, k, f(k))\big)$.

To accomplish the proof of the theorem, we conclude that (6) contradicts to (3) because $Y{\downarrow}\tau_0 = X{\downarrow}\tau_0$ by construction. □

Corollary 29.2. *Under the assumptions of Theorem 29.1, suppose that* $(*)$ *of the theorem holds for all* $\tau_0 \in \Omega$*. Then* $\mathbf{AC}_\omega$ *holds in* $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$.

Proof. Every set $P \in \mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$, $P \subseteq \omega\times\omega^\omega$, belongs to $\mathrm{OD}(\mathbf{W}_\Omega[\boldsymbol{v}], \boldsymbol{v}{\downarrow}\tau_0)$ in $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$ for a suitable $\tau_0 \in \Omega$. □

A somewhat simpler set of properties leads to **DC** in classes of the form $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$, as the next theorem shows.

Theorem 29.3. *Assume that* $\mathscr{X} \in \mathbf{NF}$ *has the fusion property,* $\Omega \subseteq \boldsymbol{\Xi}$, $\Omega \in \mathbf{L}$ *is closed in* $\mathbf{L}$ *under countable unions, and* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *is* $\mathscr{X}$*-generic. Then* **DC** *holds in* $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$.

Proof. Let $\Delta = \bigcup \Omega$; then $\Delta \in \mathbf{L}$, $\Delta \subseteq \boldsymbol{I}$, $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}]) \subseteq \mathbf{L}[\boldsymbol{v}{\downarrow}\Delta]$. We claim that $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}]) \cap \omega^\omega = \mathbf{L}[\boldsymbol{v}{\downarrow}\Delta] \cap \omega^\omega$; this proves the theorem because the full **AC** holds in $\mathbf{L}[\boldsymbol{v}{\downarrow}\Delta]$. In the nontrivial direction, let $x \in \mathbf{L}[\boldsymbol{v}{\downarrow}\Delta] \cap \omega^\omega$. It follows by Theorem 27.1(iii) that there is a ($\mathbf{L}$-countable!) $\xi \in \boldsymbol{\Xi}$, $\xi \subseteq \Delta$ satisfying $x \in \mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$. But then $\xi \in \Omega$ as Ω is closed in $\mathbf{L}$ under countable unions. Therefore $x \in \mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$ as required. □

V Choiceless generic subextensions

Thus Chapter defines and studies generic models, of the form $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$, which will be used in the proof of Theorem 1.1. The forcing notion $\mathscr{X}$ is not yet defined, so our goal here will be to determine some key properties of $\mathscr{X}$-generic arrays (the definability and even extension properties defined below) that will eventually lead to Theorem 1.1.

30 Key sets $\boldsymbol{\Omega}_e$ and permutation groups $\boldsymbol{\Gamma}_e$

Classes of the form $\mathbf{L}(\mathbf{W}_\Omega[\boldsymbol{v}])$ will serve as models for different parts of our main theorem. Here $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ will be $\mathscr{X}$-generic over $\mathbf{L}$ for a special forcing $\mathscr{X} \in \mathbf{NF} \cap \mathbf{L}$, whereas $\Omega \in \mathbf{L}$ will be selected as special subsets of $\boldsymbol{\Xi}$.

First of all, we are going to define sets $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_4 \subseteq \boldsymbol{\Xi}$ is $\mathbf{L}$. This involves the notion of *even* and *odd* tuples in $\boldsymbol{I}$ as defined in Section 13.

Definition 30.1 (in $\mathbf{L}$)**.** If $\boldsymbol{i} \subseteq \boldsymbol{j}$ belong to $\boldsymbol{I}$ then $\boldsymbol{j}$ is an *odd expansion* of $\boldsymbol{i}$, in symbol $\boldsymbol{i} \subseteq_{\mathsf{odd}} \boldsymbol{j}$, iff $\boldsymbol{j}(k)$ is an odd ordinal for all $\mathtt{lh}(\boldsymbol{i}) \le k < \mathtt{lh}(\boldsymbol{j})$.

If $\xi, \eta \in \boldsymbol{\Xi}$ then ξ is an *odd expansion* of η, in symbol $\eta \subseteq_{\mathsf{odd}} \xi$, iff $\eta \subseteq \xi$ and in addition all tuples $\boldsymbol{i} \in \xi \setminus \eta$ are odd. Put:

$$\begin{aligned}
\xi[\alpha] &= \{\boldsymbol{i} \in \xi : \boldsymbol{i}(0) = \alpha\}, \text{ for any } \alpha < \omega_1,\ \xi \subseteq \boldsymbol{I} \text{ — the } \alpha\text{-}slice \text{ of } \xi, \\
&\quad \text{in particular } \boldsymbol{I}[\alpha] = \{\boldsymbol{i} \in \boldsymbol{I} : \boldsymbol{i}(0) = \alpha\}\,; \\
\boldsymbol{\Omega}_1 &= \{\tau \in \boldsymbol{\Xi} : \exists\, m\, \forall\, \boldsymbol{i} \in \tau\, (\boldsymbol{i} \text{ is even } \implies \mathtt{lh}(\boldsymbol{i}) \le m)\}; \\
\boldsymbol{\Omega}_3 &= \text{all } \tau \in \boldsymbol{\Xi} \text{ which contain no infinite paths } \boldsymbol{i}_0 \subset \boldsymbol{i}_1 \subset \boldsymbol{i}_2 \subset \dots \\
&\quad \text{of } \textbf{even} \text{ tuples } \boldsymbol{i}_k \in \tau; \\
\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_3 &= \boldsymbol{\Pi}, \text{ all parity-preserving and } \subset\text{-preserving } \pi : \boldsymbol{I} \overset{\text{onto}}{\longrightarrow} \boldsymbol{I}. \qquad \square
\end{aligned}$$

It takes more work to define $\boldsymbol{\Omega}_2$ and $\boldsymbol{\Omega}_4$. First of all, if $\alpha, \beta < \omega_1$ then define a *shift permutation* $\pi_{\alpha\beta} \in \boldsymbol{\Pi}$ such that if $\boldsymbol{i} \in \boldsymbol{I}$ then $\boldsymbol{j} = \pi_{\alpha\beta}(\boldsymbol{i})$ satisfies $\mathtt{lh}(\boldsymbol{j}) = \mathtt{lh}(\boldsymbol{i})$ and the following:

– if $\boldsymbol{i}(0) \notin \{\alpha, \beta\}$ then $\boldsymbol{j} = \boldsymbol{i}$;

– if $\boldsymbol{i}(0) = \alpha$ then $\boldsymbol{j}(0) = \beta$ and $\boldsymbol{j}(k) = \boldsymbol{i}(k)$ for all $0 < k < \mathtt{lh}(\boldsymbol{i})$;

– if $\boldsymbol{i}(0) = \beta$ then $\boldsymbol{j}(0) = \alpha$ and $\boldsymbol{j}(k) = \boldsymbol{i}(k)$ for all $0 < k < \mathtt{lh}(\boldsymbol{i})$.

Note that $\pi_{\alpha\beta} \in \mathbf{Perm}$, and even $\pi_{\alpha\beta} \in \boldsymbol{\Pi}$ in case α, β have equal parity.

A routine proof of the next lemma is left to the reader.

Lemma 30.2 (in $\mathbf{L}$)**.** *There is a sequence* $\langle \zeta_\alpha \rangle_{\alpha < \omega_1 \text{ successor}}$ *such that*:

(i) *if* $\alpha < \omega_1$ *is a successor ordinal then* $\zeta_\alpha \in \boldsymbol{\Xi}$ *and* $\zeta_\alpha \subseteq \boldsymbol{I}[\alpha]$;

(ii) *if* $\alpha, \lambda < \omega_1$, $\eta \in \boldsymbol{\Xi}$, $\eta \subseteq \boldsymbol{I}[\alpha]$, *then there is a successor* $\beta > \lambda$ *such that* $\pi_{\alpha\beta} \bullet \eta = \zeta_\beta$ *and the ordinals* α, β *have the same parity.* $\square$

We fix such a sequence of sets ζ_α in $\mathbf{L}$.

Definition 30.3 (in $\mathbf{L}$)**.** Put $\boldsymbol{\Omega}_2 =$, resp., $\boldsymbol{\Omega}_4 =$ all $\tau \in \boldsymbol{\Xi}$ such that:

(1) if $\alpha < \omega_1$ is a successor and $\tau[\alpha] \neq \varnothing$ then $\zeta_\alpha \subseteq_{\mathsf{odd}} \tau[\alpha]$;

(2) if $\alpha < \omega_1$ is limit then $\tau[\alpha] \in \boldsymbol{\Omega}_1$, resp., $\tau[\alpha] \in \boldsymbol{\Omega}_3$.

In addition, put $\boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_4 =$ all $\pi \in \boldsymbol{\Pi}$ such that

(3) if $\pi(\boldsymbol{i}) = \boldsymbol{j}$ and $\boldsymbol{i}(0)$ is limit then so is $\boldsymbol{j}(0)$, and

(4) if $\pi(\boldsymbol{i}) = \boldsymbol{j}$ and $\alpha = \boldsymbol{i}(0)$ is a successor then $\beta = \boldsymbol{j}(0)$ is a successor either, and $\pi \bullet \zeta_\alpha = \zeta_\beta$. $\square$

To conclude, sets $\boldsymbol{\Omega}_e = \boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_4 \subseteq \boldsymbol{\Xi}$ and associated groups $\boldsymbol{\Gamma}_e \subseteq \boldsymbol{\Pi}$ have been defined in $\mathbf{L}$, mainly via conditions related to **even** tuples $\boldsymbol{i} \in \xi \in \boldsymbol{\Omega}_e$, while giving **odd** tuples much more freedom.

Some **related sets** $\Omega \subseteq \boldsymbol{\Xi}$ will also be considered.

Definition 30.4 (in $\mathbf{L}$)**.** Let $\vartheta \in \boldsymbol{\Xi}$. We first put

$$\boldsymbol{\Omega}_1^\vartheta = \boldsymbol{\Omega}_3^\vartheta = \{\tau \in \boldsymbol{\Xi} : \vartheta \subseteq_{\mathsf{odd}} \tau\} \quad \text{and} \quad \boldsymbol{\Gamma}_1^\vartheta = \boldsymbol{\Gamma}_3^\vartheta = \boldsymbol{\Pi}(\vartheta).$$

To handle the $\{2,4\}$-case, we let $\zeta_\alpha^\vartheta = \zeta_\alpha$ if $\alpha < \omega_1$ is a successor ordinal, and $\zeta_\alpha^\vartheta = \vartheta[\alpha]$ if α is limit. Now we define:

$$\begin{aligned} \boldsymbol{\Omega}_2^\vartheta = \boldsymbol{\Omega}_4^\vartheta &= \{\tau \in \boldsymbol{\Xi} : \forall\, \alpha < \omega_1 \left(\tau[\alpha] \neq \varnothing \Longrightarrow \zeta_\alpha^\vartheta \subseteq_{\mathsf{odd}} \tau[\alpha]\right)\}, \\ \boldsymbol{\Gamma}_2^\vartheta = \boldsymbol{\Gamma}_4^\vartheta &= \{\pi \in \boldsymbol{\Pi} : \forall\, \alpha, \beta \left(\pi(\langle\alpha\rangle) = \langle\beta\rangle \Longrightarrow \zeta_\beta^\vartheta = \pi \bullet \zeta_\alpha^\vartheta\right)\}. \end{aligned}$$

Put
$$\begin{aligned} \boldsymbol{\Omega}_* &= \{\tau \in \boldsymbol{\Omega}_2 \text{ (equivalently, } \boldsymbol{\Omega}_4) : \forall\, \alpha\, (\alpha \text{ is limit} \Longrightarrow \tau[\alpha] = \varnothing)\}, \\ \boldsymbol{\Gamma}_* &= \boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_4. \end{aligned}$$
$\square$

31 Invariance, isolation and other results

Recall Definition 26.1 on isolation. Recall that $\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_3 = \boldsymbol{\Pi}$.

Theorem 31.1 (in $\mathbf{L}$, summary)**.** *Let* $e = 1, 2, 3, 4$ *and* $\vartheta \in \boldsymbol{\Omega}_e$. *Then*

(i) $\boldsymbol{\Omega}_1 \subseteq \boldsymbol{\Omega}_3$, $\boldsymbol{\Omega}_2 \subseteq \boldsymbol{\Omega}_4$, $\bigcup \boldsymbol{\Omega}_1 = \bigcup \boldsymbol{\Omega}_3 = \boldsymbol{I}$, *whereas* $\bigcup \boldsymbol{\Omega}_2 = \bigcup \boldsymbol{\Omega}_4 = \{\boldsymbol{i} \in \boldsymbol{I} : \alpha = \boldsymbol{i}(0) \text{ is a successor} \Longrightarrow \zeta_\alpha \subseteq_{\mathsf{odd}} \zeta_\alpha \cup [\subseteq \boldsymbol{i}]\}$;

(ii) *if* $e = 1, 3$ *and* $\tau \in \boldsymbol{\Xi}$, $\tau \subseteq \eta \in \boldsymbol{\Omega}_e$, *then* $\tau \in \boldsymbol{\Omega}_e$ (false for $e = 2, 4$);

(iii) *$\vartheta \in \mathbf{\Omega}_e^\vartheta \subseteq \mathbf{\Omega}_e$, and if $\vartheta \in \mathbf{\Omega}_4$ then $\mathbf{\Omega}_* \subseteq \mathbf{\Omega}_2^\vartheta = \mathbf{\Omega}_4^\vartheta \subseteq \mathbf{\Omega}_2 \subseteq \mathbf{\Omega}_4$;*

(iv) *if ξ and $\eta \subseteq_{\mathsf{odd}} \tau$ belong to $\mathbf{\Xi}$, then $\eta \in \mathbf{\Omega}_e \Longrightarrow \tau \in \mathbf{\Omega}_e$, and $\eta \in \mathbf{\Omega}_e^\xi \Longrightarrow \tau \in \mathbf{\Omega}_e^\xi$;*

(v) *the sets $\mathbf{\Omega}_e$ are closed under finite unions, whereas $\mathbf{\Omega}_*$, $\mathbf{\Omega}_e^\vartheta$ are closed under countable unions* (obvious);

(vi) *$\mathbf{\Omega}_e$ is $\mathbf{\Gamma}_e$-invariant, $\mathbf{\Omega}_e^\vartheta$ is $\mathbf{\Gamma}_e^\vartheta$-invariant, $\mathbf{\Omega}_*$ is $\mathbf{\Gamma}_*$-invariant.*

(vii) *$\mathbf{\Omega}_2$ satisfies $(*)$ of Theorem 29.1 in case $\tau_0 = \varnothing$;*

(viii) *the sets $\mathbf{\Omega}_3, \mathbf{\Omega}_4$ satisfy $(*)$ of Thm 29.1 for all $\tau_0 \in \mathbf{\Omega}_3$, resp. $\tau_0 \in \mathbf{\Omega}_4$;*

(ix) *if $\xi \in \mathbf{\Omega}_*$, and $\tau \in \mathbf{\Xi}$ satisfies (1) of Definition 30.3, then there is a permutation $\pi \in \mathbf{\Pi}(\xi)$ such that $\pi{\bullet}\tau \in \mathbf{\Omega}_*$;*

(x) *if $e = 2, 4$, $\xi \in \mathbf{\Omega}_e$, $\tau \in \mathbf{\Omega}_e^\xi$, then there is a permutation $\pi \in \mathbf{\Gamma}_e^\xi$ such that $\sigma = \pi{\bullet}\tau \in \mathbf{\Omega}_*$ and $\sigma \cap \tau = \varnothing$;*

(xi) *$\mathbf{\Gamma}_e \subseteq \mathbf{Inv}(\mathbf{\Omega}_e)$, each $\tau \in \mathbf{\Omega}_e$ is $\mathbf{\Gamma}_e$-isolated;*

(xii) *$\mathbf{\Gamma}_e^\vartheta \subseteq \mathbf{Inv}(\mathbf{\Omega}_e^\vartheta)$, each $\tau \in \mathbf{\Omega}_e^\vartheta$ is $\mathbf{\Gamma}_e^\vartheta$-isolated;*

(xiii) *$\mathbf{\Gamma}_* \subseteq \mathbf{\Gamma}_2^\vartheta = \mathbf{\Gamma}_4^\vartheta$, each $\tau \in \mathbf{\Omega}_*$ is $\mathbf{\Gamma}_*$-isolated.* □

Proof (in $\mathbf{L}$). Claims (i), (ii), (iii), (iv), (v), (vi) are pretty routine.

(vii) Assume that sets $\sigma_k \in \mathbf{\Omega}_2$ are pairwise disjoint. Then $\sigma = \bigcup_k \sigma_k \in \mathbf{\Xi}$. Let $\alpha < \omega_1$ be limit. Then $\sigma[\alpha] = \sigma_k[\alpha]$ for some k by the disjointness condition. Thus $\sigma[\alpha] \in \mathbf{\Omega}_1$, as required.

(viii) Assume that $\tau_0 \in \mathbf{\Omega}_3$ and sets $\sigma_k \in \mathbf{\Omega}_2$ satisfy (*) $\sigma_k \cap \sigma_\ell = \tau_0$ for all $k \neq \ell$. Then any $\subset$-increasing sequence in $\sigma = \bigcup_k \sigma_k$ entirely belongs to one of σ_k, hence it cannot be infinite.

(ix) We can w.l.o.g. assume that $\xi \subseteq \tau$ (otherwise replace τ by $\xi \cup \tau$). Let $T = \{\boldsymbol{i}(0) : \boldsymbol{i} \in \tau\}$ and $\mu = \sup T$. If $\alpha \in T_0 = \{\alpha' \in T : \alpha' \text{ is limit}\}$ then by (ii) of Lemma 30.2 there is a countable successor ordinal $\beta(\alpha) > \mu$, of the same parity as α, such that $\pi_{\alpha,\beta(\alpha)}{\bullet}\tau[\alpha] = \zeta_{\beta(\alpha)}$. We can choose these ordinals $\beta(\alpha)$ so that $\alpha \neq \alpha' \Longrightarrow \beta(\alpha) \neq \beta(\alpha')$ for all $\alpha \in T_0$. This allows to define $\pi \in \mathbf{\Pi}$ as follows:

$$\pi(\boldsymbol{i}) = \begin{cases} \boldsymbol{i} & \text{, in case } \ \boldsymbol{i}(0) \notin T_0 \cup \{\beta(\alpha) : \alpha \in T_0\}; \\ \pi_{\alpha,\beta(\alpha)}(\boldsymbol{i}) & \text{, in case } \ \boldsymbol{i}(0) \in T_0 \cup \{\beta(\alpha) : \alpha \in T_0\}. \end{cases} \tag{1}$$

Note that $\pi \in \mathbf{\Pi}(\xi)$: if $\boldsymbol{i} \in \xi$ then $\boldsymbol{i}(0)$ is a successor because $\xi \in \mathbf{\Omega}_*$, and hence $\boldsymbol{i}(0) \notin T_0 \cup \{\beta(\alpha) : \alpha \in T_0\}$ by construction, and $\pi(\boldsymbol{i}) = \boldsymbol{i}$.

It remains to check that $\sigma = \pi{\bullet}\tau \in \mathbf{\Omega}_*$. Let $\beta < \omega_1$ and $\sigma[\beta] \neq \varnothing$.

Case 1: $\beta = \beta(\alpha)$ for some $\alpha \in T_0$. Then $\sigma[\beta] = \pi_{\alpha\beta}\bullet\tau[\alpha] = \zeta_\beta = \zeta_\beta^{\varnothing}$ by construction.

Case 2: $\beta \in T \smallsetminus T_0$, hence β is a successor. Then $\sigma[\beta] = \tau[\beta]$ by construction. Therefore $\zeta_\beta = \zeta_\beta^{\varnothing} \subseteq_{\mathsf{ev}} \sigma[\beta]$, as $\tau \in \mathbf{\Omega}_* = \mathbf{\Omega}_2^{\varnothing}$.

Combining the results in two cases, we get $\sigma \in \mathbf{\Omega}_*$.

(x) The proof is rather similar. Assuming that $\xi \subseteq \tau$ as above, we pick, for each $\alpha \in T = \{\boldsymbol{i}(0) : \boldsymbol{i} \in \tau\}$, a successor ordinal $\beta(\alpha) > \mu = \sup T$, of the same parity as α, such that $\pi_{\alpha,\beta(\alpha)}\bullet\tau[\alpha] = \zeta_{\beta(\alpha)}$. Choose $\beta(\alpha)$ so that $\alpha < \alpha' \implies \beta(\alpha) < \beta(\alpha')$. Define $\pi \in \mathbf{\Pi}$ as follows:

$$\pi(\boldsymbol{i}) = \begin{cases} \boldsymbol{i} & , \text{ in case } \quad \boldsymbol{i}(0) \notin T \cup \{\beta(\alpha) : \alpha \in T\}\,; \\ \pi_{\alpha,\beta(\alpha)}(\boldsymbol{i}) & , \text{ in case } \quad \boldsymbol{i}(0) \in T \cup \{\beta(\alpha) : \alpha \in T\}\,. \end{cases} \qquad (2)$$

(xi) To prove the isolation claim, let $\lambda < \omega_1$ be a limit ordinal $>$ all ordinals $\boldsymbol{j}(k)$, where $\boldsymbol{j} \in \xi$ and $k < \mathtt{lh}(\boldsymbol{j})$. To handle **the case** $e = 1, 3$, recall that each $\eta \in \mathbf{\Xi}$ is $\mathbf{\Pi}$-isolated by Lemma 26.2.

To handle **the case** $e = 2, 4$, prove that each $\eta \in \mathbf{\Xi}$, satisfying (1) of Definition 30.3, is $\mathbf{\Gamma}_2$-isolated. Let $\eta \subseteq \xi \in \mathbf{\Xi}$; let's define $\pi \in \mathbf{\Pi}(\xi) \cap \mathbf{\Gamma}_2$ satisfying $\xi \cap (\pi\bullet\xi) = \eta$. Splitting $\boldsymbol{I}$ into the limit and successot parts

$$\boldsymbol{I}_0 = \{\boldsymbol{i} \in \boldsymbol{I} : \boldsymbol{i}(0) \text{ is limit}\} \quad \text{and} \quad \boldsymbol{I}_1 = \{\boldsymbol{i} \in \boldsymbol{I} : \boldsymbol{i}(0) \text{ is a successor}\},$$

we accordingly put $\eta_e = \eta \cap \boldsymbol{I}_e \subseteq \xi_e = \xi \cap \boldsymbol{I}_e$, $e = 0, 1$, define permutations π_e of the domains $\boldsymbol{I}_e$ separately, and put $\pi = \pi_0 \cup \pi_1$ at the end.

Part 1. We leave it to the reader to define $\pi_0 : \boldsymbol{I}_0 \xrightarrow{\text{onto}} \boldsymbol{I}_0$ with $\pi_0 \restriction \eta_0 =$ the identity and $\xi_0 \cap (\pi_0\bullet\xi_0) = \eta_0$, following the proof of Lemma 26.2.

Part 2. We now concentrate on the construction of $\pi_1 : \boldsymbol{I}_1 \xrightarrow{\text{onto}} \boldsymbol{I}_1$.

If $\boldsymbol{i} \in \eta_1$ then put $\pi_1(\boldsymbol{i}) = \boldsymbol{i}$. **Now let** $\boldsymbol{i} \in \boldsymbol{I}_1 \smallsetminus \eta_1$. Consider the sets

$$A_1 = \{\boldsymbol{j}(0) : \boldsymbol{j} \in \eta_1\} \;\subseteq\; B_1 = \{\boldsymbol{j}(0) : \boldsymbol{j} \in \xi_1\} \;\subseteq\; \{\alpha < \omega_1 : \alpha \text{ successor}\}.$$

Following the proof of (ix) above, if $\alpha < \omega_1$ is a successor then by (ii) of Lemma 30.2 there is a successor $\beta(\alpha) > \lambda$, of the same parity as α, such that $\pi_{\alpha,\beta(\alpha)}\bullet\zeta_\alpha = \zeta_{\beta(\alpha)}$. We can choose these ordinals $\beta(\alpha)$ so that $\alpha < \alpha' \implies \beta(\alpha) < \beta(\alpha')$. Now, if $\boldsymbol{i} \in \boldsymbol{I}_1$ but $\boldsymbol{i}(0) \notin A_1$ then put

$$\pi(\boldsymbol{i}) = \begin{cases} \boldsymbol{i} & , \text{ if } \quad \boldsymbol{i}(0) \notin B_1 \cup \{\beta(\alpha) : \alpha \in (B_1 \smallsetminus A_1)\}\,; \\ \pi_{\alpha,\beta(\alpha)}(\boldsymbol{i}) & , \text{ if } \quad \boldsymbol{i}(0) \in (B_1 \smallsetminus A_1) \cup \{\beta(\alpha) : \alpha \in (B_1 \smallsetminus A_1)\}; \end{cases} \qquad (3)$$

following the idea of (1), (2) above.

Part 3. We finally define π_1 on the domain $\boldsymbol{I}_1' = \{\boldsymbol{i} \in \boldsymbol{I}_1 : \boldsymbol{i}(0) \in A_1\}$. Note that if $\alpha \in A_1$ then $\zeta_\alpha \subseteq \eta$ since η satisfies (1) of Definition 30.3.

If $\boldsymbol{i} \in \eta$ then $\pi(\boldsymbol{i}) = \boldsymbol{i}$, see above Part 2. Now suppose that $\boldsymbol{i} \in \boldsymbol{I}_1' \smallsetminus \eta$. Define $m_{\boldsymbol{i}} < \mathtt{lh}(\boldsymbol{i})$ as in the case $e = 1, 3$ above and define $\pi(\boldsymbol{i})$ as in the proof of Lemma 26.2.

Finalization. Combining the construction in Parts 1, 2, 3, we get the a transformation $\pi \in \mathbf{\Pi}(\xi) \cup \mathbf{\Gamma}_2$ that proves the result in case $e = 2, 4$.

(xii) The proof is pretty similar to Part 2 in the proof of (ix) in case $e = 2, 4$, and we left it to the reader.

(xiii) The isolation claim is case $\xi = \varnothing$ of (xii). □

We are able now to establish the following theorem.

Theorem 31.2. *Assume that $\mathscr{X} \in \mathbf{NF}$ has the fusion property, and $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic. Then:*

(i) $\mathbf{AC}_\omega(\mathrm{OD})$ *holds in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$;

(ii) *full* $\mathbf{AC}_\omega$ *holds in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$ *and in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_4}[\boldsymbol{v}])$;

(iii) *full* $\mathbf{DC}$ *holds in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e^\eta}[\boldsymbol{v}])$ *for any* $e = 1, 2, 3, 4$ *and* $\eta \in \mathbf{\Omega}_e$.

Proof. (i) We are going to apply Theorem 29.1, therefore it suffices to check its premices for $\mathbf{\Omega}_2$. We know that each $\eta \in \mathbf{\Omega}_2$ is $\mathbf{\Gamma}_2$-isolated by Theorem 31.1(xi). On the other hand, we know that $\mathbf{\Gamma}_2 \subseteq \mathbf{\Gamma}_1 = \mathbf{\Pi}$, and we have $\mathbf{\Gamma}_2 \subseteq \mathbf{Inv}(\mathbf{\Omega}_2)$ since $\mathbf{\Omega}_2$ is $\mathbf{\Gamma}_2$-invariant by Theorem 31.1(vi). This proves the isolation condition of Theorem 29.1. Moreover, $\mathbf{\Omega}_2$ satisfies $(*)$ of Theorem 29.1 in case $\tau_0 = \varnothing$ by Theorem 31.1(vii). It remains to apply Theorem 29.1.

(ii) Essentially the same argument, but with (viii) of Theorem 31.1 instead of (vii). In addition, note that each real in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$ belongs to $\mathbf{L}[\boldsymbol{v}{\downarrow}\tau]$ for some $\tau \in \mathbf{\Omega}_e$ by Corollary 26.7.

(iii) Reference to Theorem 31.1(v) and Theorem 29.3. □

32 Definability property and violation of Choice

The next definition introduces a condition leading to level-dependent violations of some forms of countable Choice in the generic models considered.

Definition 32.1. Let $n < \omega$. Say that $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ has the (n)-*definability property*, if and only if

(I) for all $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}$, $\boldsymbol{v}(\boldsymbol{i}) \in \mathbf{L}[\boldsymbol{v}(\boldsymbol{j})]$ iff $\boldsymbol{i} \subseteq \boldsymbol{j}$;

(II) if $\mathfrak{M} \subseteq \mathbf{L}[\boldsymbol{v}]$ is a transitive class closed under pairs, and $\mathbf{L}[x] \subseteq \mathfrak{M}$ for all $x \in \mathfrak{M}$, then $\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) \cap \mathfrak{M}$, $\mathbf{E}^{\mathtt{odd}}(\boldsymbol{v}) \cap \mathfrak{M}$ are $\varPi^1_{n+1}$ over $\mathfrak{M}$, where

$$\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) = \{\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle : k \geq 1 \wedge \boldsymbol{i} \in \boldsymbol{I} \text{ is even } \wedge \mathtt{lh}(\boldsymbol{i}) = k\},$$
$$\mathbf{E}^{\mathtt{odd}}(\boldsymbol{v}) = \{\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle : k \geq 1 \wedge \boldsymbol{i} \in \boldsymbol{I} \text{ is odd } \wedge \mathtt{lh}(\boldsymbol{i}) = k\}.$$

A forcing notion $\mathscr{X} \in \mathbf{NF}$ has the (n)-*definability property*, if ($\mathscr{X}$ forces over $\mathbf{L}$ that) each $\mathscr{X}$-generic $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ has the (n)-definability property. □

Remark 32.2. The class $\mathfrak{M}$ is **not** assumed to satisfy **ZF**, and the sets $\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) \cap \mathfrak{M}$ and $\mathbf{E}^{\mathtt{odd}}(\boldsymbol{v}) \cap \mathfrak{M}$ are **not** claimed to belong to $\mathfrak{M}$ in (II). In fact, the proof of Theorem 1.1 below will be related to the case when $\mathfrak{M}$ does satisfy **ZF** and accordingly the sets $\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) \cap \mathfrak{M}$ and $\mathbf{E}^{\mathtt{odd}}(\boldsymbol{v}) \cap \mathfrak{M}$ do belong to $\mathfrak{M}$. However the proof of Theorem 1.2 in Chapter XI really involves the case when $\mathfrak{M}$ is not a **ZF**-class, and in fact the sets $\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) \cap \mathfrak{M}$ and $\mathbf{E}^{\mathtt{odd}}(\boldsymbol{v}) \cap \mathfrak{M}$ will not belong to $\mathfrak{M}$ in that case. □

The construction of forcings $\mathscr{X} \in \mathbf{NF}$ with the (n)-definability property is quite a difficult task. Below, a method will be elaborated for such a construction for a given n. (Note in brackets that, for example, **IPS**-generic arrays $\boldsymbol{v}$ do not have the (n)-definability property for any n.)

Theorem 32.3. *Assume that $n \geq 1$, $\mathscr{X} \in \mathbf{NF}$, and $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is $\mathscr{X}$-generic and has the (n)-definability property. Then:*

(i) $\mathbf{AC}_\omega(\varPi^1_{n+1})$ *fails in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$, (iii) $\mathbf{DC}(\varPi^1_{n+1})$ *fails in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$,

(ii) $\mathbf{AC}_\omega(\mathbf{\Pi}^1_{n+1})$ *fails in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$, (iv) $\mathbf{DC}(\mathbf{\Pi}^1_{n+1})$ *fails in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_4}[\boldsymbol{v}])$.

Note the difference between the lightface and boldface classes.

Proof. We'll make use of the following *key sets* as counterexamples:

$$P_1 = \{\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle : k \geq 1 \wedge \boldsymbol{i} \in \boldsymbol{I} \text{ is even } \wedge \mathtt{lh}(\boldsymbol{i}) = k\},$$
$$P_2 = \{\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle : k \geq 1 \wedge \boldsymbol{i} \in \boldsymbol{I} \text{ is even } \wedge \mathtt{lh}(\boldsymbol{i}) = k \wedge \boldsymbol{i}(0) = 0\},$$
$$P_3 = \{\langle \boldsymbol{v}(\boldsymbol{i}), \boldsymbol{v}(\boldsymbol{j})\rangle : \boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I} \text{ are even } \wedge \boldsymbol{i} \subset \boldsymbol{j}\},$$
$$P_4 = \{\langle \boldsymbol{v}(\boldsymbol{i}), \boldsymbol{v}(\boldsymbol{j})\rangle : \boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I} \text{ are even } \wedge \boldsymbol{i} \subset \boldsymbol{j} \wedge \boldsymbol{i}(0) = 0\}. \quad \text{or } =1\text{ ?}$$

Lemma 32.4. *Let $e = 1, 2, 3, 4$. Then $P_e \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$ and:*

(a) *P_1 is $\varPi^1_{n+1}$ in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$;*

(b) *P_2 is $\mathbf{\Pi}^1_{n+1} \wedge \mathbf{\Sigma}^1_2$ in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$, hence just $\mathbf{\Pi}^1_{n+1}$ in case $n \geq 2$;*

(c) P_3 *is* $\mathit{\Pi}^1_{n+1} \land \mathit{\Sigma}^1_2$ *in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$*, hence just* $\mathit{\Pi}^1_{n+1}$ *in case* $n \ge 2$;

(d) P_4 *is* $\mathbf{\Pi}^1_{n+1} \land \mathbf{\Sigma}^1_2$ *in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_4}[\boldsymbol{v}])$*, hence just* $\mathbf{\Pi}^1_{n+1}$ *in case* $n \ge 2$.

By $\mathbf{\Pi}^1_{n+1} \land \mathbf{\Sigma}^1_2$ in (b) and (d) we mean the definability by a conjunction of a $\mathbf{\Pi}^1_{n+1}$ formula and a $\mathbf{\Sigma}^1_2$ formula with real parameters, and $\mathit{\Pi}^1_{n+1} \land \mathit{\Sigma}^1_2$ in (c) is understood similarly (no parameters).

Proof (Lemma)**.** If $e = 1, 2, 3, 4$ then define $\mathbf{S}_e := \mathbf{E}^{\mathtt{evn}}(\boldsymbol{v})$ (that is, $\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v})$ as in Definition 32.1 with $\mathfrak{M} = \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$), and

$$\mathbf{S}^0_e = \{\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle \in \mathbf{S}_e : \boldsymbol{i}(0) = 0\} = \{\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle \in \mathbf{S}_e : \boldsymbol{v}(\langle 0\rangle) \in \mathbf{L}[\boldsymbol{v}(\boldsymbol{i})]\}$$

(the equality holds by (I) of Definition 32.1). We may note that $\bigcup \mathbf{\Omega}_1 = \bigcup \mathbf{\Omega}_3 = \boldsymbol{I}$, whereas $\boldsymbol{I}[0] \subseteq \bigcup \mathbf{\Omega}_2 = \bigcup \mathbf{\Omega}_4 \subsetneqq \boldsymbol{I}$ by Theorem 31.1(i). It follows that $\boldsymbol{v}(\boldsymbol{i}) \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$ for all $\boldsymbol{i} \in \boldsymbol{I}$ in case $e = 1, 3$, whereas $\boldsymbol{v}(\boldsymbol{i}) \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$ for $e = 2, 4$ provided $\boldsymbol{i}(0) = 0$. Therefore, by (II) of Definition 32.1, $\mathbf{S}_e$ is $\mathit{\Pi}^1_{n+1}$ in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$ for $e = 1, 3$, but $\mathbf{S}^0_e$ is $\mathbf{\Pi}^1_{n+1}$ in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$ (with $p = \boldsymbol{v}(\langle 0\rangle) \in \mathscr{D}$ as the only parameter) in case $n \ge 2$, and is $\mathbf{\Pi}^1_{n+1} \land \mathbf{\Sigma}^1_2$ in case $n = 1$ because "$x \in \mathbf{L}[y]$" is a $\mathit{\Sigma}^1_2$ formula.

(a) We immediately conclude that $P_1 = \mathbf{S}_1$ is $\mathit{\Pi}^1_{n+1}$ in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$.

(b) Similarly $P_2 = \mathbf{S}^0_2$ is $\mathbf{\Pi}^1_{n+1} \land \mathbf{\Sigma}^1_2$ in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$.

(c) Using (I) of Definition 32.1, we observe that

$$P_3 = \{\langle x, y\rangle : \exists\, k < \ell\, (\langle k, x\rangle \in P_1 \land \langle \ell, y\rangle \in P_1 \land x \in \mathbf{L}[y] \land y \notin \mathbf{L}[x]\}.$$

Thus P_3 is $\mathit{\Pi}^1_{n+1} \land \mathit{\Sigma}^1_2$ in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$.

(d) follows from (c) similarly to (a) $\Longrightarrow$ (b). □ (Lemma)

In continuation of the proof of the theorem, we prove another lemma.

Lemma 32.5 (premices)**.** *The premices of the choice principles hold*:

$$\mathtt{dom}\, P_1 = \omega \smallsetminus \{0\}, \qquad \mathtt{dom}\, P_2 = \omega \smallsetminus \{0, 1\},$$
$$\mathtt{ran}\, P_3 \subseteq \mathtt{dom}\, P_3, \qquad \mathtt{ran}\, P_4 \subseteq \mathtt{dom}\, P_4.$$

Proof (Lemma)**.** Assume that $k \ge 1$. Let $\boldsymbol{i} = \langle 1, 1, \dots, 1\rangle$ (k terms equal to 1). Then $\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle \in P_1$, hence $k \in \mathtt{dom}\, P_1$.

If $k \ge 2$ and $\boldsymbol{i} = \langle 0, 1, 1, \dots, 1\rangle$ (0 and $k-1$ terms equal to 1), then $\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle \in P_2$, hence $k \in \mathtt{dom}\, P_2$.

Similarly, $\mathtt{ran}\, P_3 = \{\boldsymbol{v}(\boldsymbol{i}) : \boldsymbol{i} \in \boldsymbol{I} \land \mathtt{lh}(\boldsymbol{i}) \ge 2\} \subseteq \mathtt{dom}\, P_3 = \{\boldsymbol{v}(\boldsymbol{i}) : \boldsymbol{i} \in \boldsymbol{I}\}$.

Finally, we have $\mathtt{ran}\, P_4 = \{\boldsymbol{v}(\boldsymbol{i}) : \boldsymbol{i} \in \boldsymbol{I} \land \mathtt{lh}(\boldsymbol{i}) \ge 2 \land \boldsymbol{i}(0) = 0\}$, whereas $\mathtt{dom}\, P_4 = \{\boldsymbol{v}(\boldsymbol{i}) : \boldsymbol{i} \in \boldsymbol{I} \land \boldsymbol{i}(0) = 0\}$. □ (Lemma)

Coming back to Theorem 32.3, we finally show that the choice functions required do not exist is the corresponding models.

(i) We claim that there is no function $f \in \mathbf{L}(\mathbf{W}_{\boldsymbol{\Omega}_1}[\boldsymbol{v}])$ satisfying the formula $\langle k, f(k)\rangle \in P_1$ for all $k \geq 1$. Indeed suppose to the contrary that f is such a function. Corollary 26.7 implies $f \in \mathbf{L}[\boldsymbol{v}\!\downarrow\!\eta]$ for some $\eta \in \boldsymbol{\Omega}_1$. If $k \geq 1$ then by definition $f(k) = \boldsymbol{v}(\boldsymbol{i}_k)$ for some even $\boldsymbol{i}_k \in \boldsymbol{I}$ with $\mathtt{lh}(\boldsymbol{i}_k) = k$, and we have $\boldsymbol{i}_k \in \eta$ by Corollary 26.4. On the other hand, by definition there is $m < \omega$ such that $\mathtt{lh}(\boldsymbol{i}) \leq m$ for all even $\boldsymbol{i} \in \eta$, in particular, $\mathtt{lh}(\boldsymbol{i}_k) \leq m$ for all k, which contradicts the above.

To conclude, P_1 witnesses that $\mathbf{AC}_\omega(\mathit{\Pi}^1_{n+1})$ fails in $\mathbf{L}(\mathbf{W}_{\boldsymbol{\Omega}_1}[\boldsymbol{v}])$, because $\mathtt{dom}\, P_1 = \omega \smallsetminus \{0\}$ by Lemma 32.5.

(ii) A very similar argument shows that $\mathbf{AC}_\omega(\boldsymbol{\Delta}^1_{n+2})$ fails in $\mathbf{L}(\mathbf{W}_{\boldsymbol{\Omega}_2}[\boldsymbol{v}])$ via P_2. The failure of $\mathbf{AC}_\omega(\boldsymbol{\Pi}^1_{n+1})$ then follows by Lemma 2.2(iii).

(iii) We claim that no function $f \in \mathbf{L}(\mathbf{W}_{\boldsymbol{\Omega}_3}[\boldsymbol{v}])$ satisfies $\langle f(k), f(k{+}1)\rangle \in P_3$ for all k. Indeed otherwise such a function f belongs to $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\eta]$ for some $\eta \in \boldsymbol{\Omega}_3$, by Corollary 26.7. If $k < \omega$ then by definition $f(k) = \boldsymbol{v}(\boldsymbol{i}_k)$ and $f(k+1) = \boldsymbol{v}(\boldsymbol{i}_{k+1})$ for some even $\boldsymbol{i}_k, \boldsymbol{i}_{k+1} \in \eta$ with $\boldsymbol{i}_k \subset \boldsymbol{i}_{k+1}$, by Corollary 26.4. In other words, the set $\eta' = \{\boldsymbol{i} \in \eta : \boldsymbol{i} \text{ is even}\} \in \mathbf{L}$ is $\subset$-ill-founded in $\mathbf{L}(\mathbf{W}_{\boldsymbol{\Omega}_3}[\boldsymbol{v}])$. Then η' is ill-founded in $\mathbf{L}$ as well, which contradicts the definition of $\boldsymbol{\Omega}_3$.

Thus P_3 witnesses the failure of $\mathbf{DC}^*(\mathit{\Delta}^1_{n+2})$ in $\mathbf{L}(\mathbf{W}_{\boldsymbol{\Omega}_3}[\boldsymbol{v}])$, because $\mathtt{ran}\, P_3 \subseteq \mathtt{dom}\, P_3$ by Lemma 32.5. Lemma 2.2(iv) helps to improve this to the failure of $\mathbf{DC}(\mathit{\Pi}^1_{n+1})$.

(iv) The same argument with P_4. □ (Theorem 32.3)

33 Odd expansion property

Recall Theorem 31.1(iv), and the notion of *odd expansion* $\subseteq_{\mathsf{odd}}$ of Definition 30.1.

Definition 33.1. Let $n < \omega$. Say that $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ has the (n)-*odd-expansion*, or (n)-oe, property, iff for every $\eta \in \boldsymbol{\Xi}$ and a $\mathit{\Pi}^1_n$ formula $\varphi(\cdot)$, with reals in $\mathbf{L}[\boldsymbol{v}\!\downarrow\!\eta]$ as parameters, if $\exists\, x\, \varphi(x)$ is true in $\mathbf{L}[\boldsymbol{v}]$ then there is an odd expansion $\tau \in \boldsymbol{\Xi}$ of η and some $x \in \mathbf{L}[\boldsymbol{v}\!\downarrow\!\tau]$ such that $\mathbf{L}[\boldsymbol{v}] \models \varphi(x)$.

A forcing notion $\mathscr{X} \in \mathbf{NF}$ has the (n)-*oe property*, if ($\mathscr{X}$ forces over $\mathbf{L}$ that) each $\mathscr{X}$-generic array $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ has the (n)-oe property. □

This property is used through the following lemma.

Lemma 33.2. *Suppose that* $n < \omega$, $e = 1, 2, 3, 4$, *and* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *has the* (n)*-odd-expansion property. Then*

(i) $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$ *is an elementary submodel of* $\mathbf{L}[\boldsymbol{v}]$ *w.r.t. all* Σ^1_{n+1} *formulas with parameters in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e}[\boldsymbol{v}])$, *and*

(ii) *if* $\xi \in \mathbf{\Omega}_e$ *then* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e^\xi}[\boldsymbol{v}])$ *is an elementary submodel of* $\mathbf{L}[\boldsymbol{v}]$ *w.r.t. all* Σ^1_{n+1} *formulas with parameters in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_e^\xi}[\boldsymbol{v}])$.

Proof (sketch)**.** For Σ^1_2 formulas apply the Shoenfield absoluteness. The step is carried out straightforwardly using Lemma 31.1(iv). □

Remark 33.3. If $n = 1$ then (n)-odd-expansion property and Lemma 33.2 definitely hold for any $\boldsymbol{v}$ by the Shoenfield absoluteness theorem [55]. □

Now let's infer come corollaries.

Theorem 33.4. *Assume that* $\mathscr{X} \in \mathbf{NF}$ *has the fusion property,* $n \geq 1$, *and* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *is* $\mathscr{X}$*-generic and has the* (n)*-oe property. Then*

(i) $\mathbf{DC}(\mathbf{\Pi}^1_n)$ *holds in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$ *and in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$,

(ii) $\mathbf{DC}(\Pi^1_{n+1})$ (lightface!) *holds in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$ *and in* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_4}[\boldsymbol{v}])$.

Proof. (i) Consider a Π^1_n formula $\varphi(x, y)$ such that

(∗) $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}]) \models \forall\, x\, \exists\, y\, \varphi(x, y)$,

and with parameters in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$. Let $x_0 \in \omega^\omega \cap \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$. There is $\xi \in \mathbf{\Omega}_1$ such that x_0 and all parameters in φ belong to $\mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$. Consider the submodel $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1^\xi}[\boldsymbol{v}]) \subseteq \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$. Thus $\xi \in \mathbf{\Omega}_1^\xi$, and hence x_0 and all parameters in φ belong to $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1^\xi}[\boldsymbol{v}])$. However

(†) $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1^\xi}[\boldsymbol{v}])$ is an elementary submodel of $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$ w.r.t. all Σ^1_{n+1} formulas with reals in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1^\xi}[\boldsymbol{v}])$ as parameters, by Lemma 33.2.

Therefore $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1^\xi}[\boldsymbol{v}]) \models \forall\, x\, \exists\, y\, \varphi(x, y)$ by (∗). Moreover, $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1^\xi}[\boldsymbol{v}]) \models \mathbf{DC}$ by Theorem 31.2(iii). This allows to define a sequence $\langle x_k \rangle_{k<\omega} \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1^\xi}[\boldsymbol{v}])$ of reals, beginning with the x_0 given above, and satisfying $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1^\xi}[\boldsymbol{v}]) \models \varphi(x_k, x_{k+1})$, $\forall\, k$. It remains to refer to (†) in order to return to $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$.

The proof for $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$ is pretty similar.

(ii) This part involves trickier arguments contained in two lemmas.

Lemma 33.5. *Assume that* $\xi \in \mathbf{\Omega}_2$, $\varphi(y)$ *is a parameter-free* Σ^1_∞ *formula, and* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}]) \models \exists\, y\, \varphi(y)$. *Then there is* $y \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_*}[\boldsymbol{v}])$ *such that* $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}]) \models \varphi(y)$. *The same for* $\mathbf{\Omega}_4$ *and* $\mathbf{\Omega}_4^\xi$.

Proof (Lemma)**.** The $\mathscr{L}$-formula

$$\chi(U) := \exists\, y \in \omega^\omega \cap \mathbf{L}(U)\,\big(y \in \underline{\mathbf{W}}_{\mathbf{\Omega}_2^\xi} \wedge \mathbf{L}(\underline{\mathbf{W}}_{\mathbf{\Omega}_2^\xi}) \models \varphi(y)\big)$$

satisfies $\|\chi\| = \varnothing$ and $\mathbf{Inv}(\chi) = \mathbf{Inv}(\mathbf{\Omega}_2^\xi)$. Under the assumptions of the lemma, $\mathbf{L}[\boldsymbol{v}] \models \chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_2^\xi})[\boldsymbol{v}]$, via some $y \in \mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}]$. Then $y \in \mathbf{L}[\boldsymbol{v}{\downarrow}\tau]$, $\tau \in \mathbf{\Omega}_2^\xi$, by Corollary 26.7 (in which the isolation condition follows from Theorem 33.7(xii)). Thus $\mathbf{L}[\boldsymbol{v}] \models \chi(\underline{\boldsymbol{v}}{\downarrow}\tau)[\boldsymbol{v}]$. Corollary 26.5 yields a condition $X \in (\mathscr{X}{\downarrow}\tau) \cap \mathscr{G}_{\boldsymbol{v}}$ such that (1) $X \Vdash_{\mathscr{X}} \chi(\underline{\boldsymbol{v}}{\downarrow}\tau)$. We claim that

(2) $X \Vdash_{\mathscr{X}} \chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_*})$ — that obviously implies the lemma.

Suppose towards the contrary that (2) fails. Then (3) $Y \Vdash_{\mathscr{X}} \neg\,\chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_*})$ holds for some $Y \in \mathscr{X}$, $Y \downarrow\subseteq X$, but still (4) $Y \Vdash_{\mathscr{X}} \chi(\underline{\boldsymbol{v}}{\downarrow}\tau)$ by (1). We may assume that $\xi \subseteq \tau$, and that $\|Y\| = \tau$ by Theorem 26.3.

By Theorem 31.1(x), there is a permutation $\pi \in \mathbf{\Gamma}_2^\xi$ satisfying $\sigma = \pi{\bullet}\tau \in \mathbf{\Omega}_*$ and $\sigma \cap \tau = \varnothing$. We may note that $\mathbf{\Gamma}_2^\xi \subseteq \mathbf{Inv}(\mathbf{\Omega}_2^\xi) \subseteq \mathbf{\Pi}$, so that $\pi \in \mathbf{Inv}(\chi)$. Then we have from (4) by Corollary 25.4 that $S \Vdash_{\mathscr{X}} \chi(\underline{\boldsymbol{v}}{\downarrow}\sigma)$, where $S = \pi{\bullet}Y$, and further (5) $S \Vdash_{\mathscr{X}} \chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_*})$ as $\sigma \in \mathbf{\Omega}_*$.

However conditions S and Y are compatible because $\tau \cap \sigma = \varnothing$. Thus (5) contradicts to (3), which proves (2) and the lemma. □ (Lemma)

Lemma 33.6. *Assume that $\varphi(y)$ is a Σ^1_∞ formula with parameters in $\omega^\omega \cap \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_*}[\boldsymbol{v}])$, and there is $y \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$ such that $\mathbf{L}[\boldsymbol{v}] \models \varphi(y)$. Then there is $x \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_*}[\boldsymbol{v}])$ such that $\mathbf{L}[\boldsymbol{v}] \models \varphi(x)$.*

Proof (Lemma)**.** By Corollary 26.7, there is $\xi \in \mathbf{\Omega}_*$ such that all parameters in $\varphi(\cdot)$ belong to $\mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$. Then there is an $\mathscr{L}$-formula $\psi(\cdot)$ that contains only $\underline{\boldsymbol{v}}{\downarrow}\xi$ and some $\overset{\bullet}{z}$, $z \in \mathbf{L}$, as the only $\mathscr{L}$-names, and such that $\psi(\cdot)[\boldsymbol{v}]$ is identic to $\varphi(\cdot)$. Let $\chi(U)$ be the $\mathscr{L}$-formula: $(\exists\, x \in \omega^\omega \cap \mathbf{L}(U))\,\psi(x)$. Then obviously $\|\chi\| = \|\psi\| = \xi$ and $\mathbf{Inv}(\chi) = \mathbf{\Pi}$.

By definition we have $\mathbf{L}[\boldsymbol{v}] \models \chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_2})[\boldsymbol{v}]$, where $\|\chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_2})\| = \xi$ and $\mathbf{Inv}(\chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_2})) = \mathbf{\Gamma}_2$ by the above. It follows by Corollary 26.5 that there is a condition $X \in (\mathscr{X}{\downarrow}\xi) \cap \mathscr{G}_{\boldsymbol{v}}$ such that (1) $X \Vdash_{\mathscr{X}} \chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_2})$. We claim that

(2) $X \Vdash_{\mathscr{X}} \chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_*})$ — which obviously proves the lemma.

Suppose towards the contrary that (2) fails. Then (3) $Y \Vdash_{\mathscr{X}} \neg\,\chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_*})$ holds for some $Y \in \mathscr{X}$, $Y \downarrow\subseteq X$. We may assume that $\|Y\| = \xi$ by Theorem 26.3. Then $Y \subseteq X$ and $Y \Vdash_{\mathscr{X}} \chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_2})$ by (1). We conclude by Corollary 26.8 that there is a condition $Z \in \mathscr{X}$, $Z \downarrow\subseteq Y$, and $\tau \in \mathbf{\Omega}_2$, such that (4) $Z \Vdash_{\mathscr{X}} \chi(\underline{\boldsymbol{v}}{\downarrow}\tau)$. We can w.l.o.g. assume that $\xi \subseteq \tau = \|Z\|$.

Theorem 31.1(ix) yields a permutation $\pi \in \mathbf{\Pi}(\xi)$ with $\sigma = \pi{\bullet}\tau \in \mathbf{\Omega}_*$. Then we have $S \Vdash_{\mathscr{X}} \chi(\underline{\boldsymbol{v}}{\downarrow}\sigma)$ from (4) by Corollary 25.4, where $S = \pi{\bullet}Z$. We further conclude that (5) $S \Vdash_{\mathscr{X}} \chi(\underline{\mathbf{W}}_{\mathbf{\Omega}_*})$ since $\sigma \in \mathbf{\Omega}_*$.

On the other hand, $S{\downarrow}\xi = Z{\downarrow}\xi$ holds because $\pi \in \mathbf{\Pi}(\xi)$. Therefore $S \downarrow\subseteq Y$ (since $\|Y\| = \xi$ and $Z \downarrow\subseteq Y$). It follows that (3) and (5) are contradictory. The contradiction proves (2) and the lemma. □ (Lemma)

We proceed with **the proof of Theorem 33.4(ii)**. Consider a parameter-free Π^1_{n+1} formula $\varphi(\cdot,\cdot)$, satisfying $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}]) \models \forall\, x\, \exists\, y\, \varphi(x,y)$, and let $u \in \omega^\omega \cap \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$. Corollary 26.7 implies $u \in \mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$ for some $\xi \in \mathbf{\Omega}_2$. Then $\xi \in \mathbf{\Omega}_2^\xi$ and $u \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}])$.

Lemma 33.7. $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}]) \models \forall\, x\, \exists\, y\, \varphi(x,y)$.

Proof (Lemma)**.** Suppose otherwise. Then by Lemma 33.5 there is $p \in \omega^\omega \cap \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_*}[\boldsymbol{v}])$ such that $(*)$ $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}]) \models \forall\, y\, \varphi^-(p,y)$, where $\varphi^-(x,y)$ is the canonical Σ^1_{n+1} formula equivalent to $\neg\,\varphi(x,y)$.

However $p \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_*}[\boldsymbol{v}])$, and hence, we have $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}]) \models \exists\, y\, \varphi(p,y)$ in our assumptions. Then $\mathbf{L}[\boldsymbol{v}] \models \exists\, y\, \varphi(p,y)$ by Lemma 33.2. Furthermore, by Lemma 33.6, there is $q \in \omega^\omega \cap \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_*}[\boldsymbol{v}])$ satisfying $\mathbf{L}[v] \models \varphi(p,q)$. Now $p,q \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}])$ by Theorem 31.1(iii), and we have $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}]) \models \varphi(p,q)$ still by Lemma 33.2. But this contradicts $(*)$. □ (Lemma)

Now let us accomplish the proof of Theorem 33.4(ii). By the last lemma, and since $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}]) \models \mathbf{DC}$ (by Theorem 31.2(iii)), there is a sequence $\langle x_k\rangle_{k<\omega} \in \mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}])$ of reals x_k satisfying $x_0 = u$ and $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2^\xi}[\boldsymbol{v}]) \models \varphi(x_k, x_{k+1})$, $\forall\, k$. Then Lemma 33.2 implies $\varphi(x_k, x_{k+1})$, $\forall\, k$, in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$ as well, as required. □

34 Second form of the main theorem

To summarize the results achieved above, we approach our first main result (Theorem 1.1 in the introduction) by means of the following theorem.

Theorem 34.1 (in $\mathbf{L}$)**.** *Assume that $\mathfrak{n} \ge 1$. Then there is a normal forcing $\mathscr{X} \in \mathbf{NF}$ which has the fusion property, the $(\mathfrak{n})$-odd expansion property, and the $(\mathfrak{n})$-definability property.*

Proof (Thm 1.1 from Thm 34.1)**.** Assuming that $\omega_1^{\mathbf{L}}$ is countable, let $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ be an array $\mathscr{X}$-generic over $\mathbf{L}$. Then:

– $\mathbf{AC}_\omega(\mathrm{OD})$ holds in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$ whereas the full $\mathbf{AC}_\omega$ holds in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$ and in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_4}[\boldsymbol{v}])$ — by Theorem 31.2;

– $\mathbf{AC}_\omega(\mathit{\Pi}^1_{n+1})$, $\mathbf{AC}_\omega(\mathbf{\Pi}^1_{n+1})$, $\mathbf{DC}(\mathit{\Pi}^1_{n+1})$, $\mathbf{DC}(\mathbf{\Pi}^1_{n+1})$ fail in resp. $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$, $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$, $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$, $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_4}[\boldsymbol{v}])$ by Theorem 32.3;

– $\mathbf{DC}(\mathbf{\Pi}^1_n)$ holds in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$ and in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$, whereas $\mathbf{DC}(\mathit{\Pi}^1_{n+1})$ holds in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$ and in $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_4}[\boldsymbol{v}])$ — by Theorem 33.4.

Thus $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_1}[\boldsymbol{v}])$, $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_2}[\boldsymbol{v}])$, $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_3}[\boldsymbol{v}])$, $\mathbf{L}(\mathbf{W}_{\mathbf{\Omega}_4}[\boldsymbol{v}])$ are models of **ZF** in which implications resp. (1), (2), (3), (4) of Thm 1.1 fail, as required. □

Thus Theorem 34.1 implies Theorem 1.1, the first main result of this paper. Chapters VI–X below will contain the proof of Theorem 34.1, and thereby will accomplish the proof of Theorem 1.1 as well.

VI Reduction of the odd-expansion property to the completeness property

The goal of this Chapter is to reduce n-odd-expansion property of generic arrays, as in Definition 33.1, to a property of a given normal forcing notion $\mathscr{X} \subseteq \mathbf{IPS}$, called *$n$-completeness* (Section 39). This property will essentially say that $\mathscr{X}$ is an elementary substructure of $\mathbf{IPS}$ w.r.t. the forcing relation for Σ^1_n formulas. We have to begin with some technicalities, which include:

- representation of reals via (codes of) continuous maps (Section 35),
- a corresponding extension of the 2nd order Peano language and a forcing-type relation `forc` for the extended $\mathfrak{L}$-language (Section 36),
- the narrowing and odd expansion theorems for `forc` (Section 37),
- the action of projection-keeping homomorphisms on `forc` (Section 38).

Note that the content of this Chapter has no relation to the case $\mathbb{n} = 1$ of Theorems 1.1 and 34.1 because the n-odd-expansion property holds for $n = 1$ anyway.

35 Coding continuous functions

The Baire space ω^ω is separable Polish space, and such is the Cantor space $\mathscr{D} = 2^\omega \subseteq \omega^\omega$, as well as every space of the form $\mathscr{D}^\xi$ and every closed subset in such a space. In addition, the spaces $\mathscr{D}$ and $\mathscr{D}^\xi$ are compact. It follows from the compactness that a function $F : \mathscr{D}^\xi \to \omega^\omega$ is continuous ($F \in \mathbf{CF}_\xi$, Section 18), iff its *graph* $\{\langle x, F(x)\rangle : x \in \mathscr{D}^\xi\}$ (identified with F) is a closed set in $\mathscr{D}^\xi \times \omega^\omega$. Therefore, if $F : \mathscr{D}^\xi \to \omega^\omega$ is in fact continuous, and a set $X \subseteq \mathscr{D}^\xi$ is topologically dense in $\mathscr{D}^\xi$ then (the graph of) F coincides with the closure $(F \restriction X)^\#$ of the restricted map $F \restriction X$ in $\mathscr{D}^\xi \times \omega^\omega$. We take

$$\mathbf{Rat}_\xi = \{x \in \mathscr{D}^\xi : x(\boldsymbol{i})(k) = 0 \text{ for all but finite pairs } \langle \boldsymbol{i}, k\rangle \in \xi \times \omega\}$$

($\mathscr{D}^\xi$-rationals) as a *canonical* countable dense set in $\mathscr{D}^\xi$. Accordingly let

$$\begin{aligned} \mathbf{cCF}_\xi &= \{f \in \mathbf{L} : f : \mathbf{Rat}_\xi \to \omega^\omega \wedge f^\# \text{ is a continuous map } \mathscr{D}^\xi \to \omega^\omega\}; \\ \mathbf{cCF}^*_\xi &= \{f \in \mathbf{cCF}_\xi : f : \mathbf{Rat}_\xi \to \mathscr{D}, \text{ so } f^\# : \mathscr{D}^\xi \to \mathscr{D} \text{ is continuous}\}. \end{aligned}$$

If $f \in \mathbf{cCF}_\xi$ then let $\|f\| = \xi$.

We further define $\mathbf{cCF} = \bigcup_{\xi \in \Xi} \mathbf{cCF}_\xi$ and $\mathbf{cCF}^* = \bigcup_{\xi \in \Xi} \mathbf{cCF}^*_\xi$; thus $\mathbf{cCF}, \mathbf{cCF}^* \in \mathbf{L}$. Each $f \in \mathbf{cCF}$ is viewed as a *code* of the continuous map $f^\# \in \mathbf{CF}$, and each $f \in \mathbf{cCF}^*$ as a *code* of the continuous map $f^\# \in \mathbf{CF}^*$.

In the particular case $\tau = \varnothing$ we have $\mathscr{D}^{\varnothing} = \mathbf{Rat}_{\varnothing} = \{\varnothing\}$, accordingly $\mathbf{cCF}_{\varnothing}$ consists of all functions $h_x(\varnothing) = x$, $x \in \omega^\omega$, defined on $\{\varnothing\}$.

We would prefer to deal with continuous functions $F : \mathscr{D}^{\tau} \to \omega^\omega$ themselves rather than their countable codes. But as any such F is an uncountable set, this would make hardly possible to treat definability questions on the basis of definability over $\mathbf{HC} = \{$all hereditarily countable sets$\}$.

Corollary 35.1 (of Theorem 27.1(iv)). *Assume that, in* $\mathbf{L}$, $\mathscr{X} \in \mathbf{NF}$ *has the fusion property,* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *is* $\mathscr{X}$*-generic over* $\mathbf{L}$, $\tau \in \boldsymbol{\Xi}$, *and* $a \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}{\downarrow}\tau]$. *Then there is* $f \in \mathbf{cCF}_\tau$ *such that* $a = f^{\#}(\boldsymbol{v}{\downarrow}\tau)$. □

36 Forcing approximation

Corollary 35.1 enables us to introduce a special language for describing elements of ω^ω in generic extensions, using function codes in $\mathbf{cCF}$ to be names of elements of type 1 (i.e., taking values in ω^ω when interpreted).

Consider the language of 2nd order Peano arithmetic with type-0 variables k, l, m, n over ω and type-1 variables $x, y, z, \dots$ over ω^ω. The following are standard classes of formulas:

Σ^0_∞ = arithmetic formulas, *i.e.*, no type-1 quantifiers;

Σ^1_{n+1} = formulas of the form $\exists\, x\, \psi(x)$, ψ being Π^1_n (or Σ^0_∞ in case $n = 0$);

Π^1_{n+1} = formulas of the form $\forall\, x\, \psi(x)$, ψ being Σ^1_n (or Σ^0_∞ in case $n = 0$).

Let $\mathfrak{L}$ be the extension of this language by using natural numbers as type-0 parameters and function codes $f \in \mathbf{cCF}$ — as type-1 parameters. Let $\mathfrak{L}\Sigma^0_\infty$, $\mathfrak{L}\Sigma^1_n$, $\mathfrak{L}\Pi^1_n$ be the according classes of $\mathfrak{L}$-formulas.

If φ is an $\mathfrak{L}$-formula then let $\|\varphi\| = \bigcup\{\|f\| : f$ occurs in $\varphi\}$. If φ is $\mathfrak{L}\Sigma^1_n$, then φ^- denotes the result of the canonical reduction of $\neg\,\varphi$ to $\mathfrak{L}\Pi^1_n$-form; similarly for φ in $\mathfrak{L}\Pi^1_n$. If φ is $\mathfrak{L}\Sigma^0_\infty$ then φ^- is just $\neg\,\varphi$.

If φ is an $\mathfrak{L}$-formula, $\|\varphi\| \subseteq \eta \subseteq \boldsymbol{I}$ and $v \in \mathscr{D}^\eta$, then the *valuation* $\varphi\langle v\rangle$ is obtained by the substitution of $f^{\#}(v{\downarrow}\|f\|) \in \omega^\omega$ for any code $f \in \mathbf{cCF}$ in φ. Thus $\varphi\langle v\rangle$ is a usual 2nd order arithmetic formula with type-1 parameters in $\omega^\omega \cap \mathbf{L}[v{\downarrow}\|\varphi\|]$.

Definition 36.1 (in $\mathbf{L}$). Define a relation $X \;\mathtt{forc}\; \varphi$, where $X \in \mathbf{IPS}$ and φ is a closed $\mathfrak{L}$-formula in $\mathfrak{L}\Sigma^0_\infty \cup \bigcup_{k\ge 1}(\mathfrak{L}\Sigma^1_k \cup \mathfrak{L}\Pi^1_k)$, by induction.

1°. If φ is a closed formula in $\mathfrak{L}\Sigma^0_\infty \cup \mathfrak{L}\Sigma^1_1 \cup \mathfrak{L}\Pi^1_1$, and $X \in \mathbf{IPS}$, then $X \;\mathtt{forc}\; \varphi$ iff $\varphi\langle x\rangle$ holds for all $x \in X{\uparrow}\tau$, where $\tau = \|\varphi\| \cup \|X\|$.

2°. If $\varphi(x)$ is a $\mathfrak{L}\Pi^1_k$ formula, $k \ge 1$, then $X \,\texttt{forc}\, \exists\, x\, \varphi(x)$ iff $X \,\texttt{forc}\, \varphi(f)$ for some $f \in \mathbf{cCF}$.

3°. If φ is a closed $\mathfrak{L}\Pi^1_k$ formula, $k \ge 2$, $X \in \mathbf{IPS}$, then $X \,\texttt{forc}\, \varphi$ iff there exists no $Y \in \mathbf{IPS}$, $Y \downarrow\subseteq X$, such that $Y \,\texttt{forc}\, \varphi^-$. □

Lemma 36.2. (i) *If $X \,\texttt{forc}\, \varphi$, $Y \in \mathbf{IPS}$, $Y \downarrow\subseteq X$, then $Y \,\texttt{forc}\, \varphi$;*

(ii) *$X \,\texttt{forc}\, \varphi$ and $X \,\texttt{forc}\, \varphi^-$ cannot hold together*;

(iii) *if $X \in \mathbf{IPS}$, φ is a closed $\mathfrak{L}\Sigma^1_1$ formula, then there exists $Y \in \mathbf{IPS}$, $Y \downarrow\subseteq X$ such that $Y \,\texttt{forc}\, \varphi$ or $Y \,\texttt{forc}\, \varphi^-$;*

(iv) *if $X \in \mathbf{IPS}$, $k \ge 2$, φ is a closed $\mathfrak{L}\Pi^1_k$ formula, and $\neg\, X \,\texttt{forc}\, \varphi$ then there exists $Y \in \mathbf{IPS}$, $Y \downarrow\subseteq X$ such that $Y \,\texttt{forc}\, \varphi^-$;*

(v) *if $X \in \mathbf{IPS}$, $\eta = \|X\| \subseteq \tau \in \mathbf{\Xi}$, and $X{\uparrow}\tau \,\texttt{forc}\, \varphi$ then $X \,\texttt{forc}\, \varphi$.*

Proof. Here (ii),(iv) hold by definition, (i) is verified by routine induction.

To check (iii), note that the set $U = \{v \in X{\uparrow}\tau : \varphi\langle v\rangle\}$ is $\mathbf{\Sigma}^1_1$, where $\tau = \|X\| \cup \|\psi\|$, hence it has the Baire property in $X{\uparrow}\tau$. It follows by Corollary 16.3 that there exists a set $Y \in \mathbf{IPS}_\tau$ such that either $Y \subseteq U$, or $U \subseteq (X{\uparrow}\tau) \smallsetminus U$. Then accordingly $Y \,\texttt{forc}\, \varphi$ or $Y \,\texttt{forc}\, \varphi^-$, as required.

(v) Lemma 10.5 makes sure that $X{\uparrow}\tau \in \mathbf{IPS}$. The proof goes by induction, and 3° is the only nontrivial step. Suppose to the contrary that ψ is $\mathfrak{L}\Sigma^1_k$, $X{\uparrow}\tau \,\texttt{forc}\, \psi^-$, but $\neg\, X \,\texttt{forc}\, \psi^-$. There is $Y \in \mathbf{IPS}$, $Y \downarrow\subseteq X$, $Y \,\texttt{forc}\, \psi$. Let $\xi = \|Y\|$, $\zeta = \xi \cup \tau$, $Z = Y{\uparrow}\zeta$, $\eta' = \tau \cap \xi$. Then $Z \downarrow\subseteq Y$, hence $Z \,\texttt{forc}\, \psi$ by (i). However $Z{\downarrow}\tau = (Y{\downarrow}\eta'){\uparrow}\tau$ by Lemma 10.3. Here $Y{\downarrow}\eta' \subseteq X{\uparrow}\eta'$ since $Y \downarrow\subseteq X$, as clearly $\eta \subseteq \eta'$. Therefore $Z{\downarrow}\tau \subseteq X{\uparrow}\eta'{\uparrow}\tau = X{\uparrow}\tau$. Thus $Z \downarrow\subseteq X{\uparrow}\tau$. We conclude that $Z \,\texttt{forc}\, \psi^-$ by (i). Yet $Z \,\texttt{forc}\, \psi$ as well, see above. This contradicts (ii). □

Assume that $\eta, \sigma, \tau \in \mathbf{\Xi}$, $\xi = \eta \cup \sigma \cup \tau$, $f \in \mathbf{cCF}_\sigma$, $g \in \mathbf{cCF}_\eta$, $X \in \mathbf{IPS}_\tau$. Say that f, g are *valuation-equivalent*, or simply *v-equivalent* on X, iff $f^\#(x{\downarrow}\sigma) = g^\#(x{\downarrow}\eta)$ for all $x \in X{\uparrow}\xi$. Then, $\mathfrak{L}$-formulas φ, ψ are v-equivalent on X if ψ is obtained from φ by a substitution of all codes $f \in \mathbf{cCF}$ occurring in φ with codes g v-equivalent to f on X.

Lemma 36.3 (in $\mathbf{L}$, routine by induction)**.** *If $X \,\texttt{forc}\, \varphi$, and $\mathfrak{L}$-formulas φ, ψ are v-equivalent on X then $X \,\texttt{forc}\, \varphi$ iff $X \,\texttt{forc}\, \psi$.* □

Lemma 36.4 (in $\mathbf{L}$)**.** *Assume that $X \in \mathbf{IPS}$, $\varphi(x)$ is a $\mathfrak{L}\Pi^1_k$-formula, $k \ge 1$, $\tau = \|X\| \cup \|\varphi\|$, and $X \,\texttt{forc}\, \exists\, x\, \varphi(x)$. Then there is a code $g \in \mathbf{cCF}_\xi$ for some $\xi \in \mathbf{\Xi}$, $\tau \subseteq \xi$, such that $X \,\texttt{forc}\, \varphi(g)$.*

Proof. By definition we have $X \,\mathtt{forc}\, \varphi(f)$ for a code $f \in \mathbf{cCF}$. Let $\sigma = \|f\|$ and $\xi = \sigma \cup \tau$. Define $g \in \mathbf{cCF}_\xi$ by $g(x) = f(x{\downarrow}\sigma)$ for each $x \in \mathbf{Rat}_\xi$, and use Lemma 36.3. □

37 The narrowing and odd expansion theorems

Corollary 35.1 allows to view $\mathtt{forc}$ as a forcing-type relation compatible with **IPS** as the forcing notion. Yet unlike the ordinary forcing $\Vdash_{\mathbf{IPS}}$, $\mathtt{forc}$ treats the $\exists$ quantifier over ω^ω in the sense of continuous reading of names. This adds difficulty and extra work to the proof of the next theorem.

Theorem 37.1 (the narrowing theorem, in $\mathbf{L}$). *Suppose that φ is a closed $\mathfrak{L}$-formula, $\|\varphi\| = \eta \subseteq \tau \in \boldsymbol{\Xi}$, $X \in \mathbf{IPS}_\tau$, $X \,\mathtt{forc}\, \varphi$. Then $X{\downarrow}\eta \,\mathtt{forc}\, \varphi$.*

This is quite similar to Theorem 26.3, but the proof in Section 38 will be somewhat more difficult because of the mentioned difference in the treatment of $\exists$. Meanwhile, here we apply Theorem 37.1 in the proof of the following result. Recall Definition 30.1 on odd expansions.

Theorem 37.2 (the odd expansion theorem, in $\mathbf{L}$). *Let $k < \omega$, $\varphi(x)$ be an $\mathfrak{L}\Pi^1_k$-formula, $\|\varphi(x)\| = \tau_0$, $X \in \mathbf{IPS}$, $X \,\mathtt{forc}\, \exists x\, \varphi(x)$. Then there is an odd expansion $\tau \in \boldsymbol{\Xi}$ of τ_0, and $g \in \mathbf{cCF}_\tau$, such that $X \,\mathtt{forc}\, \varphi(g)$.*

The next lemma will be used in the proof of Theorem 37.2 as well.

Lemma 37.3 (in $\mathbf{L}$). *Assume that $\tau_0 \subseteq \sigma$ belong to $\boldsymbol{\Xi}$. Then there is $\pi \in \mathbf{Perm}(\tau_0)$ such that $\tau = \pi{\bullet}\sigma$ is an odd expansion of τ_0 and $\tau \cap \sigma = \varnothing$.*

Recall that **Perm** consists of all, not necessarily parity-preserving, permutations of $\boldsymbol{I}$, and $\mathbf{Perm}(\tau_0)$ contains all $\pi \in \mathbf{Perm}$ such that $\pi{\restriction}\tau_0$ is the identity.

Proof. Emulating the proof of Lemma 26.2, we let $\lambda < \omega_1$ be a limit ordinal bigger than $\sup\{\boldsymbol{i}(k) : \boldsymbol{i} \in \sigma \wedge k < \mathtt{lh}(\boldsymbol{i})\}$. For any $\alpha < \lambda$, pick an *odd* ordinal $\lambda \le \beta(\alpha) \le \lambda + \lambda$ such that $\alpha < \alpha' \Longrightarrow \beta(\alpha) < \beta(\alpha')$. If $\alpha < \omega_1$, let $B(\alpha) = B^{-1}(\alpha) = \beta(\alpha)$, whereas $B(\alpha) = \alpha$ in case $\alpha \notin \lambda \cup \{\beta(\alpha') : \alpha' < \lambda\}$. Thus B is a bijection of ω_1.

If $\boldsymbol{i} \in \boldsymbol{I}$ then define $\boldsymbol{j} = \rho(\boldsymbol{i}) \in \boldsymbol{I}$ such that $\mathtt{lh}(\boldsymbol{j}) = \mathtt{lh}(\boldsymbol{i})$ and $\boldsymbol{j}(\ell) = \boldsymbol{i}(B(\ell))$ for all $\ell < \mathtt{lh}(\boldsymbol{j}) = \mathtt{lh}(\boldsymbol{i})$, thus ρ is a permutation in **Perm**.

Now let $\boldsymbol{i} \in \boldsymbol{I}$. Take a largest number $m_{\boldsymbol{i}} \le \mathtt{lh}(\boldsymbol{i})$ such that $\boldsymbol{i}{\restriction}m_{\boldsymbol{i}} \in \tau_0$. Then $\boldsymbol{i} = (\boldsymbol{i}{\restriction}m_{\boldsymbol{i}})^\frown\boldsymbol{k}$ for some $\boldsymbol{k} \in \boldsymbol{I} \cup \{\Lambda\}$. Put $\pi(\boldsymbol{i}) = (\boldsymbol{i}{\restriction}m_{\boldsymbol{i}})^\frown B(\boldsymbol{k})$. □

Proof (Theorem 37.2 from Theorem 37.1, in **L**)**.** By Lemma 36.2(v), we can assume that $\tau_0 \subseteq \|X\|$. Then by Theorem 37.1, we assume that $\tau_0 = \|X\|$ exactly. Now, as X `forc` $\exists\, x\, \varphi(x)$, we have (*) X `forc` $\varphi(f)$ for some $f \in \mathbf{cCF}_\sigma$, $\sigma \in \boldsymbol{\Xi}$. We can w.l.o.g. assume that $\tau_0 \subseteq \sigma$ (by Lemma 36.3).

Now Lemma 37.3 implies a permutation $\pi \in \mathbf{Perm}(\tau_0)$ such that $\tau = \pi{\bullet}\sigma$ is an odd expansion of τ_0 and $\tau \cap \sigma = \varnothing$. Note that $\pi{\bullet}X = X$ as $\tau_0 = \|X\|$.

It does not take much effort to define the action of π on **cCF**. Namely if $\xi \in \boldsymbol{\Xi}$ and $\eta = \pi{\bullet}\xi$ then clearly $\mathbf{Rat}_\eta = \pi{\bullet}\mathbf{Rat}_\xi$ in the sense of Section 13. (Note that $\mathbf{Rat}_\xi \subseteq \mathscr{D}^\xi$.) Therefore if $f \in \mathbf{cCF}_\xi$ then we naturally define $g = \pi{\bullet}f \in \mathbf{cCF}_\eta$ by $g(\pi{\bullet}x) = f(x)$ for all $x \in \mathscr{D}^\xi$.

Furthermore if ψ is an $\mathfrak{L}$-formula then we let $\pi\psi$ be obtained by the substitution of $\pi{\bullet}f$ for any code $f \in \mathbf{cCF}$ in ψ. As far as the given formula $\varphi(x)$ is concerned, note that $\pi\varphi(x)$ is identic to $\varphi(x)$ since $\tau_0 = \|\varphi(x)\|$.

Lemma 37.4 (routine by induction on the complexity)**.** *If $X \in \mathbf{IPS}$ and ψ is an $\mathfrak{L}$-formula then X `forc` φ iff $\pi{\bullet}X$ `forc` $(\pi\varphi)$.* □

Applying the lemma to (*), we get $\pi{\bullet}X$ `forc` $\pi\varphi(g)$, where $g = \pi{\bullet}f \in \mathbf{cCF}_\tau$. However $\pi{\bullet}X = X$ and $\pi\varphi(x)$ is identic with $\varphi(x)$, see above. Thus X `forc` $\varphi(g)$, as required. □ (Thm 37.2 mod Thm 37.1)

38 Proof of the narrowing theorem

Proof (Theorem 37.1, in **L**)**.** Let $Y = (X{\downarrow}\eta){\uparrow}\tau$; clearly $X{\downarrow}\eta = Y{\downarrow}\eta$. Recall that the notion of projection-keeping homeomorphisms, or PKHs for brevity, was introduced by Definition 9. This will be our tool for the proof of Theorem 37.1. In particular, Lemma 10.6 implies the existence of a PKH $H : X \xrightarrow{\text{onto}} Y$ such that $H(x){\restriction}\eta = x{\restriction}\eta$ for all $x \in X$. **Fix such an** H.

As **the first step** of the proof, we extend the action of H as follows.

1*. If $\xi \subseteq \tau$, $\xi \in \boldsymbol{\Xi}$, then a PKH $H_\xi : X{\downarrow}\xi \xrightarrow{\text{onto}} Y{\downarrow}\xi$ is defined by $H_\xi(x{\downarrow}\xi) = H(x)$ for any $x \in X$.

2*. Let $\zeta \in \boldsymbol{\Xi}$ satisfy $\tau \subseteq \zeta$. If $x \in X' = X{\uparrow}\zeta$ then $y = H_\zeta(x) \in Y' = Y{\uparrow}\zeta$ is defined by $y{\downarrow}\tau = H(x{\downarrow}\tau)$ (thus $y{\downarrow}\tau \in Y$) and $y(\boldsymbol{i}) = x(\boldsymbol{i})$ for all $\boldsymbol{i} \in \zeta \smallsetminus \tau$. *We assert that $H_\zeta : X' \xrightarrow{\text{onto}} Y'$ is a PKH.*

Indeed let $\sigma \in \boldsymbol{\Xi}$, $\sigma \subseteq \zeta$, and $u, v \in X'$ satisfy $u{\downarrow}\sigma = v{\downarrow}\sigma$. Then in particular $u{\downarrow}\xi = v{\downarrow}\xi$, where $\xi = \sigma \cap \tau$, and hence, by 1*,

$$H_\zeta(u){\downarrow}\xi = H_\xi(u{\downarrow}\xi) = H_\xi(v{\downarrow}\xi) = H_\zeta(v){\downarrow}\xi.$$

But if $\boldsymbol{i} \in \sigma \smallsetminus \xi$ then $\boldsymbol{i} \in \zeta \smallsetminus \tau$, so $H_\zeta(u)(\boldsymbol{i}) = u(\boldsymbol{i}) = v(\boldsymbol{i}) = H_\zeta(v)(\boldsymbol{i})$. Overall, $H_\zeta(u){\downarrow}\sigma = H_\zeta(v){\downarrow}\sigma$, as required.

We may note that $H_\zeta(x){\downarrow}\eta = x{\downarrow}\eta$ since H itself has this property.

Definition 38.1. If still $\tau \subseteq \zeta \in \boldsymbol{\Xi}$ and $x \in X{\uparrow}\zeta$, then put $H^\sim x = H_\zeta(x)$, and define $H^\sim Z = \{H^\sim x : x \in Z\}$ for any $Z \in \mathbf{IPS}_\zeta$, $Z \subseteq X{\uparrow}\zeta$.

(1) By 2* and Lemma 9.4 the map $Z \mapsto H^\sim Z$ is a $\downarrow\subseteq$-preserving and $\|...\|$-preserving bijection from $\mathbf{IPS}_{\downarrow\subseteq X} = \{Z \in \mathbf{IPS} : Z \downarrow\subseteq X\}$ onto $\mathbf{IPS}_{\downarrow\subseteq Y} = \{Z \in \mathbf{IPS} : Z \downarrow\subseteq Y\}$.

(2) $(H^\sim Z){\downarrow}\eta = Z{\downarrow}\eta$ for all $Z \in \mathbf{IPS}_{\downarrow\subseteq X}$ by the above. □

The action of H on $\mathbf{cCF}$ is somewhat less natural because the domain of the given H is a set $X \in \mathbf{IPS}_\tau$, perhaps a proper subset of $\mathscr{D}^\tau$.

Lemma 38.2. *Under the assumptions above, let $\sigma \in \boldsymbol{\Xi}$, and $\sigma \subseteq \eta$ or $\tau \subseteq \sigma$. Then for any code $f \in \mathbf{cCF}_\sigma$ there is $g = H^\sim f \in \mathbf{cCF}_\sigma$ satisfying*:

(i) *$g = f$ and $g^\#(H_\sigma(x)) = f^\#(x)$ for all $x \in X{\downarrow}\sigma$ — in case $\sigma \subseteq \eta$;*

(ii) *$g^\#(H_\sigma(x)) = f^\#(x)$ for all $x \in X{\uparrow}\sigma$, in case $\tau \subseteq \sigma$.*

Moreover, if $h \in \mathbf{cCF}_\sigma$ then there exists $f \in \mathbf{cCF}_\sigma$ such that h is v-equivalent to $g = H^\sim f$ on Y, that is, $g^\#(y) = h^\#(y)$ for all $y \in Y{\uparrow}\sigma$.

Proof. (i) The code $g = f$ satisfies $g^\#(H_\sigma(x)) = f^\#(x)$ for all $x \in X{\downarrow}\sigma$, because $\sigma \subseteq \eta$ and $H(x){\restriction}\eta = x{\restriction}\eta$ for all $x \in X$.

(ii) As $\tau \subseteq \sigma$, $H_\sigma : X{\uparrow}\sigma \xrightarrow{\text{onto}} Y{\uparrow}\sigma$ is a PKH, see 2* above, in particular, a homeomorphism. If $y \in Y{\uparrow}\sigma$ then let $G'(y) = f^\#(H_\sigma^{-1}(y))$, thus $G' : Y{\uparrow}\sigma \to \omega^\omega$ is continuous. It has a continuous extension $G : \mathscr{D}^\sigma \to \omega^\omega$. Let $g = G{\restriction}\mathbf{Rat}_\sigma$, so that $G = g^\#$ and $g \in \mathbf{cCF}_\sigma$. Thus $g^\#(H_\sigma(x)) = f^\#(x)$ holds for all $x \in X{\uparrow}\sigma$. To be more specific, we let $H^\sim f$ to be the Gödel-least one of all $g \in \mathbf{cCF}_\sigma$ with this property. Thus $g = H^\sim f \in \mathbf{cCF}_\sigma$ is defined, satisfying $g^\#(H_\sigma(x)) = f^\#(x)$ for all $x \in X{\uparrow}\sigma$.

Finally to prove the 'moreover' claim, note that $F'(x) = h^\#(H_\sigma(x))$ is a continuous map $X{\uparrow}\sigma \to \omega^\omega$, extend it to a continuous $F = f^\# : \mathscr{D}^\sigma \to \omega^\omega$, where $f \in \mathbf{cCF}_\sigma$, and let $g = H^\sim f$. □ (Lemma)

The next definition and lemma continue the proof of Theorem 37.1.

Definition 38.3. If Φ is a $\mathfrak{L}$-formula such that any $f \in \mathbf{cCF}$ in Φ satisfies $\|f\| \subseteq \eta$ or $\tau \subseteq \|f\|$, then $H\Phi$ is the result of substitution of $H^\sim f$ for any $f \in \mathbf{cCF}$ occurring in Φ. □

Lemma 38.4. *Let Φ be a closed $\mathfrak{L}$-formula as in Definition 38.3, and $Z \in \mathbf{IPS}_{\downarrow\subseteq X}$. Then $Z\,\mathtt{forc}\,\Phi$ iff $H\tilde{\ }Z\,\mathtt{forc}\,H\Phi$.*

Proof. The case of Φ as in 1° of Definition 36.1, as the basis of induction, routinely follow from the equality $g^{\#}(H_\sigma(x)) = f^{\#}(x)$ of Lemma 38.2 because $Z \downarrow\subseteq X$. It remains to take care of the steps 2°,3°.

2°. Let Φ be $\exists\, x\, \psi(x)$. Assume $Z\,\mathtt{forc}\,\exists\, x\, \psi(x)$, so that $Z\,\mathtt{forc}\,\psi(f)$ for some $f \in \mathbf{cCF}_\sigma$, $\sigma \in \mathbf{\Xi}$. By Lemma 36.4, we can assume that $\tau \subseteq \|f\|$, so $\psi(f)$ is still of the form as in Definition 38.3. Then $H\tilde{\ }Z\,\mathtt{forc}\,H(\psi(f))$ by the inductive hypothesis, meaning that $H\tilde{\ }Z\,\mathtt{forc}\,(H\psi)(g)$, where $g = H\tilde{\ }f$, and hence $H\tilde{\ }Z\,\mathtt{forc}\,\exists\, x\,(H\psi)(x)$, and $H\tilde{\ }Z\,\mathtt{forc}\,H\Phi$.

To prove the inverse, we suppose that $H\tilde{\ }Z\,\mathtt{forc}\,\exists\, x\,(H\psi)(x)$, that is, $H\tilde{\ }Z\,\mathtt{forc}\,(H\psi)(h)$, for some $h \in \mathbf{cCF}_\sigma$, $\tau \subseteq \sigma \in \mathbf{\Xi}$. By Lemma 38.2, there exists $f \in \mathbf{cCF}_\sigma$ such that h is v-equivalent to $g = H\tilde{\ }f$ on Y, and hence on $H\tilde{\ }Z \downarrow\subseteq Y$ as well. Then $H\tilde{\ }Z\,\mathtt{forc}\,(H\psi)(g)$ by Lemma 36.3, and hence $Z\,\mathtt{forc}\,\psi(f)$ by the inductive hypothesis, and $Z\,\mathtt{forc}\,\Phi$, as required.

3°. Let Φ be ψ^-, where ψ is a $\mathfrak{L}\Sigma^1_n$ formula. Assume that $Z\,\mathtt{forc}\,\Phi$ **fails**. By definition there is a condition $Z' \downarrow\subseteq Z$, $Z'\,\mathtt{forc}\,\psi$. The inductive hypothesis implies $H\tilde{\ }Z'\,\mathtt{forc}\,H\psi$. However $H\tilde{\ }Z' \downarrow\subseteq H\tilde{\ }Z'$, hence we conclude that $H\tilde{\ }Z\,\mathtt{forc}\,\Phi$ **fails**. The converse is similar. □ (Lemma)

Now we return to the formula φ of Theorem 37.1. It satisfies $\|\varphi\| \subseteq \eta$, and $X\,\mathtt{forc}\,\varphi$. Lemma 38.4 is applicable, so that $Y\,\mathtt{forc}\,\varphi$, because $H\varphi$ is identic to φ since $\|\varphi\| \subseteq \eta$. This implies $X{\downarrow}\eta\,\mathtt{forc}\,\varphi$ by Lemma 36.2(v).

□ (Theorems 37.1 and 37.2)

Corollary 38.5. *Let $X \in \mathbf{IPS}$, $k < \omega$, φ is a closed $\mathfrak{L}$ formula, $\eta = \|X\| \cup \|\varphi\|$, $\neg\, X\,\mathtt{forc}\,\varphi$. Then there is $Z \in \mathbf{IPS}_\eta$, $Z \downarrow\subseteq X$, $Z\,\mathtt{forc}\,\varphi^-$.*

Proof. Lemma 36.2(iv),(iv) yields $Y \in \mathbf{IPS}$ such that $\eta \subseteq \|Y\|$, $Y \downarrow\subseteq X$, and $Y\,\mathtt{forc}\,\varphi^-$. Now let $Z = Y{\downarrow}\eta$ and apply Theorem 37.1. □

39 Complete normal forcing notions

After working out some technical issues with `forc`, we'll prove the truth theorem for this forcing-type relation. It is based on the next definition.

Definition 39.1 (in **L**). A normal forcing notion $\mathscr{X} \subseteq \mathbf{IPS}$ is *n-complete* if for any closed formula φ in $\bigcup_{k \le n} \mathfrak{L}\Sigma^1_k$ the set

$$\mathbf{Forc}_\varphi = \{X \in \mathscr{X} : X\,\mathtt{forc}\,\varphi \text{ or } X\,\mathtt{forc}\,\varphi^-\}$$

is dense in $\mathscr{X}$. □

For example, the set **IPS** is n-complete for each n by Lemma 36.2, (iii) and (iv). We will not use this fact, but it is useful to keep it in mind. In its light, n-complete normal forcing notions $\mathscr{X} \subseteq \mathbf{IPS}$ can be viewed as "similar to **IPS** up to level n in the sense of $\texttt{forc}$". Let us now prove the theorem connecting $\texttt{forc}$ and truth in generic extensions.

Theorem 39.2 (truth theorem)**.** *Assume that $n \ge 1$ and, in $\mathbf{L}$, a normal forcing $\mathscr{X} \subseteq \mathbf{IPS}$ is n-complete and has the fusion property. Let $\boldsymbol{v}$ be an $\mathscr{X}$-generic array over $\mathbf{L}$. Let φ be a closed formula in $\mathfrak{L}\Sigma^1_k$, $k \le n+1$. Then $\mathbf{L}[\boldsymbol{v}] \models \varphi\langle\boldsymbol{v}\rangle$ iff there exists a condition $X \in \mathscr{X} \cap \mathscr{G}_{\boldsymbol{v}}$, $X \,\texttt{forc}\, \varphi$.*

Proof. We argue by induction on $k \le n+1$. *Starting with* $k=1$, suppose that φ is a $\mathfrak{L}\Sigma^1_1$ formula. By the n-completeness of $\mathscr{X}$ and the genericity of $\boldsymbol{v}$, there exists a condition $X \in \mathscr{X} \cap \mathscr{G}_{\boldsymbol{v}}$, $X \,\texttt{forc}\, \varphi$ or $X \,\texttt{forc}\, \varphi^-$. Assume that $X \,\texttt{forc}\, \varphi$. This claim can be naturally converted into a Π^1_2 sentence with parameters in $\mathbf{L}$, true in $\mathbf{L}$. Then $\mathbf{L}[\boldsymbol{v}] \models \varphi\langle\boldsymbol{v}\rangle$ by the Shoenfield absoluteness. Similarly, if $X \,\texttt{forc}\, \varphi^-$ (a Π^1_1 sentence) then $\mathbf{L}[\boldsymbol{v}] \models \varphi^-\langle\boldsymbol{v}\rangle$, by the same absoluteness argument.

Step $k \to k+1$. Suppose that $\varphi := \exists\, x\, \psi^-(x)$ is a $\mathfrak{L}\Sigma^1_{k+1}$ formula, $\psi(x)$ being $\mathfrak{L}\Sigma^1_k$, and $k \le n$.

Direction $\Longleftarrow$. Assume that $\mathbf{L}[\boldsymbol{v}] \models \varphi\langle\boldsymbol{v}\rangle$, that is, $\mathbf{L}[\boldsymbol{v}] \models \psi^-\langle\boldsymbol{v}\rangle(p)$, for a suitable real $p \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}]$. Then $p = f^\#(\boldsymbol{v}{\downarrow}\xi)$ for some $f \in \mathbf{cCF}_\xi$, $\xi \in \boldsymbol{\Xi}$, by Corollary 35.1. Thus $\mathbf{L}[\boldsymbol{v}] \models \psi^-(f)\langle\boldsymbol{v}\rangle$, and hence, by the inductive hypothesis, no condition $X \in \mathscr{X} \cap \mathscr{G}_{\boldsymbol{v}}$ satisfies $X \,\texttt{forc}\, \psi(f)$. We conclude by the n-completeness that there is a condition $X \in \mathscr{X} \cap \mathscr{G}_{\boldsymbol{v}}$ with $X \,\texttt{forc}\, \psi^-(f)$, and then $X \,\texttt{forc}\, \varphi$ by 2° of Definition 36.1.

Direction $\Longrightarrow$. Assume that $X \,\texttt{forc}\, \varphi$, that is, $X \,\texttt{forc}\, \psi^-(f)$ for some $f \in \mathbf{cCF}_\xi$, $\xi \in \boldsymbol{\Xi}$, still by 2° of Definition 36.1. Then no condition $X \in \mathscr{X} \cap \mathscr{G}_{\boldsymbol{v}}$ satisfies $X \,\texttt{forc}\, \psi(f)$. Hence, by the inductive hypothesis, $\mathbf{L}[\boldsymbol{v}] \models \neg\psi\langle\boldsymbol{v}\rangle(p)$, where $p = f^\#(\boldsymbol{v}{\downarrow}\xi) \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}]$. We conclude that $\mathbf{L}[\boldsymbol{v}] \models \varphi\langle\boldsymbol{v}\rangle$, as required. □

Now we apply the truth theorem just proved, to show that the completeness of a normal forcing implies the odd expansion property, via the odd expansion theorem (Theorem 37.2).

Theorem 39.3 (in $\mathbf{L}$)**.** *Assume that $n \ge 1$ and a normal forcing $\mathscr{X} \subseteq$ **IPS** is n-complete and has the fusion property. Then $\mathscr{X}$ has the n-odd-expansion property of Definition 33.1.*

Proof. Let $\boldsymbol{v}$ be an $\mathscr{X}$-generic array over $\mathbf{L}$. Suppose that $\eta \in \boldsymbol{\Xi}$ and $\varphi(\cdot)$ is a $\mathit{\Pi}^1_n$ formula, with reals in $\mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$ as parameters, and $\mathbf{L}[\boldsymbol{v}] \models \exists\, x\, \varphi(x)$.

We have to find an odd expansion $\tau \in \mathbf{\Xi}$ of η, and some $q \in \mathbf{L}[\boldsymbol{v}{\downarrow}\tau]$, such that $\mathbf{L}[\boldsymbol{v}] \models \varphi(q)$.

If $p \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}{\downarrow}\eta]$ occurs in φ then Corollary 35.1 yields a code $f_p \in \mathbf{cCF}_\eta$ such that $p = f_p^{\#}(\boldsymbol{v}{\downarrow}\eta)$. Change each p to f_p in $\varphi(\cdot)$, and let $\psi(\cdot)$ be the $\mathfrak{L}$-formula obtained. Then $\varphi(\cdot)$ is identic to $\psi(\cdot)\langle\boldsymbol{v}\rangle$ and $\|\psi\| = \eta$.

By Theorem 39.2, there is a condition $X \in \mathscr{G}_{\boldsymbol{v}} \cap \mathscr{X}$ satisfying X $\texttt{forc}$ $\exists\, x\, \psi(x)$. Then by Theorem 37.2 there is an odd expansion $\tau \in \mathbf{\Xi}$ of η, and $g \in \mathbf{cCF}_\tau$, such that X $\texttt{forc}$ $\psi(g)$. Then $\mathbf{L}[\boldsymbol{v}] \models \psi(g)\langle\boldsymbol{v}\rangle$, that is, $\mathbf{L}[\boldsymbol{v}] \models \varphi(q)$, where $q = g^{\#}(v{\downarrow}\tau) \in \omega^\omega \cap \mathbf{L}[\boldsymbol{v}{\downarrow}\tau]$, as required. □

This theorem will allow us to replace the $\mathfrak{n}$-odd expansion condition in Theorem 34.1 by the $\mathfrak{n}$-completeness of $\mathscr{X}$ in $\mathbf{L}$.

VII The construction of the final forcing begins

The goal of the final Chapters VII–X is to define a normal forcing $\mathscr{X} \in \mathbf{L}$ satisfying requirements of Theorem 34.1. This will be a difficult task.

As mentioned in the end of Section 22, in principle it suffices to first define an auxiliary $\boldsymbol{I}[<2]$-kernel $\mathcal{K}$ and then let $\mathscr{X} = \mathbf{NH}(\mathcal{K}^{\mathsf{ex}})$ by Lemmas 22.4 and 22.3. Unfortunately it does not seem to work that simple way. Instead, following [29], we'll make use of certain transfinite sequences of countable sets $\mathcal{P} \subseteq \mathbf{IPS}$ called *rudiments*. This construction realizes the idea of generalized $\boldsymbol{I}$-iteration of Jensen's forcing somewhat differently than in [13, 15, 59], in particular, the CCC property will not be achieved.

We argue in L in this Chapter.

40 Rudiments

Planning to maintain a construction of normal forcing notions in the form $\mathscr{X} = \mathbf{NH}(\bigcup_{\alpha<\omega_1} \mathcal{P}_\alpha)$, where each $\mathcal{P}_\alpha$ is countable, we may note that the summands $\mathcal{P}_\alpha$ cannot be normal forcing notions themselves, because each of conditions 3°, 6°, 5° of Section 21 implies the uncountability of any normal forcing. Thus we have to somehow reduce the generality of those conditions. This is the content of this section. We begin with two auxiliary notes.

First, suppose that $\eta \subseteq \xi$ belong to $\boldsymbol{\Xi}$. Say that η is a *finite-type* in ξ, in symbol $\eta \in \mathbf{FT}(\xi)$, if η is obtained from sets of the form

$$\xi \text{ itself}, \quad [\subseteq \boldsymbol{i}] = \{\boldsymbol{j} \in \boldsymbol{I} : \boldsymbol{j} \subseteq \boldsymbol{i}\}, \quad \text{and} \quad \xi \cap \boldsymbol{I}[<\alpha],$$

where

$$\alpha < \omega_1, \quad \boldsymbol{i} \in \xi, \quad \text{and} \quad \boldsymbol{I}[<\alpha] = \{\boldsymbol{i} \in \boldsymbol{I} : \mathtt{ran}\, \boldsymbol{i} \subseteq \alpha\} = \alpha^{<\omega} \smallsetminus \{\Lambda\},$$

by a finite number of operations of set difference $\smallsetminus$ and (finite) $\cup$ and $\cap$. Clearly $\mathbf{FT}(\xi)$ is at most countable Boolean algebra, whereas there can be uncountably many arbitrary initial segments $\eta \subseteq \xi$.

Second, if $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$ belong to $\boldsymbol{I}$, then there exists a canonical permutation $\pi_{\boldsymbol{ij}} \in \boldsymbol{\Pi}$ satisfying $\pi_{\boldsymbol{ij}}(\boldsymbol{i}) = \boldsymbol{j}$ and $\pi_{\boldsymbol{ij}} = \pi_{\boldsymbol{ij}}^{-1}$, see Example 13.1.

Definition 40.1. Let $\alpha < \omega_1$. A set $\mathcal{P}$ is a *rudiment of width* α, in symbol $\mathcal{P} \in \mathbf{Rud}_\alpha$, if $\mathcal{P}$ satisfies the following conditions $1^\dagger$–$4^\dagger$.

$1^\dagger$. $\varnothing \neq \mathcal{P} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$, where, we recall, $\boldsymbol{I}[<\alpha] = \{\boldsymbol{i} \in \boldsymbol{I} : \mathtt{ran}\, \boldsymbol{i} \subseteq \alpha\}$.

$2^\dagger$. If $\eta \in \boldsymbol{\Xi}$, $\eta \subseteq \boldsymbol{I}[<\alpha]$ is finite-type in $\boldsymbol{I}[<\alpha]$, $X, Y \in \mathcal{P}$, and $Y{\downarrow}\eta \subseteq X{\downarrow}\eta$, then the set $X \cap (Y{\downarrow}\eta{\uparrow}\boldsymbol{I}[<\alpha])$ belongs to $\mathcal{P}$.

$3^\dagger$. If $X \in \mathcal{P}$, $Y \in \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$, $Y \subseteq X$, Y is clopen in X, then $Y \in \mathcal{P}$.

$4^\dagger$. Invariance: if $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}[<\alpha]$, $\boldsymbol{i} \approx_{\mathsf{par}} \boldsymbol{j}$, and $X \in \mathcal{P}$, then $\pi_{\boldsymbol{ij}} \bullet X \in \mathcal{P}$.

If $\mathcal{P}$ is such, and $\eta \in \boldsymbol{\Xi}$, $\eta \subseteq \boldsymbol{I}[<\alpha]$, then we let $\mathcal{P}{\downarrow}\eta = \{X{\downarrow}\eta : X \in \mathcal{P}\}$. In particular, if $\boldsymbol{i} \in \boldsymbol{I}[<\alpha]$ then put $\mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}} = \{X{\downarrow}_{\subseteq \boldsymbol{i}} : X \in \mathcal{P}\}$. □

Thus if $\mathscr{X} \in \mathbf{NF}$ then $\mathscr{X}{\downarrow}\boldsymbol{I}[<\alpha] = \{X{\downarrow}\boldsymbol{I}[<\alpha] : X \in \mathscr{X}\} \in \mathbf{Rud}_\alpha$.

The set $\mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$ belongs to $\mathbf{Rud}_\alpha$ by Lemmas 10.4, 10.5, 11.3. The set of all clopen sets $X \in \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$ belongs to $\mathbf{Rud}_\alpha$, too.

The following lemma clarifies the connections between kernels, rudiments, and normal forcings.

Lemma 40.2. *Assume that* $\mathcal{P} \in \mathbf{Rud}_\alpha$, $2 \le \alpha < \omega_1$, $\mathscr{D}^{\boldsymbol{I}[<\alpha]} \in \mathcal{P}$, *Then* $\mathbf{Ker}(\mathcal{P}) = \langle \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}[<\alpha]}$ *is a strong* $\boldsymbol{I}[<\alpha]$*-kernel,* $\mathscr{X} = \mathbf{NH}(\mathcal{P}) \in \mathbf{NF}$*, and* $\mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}} = \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$ *for all* $\boldsymbol{i} \in \boldsymbol{I}[<\alpha]$.

Conversely, if $\mathcal{K} = \langle \mathcal{K}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}[<\alpha]}$ *is an* $\boldsymbol{I}[<\alpha]$*-kernel, then the set*

$$\mathcal{P} = \mathcal{P}(\mathcal{K}) := \{X \in \boldsymbol{I}[<\alpha] : \forall\, \boldsymbol{i} \in \boldsymbol{I}[<\alpha]\, (X{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}})\}$$

belongs to $\mathbf{Rud}_\alpha$*, and* $\mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}} = \mathcal{K}_{\boldsymbol{i}}$ *for all* $\boldsymbol{i} \in \boldsymbol{I}[<\alpha]$.

Proof. Recall the notion of kernel in Section 22. Conditions 1^*, 2^* of Section 22 for $\mathbf{Ker}(\mathcal{P})$ are clear, and 5^* holds by $4^\dagger$ of Definition 40.1 for $\mathcal{P}$.

To verify 3^* of Section 22 for $\mathbf{Ker}(\mathcal{P})$, let $\boldsymbol{j} \subset \boldsymbol{i}$ belong to $\xi = \boldsymbol{I}[<\alpha]$, $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}}$, and $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{j}}$. Check $Z = X \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}}) \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$. By definition, $Y = Y'{\downarrow}_{\subseteq \boldsymbol{j}}$ and $X = X'{\downarrow}_{\subseteq \boldsymbol{j}}$ for some $X', Y' \in \mathcal{P}$. And we have $Y'{\downarrow}_{\subseteq \boldsymbol{j}} = Y \subseteq X'{\downarrow}_{\subseteq \boldsymbol{j}}$. Therefore the set

$$Z' = X' \cap (Y'{\downarrow}_{\subseteq \boldsymbol{j}}{\uparrow}\xi) = X' \cap (Y{\uparrow}\xi)$$

belongs to $\mathcal{P}$ by $2^\dagger$. Then $Z'{\downarrow}_{\subseteq \boldsymbol{i}} = (X'{\downarrow}_{\subseteq \boldsymbol{i}}) \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}}) = X \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}}) = Z$, hence $Z \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, as required.

To check 4^* of Section 22, assume that $\boldsymbol{i} \in \boldsymbol{I}[<\alpha]$, $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, $\varnothing \neq Y \subseteq X$ is clopen in X, and prove that $Y \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$. We have $Y \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ by Lemma 11.3. By definition, $X = X'{\downarrow}_{\subseteq \boldsymbol{i}}$ for some $X' \in \mathcal{P}$. It follows by Lemma 10.5 that the set $Y' = X' \cap (Y{\uparrow}\xi)$ belongs to $\mathbf{IPS}_\xi$, and Y' is clopen in X' by the choice of W. It follows that $Y' \in \mathcal{P}$ by $3^\dagger$ of Definition 40.1. Therefore $Y = Y'{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, as required.

Thus indeed $\mathcal{K} = \mathbf{Ker}(\mathcal{P})$ is a strong $\boldsymbol{I}[<\alpha]$-kernel. Then the expanded system $\mathcal{K}^{\mathsf{ex}}$ is a strong $\boldsymbol{I}$-kernel by Lemma 22.4. It follows by Lemma 22.3 that $\mathscr{Z} = \mathbf{NH}(\mathcal{K}^{\mathsf{ex}})$ is a normal forcing with $\mathscr{Z}{\downarrow}_{\subseteq \boldsymbol{i}} = \mathcal{K}^{\mathsf{ex}}{}_{\boldsymbol{i}} = \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \boldsymbol{I}$ and accordingly $\mathscr{Z}{\downarrow}_{\subseteq \boldsymbol{i}} = \mathcal{K}_{\boldsymbol{i}} = \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \boldsymbol{I}[<\alpha]$. Therefore $\mathcal{P} \subseteq \mathscr{Z}$ by 6° of Section 21 for $\mathscr{Z}$, hence $\mathscr{X} \subseteq \mathscr{Z}$ by the minimality of $\mathscr{X}$.

We similarly get the inverse inclusion $\mathscr{Z} \subseteq \mathscr{X}$ by the minimality of $\mathscr{Z}$. We conclude that $\mathscr{X} = \mathscr{Z}$, and hence the equality $\mathscr{X}\!\downarrow_{\subseteq \boldsymbol{i}} = \mathcal{P}\!\downarrow_{\subseteq \boldsymbol{i}}$ holds for all $\boldsymbol{i} \in \boldsymbol{I}[<\alpha]$ by the above.

The proof of the converse claim goes pretty similar to the proof of Lemma 22.3, and hence we leave the details for the reader. □

41 Hulls, liftings and restrictions of rudiments

For any $\alpha < \omega_1$, if $\varnothing \neq \mathcal{U} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$ then there exists a least set $\mathcal{P} \in \mathbf{Rud}_\alpha$ with $\mathcal{U} \subseteq \mathcal{P}$. This $\mathcal{P}$ will be denoted by $\mathbf{RH}(\mathcal{U})$, the *rudimentary hull* of $\mathcal{U}$. Note that the number of finite-type sets $\eta \subseteq \boldsymbol{I}[<\alpha]$ is countable, and so is the number of clopen subsets. Therefore we have the following lemma:

Lemma 41.1. *If $\alpha < \omega_1$ and $\varnothing \neq \mathcal{U} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$ is countable then $\mathbf{RH}(\mathcal{U})$ is countable as well.* □

Several next lemmas study *liftings* of rudiments to bigger domains. Recall that if $\gamma < \alpha < \omega_1$ and $\mathcal{P} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\gamma]}$ then $\mathcal{P}\!\uparrow\!\boldsymbol{I}[<\alpha] = \{X\!\uparrow\!\boldsymbol{I}[<\alpha] : X \in \mathcal{P}\}$, where $X\!\uparrow\!\boldsymbol{I}[<\alpha] \in \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$ (lifting) is defined as in Section 8. If $\mathcal{P} \in \mathbf{Rud}_\gamma$ then $\mathcal{P}\!\uparrow\!\boldsymbol{I}[<\alpha]$ is not a rudiment, but $\mathbf{RH}(\mathcal{P}\!\uparrow\!\boldsymbol{I}[<\alpha]) \in \mathbf{Rud}_\alpha$, of course. It is not that easy to clearly describe the structure of $\mathbf{RH}(\mathcal{P}\!\uparrow\!\boldsymbol{I}[<\alpha])$. Yet the next lemma at least claims that small projections do not change.

Lemma 41.2. *Assume that $2 \le \gamma < \alpha < \omega_1$ and $\mathcal{P} \in \mathbf{Rud}_\gamma$. Let $\mathcal{R} = \mathbf{RH}(\mathcal{P}\!\uparrow\!\boldsymbol{I}[<\alpha])$. Then $\mathcal{R}\!\downarrow_{\subseteq \boldsymbol{i}} = \mathcal{P}\!\downarrow_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \boldsymbol{I}[<\gamma]$.*

Proof. If $\boldsymbol{i} \in \boldsymbol{I}[<\alpha]$ then let $\underline{\boldsymbol{i}} \in \boldsymbol{I}[<2]$ be the only tuple in $\boldsymbol{I}[<2]$ with $\boldsymbol{i} \approx_{\text{par}} \underline{\boldsymbol{i}}$. Put $\mathcal{K}_{\boldsymbol{i}} = \pi_{\boldsymbol{i},\underline{\boldsymbol{i}}} {\bullet} \mathcal{P}\!\downarrow_{\subseteq \boldsymbol{i}}$. The system $\langle \mathcal{P}\!\downarrow_{\subseteq \boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}[<\gamma]}$ is an $\boldsymbol{I}[<\gamma]$-kernel (see the proof of Lemma 40.2). It easily follows by $4^\dagger$ of Definition 40.1 that $\langle \mathcal{K}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \boldsymbol{I}[<\alpha]}$ is an $\boldsymbol{I}[<\alpha]$-kernel, and (*) $\mathcal{K}_{\boldsymbol{i}} = \mathcal{P}\!\downarrow_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i}$ in the old domain $\boldsymbol{I}[<\gamma]$. Then $\mathcal{Q} = \{X \in \boldsymbol{I}[<\alpha] : \forall\, \boldsymbol{i} \in \boldsymbol{I}[<\alpha](X\!\downarrow_{\subseteq \boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}})\} \in \mathbf{Rud}_\alpha$. Therefore $\mathcal{R} \subseteq \mathcal{Q}$. But $\mathcal{Q}\!\downarrow_{\subseteq \boldsymbol{i}} = \mathcal{K}_{\boldsymbol{i}} = \mathcal{P}\!\downarrow_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \boldsymbol{I}[<\gamma]$ by (*). □

Lemma 41.3. *If $\gamma < \alpha < \omega_1$ and $\mathcal{U} \in \mathbf{Rud}_\alpha$ then the set $\mathcal{U}\!\downarrow\!\boldsymbol{I}[<\gamma] = \{X\!\downarrow\!\boldsymbol{I}[<\gamma] : X \in \mathcal{U}\}$ belongs to $\mathbf{Rud}_\gamma$.*

Proof. To check $2^\dagger$ of Definition 40.1 for $\mathcal{U}\!\downarrow\!\boldsymbol{I}[<\gamma]$, suppose that $X' = X\!\downarrow\!\boldsymbol{I}[<\gamma]$, $Y' = Y\!\downarrow\!\boldsymbol{I}[<\gamma]$, where $X, Y \in \mathcal{U}$, and $\eta \in \mathbf{FT}(\boldsymbol{I}[<\gamma])$, $Y'\!\downarrow\!\eta \subseteq X'\!\downarrow\!\eta$. We have to prove that $Z' = X' \cap (Y'\!\downarrow\!\eta\!\uparrow\!\boldsymbol{I}[<\gamma])$ belongs to $\mathcal{U}\!\downarrow\!\boldsymbol{I}[<\gamma]$. Note that $\eta \in \mathbf{FT}(\boldsymbol{I}[<\alpha])$ as well because $\boldsymbol{I}[<\gamma]$ itself belongs to $\mathbf{FT}(\boldsymbol{I}[<\alpha])$. It follows that $Z = X \cap (Y\!\downarrow\!\eta\!\uparrow\!\boldsymbol{I}[<\alpha])$ belongs to $\mathcal{U}$. However easily $Z' = Z\!\downarrow\!\boldsymbol{I}[<\gamma]$.

Conditions $3^\dagger$ and $4^\dagger$ are verified by similar routine arguments. □

Corollary 41.4. *If $\gamma < \alpha < \omega_1$ and $\mathscr{D}^{\boldsymbol{I}[<\gamma]} \in \mathscr{X} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\gamma]}$, $\mathcal{P} = \mathbf{RH}(\mathscr{X})$, then the sets $\mathcal{Q}' = \mathbf{RH}(\mathscr{X}{\uparrow}\boldsymbol{I}[<\alpha])$ and $\mathcal{Q} = \mathbf{RH}(\mathcal{P}{\uparrow}\boldsymbol{I}[<\alpha])$ coincide.*

Proof. Clearly $\mathcal{Q}' \subseteq \mathcal{Q}$. To prove the converse, note that $\mathcal{P}' = \mathcal{Q}'{\downarrow}\boldsymbol{I}[<\gamma] \in \mathbf{Rud}_\gamma$ by Lemma 41.3, and obviously $\mathscr{X} \subseteq \mathcal{P}'$. Therefore $\mathcal{P} \subseteq \mathcal{P}'$. On the other hand, $\mathcal{P}'{\uparrow}\boldsymbol{I}[<\alpha] \subseteq \mathcal{Q}'$ because if $Y \in \mathcal{Q}'$ and $X = Y{\downarrow}\boldsymbol{I}[<\gamma] \in \mathcal{Q}'$ then $X{\uparrow}\boldsymbol{I}[<\alpha] = \mathscr{D}^{\boldsymbol{I}[<\alpha]} \cap Y{\downarrow}\boldsymbol{I}[<\gamma]{\uparrow}\boldsymbol{I}[<\alpha] \in \mathcal{Q}'$. (Note that $\mathscr{D}^{\boldsymbol{I}[<\alpha]} \in \mathcal{Q}'$ since $\mathscr{D}^{\boldsymbol{I}[<\gamma]} \in \mathscr{X}$.) To conclude, $\mathcal{Q} = \mathbf{RH}(\mathcal{P}{\uparrow}\boldsymbol{I}[<\alpha]) \subseteq \mathbf{RH}(\mathcal{P}'{\uparrow}\boldsymbol{I}[<\alpha]) \subseteq \mathbf{RH}(\mathcal{Q}') = \mathcal{Q}'$. □

Lemma 41.5. *Assume that $\lambda < \omega_1$ is limit, $\mathcal{P}_\gamma \in \mathbf{Rud}_\gamma$ for all $\gamma < \lambda$, and $\mathcal{P}_\gamma{\uparrow}\boldsymbol{I}[<\alpha] \subseteq \mathcal{P}_\alpha$ for all $\gamma < \alpha < \lambda$. Then $\mathcal{P} = \bigcup_{\gamma<\lambda}(\mathcal{P}_\gamma{\uparrow}\boldsymbol{I}[<\lambda]) \in \mathbf{Rud}_\lambda$.*

Proof. $\mathcal{P} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\lambda]}$ holds by Lemma 10.5.

We check $3^\dagger$ of Definition 40.1. Let $Y \in \mathbf{IPS}_{\boldsymbol{I}[<\lambda]}$, $Y \subseteq X \in \mathcal{P}$, Y be clopen in X; prove $Y \in \mathcal{P}$. By compactness, any clopen set is a finite union of basic clopen sets, hence there is $\gamma < \lambda$ such that $X = X'{\uparrow}\boldsymbol{I}[<\lambda]$ and $Y = Y'{\uparrow}\boldsymbol{I}[<\lambda]$, where $X' = X{\downarrow}\boldsymbol{I}[<\gamma] \in \mathcal{P}_\gamma$ and $Y' = Y{\downarrow}\boldsymbol{I}[<\gamma]$. However $Y' \in \mathbf{IPS}_{\boldsymbol{I}[<\gamma]}$ by Lemma 10.4 and Y' is clopen in X' by Lemma 11.4. Thus $Y' \in \mathcal{P}_\gamma$ by $3^\dagger$ of Definition 40.1 for $\mathcal{P}_\gamma$. Therefore $Y = Y'{\uparrow}\boldsymbol{I}[<\lambda] \in \mathcal{P}$.

We check $2^\dagger$. Assume that $\eta \in \boldsymbol{\Xi}$, $\eta \subseteq \boldsymbol{I}[<\lambda]$ is finite-type in $\boldsymbol{I}[<\lambda]$, $X, Y \in \mathcal{P}$, and $Y{\downarrow}\eta \subseteq X{\downarrow}\eta$; prove that the set $Z = X \cap (Y{\downarrow}\eta{\uparrow}\boldsymbol{I}[<\lambda])$ belongs to $\mathcal{P}$. As above, there is $\gamma < \lambda$ such that $X = X'{\uparrow}\boldsymbol{I}[<\lambda]$ and $Y = Y'{\uparrow}\boldsymbol{I}[<\lambda]$, where $X' = X{\downarrow}\boldsymbol{I}[<\gamma]$, $Y' = Y{\downarrow}\boldsymbol{I}[<\gamma]$, and $X', Y' \in \mathcal{P}_\gamma$. Further, $\eta' = \eta \cap \boldsymbol{I}[<\gamma] \in \boldsymbol{\Xi}$ and η' is of finite-type in $\boldsymbol{I}[<\gamma]$, and clearly $Y'{\downarrow}\eta' = Y{\downarrow}\eta' \subseteq X'{\downarrow}\eta'$. It follows by $2^\dagger$ for $\mathcal{P}_\gamma$ that the set $Z' = X' \cap (Y'{\downarrow}\eta'{\uparrow}\boldsymbol{I}[<\gamma])$ belongs to $\mathcal{P}_\gamma$. On the other hand, $Z{\downarrow}\boldsymbol{I}[<\gamma] = (X{\downarrow}\boldsymbol{I}[<\gamma]) \cap (Y{\downarrow}\eta'{\uparrow}\boldsymbol{I}[<\gamma])$ by Lemma 10.3, so that $Z{\downarrow}\boldsymbol{I}[<\gamma] = Z' \in \mathcal{P}_\gamma$. Therefore $Z = Z'{\uparrow}\boldsymbol{I}[<\lambda] \in \mathcal{P}$.

$4^\dagger$. Take $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$ in $\boldsymbol{I}[<\lambda]$, and $X \in \mathcal{P}$; show that $Y = \pi_{\boldsymbol{ij}}{\bullet}X \in \mathcal{P}$. By construction, there is an index $\gamma < \lambda$ such that $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}[<\gamma]$, and $X = X'{\uparrow}\boldsymbol{I}[<\lambda]$, where $X' = X{\downarrow}\boldsymbol{I}[<\gamma] \in \mathcal{P}_\gamma$. Then $Y' = \pi_{\boldsymbol{ij}}{\bullet}X' \in \mathcal{P}_\gamma$ by $4^\dagger$ for $\mathcal{P}_\gamma$, and on the other hand easily $Y = Y'{\uparrow}\boldsymbol{I}[<\lambda] \in \mathcal{P}$, as required. □

42 Refining rudimentary forcings

Definition 42.1 (refinement)**.** Let $\mathcal{P}, \mathcal{Q} \in \mathbf{Rud}_\alpha$, $\xi = \boldsymbol{I}[<\alpha]$. Say that $\mathcal{Q}$ is a *refinement* of $\mathcal{P}$, in symbol $\mathcal{P} \sqsubset \mathcal{Q}$, if the next three conditions hold:

$5^\dagger$. $\mathscr{D}^\xi \in \mathcal{P}$.

$6^\dagger$. If $\eta \in \mathbf{FT}(\xi)$, $X \in \mathcal{P}$, $Y \in \mathcal{Q}$, $Y{\downarrow}\eta \subseteq X{\downarrow}\eta$, then there is $Z \in \mathcal{Q}$ such that $Z \subseteq X$ and $Z{\downarrow}\eta = Y{\downarrow}\eta$ — in particular ($\eta = \varnothing$) if $X \in \mathcal{P}$ then there is $Z \in \mathcal{Q}$ such that $Z \subseteq X$.

$7^\dagger$. If $\boldsymbol{i} \in \xi$, $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$, then $X \cap Y$ is *clopen* in Y, hence if in addition $X \cap Y \neq \varnothing$ then $X \cap Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$ by $3^\dagger$ of Definition 40.1. □

The transitivity of $\sqsubset$ does not necessarily hold.

Lemma 42.2. *Assume that $\alpha < \omega_1$, $\mathcal{P} \sqsubset \mathcal{Q}$ belong to $\mathbf{Rud}_\alpha$ and $\boldsymbol{j} \subset \boldsymbol{i}$ belong to $\boldsymbol{I}[{<}\alpha]$. Then*

(a) *if $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}}$, then there is $Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$, $Y \subseteq X$;*

(b) *if $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$, $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{j}}$, then there is $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$ such that $Z \subseteq X$ and $Z{\downarrow}_{\subseteq \boldsymbol{j}} = Y$;*

(c) *if $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$, $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{j}}$, $W \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$, and the set $Z = X \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}})$ satisfies $Z \subseteq W$, then $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$.*

Proof. (a) By definition, there exists $X' \in \mathcal{P}$ with $X = X'{\downarrow}_{\subseteq \boldsymbol{j}}$. By $6^\dagger$ of Definition 42.1 (with $\eta = \varnothing$), there is $Y' \in \mathcal{Q}$, $Y' \subseteq X'$. Put $Y = Y'{\downarrow}_{\subseteq \boldsymbol{j}}$.

(b) There exist $X' \in \mathcal{P}$, $Y' \in \mathcal{Q}$ with $X = X'{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y = Y'{\downarrow}_{\subseteq \boldsymbol{j}}$. Thus $Y'{\downarrow}_{\subseteq \boldsymbol{j}} \subseteq X'{\downarrow}_{\subseteq \boldsymbol{j}}$. By $6^\dagger$ of Definition 42.1 (with $\eta = [\subseteq \boldsymbol{j}]$), there is $Z' \in \mathcal{Q}$, $Z' \subseteq X'$, such that $Z'{\downarrow}_{\subseteq \boldsymbol{j}} = Y'{\downarrow}_{\subseteq \boldsymbol{j}} = Y$. Put $Z = Z'{\downarrow}_{\subseteq \boldsymbol{i}}$.

(c) We have $Z{\downarrow}_{\subseteq \boldsymbol{j}} = Y \subseteq W{\downarrow}_{\subseteq \boldsymbol{j}}$, therefore $U = W \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}}) \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$ as $\mathbf{Ker}(\mathcal{Q})$ is a kernel by Lemma 40.2. Yet $Z = U \cap X$, hence Z is clopen in U by $7^\dagger$ of Definition 42.1. Thus $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$ by $3^\dagger$ of Section 40. □

The next theorem deals with the set $\mathbf{RH}(\mathcal{P} \cup \mathcal{Q})$ (the rudimentary hull) in case $\mathcal{P} \sqsubset \mathcal{Q}$. We expect that $\mathcal{Q}$ is $\subseteq$-dense in $\mathbf{RH}(\mathcal{P} \cup \mathcal{Q})$, in this case, but thus turns out to be too hard a problem. Still a result of this form holds in a local form as claim (I) of the next theorem shows.

Theorem 42.3. *Assume that $\mathcal{P} \sqsubset \mathcal{Q}$ belong to $\mathbf{Rud}_\alpha$ and $\mathcal{R} = \mathbf{RH}(\mathcal{P} \cup \mathcal{Q})$. Then, for any $\boldsymbol{i} \in \boldsymbol{I}[{<}\alpha]$, $\mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$ is $\subseteq$-open-dense in $\mathcal{R}{\downarrow}_{\subseteq \boldsymbol{i}}$, that is,*

(I) $\forall\, Z \in \mathcal{R}{\downarrow}_{\subseteq \boldsymbol{i}}\, \exists\, Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}\, (Y \subseteq Z)$, *and*

(II) $\forall\, Z \in \mathcal{R}{\downarrow}_{\subseteq \boldsymbol{i}}\, \forall\, Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}\, (Z \subseteq Y \Longrightarrow Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}})$.

Proof. Define sets $\mathcal{Z}_{\boldsymbol{i}} \subseteq \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ by induction on $\mathtt{lh}(\boldsymbol{i})$ as follows:

(A) if $\mathtt{lh}(\boldsymbol{i}) = 1$ then simply $\mathcal{Z}_{\boldsymbol{i}} = \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}} \cup \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$;

(B) if $\mathtt{lh}(\boldsymbol{i}) = n+1 \geq 2$ and $\boldsymbol{j} = \boldsymbol{i} {\restriction} n$ then $\mathcal{Z}_{\boldsymbol{i}}$ contains all $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$ and all sets $X \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}})$, where $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y \in \mathcal{Z}_{\boldsymbol{j}}$, $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{j}}$.

Let $\boldsymbol{j} \in \boldsymbol{I}[<\alpha]$. We prove the following list of claims, one by one:

(1) $\mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}} \cup \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}} \subseteq \mathcal{Z}_{\boldsymbol{j}} \subseteq \mathbf{IPS}_{\subseteq \boldsymbol{j}}$;

(2) if $Z \in \mathcal{Z}_{\boldsymbol{j}}$ then either $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ or $Z \subseteq X$ for some $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}}$;

(3) if $Z \in \mathcal{Z}_{\boldsymbol{j}}$, and $Z \subseteq W \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$, then $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$;

(4) if $\boldsymbol{j} \subset \boldsymbol{i}$, $Z \in \mathcal{Z}_{\boldsymbol{i}}$, $W \in \mathcal{Z}_{\boldsymbol{j}}$, $W \subseteq Z{\downarrow}_{\subseteq \boldsymbol{j}}$, then $P = Z \cap (W{\uparrow}^{\subseteq \boldsymbol{i}}) \in \mathcal{Z}_{\boldsymbol{i}}$;

(5) if $X \in \mathcal{Z}_{\boldsymbol{j}}$, $\varnothing \neq Y \subseteq X$, Y is clopen in X, then $Y \in \mathcal{Z}_{\boldsymbol{j}}$;

(6) if $\boldsymbol{j}, \boldsymbol{k} \in \boldsymbol{I}[<\alpha]$, $\boldsymbol{k} \approx_{\mathtt{par}} \boldsymbol{j}$, and $X \in \mathcal{Z}_{\boldsymbol{j}}$, then $\pi_{\boldsymbol{jk}} {\bullet} X \in \mathcal{Z}_{\boldsymbol{k}}$;

(7) $\mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ is dense in $\mathcal{Z}_{\boldsymbol{j}}$: if $Z \in \mathcal{Z}_{\boldsymbol{j}}$ then there is $X \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}, X \subseteq Z$;

(8) $\mathcal{Z}_{\boldsymbol{i}} = \mathcal{R}{\downarrow}_{\subseteq \boldsymbol{i}}$.

(1) $\mathcal{Z}{\downarrow}_{\subseteq \boldsymbol{j}} \subseteq \mathbf{IPS}_{\subseteq \boldsymbol{j}}$ goes by induction on $\mathtt{lh}(\boldsymbol{j})$, and the induction step via (B) above is carried out by Lemma 10.5. $\mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}} \subseteq \mathcal{Z}_{\boldsymbol{j}}$ holds directly by the first option of (B), whereas $\mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}} \subseteq \mathcal{Z}{\downarrow}_{\subseteq \boldsymbol{j}}$ is proved by induction using (B) and still Lemma 10.5. Claim (2) are rather easy.

(3) Argue by induction on $\mathtt{lh}(\boldsymbol{j})$. If $\mathtt{lh}(\boldsymbol{j}) = 1$ then use (A) and $7^\dagger$ of Definition 42.1. Suppose that $\mathtt{lh}(\boldsymbol{j}) = n+1 \geq 2$ and $\boldsymbol{k} = \boldsymbol{j} {\restriction} n$. Then either $X \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ and we are done, or $Z = X \cap (Y{\uparrow}^{\subseteq \boldsymbol{j}})$ where $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}}$, $Y \in \mathcal{Z}{\downarrow}_{\subseteq \boldsymbol{k}}$, $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{k}}$. It follows that $Y = Z{\downarrow}_{\subseteq \boldsymbol{k}} \subseteq W{\downarrow}_{\subseteq \boldsymbol{k}} \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{k}}$. Then $Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{k}}$ by the inductive hypothesis. Mow $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ by Lemma 42.2(c).

(4) If $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$ then $Z' = Z{\downarrow}_{\subseteq \boldsymbol{j}} \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$, hence $W \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ by (3), and we are done. Consider the second case of (B), that is, $\mathtt{lh}(\boldsymbol{i}) = n+1 \geq 2$, $\boldsymbol{k} = \boldsymbol{i} {\restriction} n$, and $Z = X \cap (Y{\uparrow}^{\subseteq \boldsymbol{i}})$, where $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{k}}$, $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{k}}$. Then $W \subseteq Z{\downarrow}_{\subseteq \boldsymbol{j}} = Y{\downarrow}_{\subseteq \boldsymbol{j}} \in \mathcal{Q}_{\boldsymbol{j}}$, hence $W \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ by (3). It follows that $U = Y \cap (W{\uparrow}^{\subseteq \boldsymbol{k}}) \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{k}}$. Finally $P = X \cap (U{\uparrow}^{\subseteq \boldsymbol{i}}) \in \mathcal{Z}{\downarrow}_{\subseteq \boldsymbol{i}}$.

(5) Argue by induction. If $\mathtt{lh}(\boldsymbol{j}) = n+1 \geq 2$ and $Z = U \cap (Z'{\uparrow}^{\subseteq \boldsymbol{j}})$, where $U \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}}$, $Z' \in \mathcal{Z}{\downarrow}_{\subseteq \boldsymbol{k}}$, $\boldsymbol{k} = \boldsymbol{j} {\restriction} n$, $Z' \subseteq U{\downarrow}_{\subseteq \boldsymbol{k}}$, use Lemma 11.6 and then use the inductive hypothesis.

(6) A routine induction on (A), (B), based on $4^\dagger$ of Definition 40.1.

(7) Argue by induction on $\mathtt{lh}(\boldsymbol{j})$. If $\mathtt{lh}(\boldsymbol{j}) = 1$, *i.e.*, $Z \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}} \cup \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$, then in case $Z \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}}$ apply Lemma 42.2(a). Assume that $\mathtt{lh}(\boldsymbol{j}) = n+1 \geq 2$. If $Z \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ then there is nothing to prove. Suppose now that $Z = U \cap (Z'{\uparrow}^{\subseteq \boldsymbol{j}})$, where $U \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{j}}$, $Z' \in \mathcal{Z}{\downarrow}_{\subseteq \boldsymbol{k}}$, $\boldsymbol{k} = \boldsymbol{j} {\restriction} n$, $Z' \subseteq U{\downarrow}_{\subseteq \boldsymbol{k}}$. By the inductive hypothesis there is $X' \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{k}}$ such that $X' \subseteq Z'$. Applying Lemma 42.2(b), we get a set $X \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ with $X \subseteq U$ and $X{\downarrow}_{\subseteq \boldsymbol{k}} = X'$.

(8) *Prove* $\subseteq$ *by induction on* $\mathtt{lh}(\boldsymbol{i})$. As case (A) is obvious, consider the step (B). Thus suppose that $\mathtt{lh}(\boldsymbol{i}) = n+1 \geq 2$, $\boldsymbol{j} = \boldsymbol{i} {\restriction} n$, $Z = X \cap$

$(Y{\uparrow}^{\subseteq \boldsymbol{i}}) \in \mathscr{Z}_{\boldsymbol{i}}$, where $X \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y \in \mathscr{Z}_{\boldsymbol{j}}$, $Y \subseteq X{\downarrow}_{\subseteq \boldsymbol{j}}$, and in addition $Z \subseteq W \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{i}}$. Then $Y \subseteq W{\downarrow}_{\subseteq \boldsymbol{j}} \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$, hence $Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$ by the inductive hypothesis. Thus $Y = Y'{\downarrow}_{\subseteq \boldsymbol{j}}$, $X = X'{\downarrow}_{\subseteq \boldsymbol{i}}$, $X' \in \mathcal{P}$, $Y' \in \mathcal{Q}$, and $Y'{\downarrow}_{\subseteq \boldsymbol{j}} \subseteq X'{\downarrow}_{\subseteq \boldsymbol{j}}$. As $X', Y' \in \mathcal{R}$, the set $Z' = X' \cap (Y'{\downarrow}_{\subseteq \boldsymbol{j}}{\uparrow}\boldsymbol{I}[{<}\alpha])$ belongs to $\mathcal{R}$ by $2^{\dagger}$ of Definition 40.1. On the other hand, we have $Z'{\downarrow}_{\subseteq \boldsymbol{i}} = Z$ by Lemma 10.3. Thus $Z \in \mathcal{R}{\downarrow}_{\subseteq \boldsymbol{i}}$, as required. *Now prove the direction* $\supseteq$.

Consider the collection $\mathscr{Z}$ of all sets $X \in \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$ satisfying $X{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathscr{Z}_{\boldsymbol{i}}$ for all $\boldsymbol{i} \in \boldsymbol{I}[{<}\alpha]$. Thus $\mathcal{P} \cup \mathcal{Q} \subseteq \mathscr{Z}$ by (1). We claim that $\mathscr{Z} \in \mathbf{Rud}_{\alpha}$.

Indeed, if $Y \in \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$, $Y \subseteq X \in \mathscr{Z}$, Y is clopen in X, then $Y{\downarrow}_{\subseteq \boldsymbol{i}}$ is clopen in $X{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathscr{Z}_{\boldsymbol{i}}$ for any $\boldsymbol{i} \in \boldsymbol{I}[{<}\alpha]$ by Lemma 11.4, so that $X{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathscr{Z}_{\boldsymbol{i}}$ by (5), and we conclude that $Y \in \mathscr{Z}$. Thus $\mathscr{Z}$ satisfies $3^{\dagger}$ of Definition 40.1.

To check that $\mathscr{Z}$ also satisfies $2^{\dagger}$ of Definition 40.1, assume that $\eta \in \boldsymbol{\Xi}$, $\eta \subseteq \boldsymbol{I}[{<}\alpha]$, $X, Y \in \mathscr{Z}$, and $Y{\downarrow}\eta \subseteq X{\downarrow}\eta$; let's prove that the set $Z = X \cap (Y{\downarrow}\eta{\uparrow}\boldsymbol{I}[{<}\alpha])$ belongs to $\mathscr{Z}$. If $\boldsymbol{i} \in \eta$ then $Z{\downarrow}_{\subseteq \boldsymbol{i}} = Y{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathscr{Z}_{\boldsymbol{i}}$. If $\boldsymbol{i} \in \boldsymbol{I}[{<}\alpha] \smallsetminus \eta$ and $\sigma = \eta \cap [{\subseteq}\boldsymbol{i}]$ then $Z{\downarrow}_{\subseteq \boldsymbol{i}} = X{\downarrow}_{\subseteq \boldsymbol{i}} \cap (Y{\downarrow}\eta){\uparrow}^{\subseteq \boldsymbol{i}}$ by Lemma 10.3, hence yet again $Z{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathscr{Z}_{\boldsymbol{i}}$, as required.

Now to check that $\mathscr{Z}$ satisfies $4^{\dagger}$ of Definition 40.1, make use of (6).

To conclude, $\mathscr{Z} \in \mathbf{Rud}_{\alpha}$, and hence $\mathcal{R} \subseteq \mathscr{Z}$ and $\mathcal{R}_{\boldsymbol{i}} \subseteq \mathscr{Z}{\downarrow}_{\subseteq \boldsymbol{i}}$, as required.

Finally to prove claims (I), (II) of the theorem, make use of (8), and also of (7) and (3). For instance, to check (I), note that $Z \in \mathscr{Z}_{\boldsymbol{i}}$ by (8), and hence there is $Y \in \mathcal{Q}{\downarrow}_{\subseteq \boldsymbol{j}}$, $Y \subseteq Z$ by (7). □

43 Rudimentary sequences

The next definition introduces the notion of transfinite sequences of rudiments, "$\sqsubset$-increasing" in the sense that each term is a $\sqsubset$-successor of the rudimentary hull of the union of all previous terms, by condition (D) of Definition 43.1 below. We use quotation marks because $\sqsubseteq$ is not claimed to be a transitive relation.

We still argue in L.

Definition 43.1. Let a *rudimentary sequence* (or **Rud** sequence) of length $3 \le \lambda \le \omega_1$ be any sequence $\boldsymbol{\varphi} = \langle \mathcal{Q}_{\alpha} \rangle_{\alpha < \lambda}$, satisfying (A),(B),(C),(D) below:

(A) $\mathcal{Q}_0 = \mathcal{Q}_1 = \mathcal{Q}_2 = \{$all clopen sets $X \in \mathbf{IPS}_{\boldsymbol{I}[<2]}\} \in \mathbf{Rud}_2$;

(B) if $\nu < \lambda$ then $\mathcal{Q}_{\nu} \in \mathbf{Rud}_{\nu}$ is at most countable;

(C) if $\alpha < \nu < \lambda$, $\boldsymbol{i} \in \boldsymbol{I}[{<}\alpha]$, and $X \in \mathcal{Q}_{\nu}{\downarrow}_{\subseteq \boldsymbol{i}}$ then $X \subseteq^{\mathtt{fin}} \bigcup(\mathcal{Q}_{\alpha}{\downarrow}_{\subseteq \boldsymbol{i}})$ in the sense of Section 24.

For any such $\boldsymbol{\varphi}$ we put $\bigcup \boldsymbol{\varphi} = \bigcup_{\alpha<\lambda} \mathcal{Q}_\alpha$ and $\mathbf{NH}(\boldsymbol{\varphi}) = \mathbf{NH}(\bigcup \boldsymbol{\varphi})$; thus $\bigcup \boldsymbol{\varphi} \subseteq \mathbf{IPS}$ and $\mathbf{NH}(\boldsymbol{\varphi}) \in \mathbf{NF}$ is a normal forcing.

If $\lambda < \omega_1$ strictly then we define $\bigsqcup \boldsymbol{\varphi} = \bigsqcup_{\alpha<\lambda} \mathcal{Q}_\alpha := \bigcup_{\alpha<\lambda} (\mathcal{Q}_\alpha \!\uparrow\! \boldsymbol{I}[{<}\lambda])$; thus $\bigsqcup \boldsymbol{\varphi} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\lambda]}$, and hence $\mathbf{RH}(\boldsymbol{\varphi}) := \mathbf{RH}(\bigsqcup \boldsymbol{\varphi}) \in \mathbf{Rud}_\lambda$ in this case.

Now we add the last condition.

(D) if $3 \le \nu < \lambda$ then $\mathbf{RH}(\boldsymbol{\varphi} \restriction \nu) \sqsubset \mathcal{Q}_\nu$ in the sense of Definition 42.1; here $\mathbf{RH}(\boldsymbol{\varphi} \restriction \nu) = \mathbf{RH}(\bigsqcup (\boldsymbol{\varphi} \restriction \nu)) = \mathbf{RH}(\bigcup_{\alpha<\nu} (\mathcal{Q}_\alpha \!\uparrow\! \boldsymbol{I}[{<}\nu]))$.

Let $\mathbf{RudS}_\lambda =$ all $\mathbf{Rud}$ sequences of length λ, $\mathbf{RudS} = \bigcup_{\lambda<\omega_1} \mathbf{RudS}_\lambda$. $\square$

Theorem 43.2 (in $\mathbf{L}$). *Let* $\boldsymbol{\varphi} = \langle \mathcal{Q}_\alpha \rangle_{\alpha<\lambda} \in \mathbf{RudS}_\lambda$, $3 \le \lambda \le \omega_1$. *Then*

(i) $\mathcal{R} = \mathbf{RH}(\boldsymbol{\varphi}) \in \mathbf{Rud}_\lambda$, $\mathscr{D}^{\boldsymbol{I}[<\lambda]} \in \mathcal{R}$, *and if* $\lambda < \omega_1$ *then* $\mathcal{R}$ *is countable*;

(ii) *if* $\alpha < \lambda$ *then*: (a) *the set* $\mathcal{P}_\alpha = \mathbf{RH}(\boldsymbol{\varphi} \restriction \alpha) \in \mathbf{Rud}_\alpha$ *is countable,*
(b) $\mathscr{D}^{\boldsymbol{I}[<\alpha]} \in \mathcal{P}_\alpha$,
(c) $\mathcal{Q}^-_\alpha \subseteq \mathcal{P}_\alpha = \mathbf{RH}(\mathcal{Q}^-_\alpha)$, *where* $\mathcal{Q}^-_\alpha = \bigsqcup (\boldsymbol{\varphi} \restriction \alpha) = \bigcup_{\gamma<\alpha} (\mathcal{Q}_\gamma \!\uparrow\! \boldsymbol{I}[{<}\alpha])$,
(d) $\forall X \in \mathcal{P}_\alpha \, \exists Y \in \mathcal{Q}_\alpha \, (Y \subseteq X)$;

(iii) *if* $\gamma < \alpha < \lambda$ *then* $(\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma) \!\uparrow\! \boldsymbol{I}[{<}\alpha] \subseteq \mathcal{P}_\alpha$;

(iv) *if* $\lambda = \gamma{+}1$ *then* $\mathcal{R} = \mathbf{RH}((\mathcal{Q}^-_\gamma \cup \mathcal{Q}_\gamma) \!\uparrow\! \boldsymbol{I}[{<}\lambda]) = \mathbf{RH}((\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma) \!\uparrow\! \boldsymbol{I}[{<}\lambda])$;

(v) *if* $\lambda = \gamma + 1$ *and* $\boldsymbol{j} \in \boldsymbol{I}[{<}\gamma]$ *then* $\mathcal{R} \!\downarrow_{\subseteq \boldsymbol{j}} = (\mathbf{RH}(\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma)) \!\downarrow_{\subseteq \boldsymbol{j}}$;

(vi) *if* $\lambda < \omega_1$ *is a limit ordinal then* $\mathcal{R} = \bigcup_{\alpha<\lambda} (\mathcal{P}_\alpha \!\uparrow\! \boldsymbol{I}[{<}\lambda])$, *and the set* $(\bigcup \boldsymbol{\varphi}) \!\uparrow\! \boldsymbol{I}[{<}\lambda]$ *is* $\subseteq$*-dense in* $\mathcal{R}$;

(vii) *if* $\boldsymbol{j} \in \boldsymbol{I}[{<}\alpha]$, $\alpha < \lambda < \omega_1$, *then the set* $\bigcup_{\alpha \le \beta < \lambda} (\mathcal{Q}_\beta \!\downarrow_{\subseteq \boldsymbol{j}})$ *is* $\subseteq$*-dense in* $\mathcal{R} \!\downarrow_{\subseteq \boldsymbol{j}}$;

(viii) *if* $\boldsymbol{j} \in \boldsymbol{I}[{<}\alpha]$, $2 \le \alpha < \lambda < \omega_1$, *then* $\mathcal{Q}_\alpha \!\downarrow_{\subseteq \boldsymbol{j}}$ *is* $\subseteq$*-predense in* $\mathcal{R} \!\downarrow_{\subseteq \boldsymbol{j}}$;

(ix) *if* $\lambda < \omega_1$, $\eta \in \boldsymbol{\Xi}$, $\eta \subseteq \boldsymbol{I}[{<}\lambda]$, *and* $X \in \mathbf{IPS}_\eta$, *then* $X \in \mathscr{X} := \mathbf{NH}(\boldsymbol{\varphi})$ *iff* $X \!\downarrow_{\subseteq \boldsymbol{j}} \in \mathcal{R} \!\downarrow_{\subseteq \boldsymbol{j}}$ *for all* $\boldsymbol{j} \in \eta$, *where* $\mathcal{R} = \mathbf{RH}(\boldsymbol{\varphi})$ *by* (i).

(x) *therefore, by* (ix), *if* $\lambda < \omega_1$ *and* $\boldsymbol{i} \in \boldsymbol{I}[{<}\lambda]$, *then the sets* $\mathcal{R} = \mathbf{RH}(\boldsymbol{\varphi})$ *and* $\mathscr{X} = \mathbf{NH}(\boldsymbol{\varphi})$ *satisfy* $\mathscr{X} \!\downarrow_{\subseteq \boldsymbol{j}} = \mathcal{R} \!\downarrow_{\subseteq \boldsymbol{j}}$.

Proof. (i), (ii) are easy: $\mathscr{D}^{\boldsymbol{I}[<\lambda]} \in \mathcal{R}$ and (ii)(b) hold by (A) of Definition 43.1, (ii)(d) holds by (D) and $6^\dagger$ of Section 42 (the particular case).

(iii) We have $\mathcal{P}_\gamma \!\uparrow\! \boldsymbol{I}[{<}\alpha] \subseteq \mathbf{RH}(\mathcal{Q}^-_\gamma \!\uparrow\! \boldsymbol{I}[{<}\alpha])$ by Corollary 41.4, hence

$$(\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma) \!\uparrow\! \boldsymbol{I}[{<}\alpha] \subseteq \mathbf{RH}(\mathcal{Q}^-_\gamma \!\uparrow\! \boldsymbol{I}[{<}\alpha]) \cup (\mathcal{Q}_\gamma \!\uparrow\! \boldsymbol{I}[{<}\alpha]) \subseteq \mathbf{RH}(\mathcal{Q}^-_\alpha) = \mathcal{P}_\alpha,$$

as required. ($\mathscr{D}^{\boldsymbol{I}[<\gamma]} \in \mathcal{P}_\gamma$ holds by (ii)(b).)

(iv) Let $\mathcal{U} = \mathcal{Q}_\gamma^-$. Then $\mathcal{U} \subseteq \mathcal{P}_\gamma = \mathbf{RH}(\mathcal{U})$ and

$$\begin{array}{rclcll} \mathcal{R} & = & \mathbf{RH}((\mathcal{U} \cup \mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda]) & \subseteq & \mathbf{RH}((\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda]) & \subseteq \\ & & & \subseteq & \mathbf{RH}(\mathbf{RH}(\mathcal{U} \cup \mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda]) & , \end{array}$$

because $\mathcal{U} \cup \mathcal{Q}_\gamma \subseteq \mathcal{P}_\gamma \cup \mathcal{Q}_\gamma \subseteq \mathbf{RH}(\mathcal{U} \cup \mathcal{Q}_\gamma)$. On the other hand, by Corollary 41.4, we have $\mathbf{RH}((\mathcal{U}\cup\mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda]) = \mathbf{RH}(\mathbf{RH}(\mathcal{U}\cup\mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda])$, so that both inclusions in the displayed formula are equalities, and we are done.

(v) We have $\mathcal{R} = \mathbf{RH}((\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda]) = \mathbf{RH}(\mathbf{RH}(\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda])$, see the proof of (iv). Therefore $\mathcal{R}{\downarrow}_{\subseteq \boldsymbol{j}} = \mathbf{RH}(\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma){\downarrow}_{\subseteq \boldsymbol{j}}$ by Lemma 41.2.

(vi) As $\mathbf{RH}(\mathcal{Q}_\alpha^-) = \mathcal{P}_\alpha$ by (ii), $\mathcal{P}_\alpha{\uparrow}\boldsymbol{I}[{<}\lambda] \subseteq \mathbf{RH}(\mathcal{Q}_\alpha^-{\uparrow}\boldsymbol{I}[{<}\lambda]) \subseteq \mathcal{R}$ by Corollary 41.4, hence the set $\mathcal{R}' = \bigcup_{\alpha<\lambda}(\mathcal{P}_\alpha{\uparrow}\boldsymbol{I}[{<}\lambda])$ satisfies $\mathcal{R}' \subseteq \mathcal{R}$. Yet $\mathcal{R}' = \mathbf{RH}(\mathcal{R}')$ by Lemma 41.5 and (iii). Then, as $\mathcal{Q}_\alpha^- \subseteq \mathcal{P}_\alpha$, we have

$$\mathcal{R} = \mathbf{RH}(\textstyle\bigcup_{\alpha<\lambda}(\mathcal{Q}_\alpha^-{\uparrow}\boldsymbol{I}[{<}\lambda])) \subseteq \mathbf{RH}(\bigcup_{\alpha<\lambda}(\mathcal{P}_\alpha{\uparrow}\boldsymbol{I}[{<}\lambda])) = \mathbf{RH}(\mathcal{R}') = \mathcal{R}',$$

and clearly $\mathcal{R}' \subseteq \mathcal{R}$, so that $\mathcal{R} = \mathcal{R}' = \bigcup_{\alpha<\lambda}(\mathcal{P}_\alpha{\uparrow}\boldsymbol{I}[{<}\lambda])$, as required.

To prove the density in (vi), let $X \in \mathcal{R}$. Then $X = Y{\uparrow}\boldsymbol{I}[{<}\lambda]$, where $Y \in \mathcal{P}_\alpha$ and $\alpha < \lambda$, by the above. However $\mathcal{P}_\alpha = \mathbf{RH}(\varphi{\restriction}\alpha) \sqsubset \mathcal{Q}_\alpha$ by (D) of Definition 43.1. Therefore there iz $Y' \in \mathcal{Q}_\alpha$, $Y' \subseteq Y$, by $6^\dagger$ of Definition 42.1. It remains to take $X' = Y'{\uparrow}\boldsymbol{I}[{<}\lambda]$.

The limit case in (vii) easily follows from (vi). Therefore suppose that $\lambda = \gamma + 1$ in (vii). Then $\alpha \le \gamma$, $\mathcal{R} = \mathbf{RH}((\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda])$ by (iv), $\boldsymbol{j} \in \boldsymbol{I}[{<}\gamma]$. We convert this to $\mathcal{R} = \mathbf{RH}(\mathbf{RH}(\mathcal{P}_\gamma\cup\mathcal{Q}_\gamma){\uparrow}\boldsymbol{I}[{<}\lambda])$ by Corollary 41.4. Therefore $\mathcal{R}{\downarrow}_{\subseteq \boldsymbol{j}} = (\mathbf{RH}(\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma)){\downarrow}_{\subseteq \boldsymbol{j}}$ by Lemma 41.2. However $\mathcal{Q}_\gamma{\downarrow}_{\subseteq \boldsymbol{j}}$ is dense in $(\mathbf{RH}(\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma)){\downarrow}_{\subseteq \boldsymbol{j}}$ by Theorem 42.3(I).

(viii) Let λ be limit and $X \in \mathcal{R}{\downarrow}_{\subseteq\lambda}$. Then by (vii) there is an ordinal β, $\alpha < \beta < \lambda$, and $Y \in \mathcal{Q}_\beta{\downarrow}_{\subseteq \boldsymbol{j}}$, such that $Y \subseteq X$. Then $Y \subseteq^{\mathtt{fin}} \bigcup(\mathcal{Q}_\alpha{\downarrow}_{\subseteq \boldsymbol{j}})$ by (C) of Definition 43.1. We conclude that there is $Z \in \mathcal{Q}_\alpha{\downarrow}_{\subseteq \boldsymbol{j}}$ such that $Y \cap Z$ is not meager in Y. Therefore there us a set $\varnothing \ne U \subseteq Y \cap Z$ clopen in Y. Then $U \in \mathcal{Q}_\beta{\downarrow}_{\subseteq \boldsymbol{j}}$ by $3^\dagger$ of Definition 40.1, and we are done.

Now let $\lambda = \gamma + 1$ in (viii). Suppose that $X \in \mathcal{R}{\downarrow}_{\subseteq \boldsymbol{j}}$, where $\mathcal{R}{\downarrow}_{\subseteq \boldsymbol{j}} = (\mathbf{RH}(\mathcal{P}_\gamma \cup \mathcal{Q}_\gamma)){\downarrow}_{\subseteq \boldsymbol{j}}$ by (v). It follows from Theorem 42.3(I) that there is a set $Y \in \mathcal{Q}_\gamma{\downarrow}_{\subseteq \boldsymbol{j}}$ with $Y \subseteq X$. Then proceed as in the limit case.

Finally check (ix). By definition the set $\mathscr{X} = \mathbf{NH}(\varphi)$ satisfies $\mathscr{X} = \mathbf{NH}(\bigcup_{\alpha<\lambda}\mathcal{Q}_\alpha)$. As obviously $\mathcal{R} = \mathbf{RH}(\varphi) \subseteq \mathbf{NH}(\varphi)$, we have $\mathscr{X} = \mathbf{NH}(\mathcal{R})$ as well. It follows that $\mathscr{X}{\downarrow}_{\subseteq \boldsymbol{j}} = \mathcal{R}{\downarrow}_{\subseteq \boldsymbol{j}}$ for all $\boldsymbol{j} \in \boldsymbol{I}[{<}\lambda]$ by Lemma 40.2. It remains to refer to 6° of Section 21 for $\mathscr{X}$. □

VIII Specifying rudimentary sequences

The goal of this Chapter is to spesify a list of conditions which imply that the normal forcing $\mathscr{X} = \mathbf{RH}(\varphi)$, generated by a given **Rud** sequence $\varphi \in \mathbf{L}$ of length ω_1, satisfies Theorem 34.1.

We still argue in L in this Chapter.

44 Coding iterated perfect sets

Further study of **Rud** sequences will involve a coding system of iterated perfect sets based on codes in $\mathbf{HC}$ = all hereditarily countable sets.

Clearly any set X in some $\mathbf{IPS}_\xi$, $\xi \neq \varnothing$, is of cardinality continuum, hence X does not belong to $\mathbf{HC}$. This makes it difficult to evaluate the complexity of different collections of sets X of such kind. To fix this problem, we make use of a coding by countable dense subsets.

Definition 44.1 (codes)**.** If $\xi \in \boldsymbol{\Xi}$ then let $\mathbf{cIPS}_\xi$ (**c** from 'codes') consist of all *at most countable* sets $A \subseteq \mathscr{D}^\xi$ such that the closure $A^\#$ in $\mathscr{D}^\xi$ belongs to $\mathbf{IPS}_\xi$. We put $\mathbf{cIPS} = \bigcup_{\xi\in\boldsymbol{\Xi}} \mathbf{cIPS}_\xi$; thus $\mathbf{cIPS} \subseteq \mathbf{HC}$.

If $\mathcal{A} \subseteq \mathbf{cIPS}$ then let $\mathcal{A}^\sharp = \{A^\# : A \in \mathcal{A}\}$. □

In the trivial case $\xi = \varnothing$, the collection $\mathbf{cIPS}_\varnothing = \mathbf{IPS}_\varnothing$ contains the only one element $\mathbb{1} = \{\varnothing\}$, see Remark 9.3, and $\mathbb{1}^\# = \mathbb{1}$.

45 Getting density

This section is intended to define a condition which implies, for a given sequence $\varphi = \langle \mathcal{Q}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{RudS}_{\omega_1}$, that the set $\mathcal{Q} = \bigcup \varphi = \bigcup_{\alpha<\omega_1} \mathcal{Q}_\alpha$ is $\downarrow\subseteq$*-dense* in $\mathscr{X} = \mathbf{NH}(\varphi) := \mathbf{NH}(\bigcup\varphi)$, that is, $\forall\, X \in \mathscr{X}\, \exists\, U \in \mathcal{Q}\, (U \downarrow\subseteq X)$. This condition will be of *stepwise form*, that is, in the form of a relation between each term $\mathcal{Q}_\alpha$ and the sequence $\varphi\restriction\alpha$ obtained before α.

We continue to argue in L. Under this assumption, the set $\mathbf{HC}$ of all *hereditarily countable sets* satisfies $\mathbf{HC} = \mathbf{L}_{\omega_1}$, and hence $\mathbf{HC}$ is well-ordered by the canonical Gödel relation $\leqslant_{\mathbf{L}}$. Thus $\mathbf{HC} = \{\mathbb{c}_\alpha : \alpha < \omega_1\}$ in $\mathbf{L}$, where $\mathbb{c}_\alpha$ is the αth element of $\mathbf{HC}$ via $\leqslant_{\mathbf{L}}$. Recall that $\mathbf{HC}_{<\alpha} = \{\mathbb{c}_\gamma : \gamma < \alpha\}$. See Section 7 on details. We let

$$\mathbf{cIPS}^{<\alpha} = \mathbf{cIPS} \cap \mathbf{HC}_{<\alpha} \quad \text{and} \quad \mathbf{IPS}^{<\alpha} = \{B^\# : B \in \mathbf{cIPS}^{<\alpha}\}.$$

To provide the density property as above, we add some definitions based on the sequence of sets $\overline{\boldsymbol{S}}_\alpha \subseteq \mathbf{HC}_{<\alpha}$, $\alpha < \omega_1$, satisfying Proposition 7.3.

(I) Let $\alpha < \omega_1$. If there is a unique triple of $M \in \mathbf{cIPS}$ and $M', M'' \in \mathbf{HC}$ such that $\langle \omega, M, M', M'' \rangle \in \overline{\boldsymbol{S}}_\alpha$ then put $\mathbf{M}_\alpha = M$, $\mathbf{M}'_\alpha = M'$, $\mathbf{M}''_\alpha = M''$. Otherwise let $\mathbf{M}_\alpha = \mathbb{1} = \{\varnothing\} \in \mathbf{IPS}_\varnothing = \mathbf{cIPS}_\varnothing$ and $\mathbf{M}'_\alpha = \mathbf{M}''_\alpha = \varnothing$. Note that $\mathbf{M}_\alpha, \mathbf{M}'_\alpha, \mathbf{M}''_\alpha \in \mathbf{HC}_{<\alpha}$ and $\mathbf{M}_\alpha \in \mathbf{cIPS}^{<\alpha}$.

(II) Let $\mathbf{B}_{\alpha k} = \{B \in \mathbf{cIPS}^{<\alpha} : \langle k, B \rangle \in \overline{\boldsymbol{S}}_\alpha\}$ and $\mathbf{B}_{\alpha k}{}^\sharp = \{B^\# : B \in \mathbf{B}_{\alpha k}\}$ for any k. Thus $\mathbf{B}_{\alpha k} \subseteq \mathbf{cIPS}^{<\alpha}$, $\mathbf{B}_{\alpha k}{}^\sharp \subseteq \mathbf{IPS}^{<\alpha}$ are countable.

Lemma 45.1 (in $\mathbf{L}$). *If $M \in \mathbf{cIPS}$, $M', M'' \in \mathbf{HC}$, and $P_k \subseteq \mathbf{cIPS}$, $\forall\, k$, then the following set $\mathbf{W}$ is stationary in ω_1:*

$$\mathbf{W} = \{\alpha : \mathbf{M}_\alpha = M \wedge \mathbf{M}'_\alpha = M' \wedge \mathbf{M}''_\alpha = M'' \wedge \forall\, k\, (P_k \cap \mathbf{cIPS}^{<\alpha} = \mathbf{B}_{\alpha k})\}.$$

The sequences $\langle \langle \mathbf{M}_\alpha, \mathbf{M}'_\alpha, \mathbf{M}''_\alpha \rangle \rangle_{\alpha<\omega_1}$ and $\langle \mathbf{B}_{\alpha k} \rangle_{k<\omega, \alpha<\omega_1}$ belong to $\varDelta_1^{\mathbf{HC}}$.

Proof. Applying Proposition 7.3 for the set

$$\overline{\boldsymbol{S}} = \{\langle \omega, M, M', M'' \rangle\} \cup \{\langle k, B \rangle : k < \omega \wedge B \in P_k\},$$

we conclude that $D := \{\alpha < \omega_1 : \overline{\boldsymbol{S}} \cap \mathbf{HC}_{<\alpha} = \overline{\boldsymbol{S}}_\alpha\}$ is stationary in ω_1. On the other hand, the set W' of all $\alpha < \omega_1$, such that

$$\overline{\boldsymbol{S}} \cap \mathbf{HC}_{<\alpha} = \{\langle \omega, M, M', M'' \rangle\} \cup \{\langle k, B \rangle : k < \omega \wedge B \in P_k \cap \mathbf{cIPS}^{<\alpha}\},$$

is a club. Thus $W' \cap D$ is still stationary. However $W' \cap D \subseteq \mathbf{W}$ by construction. To prove the definability claim apply Proposition 7.3 yet again. □

Now we are sufficiently equipped to consider the density property.

Lemma 45.2 (in $\mathbf{L}$). *Assume that $\boldsymbol{\varphi} = \langle \mathcal{Q}_\alpha \rangle_{\alpha<\omega_1}$ is a* **Rud** *sequence, satisfying the following condition*:

$\mathfrak{P}_1$: for any $\lambda < \omega_1$, if $\mathbf{M}^\#_\lambda \in \mathbf{NH}(\boldsymbol{\varphi} \restriction \lambda)$ and $\|\mathbf{M}^\#_\lambda\| \subseteq \boldsymbol{I}[{<}\lambda]$, then there is $Y \in \mathcal{Q}_\lambda$ satisfying $Y \mathbin{\downarrow\subseteq} \mathbf{M}^\#_\lambda$.

Then the set $\mathcal{Q} = \bigcup \boldsymbol{\varphi} = \bigcup_{\alpha<\omega_1} \mathcal{Q}_\alpha$ is $\downarrow\subseteq$-dense in $\mathbf{NH}(\boldsymbol{\varphi})$.

Proof. Let $X \in \mathbf{NH}(\boldsymbol{\varphi})$. Then obviously $X \in \mathbf{NH}(\boldsymbol{\varphi} \restriction \lambda)$ and $\|\mathbf{M}^\#_\lambda\| \subseteq \boldsymbol{I}[{<}\lambda]$ for all λ larger than some $\lambda_0 < \omega_1$. The set $\mathbf{W} = \{\alpha : \mathbf{M}^\#_\alpha = X\}$ is stationary by Lemma 45.1, hence there is a limit ordinal $\lambda \in \mathbf{W}$, $\lambda \ge \lambda_0$. Applying $\mathfrak{P}_1$, we complete the proof. □

46 Getting fusion

The next lemma provides another stepwise condition which implies the fusion property as in Section 27.

Lemma 46.1 (in $\mathbf{L}$). *Assume that* $\boldsymbol{\varphi} = \langle \mathscr{Q}_\alpha \rangle_{\alpha<\omega_1}$ *is a* **Rud** *sequence, satisfying both* $\mathfrak{P}_1$ *of Lemma 45.2 and the following condition*:

$\mathfrak{P}_2$: for any **limit** $\alpha < \omega_1$, if $\mathbf{M}_\alpha^{\#} \in \mathscr{Q}_{<\alpha} := \bigcup_{\gamma<\alpha} \mathscr{Q}_\gamma$ then there is $X \in \mathscr{Q}_\alpha$ satisfying $X \downarrow\subseteq \mathbf{M}_\alpha^{\#}$ and $X \subseteq^{\mathtt{fd}} \bigcup \mathbf{B}_{\alpha k}{}^{\sharp}$ for all $k < \omega$ such that $\mathbf{B}_{\alpha k}{}^{\sharp} \subseteq \mathscr{Q}_{<\alpha}$ and $\mathbf{B}_{\alpha k}{}^{\sharp}$ is dense in $\mathscr{Q}_{<\alpha}$.

Then the set $\mathbf{NH}(\boldsymbol{\varphi})$ *has the fusion property of Section 27.*

Proof. We argue in $\mathbf{L}$. Let $X_0 \in \mathscr{X} := \mathbf{NH}(\boldsymbol{\varphi})$. Consider a sequence of dense sets $\mathscr{Y}_m \subseteq \mathscr{X}$. We have to find a set $Y \in \mathscr{X}$ satisfying $Y \downarrow\subseteq X_0$ and $Y \subseteq^{\mathtt{fd}} \bigcup \mathscr{Y}_m$ for all m. Assume that $X_0 \in \mathscr{Q} := \bigcup_{\alpha<\omega_1} \mathscr{Q}_\alpha$, by Lemma 45.2.

We may w.l.o.g. assume that each $\mathscr{Y}_m$ is in fact open-dense; then, still by Lemma 45.2, (*) each set $\mathscr{Z}_m := \mathscr{Y}_m \cap \mathscr{Q}$ is open dense in $\mathscr{Q}$. We let $P_m = \{B \in \mathbf{cIPS} : B^{\#} \in \mathscr{Z}_m\}$, so that $\mathscr{Z}_m = \{B^{\#} : B \in P_m\}$, $\forall\, m$. Pick a set $C \in \mathbf{cIPS}$ satisfying $X_0 = C^{\#}$. By Lemma 45.1, the set

$$\mathbf{W} = \{\alpha < \omega_1 : \mathbf{M}_\alpha = C \wedge \forall\, m\, (P_{\alpha m} = \mathbf{B}_{\alpha m})\}$$

is stationary, where $P_{\alpha m} = P_m \cap \mathbf{cIPS}^{<\alpha}$. Let $\mathscr{Z}_{\alpha m} = \{B^{\#} : B \in P_{\alpha m}\}$. Recall that $\mathscr{Q}_{<\alpha} := \bigcup_{\gamma<\alpha} \mathscr{Q}_\gamma$. Note that the set

$$E = \{\alpha < \omega_1 : X_0 = C^{\#} \in \mathscr{Q}_{<\alpha} \wedge \forall\, m\, (\mathscr{Z}_{\alpha m} \text{ is open dense in } \mathscr{Q}_{<\alpha})\}$$

is a club by (*) above. Thus there exists an ordinal $\alpha \in E \cap \mathbf{W}$.

Then we have $\mathbf{M}_\alpha^{\#} = C^{\#} \in \mathscr{Q}_{<\alpha}$, and in addition each $\mathbf{B}_{\alpha m}{}^{\sharp}$ is dense in $\mathscr{Q}_{<\alpha}$. Therefore by $\mathfrak{P}_2$ of the lemma there exists $X \in \mathscr{Q}_\alpha$ satisfying $X \downarrow\subseteq \mathbf{M}_\alpha^{\#} = X_0$ and $X \subseteq^{\mathtt{fd}} \bigcup \mathbf{B}_{\alpha m}{}^{\sharp}$ for all $m < \omega$. However $\mathbf{B}_{\alpha m}{}^{\sharp} = \mathscr{Z}_m \subseteq \mathscr{Y}_m$ by construction. □

47 Getting completeness

Here we introduce another stepwise condition on a **Rud** sequence $\boldsymbol{\varphi}$ which implies the n-completeness property of Definition 39.1 for the according normal hull $\mathbf{NH}(\boldsymbol{\varphi})$.

Lemma 47.1 (in $\mathbf{L}$). *Assume that* $n \geq 2$ *and* $\boldsymbol{\varphi} = \langle \mathscr{Q}_\alpha \rangle_{\alpha<\omega_1}$ *is a* **Rud** *sequence, satisfying* $\mathfrak{P}_1$ *of Lemma 45.2 and the following condition*:

$\mathfrak{P}_3^n$: if $\mathbb{n} \geq 2$ then for any $\lambda < \omega_1$, if $\mathbf{M}_\lambda^\# \in \mathcal{Q}_{<\lambda} := \bigcup_{\gamma<\lambda} \mathcal{Q}_\gamma$, and $\mathbf{M}'_\lambda$ is a closed formula φ in $\bigcup_{k\leq n} \mathfrak{L}\Sigma_k^1$, then there is $X \in \mathcal{Q}_\lambda$ satisfying $X \downarrow\subseteq \mathbf{M}_\lambda^\#$ and either $X \, \texttt{forc} \, \varphi$ or $X \, \texttt{forc} \, \varphi^-$.

Then the set $\mathbf{NH}(\boldsymbol{\varphi})$ *is* $\mathbb{n}$*-complete.*

We underline that condition $\mathfrak{P}_3^{\mathbb{n}}$ is void in case $\mathbb{n} = 1$.

Proof. We argue in $\mathbf{L}$. Given $X_0 \in \mathscr{X} := \mathbf{NH}(\boldsymbol{\varphi})$ and a closed formula φ in $\bigcup_{k\leq n} \mathfrak{L}\Sigma_k^1$, we have to find a set $Y \in \mathscr{X}$ satisfying $Y \downarrow\subseteq X_0$ and either $X \, \texttt{forc} \, \varphi$ or $X \, \texttt{forc} \, \varphi^-$. We can w.l.o.g. assume that $X_0 \in \mathcal{Q} := \bigcup_{\alpha<\omega_1} \mathcal{Q}_\alpha$, by Lemma 45.2. Pick a set $C \in \mathbf{cIPS}$ satisfying $X_0 = C^\#$.

The set $\mathbf{W} = \{\alpha < \omega_1 : \mathbf{M}_\alpha = C \wedge \mathbf{M}'_\alpha = \varphi\}$ is stationary by Lemma 45.1, whereas the set $E = \{\alpha < \omega_1 : X_0 = C^\# \in \mathcal{Q}_{<\alpha}\}$ is obviously a club. Thus there exists a limit ordinal $\lambda \in E \cap \mathbf{W}$. Then we have $\mathbf{M}_\lambda^\# = C^\# \in \mathcal{Q}_{<\lambda}$. Therefore by $\mathfrak{P}_3^n$ there exists a set $X \in \mathcal{Q}_\lambda$ satisfying $X \downarrow\subseteq \mathbf{M}_\lambda^\# = X_0$ and either $X \, \texttt{forc} \, \varphi$ or $X \, \texttt{forc} \, \varphi^-$, as required. □

48 Uniform sets

Our next goal will be to attack the (n)-definability property of Definition 32.1, We are going to define a group of three conditions which will imply that a normal forcing of the form $\mathscr{X} = \mathbf{NH}(\boldsymbol{\varphi})$ satisfies that property.

Here we consider part (I) of Definition 32.1. A condition related to uniform sets will be proposed to fulfill this requirement.

Recall that a set $X \in \mathbf{IPS}_\xi$ is *uniform* (Section 17), if for any pair of tuples $\boldsymbol{i} \subset \boldsymbol{j}$ in ξ and any $x, y \in X$, we have $x(\boldsymbol{j}) = y(\boldsymbol{j}) \Longrightarrow x(\boldsymbol{i}) = y(\boldsymbol{i})$.

Lemma 48.1 (in $\mathbf{L}$). *Assume that* $\boldsymbol{\varphi} = \langle \mathcal{Q}_\alpha \rangle_{\alpha<\omega_1}$ *is a* **Rud** *sequence, satisfying both* $\mathfrak{P}_1$ *of Lemma 45.2 and the following condition*:

$\mathfrak{P}_4$: for any $\lambda < \omega_1$, if $\mathbf{M}_\lambda^\# \in \mathcal{Q}_{<\lambda} := \bigcup_{\gamma<\lambda} \mathcal{Q}_\gamma$, then there is a uniform set $X \in \mathcal{Q}_\lambda$, $X \downarrow\subseteq \mathbf{M}_\lambda^\#$.

Then the set $\mathscr{X} = \mathbf{NH}(\boldsymbol{\varphi})$ *satisfies part* (I) *of the* (n)*-definability property as in Definition 32.1.*

Proof. Consider a pair of tuples $\boldsymbol{i} \subset \boldsymbol{j}$ in $\boldsymbol{I}$. We claim that the set

$$C_{\boldsymbol{ij}} = \{X \in \mathcal{Q} : X \text{ is uniform} \wedge \boldsymbol{i}, \boldsymbol{j} \in \|X\|\}$$

is $\downarrow\subseteq$-dense in $\mathscr{Q} = \bigcup_{\alpha<\omega_1} \mathscr{Q}_\alpha$. Indeed suppose that $Z \in \mathscr{Q}$. The set

$$\mathbf{W} = \{\alpha < \omega_1 : \boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}[{<}\alpha] \wedge \mathbf{M}^{\#}_{\alpha} = Z\}$$

is stationary by Lemma 45.1. Therefore there is a limit $\lambda \in \mathbf{W}$ with $Z = \mathbf{M}^{\#}_{\lambda} \in \mathscr{Q}$ and $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}[{<}\lambda] = \|Z\|$. Then $\mathfrak{P}_4$ yields a set $X \in C_{\boldsymbol{ij}}$, $X \subseteq Z$, as required.

It follows by the density that there is a set $X \in C_{\boldsymbol{ij}}$ such that $\boldsymbol{i}, \boldsymbol{j} \in \xi = \|X\|$ and $\boldsymbol{v}{\downarrow}\xi \in X^{\#}$. Then X is uniform, hence there is a continuous map $F : \mathscr{D} \to \mathscr{D}$ coded in $\mathbf{L}$ such that $f^{\#}(x(\boldsymbol{j})) = x(\boldsymbol{i})$ for all $x \in X^{\#}$. Then $\boldsymbol{v}(\boldsymbol{i}) = f^{\#}(\boldsymbol{v}(\boldsymbol{j}) \in \mathbf{L}[\boldsymbol{v}(\boldsymbol{j})]$, as required.

On the other hand, if $\boldsymbol{i} \not\subseteq \boldsymbol{j}$, then $\boldsymbol{i} \notin [\subseteq\boldsymbol{j}]$, and $\boldsymbol{v}(\boldsymbol{i}) \notin \mathbf{L}[\boldsymbol{v}(\boldsymbol{j})]$ follows from Corollary 26.4. □

49 Key formulas

Approaching part (II) of Definition 32.1, here formulas are introduced which will define sets in Definition 32.1(II).

Recall that if $X \in \mathbf{IPS}$ and $\boldsymbol{i} \in \|X\|$ then $X{\Downarrow}\boldsymbol{i} = \{x(\boldsymbol{i}) : x \in X\}$, and if $\mathscr{X} \subseteq \mathbf{IPS}$ then $\mathscr{X}{\Downarrow}\boldsymbol{i} = \{X{\Downarrow}\boldsymbol{i} : X \in \mathscr{X} \wedge \boldsymbol{i} \in \|X\|\}$. Suppose, that, **in L**,

(∗) $\boldsymbol{\varphi} = \langle \mathscr{Q}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{L}$ is a **Rud** sequence and $\mathscr{X} = \mathbf{NH}(\boldsymbol{\varphi})$ (as in Definition 43.1), so that $\mathscr{X} \in \mathbf{NF}$ is a normal forcing.

The following formulas based on $\boldsymbol{\varphi} = \langle \mathscr{Q}_\alpha \rangle_{\alpha<\omega_1} \in \mathbf{L}$ are considered.

$$\mathfrak{B}^{\mathsf{evn}}_{\boldsymbol{\varphi}}(k,x)\textbf{:}\;\; k < \omega \wedge x \in \mathscr{D} \wedge \exists\, \boldsymbol{j} \in \boldsymbol{I}[{<}2]\big(\mathtt{lh}(\boldsymbol{j}) = k \wedge \boldsymbol{j} \text{ is even} \wedge \forall\, \alpha < \omega_1\, \exists\, Z \in \mathscr{Q}_\alpha{\Downarrow}\boldsymbol{j}\, (x \in Z^{\#})\big);$$

$$\mathfrak{B}^{\mathsf{odd}}_{\boldsymbol{\varphi}}(k,x)\textbf{:}\;\; k < \omega \wedge x \in \mathscr{D} \wedge \exists\, \boldsymbol{j} \in \boldsymbol{I}[{<}2]\big(\mathtt{lh}(\boldsymbol{j}) = k \wedge \boldsymbol{j} \text{ is odd} \wedge \forall\, \alpha < \omega_1\, \exists\, Z \in \mathscr{Q}_\alpha{\Downarrow}\boldsymbol{j}\, (x \in Z^{\#})\big).$$

We'll prove that these formulas define sets in Definition 32.1(II) in $\mathscr{X}$-generic extensions of $\mathbf{L}$ — provided the basic **Rud** sequence $\boldsymbol{\varphi}$ satisfies certain conditions. The next lemma proves this result in one direction.

Lemma 49.1. *Assume* (∗) *in* $\mathbf{L}$ *as above. Let* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *be a* $\mathscr{X}$*-generic array over* $\mathbf{L}$*,* $\boldsymbol{i} \in \boldsymbol{I}$*,* $k = \mathtt{lh}(\boldsymbol{i})$*, and* $x = \boldsymbol{v}(\boldsymbol{i})$*. Then* $\mathbf{L}[x] \models \mathfrak{B}^{\mathsf{evn}}_{\boldsymbol{\varphi}}(k,x)$*, resp.,* $\mathfrak{B}^{\mathsf{odd}}_{\boldsymbol{\varphi}}(k,x)$*, provided* $\boldsymbol{i}$ *is resp. even, odd.*

Proof. Let $\boldsymbol{j} = \underline{\boldsymbol{i}} \in \boldsymbol{I}[{<}2]$ (see Section 22), so that $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$ (the parity-equivalence, Section 13), and $\pi_{\boldsymbol{ij}} \in \boldsymbol{\Pi}$ is parity-preserving. We claim that

(1) if $\alpha < \omega_1$ then $\mathscr{Q}_\alpha{\downarrow}_{\subseteq \boldsymbol{j}}$ is pre-dense in $\mathscr{X}{\downarrow}_{\subseteq \boldsymbol{j}}$.

As clearly $\mathscr{X} = \bigcup_{\lambda<\omega_1} \mathscr{X}_\lambda$, where $\mathscr{X}_\lambda = \mathbf{NH}(\boldsymbol{\varphi}{\restriction}\lambda)$, it suffices to check that

(2) if $\alpha < \lambda < \omega_1$ and λ is limit then $\mathscr{Q}_\alpha{\downarrow}_{\subseteq \boldsymbol{j}}$ is pre-dense in $\mathscr{X}_\lambda{\downarrow}_{\subseteq \boldsymbol{j}}$.

However $\mathscr{X}_\lambda{\downarrow}_{\subseteq \boldsymbol{j}} = \mathscr{P}_\lambda{\downarrow}_{\subseteq \boldsymbol{j}}$ by Theorem 43.2(x), where $\mathscr{P}_\lambda = \mathbf{RH}(\boldsymbol{\varphi}{\restriction}\lambda) \in \mathbf{Rud}_\lambda$. On the other hand, the set $\mathscr{Q}_\alpha{\downarrow}_{\subseteq \boldsymbol{j}}$ is pre-dense in $\mathscr{P}_\lambda{\downarrow}_{\subseteq \boldsymbol{j}}$ by Theorem 43.2(viii). This implies (2) and (1).

Now assume that $\alpha < \omega_1$ (in $\mathbf{L}$), and let $\boldsymbol{v}' = \pi_{\boldsymbol{ij}} {\bullet} \boldsymbol{v}$. Then $\boldsymbol{v}' \in \mathscr{D}^{\boldsymbol{I}}$ is still $\mathscr{X}$-generic over $\mathbf{L}$ along with $\boldsymbol{v}$ since $\pi_{\boldsymbol{ij}} \in \boldsymbol{\Pi}$ and $\mathscr{X}$ is necessarily $\boldsymbol{\Pi}$-invariant. It follows from (1) that $\boldsymbol{v}'{\downarrow}_{\subseteq \boldsymbol{j}} \in P^\#$ for some $P \in \mathscr{Q}_\alpha{\downarrow}_{\subseteq \boldsymbol{j}}$ by Lemma 23.2(ii), and hence obviously $\boldsymbol{v}'(\boldsymbol{j}) \in Z^\#$ for $Z = P{\Downarrow}\boldsymbol{j} \in \mathscr{Q}_\alpha{\Downarrow}\boldsymbol{j}$.

To conclude, the real $x = \boldsymbol{v}(\boldsymbol{i}) = \boldsymbol{v}'(\boldsymbol{j})$ satisfies $\mathfrak{B}^{\mathsf{evn}}_{\boldsymbol{\varphi}}(k,x)$, resp., $\mathfrak{B}^{\mathsf{odd}}_{\boldsymbol{\varphi}}(k,x)$ in $\mathbf{L}[x]$, in case $\boldsymbol{i}$ (and then $\boldsymbol{j}$ as well) is even, resp., odd. □

50 The inverse of the lemma

The condition $\mathfrak{P}_5$ defined below will allow us to reverse Lemma 49.1. This condition involves a special notation. Recall definitions in Sections 18, 35.

Definition 50.1 (in $\mathbf{L}$)**.** Let $\alpha < \omega_1$. If $\mathbf{M}''_\alpha \in \mathbf{cCF}^*$ and $\delta_\alpha := \|\mathbf{M}''_\alpha\| \subseteq \boldsymbol{I}[{<}\alpha]$ then define $]\alpha \in \mathbf{cCF}^*_{\boldsymbol{I}[<\alpha]}$ by $\mathbb{f}_\alpha(x) = \mathbf{M}''_\alpha(x{\downarrow}\delta_\alpha)$ for all $x \in \mathbf{Rat}_{\boldsymbol{I}[<\alpha]}$. Otherwise define $\mathbb{F}_\alpha \in \mathbf{cCF}^*_{\boldsymbol{I}[<\alpha]}$ by $\mathbb{F}_\alpha(x) = \omega \times 0$ for all $x \in \mathbf{Rat}_{\boldsymbol{I}[<\alpha]}$.

In both cases define $\mathbb{F}_\alpha = \mathbb{f}^\#_\alpha \in \mathbf{CF}^*_{\boldsymbol{I}[<\alpha]}$.

Let $\boldsymbol{\varphi} = \langle \mathscr{Q}_\alpha \rangle_{\alpha<\omega_1}$ be a $\mathbf{Rud}$ sequence. Define the following condition:

$\mathfrak{P}_5$: For any $\lambda < \omega_1$, if $\mathbf{M}^\#_\lambda \in \mathscr{Q}_{<\lambda} = \bigcup_{\gamma<\lambda} \mathscr{Q}_\gamma$ then there is a set $Y \in \mathscr{Q}_\lambda$, $Y \mathbin{\downarrow\subseteq} \mathbf{M}^\#_\lambda$, such that one of the two following claims holds:

(a) $\mathbb{F}_\lambda$ avoids every $E \in \mathscr{Q}_\alpha{\Downarrow}\boldsymbol{i}$ on Y for all $\boldsymbol{i} \in \boldsymbol{I}[{<}\lambda]$;

(b) there is $\boldsymbol{j} \in \boldsymbol{I}[{<}\lambda]$ such that $\mathbb{F}_\lambda$ is an $\boldsymbol{j}$-axis map on Y and $\mathbb{F}_\lambda$ avoids each $E' \in \mathscr{Q}_\lambda{\Downarrow}\boldsymbol{i}$ on Y for all $\boldsymbol{i} \in \boldsymbol{I}[{<}\lambda]$ with $\boldsymbol{i} \not\approx_{\mathsf{par}} \boldsymbol{j}$. □

Theorem 50.2. *Assume that* (∗) *of Section* 49 *holds, and* $\boldsymbol{\varphi}$ *satisfies* $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_5$ *in* $\mathbf{L}$. *Let* $\boldsymbol{v}$ *be* $\mathscr{X}$*-generic over* $\mathbf{L}$. *Then*

$$\mathbf{E}^{\mathsf{evn}}(\boldsymbol{v}) = \{\langle k,x\rangle : x \in \mathbf{L}[\boldsymbol{v}] \wedge \mathbf{L}[x] \models \mathfrak{B}^{\mathsf{evn}}_{\boldsymbol{\varphi}}(k,x)\}.$$

and the same for the 'odd' case.

Proof. The inclusions $\subseteq$ in both cases follow from Lemma 49.1. To establish the inverse inclusions, let $k \geq 1$, $x \in \mathbf{L}[\boldsymbol{v}] \cap \mathscr{D}$, and $\mathbf{L}[x] \models \mathfrak{B}^{\mathsf{evn}}_{\boldsymbol{\varphi}}(k, x)$, so that there is an even tuple $\boldsymbol{i} \in \boldsymbol{I}[{<}2]$ with $\mathtt{lh}(\boldsymbol{i}) = k$, satisfying

$$\forall\, \alpha < \omega_1 = \omega_1^{\mathbf{L}}\, \exists\, A \in \mathscr{Q}_\alpha {\Downarrow} \boldsymbol{i}\, (x \in A^{\#}). \tag{4}$$

We have to prove that $\langle k, x\rangle \in \mathbf{E}^{\mathsf{evn}}(\boldsymbol{v})$.

By $\mathfrak{P}_2$ and Lemma 46.1, the set $\mathscr{X} = \mathbf{NH}(\boldsymbol{\varphi}) \in \mathbf{NF}$ has the fusion property. It follows, by Theorem 27.1(iii) and Corollary 35.1, that $x = f^{\#}(\boldsymbol{v}{\downarrow}\sigma)$ for some $\sigma = \boldsymbol{I}[{<}\alpha_0]$, $\alpha_0 < \omega_1$, and $f \in \mathbf{cCF}^*_\sigma$. We claim that the set $D_f = \bigcup_{\alpha_0 < \lambda < \omega_1} D_{f\lambda}$ is ${\downarrow}{\subseteq}$-dense in $\mathscr{Q} = \bigcup_{\lambda < \omega_1} \mathscr{Q}_\lambda$, where

$$D_{f\lambda} = \{Y \in \mathscr{Q}_\lambda : Y \text{ satisfies } \mathfrak{P}_5\text{a or } \mathfrak{P}_5\text{b in Definiton 50.1}\}$$

Indeed suppose that $Z \in \mathscr{Q}$. The set $\mathbf{W} = \{\lambda < \omega_1 : \mathbf{M}^{\#}_\lambda = Z \wedge \mathbf{M}''_\lambda = f\}$ is stationary by Lemma 45.1. Therefore there exists a limit ordinal $\lambda \in \mathbf{W}$ satisfying $\alpha_0 < \lambda$, hence $\sigma \subseteq \boldsymbol{I}[{<}\lambda]$, $Z = \mathbf{M}^{\#}_\lambda \in \bigcup_{\gamma<\lambda} \mathscr{Q}_\gamma$, and $f = \mathbf{M}''_\lambda$. Then $\mathfrak{P}_5$ yields a set $Y \in D_f$, $Y \subseteq Z$, as required.

By the density just proved, there exist $\lambda < \omega_1$ and $Y \in D_{f\lambda}$ satisfying $\boldsymbol{v} \restriction \boldsymbol{I}[{<}\lambda] \in Y^{\#}$. (Note that $\|Y\| = \boldsymbol{I}[{<}\lambda]$ since $Y \in \mathscr{Q}_\lambda$.) We conclude from (4) and the choice of $f = \mathbf{M}''_\lambda$ that $\mathbb{F}_\lambda$ does **not** avoid some $E \in \mathscr{Q}_\lambda {\Downarrow} \boldsymbol{i}$ on Y. It follows that $\mathfrak{P}_5$a definitely fails, and hence $\mathfrak{P}_5$b holds for some $\boldsymbol{j} \in \boldsymbol{I}[{<}\lambda]$ such that $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$. In particular, $\mathbb{F}_\lambda$ is a $\boldsymbol{j}$-axis map on Y, meaning that $\mathbb{F}_\lambda(y{\downarrow}\boldsymbol{I}[{<}\lambda]) = y(\boldsymbol{j})$ for all $y \in Y$, and hence $x = \mathbb{F}_\lambda(\boldsymbol{v}{\downarrow}\boldsymbol{I}[{<}\lambda]) = \boldsymbol{v}(\boldsymbol{j})$. It remains to note that $\boldsymbol{j}$ is even and $\mathtt{lh}(\boldsymbol{j}) = k$ by the choice of $\boldsymbol{i}$, because $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$. Thus $\langle k, x\rangle \in \mathbf{E}^{\mathsf{evn}}(\boldsymbol{v})$, as required. □

51 Getting (n)-definability

Here we introduce another property, related to the definability of a **Rud** sequence as a whole, which will help us to reduce the formulas $\mathfrak{B}^{\mathsf{evn}}_{\boldsymbol{\varphi}}(k, x)$, $\mathfrak{B}^{\mathsf{odd}}_{\boldsymbol{\varphi}}(k, x)$ to $\mathit{\Pi}^1_{n+1}$ as required by Definition 32.1, and thereby to fully establish the (n)-definability property of the ensuing normal forcing.

Definition 51.1 (in $\mathbf{L}$)**.** Say that a sequence $\beta = \langle \mathscr{B}_\alpha \rangle_{\alpha<\lambda}$ is a *coded* **Rud** *sequence*, if each $\mathscr{B}_\alpha \subseteq \mathbf{cIPS}$ is at most countable and the sets $\mathscr{Q}_\alpha = {\mathscr{B}_\alpha}^{\sharp} := \{A^{\#} : A \in \mathscr{B}_\alpha\}$ form a **Rud** sequence $\boldsymbol{\varphi} = \langle \mathscr{Q}_\alpha \rangle_{\alpha<\lambda}$.

We write $\boldsymbol{\varphi} = \beta^{\sharp}$ in this case. □

Lemma 51.2 (in $\mathbf{L}$)**.** *Let $n \geq 1$ and $\boldsymbol{\varphi} = \langle \mathscr{Q}_\alpha \rangle_{\alpha<\omega_1}$ be a* **Rud** *sequence, satisfying conditions $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_4$, $\mathfrak{P}_5$, and the following condition*:

$\mathfrak{P}_6^n$: it is true in $\mathbf{L}$ that there is a coded **Rud** sequence $\beta = \langle \mathcal{B}_\alpha \rangle_{\alpha<\omega_1}$ for $\boldsymbol{\varphi}$, of the definability class $\Sigma_{\mathfrak{n}}^{\mathbf{HC}}$, such that $\boldsymbol{\varphi} = \beta^\sharp$.

Then $\mathscr{X} = \mathbf{NH}(\boldsymbol{\varphi})$ *satisfies the* $(\mathfrak{n})$*-definability property of Definition 32.1.*

Proof. By Lemma 48.1, we can concentrate on part (II) of Definition 32.1. We have to estimate the complexity of the relations $\mathbf{L}[x] \models \mathfrak{B}_{\boldsymbol{\varphi}}^{\mathsf{evn}}(k,x)$ and $\mathbf{L}[x] \models \mathfrak{B}_{\boldsymbol{\varphi}}^{\mathsf{odd}}(k,x)$ as in Theorem 50.2.

By $\mathfrak{P}_6^n$, there exists a concrete parameter-free Σ_n formula $\varphi(\cdot,\cdot)$ such that $\mathcal{Q} = \mathcal{Q}_\alpha$ iff $\alpha, \mathcal{Q} \in \mathbf{L}_{\omega_1}$ and $\mathbf{L}_{\omega_1} = (\mathbf{HC})^{\mathbf{L}} \models \varphi(\alpha, \mathcal{Q})$. Let

$$\begin{aligned}\Phi^{\mathsf{evn}}(k,x) \; := \; & \forall\,\alpha\,\forall\,\mathcal{Q}\,\big[\alpha,\mathcal{Q}\in\mathbf{L}\wedge\varphi(\alpha,\mathcal{Q})^{\mathbf{L}} \implies \exists\,\boldsymbol{j}\in\boldsymbol{I}[{<}2] \\ & \big(\mathtt{lh}(\boldsymbol{j}) = k \wedge \boldsymbol{j} \text{ is even } \wedge \exists\,A\in\mathcal{Q}{\Downarrow}\boldsymbol{j}\,(x\in A^{\#})\big)\big],\end{aligned}$$

where $\varphi(\alpha,\mathcal{Q})^{\mathbf{L}}$ means the formal relativization of all unbounded quantifiers to $\mathbf{L}$. (Compare to the formulas $\mathfrak{B}_{\boldsymbol{\varphi}}^{\mathsf{evn}}(k,x)$ in Section 49.)

Consider any $\mathscr{X}$-generic array $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ over $\mathbf{L}$, $k<\omega$, and $x \in \mathbf{L}[\boldsymbol{v}] \cap \mathscr{D}$. Recall that $\mathbf{L}[\boldsymbol{v}]$ preserves $\omega_1^{\mathbf{L}}$ by Theorem 27.1(ii), and hence using $\omega_1 = \omega_1^{\mathbf{L}} = \omega_1^{\mathbf{L}[\boldsymbol{v}]}$ does not lead to an ambiguity. Theorem 50.2 implies that

$$\langle k,x\rangle \in \mathbf{E}^{\mathsf{evn}}(\boldsymbol{v}) \iff \mathbf{L}_{\omega_1}[x] \models \Phi^{\mathsf{evn}}(k,x). \tag{1}$$

Now assume that $\mathfrak{M} \subseteq \mathbf{L}[\boldsymbol{v}]$ is a transitive class, closed under pairs, and $\mathbf{L}[x] \subseteq \mathfrak{M}$ for all $x \in \mathfrak{M}$, as in part (II) of Definition 32.1. Then we have

$$\mathbf{E}^{\mathsf{evn}}(\boldsymbol{v}) \cap \mathfrak{M} \;=\; \big\{\langle k,x\rangle \in \mathfrak{M} : \mathfrak{M} \models \Phi^{\mathsf{evn}}(k,x)^{\mathbf{L}_{\omega_1}[x]}\big\} \tag{2}$$

by (1), where the upper index $^{\mathbf{L}_{\omega_1}[x]}$ means the formal relativization of all unbounded quantifiers in $\Phi^{\mathsf{evn}}(k,x)$ to $\mathbf{L}_{\omega_1}[x]$.

Now note that φ is Σ_n, and hence so is $\varphi(\alpha,\mathcal{Q})^{\mathbf{L}}$ because "$x \in \mathbf{L}$" is Σ_1 by Gödel. We conclude that $\Phi^{\mathsf{evn}}(k,x)$ is essentially a Π_n formula. It follows that $\mathfrak{M} \models \Phi^{\mathsf{evn}}(k,x)^{\mathbf{L}_{\omega_1}[x]}$ defines a Π_n relation over $(\mathbf{HC})^{\mathfrak{M}}$ since $y \in \mathbf{L}_{\omega_1}[x]$ is still a Σ_1 relation over $(\mathbf{HC})^{\mathfrak{M}}$ by Gödel. It follows by (2) that $\mathbf{E}^{\mathsf{evn}}(\boldsymbol{v}) \cap \mathfrak{M}$ is a $\Pi_n^{\mathbf{HC}}$ set in $\mathfrak{M}$, hence a Π_{n+1}^1 set by Proposition 7.1, as required. The "odd" case is considered similarly. □

52 Third form of the main theorem

To summarize the results achieved above, we now formulate another form of Theorem 1.1 in the introduction, that further develops the previous form given by Theorem 34.1. This is based on the next definition, that gathers the stepwise properties $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3^{\mathfrak{n}}$, $\mathfrak{P}_4$, $\mathfrak{P}_5$ in a single stepwise property.

Definition 52.1. Let $\lambda < \omega_1$, $\mathtt{n} \ge 1$. Say that a term $\mathscr{Q}_\lambda$ is a *1-5-*$\mathtt{n}$ *extension* of a **Rud** sequence $\boldsymbol{\varphi} = \langle \mathscr{Q}_\gamma \rangle_{\gamma<\lambda}$ if the following (A),(B),(C) hold:

(A) the extended sequence $\boldsymbol{\varphi}^\frown \mathscr{Q}_\lambda = \langle \mathscr{Q}_\gamma \rangle_{\gamma \le \lambda}$ is still a **Rud** sequence;

(B) as in $\mathfrak{P}_1$, if $\mathbf{M}^\#_\lambda \in \mathbf{NH}(\boldsymbol{\varphi})$ and $\|\mathbf{M}^\#_\lambda\| \subseteq \boldsymbol{I}[<\lambda]$ then there is $Y \in \mathscr{Q}_\lambda$, $Y \downarrow\subseteq \mathbf{M}^\#_\lambda$;

(C) if $\mathbf{M}^\#_\lambda \in \mathscr{Q}_{<\lambda} := \bigcup_{\gamma<\lambda} \mathscr{Q}_\gamma$ then there is a set $Y \in \mathscr{Q}_\lambda$ satisfying $Y \downarrow\subseteq \mathbf{M}^\#_\lambda$ and the following conditions (C2)–(C5):

(C2) as in $\mathfrak{P}_2$, if λ is limit then $Y \subseteq^{\mathtt{fd}} \bigcup \mathbf{B}_{\lambda k}{}^\sharp$ holds for all $k < \omega$ such that $\mathbf{B}_{\lambda k}{}^\sharp \subseteq \mathscr{Q}_{<\lambda}$ and $\mathbf{B}_{\lambda k}{}^\sharp$ is dense in $\mathscr{Q}_{<\lambda}$;

(C3) as in $\mathfrak{P}_3^{\mathtt{n}}$, if $\mathtt{n} \ge 2$ and $\mathbf{M}'_\lambda$ is a closed formula φ in $\bigcup_{k \le \mathtt{n}} \mathfrak{L}\Sigma^1_k$ then $Y \,\mathtt{forc}\, \varphi$ or $Y \,\mathtt{forc}\, \varphi^-$ — *void in case* $\mathtt{n} = 1$;

(C4) as in $\mathfrak{P}_4$, Y is a uniform set;

(C5) as in $\mathfrak{P}_5$ of Definition 50.1,

either (a) $\mathbb{F}_\lambda$ avoids every $E \in \mathscr{Q}_\lambda \Downarrow \boldsymbol{i}$ on Y for all $\boldsymbol{i} \in \boldsymbol{I}[<\lambda]$,

or (b) there is $\boldsymbol{j} \in \boldsymbol{I}[<\lambda]$ such that $\mathbb{F}_\lambda$ is an $\boldsymbol{j}$-axis map on Y but $\mathbb{F}_\lambda$ avoids each $E' \in \mathscr{Q}_\lambda \Downarrow \boldsymbol{i}$ on Y for all $\boldsymbol{i} \in \boldsymbol{I}[<\lambda]$ satisfying $\boldsymbol{i} \not\approx_{\mathtt{par}} \boldsymbol{j}$. □

Theorem 52.2 (in **L**)**.** *Assume that* $\mathtt{n} \ge 1$. *Then there is a* **Rud** *sequence* $\boldsymbol{\varphi} = \langle \mathscr{Q}_\alpha \rangle_{\alpha<\omega_1}$ *satisfying the global definability condition* $\mathfrak{P}_6^{\mathtt{n}}$ *and such that, for any ordinal* $\lambda < \omega_1$, $\mathscr{Q}_\lambda$ *is a a 1-5-*$\mathtt{n}$ *extension of* $\boldsymbol{\varphi}\restriction\lambda$.

Proof (Theorem 1.1 from Theorem 52.2)**.** Let $\boldsymbol{\varphi}$ be such a **Rud** sequence as in Theorem 52.2. Consider the associated normal forcing $\mathscr{X} = \mathbf{NH}(\boldsymbol{\varphi}) \in \mathbf{NF}$.

Lemma 46.1 implies that $\mathscr{X}$ has the fusion property.

Lemma 47.1 implies that the set $\mathscr{X} = \mathbf{NH}(\boldsymbol{\varphi})$ is $\mathtt{n}$-complete, and then $\mathscr{X}$ has the ($\mathtt{n}$)-odd-expansion property by Theorem 39.3.

Finally, $\mathscr{X}$ satisfies the ($\mathtt{n}$)-definability property by Theorem 51.2.

Thus $\mathscr{X}$ is as required by Theorem 34.1.

But Theorem 34.1 implies Theorem 1.1, see Section 34.

□ (Thms 34.1 and 1.1 from Thm 52.2)

Theorem 52.2 will be the goal of the two following chapters.

IX The existence of 1-5-$\Bbbk$ extensions

Working towards the proof of Theorem 52.2, the goal of this Chapter will be the existence of 1-5-n extensions of **Rud** sequences of countable length.

53 The existence theorem and basic notation

Theorem 53.1 (in **L**). *Let $\lambda < \omega_1$ and $\Bbbk \ge 1$. Then every* **Rud** *sequence $\varphi = \langle \mathcal{Q}_\alpha \rangle_{\alpha<\lambda}$ admits a 1-5-$\Bbbk$ extension $\mathcal{Q}_\lambda$.*

Notation, in L. We fix $\lambda, \Bbbk, \varphi, \mathcal{Q}_\alpha$ as in the theorem. Put

$$\mathcal{Q}_{<\lambda} = \bigcup_{\alpha<\lambda} \mathcal{Q}_\alpha, \quad \Bbbk = \boldsymbol{I}[{<}\lambda], \quad \mathcal{U}_\lambda = \mathbf{RH}(\mathcal{Q}_{<\lambda}\!\uparrow\!\Bbbk), \quad \mathscr{X}_\lambda = \mathbf{NH}(\mathcal{Q}_{<\lambda}).$$

Remark 53.2. $\mathcal{U}_\lambda \in \mathbf{Rud}_\lambda$ is a countable rudiment, $\mathscr{X}_\lambda \in \mathbf{NF}$ is a normal forcing, $\mathcal{Q}_{<\lambda}\!\uparrow\!\Bbbk \subseteq \mathcal{U}_\lambda$, $\mathscr{D}^{\Bbbk} \in \mathcal{U}_\lambda$. In addition, $\mathcal{U}_\lambda \subseteq \mathscr{X}_\lambda$, and $\mathscr{X}_\lambda\!\downarrow_{\subseteq \boldsymbol{i}} = \mathcal{U}_\lambda\!\downarrow_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \Bbbk$ by Lemma 40.2. □

We'll use the sets $\mathbf{M}_\lambda \in \mathbf{cIPS}^{<\lambda}$; $\mathbf{M}'_\lambda, \mathbf{M}''_\lambda \in \mathbf{HC}_{<\lambda}$; $\mathbf{B}_{\lambda k} \subseteq \mathbf{cIPS}^{<\lambda}$ and ${\mathbf{B}_{\lambda k}}^\sharp \subseteq \mathbf{IPS}^{<\lambda}$ (both countable sets); defined in (I),(II) of Section 45.

(1) If $\mathbf{M}^{\#}_\lambda \in \mathcal{Q}_{<\lambda}$ then put $\overline{X} = \mathbf{M}^{\#}_\lambda\!\uparrow\!\Bbbk$, otherwise let $\overline{X} = \mathscr{D}^{\Bbbk}$, so that $\overline{X} \in \mathcal{U}$ in both cases.

(2) If $\mathbf{M}'_\lambda$ is a closed formula in $\bigcup_{k\le\Bbbk} \mathfrak{L}\Sigma^1_k$, then let ϕ_λ be that formula, otherwise let ϕ_λ be say $0 = 0$.

(3) Use $\mathbf{M}''_\lambda$ to define $\mathbb{f}_\lambda \in \mathbf{cCF}^*_{\Bbbk}$ and $\mathbb{F}_\lambda \in \mathbf{CF}^*_{\Bbbk}$ as in Definition 50.1.

On the basis of this notation, our proof of Theorem 53.1 will proceerd as follows. We define the notion of *generic iterated perfect sets* and prove the existence lemma and some properties of such sets in Section 54. Then we pick a generic set $Y_0 \subseteq \overline{X}$ in Section 55 and then shrink it to a set $\overline{Y} \subseteq Y_0$ satisfying some conditions related to (C2), (C3), (C4), (C5) of Definition 52.1 above. The next step is the lifting theorem of Section 56; it says roughly that any generic set in $\mathbf{IPS}_{\subset \boldsymbol{i}}$ can be extended to a generic set in $\mathbf{IPS}_{\subseteq \boldsymbol{i}}$. This theorem allows us to define a rudiment $\mathcal{P} \subseteq \mathbf{IPS}_{\Bbbk}$ of all sets $X \in \mathbf{IPS}_{\subseteq \Bbbk}$ whose all projections $X\!\downarrow_{\subseteq \boldsymbol{i}}$ are generic (but not necessarily X itself). This rudiment contains $\overline{Y}$ and refines $\mathcal{U}_\lambda$ (Section 57). After a short but necessary work related to condition (B), we then take a suitable countable sub-rudiment of $\mathcal{P}$ to be the layer $\mathcal{Q}_\lambda$ for Theorem 53.1.

54 Generic perfect sets

We continue to argue in $\mathbf{L}$. Consider the set $\mathbf{H}_{\omega_2} = \mathbf{L}_{\omega_2}$, and define the following countable sets:

$$\mathfrak{C} = \daleth \cup \mathcal{U}_\lambda \cup \{\lambda, \daleth, \varrho, \omega_1, \mathbf{HC}\} \subseteq \mathbf{H}_{\omega_2} = \mathbf{L}_{\omega_2};$$

$$\mathfrak{D} = \{\text{all sets } X \subseteq \mathbf{H}_{\omega_2} \ \in\text{-definable over } \mathbf{H}_{\omega_2} \text{ with parameters in } \mathfrak{C}\}.$$

Remark 54.1. Such sets as $\omega_1, \mathbf{HC}, \mathbf{IPS}, \mathbf{cIPS}$, as well as many sets related to ϱ this or another way, like $\mathcal{Q}_{<\lambda}, \mathcal{U}_\lambda, \mathscr{X}_\lambda, \langle \mathbf{B}_{\lambda k}\rangle_{k<\omega}, \langle \mathbf{B}_{\lambda k}{}^\sharp\rangle_{k<\omega}$, *etc.* belong to $\mathfrak{D} \cap \mathbf{H}_{\omega_2}$, and can be used as parameters to define sets in $\mathfrak{D}$. □

Definition 54.2. Assume that $\eta \in \Xi$, $\eta \subseteq \daleth$. A set $X \in \mathbf{IPS}_\eta$ is $\mathfrak{D}$-*generic* iff $X \subseteq^{\mathtt{fin}} \bigcup D$ holds for any set $D \in \mathfrak{D}$, $D \subseteq \mathcal{U}_\lambda{\downarrow}\eta$, dense in $\mathcal{U}_\lambda{\downarrow}\eta$.

Recall that $\mathcal{U}_\lambda{\downarrow}\eta = \{Y{\downarrow}\eta : Y \in \mathcal{U}_\lambda\}$. See Section 24 on $\subseteq^{\mathtt{fin}}, \subseteq^{\mathtt{fd}}$. □

Lemma 54.3. *If $U \in \mathcal{U}_\lambda$ then there is a $\mathfrak{D}$-generic set $X \in \mathbf{IPS}_\daleth$, $X \subseteq U$.*

Proof. Fix any $\daleth$-admissible map $\phi : \omega \xrightarrow{\text{onto}} \daleth$. The next claim is a consequence of property $2^\dagger$ of the rudiment $\mathcal{U}_\lambda$, the density, and Corollary 15.3 applied consecutively enough many times:

(1) If $m < \omega$ and a set $D \in \mathfrak{D}$, $D \subseteq \mathcal{U}_\lambda$, is dense in $\mathcal{U}_\lambda$ then any ϕ-split system $\langle X_u\rangle_{u\in 2^m}$ of sets $X_u \in \mathcal{U}_\lambda$ admits a narrowing $\langle X'_u\rangle_{u\in 2^m}$ in $\mathcal{U}_\lambda$ such that $X'_u \in D$ for all $u \in 2^m$.

Using (1) and the countability of $\mathfrak{D}$, we get a fusion sequence $\langle X_u\rangle_{u\in 2^{<\omega}}$ of sets in $\mathcal{U}_\lambda$, such that $X_\Lambda \subseteq U$, and, for each $D \in \mathfrak{D}$ dense in $\mathcal{U}_\lambda$, there is $m < \omega$ with $X_u \in D$ for all $u \in 2^m$. Then $X = \bigcap_m \bigcup_{u\in 2^m} X_u \in \mathbf{IPS}_\daleth$, $X \subseteq U$, and $X \subseteq^{\mathtt{fd}} \bigcup D$ for each set $D \in \mathfrak{D}$, $D \subseteq \mathcal{U}_\lambda$, dense in $\mathcal{U}_\lambda$. □

The next theorem provides some basic properties of $\mathfrak{D}$-generic sets.

Theorem 54.4. (i) *If $X \in \mathbf{IPS}_\daleth$ is $\mathfrak{D}$-generic and $\eta \in \mathbf{FT}(\daleth)$* (an initial segment of finite type, Section 40) *then $X{\downarrow}\eta$ is $\mathfrak{D}$-generic as well;*

(ii) *moreover, if, in (i), $D \in \mathfrak{D}$, $D \subseteq \mathcal{U}_\lambda{\downarrow}\eta$, D is pre-dense in $\mathcal{U}_\lambda{\downarrow}\eta$, then $X \subseteq^{\mathtt{fd}} \bigcup D$;*

(iii) *if $\alpha < \lambda$, $\boldsymbol{i} \in \boldsymbol{I}[{<}\alpha]$, $X \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ is $\mathfrak{D}$-generic, then $X \subseteq^{\mathtt{fd}} \bigcup(\mathcal{Q}_\alpha{\downarrow}_{\subseteq \boldsymbol{i}})$;*

(iv) *if $\eta \in \mathbf{FT}(\daleth)$, $U \in \mathcal{U}_\lambda{\downarrow}\eta$, and $X \in \mathbf{IPS}_\eta$ is $\mathfrak{D}$-generic then $X \cap U$ is clopen in X;*

(v) *if $\boldsymbol{i} \approx_{\mathtt{par}} \boldsymbol{j}$ belong to $\daleth$ and $X \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ is $\mathfrak{D}$-generic then so is $\pi_{\boldsymbol{ij}}{\bullet}X$.*

Proof. (i) Assume that $D \in \mathfrak{D}$, $D \subseteq \mathcal{U}_\lambda{\downarrow}\eta$, is dense in $\mathcal{U}_\lambda{\downarrow}\eta$; prove that $X{\downarrow}\eta \subseteq^{\mathtt{fd}} \bigcup D$. It follows from property $2^\dagger$ of the rudiment $\mathcal{U}_\lambda$ that the set $D' = \{U \in \mathcal{U}_\lambda : U{\downarrow}\eta \in D\}$ is dense in $\mathcal{U}_\lambda$. Moreover D' belongs to $\mathfrak{D}$ because so do D and $\eta \in \mathbf{FT}(\mathbb{I})$. (Not necessarily true for an arbitrary $\eta \in \mathbf{\Xi}, \eta \subseteq \mathbb{I}$.) Thus $X \subseteq^{\mathtt{fin}} \bigcup D'$ by the genericity, hence $X{\downarrow}\eta \subseteq^{\mathtt{fin}} \bigcup D$.

(ii) Apply (i) for the dense set $D_1 = \{V \in \mathcal{U}{\downarrow}\eta : \exists U \in D\,(V \subseteq U)\}$.

(iii) We know that $\mathcal{Q}_\alpha{\downarrow}_{\subseteq \boldsymbol{i}}$ is predense in $\mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$ by Theorem 43.2(viii). It remains to apply (ii) with $\eta = [\subseteq \boldsymbol{i}]$.

(iv) Recall that $\mathcal{U}_\lambda$ is a rudiment, hence it satisfies $3^\dagger$ of Section 40. It easily follows that $\mathcal{U}_\lambda{\downarrow}\eta$ satisfies $3^\dagger$ as well: if $\varnothing \neq Z \subseteq Y \in \mathcal{U}_\lambda{\downarrow}\eta$, $Z \in \mathbf{IPS}_\eta$, and Z is clopen in Y then $Z \in \mathcal{U}_\lambda{\downarrow}\eta$. (Indeed if $Y = U{\downarrow}\eta$, $U \in \mathcal{U}_\lambda$, then $U' = U \cap (Z{\uparrow}\mathbb{I}) \in \mathbf{IPS}_{\mathbb{I}}$ by Lemma 10.5, and U' is clopen in U by the choice of Z — thus $U' \in \mathcal{U}_\lambda$. But $Z = U'{\downarrow}\eta$.) We conclude that the set D of all $Y \in \mathcal{U}_\lambda{\downarrow}\eta$, satisfying $Y \subseteq U$ or $Y \cap U = \varnothing$, is dense in $U_\lambda{\downarrow}\eta$. We conclude that $X \subseteq^{\mathtt{fin}} \bigcup D$ by the genericity, in other words, $X \subseteq \bigcup D'$, where $D' \subseteq D$ is finite. Thus $D' = D'_1 \cup D'_2$, where $D'_1 = \{Y \in D' : Y \subseteq U\}$ and $D'_2 = \{Y \in D' : Y \cap U = \varnothing\}$. Thus $X \subseteq Y_1 \cup Y_2$, where $Y_e = \bigcup D'_e$ are two disjoint closed sets. Finally, $X \cap U = X \cap Y_1 = X \smallsetminus Y_2$, which implies the result required.

(v) This is clear as $\pi_{\boldsymbol{ij}} \in \mathfrak{D}$. $\square$

55 The choice of $\overline{Y}$

Using Lemma 54.3, fix a $\mathfrak{D}$-generic set $Y_0 \in \mathbf{IPS}_{\mathbb{I}}$, $Y_0 \subseteq \overline{X}$. Using consecutively Lemma 36.2(iii), Lemma 17.1, Theorem 19.1, and Theorem 20.1, we obtain a set $\overline{Y} \in \mathbf{IPS}_{\mathbb{I}}$, $\overline{Y} \subseteq Y_0 \subseteq \overline{X}$, satisfying the following $3^\vartriangle$ – $6^\vartriangle$:

- $3^\vartriangle$: $\overline{Y}$ $\mathtt{forc}$ $\oint_\lambda$ or $\overline{Y}$ $\mathtt{forc}$ $\oint_\lambda^-$;
- $4^\vartriangle$: $\overline{Y}$ is uniform;
- $5^\vartriangle$: either (a) $\mathbb{F}_\lambda$ avoids $\overline{Y}{\Downarrow}\boldsymbol{i}$ on $\overline{Y}$ for all $\boldsymbol{i} \in \mathbb{I}$, or (b) $\mathbb{F}_\lambda$ is a $\mathbb{j}$-axis map on $\overline{Y}$ for some $\mathbb{j} \in \mathbb{I}$, and $\mathbb{F}_\lambda$ avoids $\overline{Y}{\Downarrow}\boldsymbol{i}$ on $\overline{Y}$ for all $\boldsymbol{i} \in \mathbb{I} \smallsetminus \{\mathbb{j}\}$;
- $6^\vartriangle$: the image $S = \mathbb{F}_\lambda``\overline{Y}$ is U-avoidable on $\boldsymbol{i}$ for all $\boldsymbol{i} \in \mathbb{I}$, $U \in \mathcal{U}_\lambda{\downarrow}_{\subseteq i}$.

Remark 55.1. The set $\overline{Y} \subseteq \overline{X}$ is $\mathfrak{D}$-generic along with $\overline{X}$, and hence

- $2^\vartriangle$: if λ is limit, $k < \omega$, $\mathbf{B}_{\lambda k}{}^\sharp \subseteq \mathcal{Q}_{<\lambda}$, and $\mathbf{B}_{\lambda k}{}^\sharp$ is dense in $\mathcal{Q}_{<\lambda}$, then $\overline{Y} \subseteq^{\mathtt{fin}} \bigcup \mathbf{B}_{\lambda k}{}^\sharp$.

This needs some work. By the density assumption, the derived set $\Phi_{\lambda k} := \mathbf{B}_{\lambda k}{}^\sharp{\uparrow}\mathbb{I}$ is dense in $\mathcal{U}' = \mathcal{Q}_{<\lambda}{\uparrow}\mathbb{I}$. Howevere $\mathcal{U}'$ itself is dense in $\mathcal{U}_\lambda =$

$\mathbf{RH}(\mathcal{U}')$ by Theorem 43.2(vi) — here we use that λ is limit. Thus $\Phi_{\lambda k}$ is dense in $\mathcal{U}_\lambda$. It follows that $\overline{Y} \subseteq \overline{X} \subseteq^{\mathtt{fin}} \bigcup \Phi_{\lambda k}$, by the $\mathfrak{D}$-genericity. ($\Phi_{\lambda k} \in \mathfrak{D}$ holds since $\mathbf{B}_{\lambda k}{}^\sharp \in \mathfrak{D}$.) This implies $\overline{Y} \subseteq^{\mathtt{fin}} \bigcup \mathbf{B}_{\lambda k}{}^\sharp$ as well. □

Remark 55.2. A certain oddity in the numbering above is caused by the fact that we want to indicate a connection with the numbering of items in Definition 52.1. Thus say $3^\vartriangle$ corresponds to condition (C3) in 52.1, *etc.* In addition, $6^\vartriangle$ will assist $5^\vartriangle$ in getting to (C5) in 52.1, whereas (B) will be considered in Section 58 below by means not related to $\overline{Y}$. □

Remark 55.3. Coming back to $5^\vartriangle$, we may note that $\mathbb{j}$ is unique in case (b) by (1) in the proof of Theorem 19.1. Moreover (a) and (b) are incompatible. (If (b) holds then take $\boldsymbol{i} = \mathbb{j}$ in (a), getting a contradiction.) This allows us to define $\mathbb{\delta} = \mathbb{\tau}$ in case (a) of $5^\vartriangle$, and $\mathbb{\delta} = \{\boldsymbol{i} \in \mathbb{\tau} : \boldsymbol{i} \not\approx_{\mathtt{par}} \mathbb{j}\}$ in case (b). □

Let $\eta \in \mathbf{\Xi}, \eta \subseteq \mathbb{\tau}$. Say that $Z \in \mathbf{IPS}_\eta$ is a $\mathbb{\delta}$-*set* iff it is similar to $\overline{Y}$ in the sense that $\mathbb{F}_\lambda$ avoids $Z{\Downarrow}\boldsymbol{i}$ on $\overline{Y}$ for all $\boldsymbol{i} \in \mathbb{\delta} \cap \eta$.

Lemma 55.4. *$\overline{Y}$ is a $\mathfrak{D}$-generic $\mathbb{\delta}$-set.*

Proof. $\overline{Y}$ is $\mathfrak{D}$-generic since Y_0 is such and $\overline{Y} \subseteq Y_0$. $\overline{Y}$ is a $\mathbb{\delta}$-set by $5^\vartriangle$. □

56 Lifting theorem

Our further major goal will be to include $\overline{Y}$ in a suitable rudiment, by Corollary 57.2 below. The following is the key technical result.

Theorem 56.1 (in $\mathbf{L}$). *Let $\boldsymbol{i} \in \mathbb{\tau}$, $U \in \mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $X \in \mathbf{IPS}_{\subset \boldsymbol{i}}$ be a $\mathfrak{D}$-generic $\mathbb{\delta}$-set, and $X \subseteq U{\downarrow}_{\subset \boldsymbol{i}}$. Then there exists a $\mathfrak{D}$-generic $\mathbb{\delta}$-set $X' \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$, such that $X' \subseteq U$, $X'{\downarrow}_{\subset \boldsymbol{i}} = X$.*

Proof. This is a rather long argument. We fix $\boldsymbol{i}, U, X$ during the course of the proof. We can assume, by $6^\vartriangle$, that

(∗) $\mathbb{F}_\lambda$ avoids $U{\Downarrow}\boldsymbol{i}$ on $\overline{Y}$.

Let *an atom* be any set of the form $V = W \cap (P{\uparrow}^{\subseteq \boldsymbol{i}})$, where $\varnothing \neq P \subseteq X$ is clopen in X (then $P \in \mathbf{IPS}_{\subset \boldsymbol{i}}$), $W \in \mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $W \subseteq U$, and $P \subseteq W{\downarrow}_{\subset \boldsymbol{i}}$. Let $\mathcal{Q}$ = all finite non-empty unions of atoms. We claim that

(A) If $Q \in \mathcal{Q}$ then $Q{\downarrow}_{\subset \boldsymbol{i}} \subseteq X$ and $Q{\downarrow}_{\subset \boldsymbol{i}}$ is clopen in X (as a finite union of relatively clopen sets);

(B) $\mathcal{Q} \subseteq \mathbf{IPS}_{\subseteq \boldsymbol{i}}$;

(C) if $\varnothing \neq Q' \subseteq Q \in \mathscr{Q}$, Q' is clopen in Q, then $Q' \in \mathscr{Q}$.

(D) if $Q, Q' \in \mathscr{Q}$, $\eta \in \boldsymbol{\Xi}$, $\eta \subseteq [\subset \boldsymbol{i}]$, $Q{\downarrow}\eta \subseteq Q'{\downarrow}\eta$, then the set $Q'' = Q' \cap (Q{\downarrow}\eta{\uparrow}^{\subseteq \boldsymbol{i}})$ belongs to $\mathscr{Q}$.

To prove (B), assume that $Q = V_1 \cup \ldots \cup V_n \in \mathscr{Q}$, where each $V_e = W_e \cap (P_e{\uparrow}^{\subseteq \boldsymbol{i}})$ is an atom, so that $\varnothing \neq P_e \subseteq X$ is clopen in X (then $P_e \in \mathbf{IPS}_{\subset \boldsymbol{i}}$ is $\mathfrak{D}$-generic), $W_e \in \mathscr{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $W_e \subseteq U$, and $P_e \subseteq W_e{\downarrow}_{\subset \boldsymbol{i}}$. We have $V_e \in\in \mathbf{IPS}_{\subset \boldsymbol{i}}$ by Lemma 10.5, and obviously $V_e{\downarrow}_{\subset \boldsymbol{i}} = P_e$.

Let $e = 1, \ldots, n$. Coming back to Section 11, put $\mathcal{T}_e(x) = \mathbf{tree}(\mathbf{D}_{V_e,x}(\boldsymbol{i}))$ for all $x \in P_e = V_e{\downarrow}_{\subset \boldsymbol{i}}$, so that $\mathcal{T}_e : P_e \to \mathbf{PT}$ is continuous by Lemma 11.2. Define the extended map $\mathcal{T}'_e : X \to \mathbf{PT}$ by $\mathcal{T}'_e(x) := \mathcal{T}_e(x)$ for $y \in P_e$ and $\mathcal{T}'_e(x) := \varnothing$ for $x \in X \smallsetminus P_e$. Then $\mathcal{T}'_e$ is continuous since P_e is clopen in X.

We conclude that $\mathcal{T}(x) := \mathcal{T}_1(x) \cup \ldots \cup \mathcal{T}_n(x) : X \to \mathbf{PT}$ is continuous. It follows by Lemma 11.1 that the set

$$Q' = \{z \in \mathscr{D}^{[\subseteq \boldsymbol{i}]} : z{\downarrow}_{\subset \boldsymbol{i}} \in P = P_1 \cup \ldots \cup P_n \wedge z(i) \in [\mathcal{T}(x)]\}$$

belongs to $\mathbf{IPS}_{\subseteq \boldsymbol{i}}$. On the other hand easily $Q' = Q$.

It suffices to prove (C) in case when $Q = W \cap (P{\uparrow}^{\subseteq \boldsymbol{i}})$ is an atom, so that $\varnothing \neq P \subseteq X$ is clopen in X, $W \in \mathscr{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $W \subseteq U$, and $P \subseteq W{\downarrow}_{\subset \boldsymbol{i}}$. By Lemma 11.6 we have $Q' = W' \cap (P'{\uparrow}^{\subseteq \boldsymbol{i}})$, where $W' \subseteq W$ and $P' \subseteq P$ are relatively clopen and still $P \subseteq W{\downarrow}_{\subset \boldsymbol{i}}$. Thus Q' is an atom as well.

To prove (D) note that the sets $Q{\downarrow}\eta$ and $Q'{\downarrow}\eta$ are clopen in $U{\downarrow}\eta$ by Lemma 11.4. Thus Q'' is clopen in Q'. It remains to refer to (C).

Lemma 56.2. *Let $\mathscr{Y} \in \mathfrak{D}$, $\mathscr{Y} \subseteq \mathscr{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $\mathscr{Y}$ is dense in $\mathscr{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, and $Q \in \mathscr{Q}$. Then there is $Q' \in \mathscr{Q}$, $Q' \subseteq Q$, such that $Q'{\downarrow}_{\subset \boldsymbol{i}} = Q{\downarrow}_{\subset \boldsymbol{i}}$ and $Q' \subseteq^{\mathtt{fin}} \bigcup \mathscr{Y}$.*

Proof (Lemma)**.** We w.l.o.g. assume that $Q = W \cap (P{\uparrow}^{\subseteq \boldsymbol{i}})$ is an atom, where $\varnothing \neq P \subseteq X$ is clopen in X, $W \in \mathscr{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $W \subseteq U$, $P \subseteq W{\downarrow}_{\subset \boldsymbol{i}}$. Then

(1) $Q{\downarrow}_{\subset \boldsymbol{i}} = P \subseteq X$ and $Q{\downarrow}_{\subset \boldsymbol{i}}$ is clopen in X;

(2) $Q{\downarrow}_{\subset \boldsymbol{i}} = P \subseteq X$ is a $\mathfrak{D}$-generic $\mathfrak{S}$-set (because such is X).

We claim that the set

$$\mathscr{Y}_1 \;=\; \{A{\downarrow}_{\subset \boldsymbol{i}} : A \in \mathscr{Y} \wedge A \subseteq W\} \;\cup\; \{Z \in \mathscr{U}_\lambda{\downarrow}_{\subset \boldsymbol{i}} : Z \cap W{\downarrow}_{\subset \boldsymbol{i}} = \varnothing\}$$

is dense in $\mathscr{U}_\lambda{\downarrow}_{\subset \boldsymbol{i}}$. Indeed let $S \in \mathscr{U}_\lambda{\downarrow}_{\subset \boldsymbol{i}}$; we have to find $Z \in \mathscr{Y}_1$, $Z \subseteq S$.

Case 1: $S \subseteq W{\downarrow}_{\subset \boldsymbol{i}}$. Then the set $W' = W \cap (S{\uparrow}^{\subseteq \boldsymbol{i}})$ belongs to $\mathscr{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$ as $\mathscr{U}_\lambda$ is a rudiment. Thus, by the density of $\mathscr{Y}$, there is a set $A \in \mathscr{Y}$, $A \subseteq W'$. Then $Z = A{\downarrow}_{\subset \boldsymbol{i}} \in \mathscr{Y}_1$ is as required.

Case 2: $S' = S \smallsetminus (W{\downarrow}_{\subset \boldsymbol{i}}) \neq \varnothing$. Then there is a set $\varnothing \neq Z \subseteq S'$ clopen in S. As $\mathcal{U}_\lambda$ is a rudiment, we have $Z \in \mathcal{U}_\lambda{\downarrow}_{\subset \boldsymbol{i}}$. Thus $Z \in \mathscr{Y}_1$, as required.

The density of $\mathscr{Y}_1$ is established. As obviously $\mathscr{Y}_1 \in \mathfrak{D}$, it follows that $P \subseteq^{\mathtt{fin}} \bigcup \mathscr{Y}_1$ by (2), hence $P \subseteq Z_1 \cup \ldots \cup Z_m$, $Z_e \in \mathscr{Y}_1$, $\forall\, e$. By the choice of P, we can w.l.o.g. assume that each Z_e belongs to the first part of $\mathscr{Y}_1$, *i.e.*, $Z_e = A_e{\downarrow}_{\subset \boldsymbol{i}} \in \mathcal{U}_\lambda{\downarrow}_{\subset \boldsymbol{i}}$, where $A_e \in \mathscr{Y}$, $A_e \subseteq W$. Let $P_e = P \cap Z_e$.

As P is $\mathfrak{D}$-generic, each P_e is clopen in P by Theorem 54.4(iv), and hence clopen in X by (1). It follows that each $V_e = A_e \cap (P_e{\uparrow}^{\subseteq \boldsymbol{i}})$ is an atom (or $\varnothing$). Therefore $Q' = V_1 \cup \ldots \cup V_n \in \mathscr{Q}$, $Q' \subseteq^{\mathtt{fin}} \bigcup \mathscr{Y}$ (as each A_e belongs to $\mathscr{Y}$), and $Q'{\downarrow}_{\subseteq \boldsymbol{i}} = P_1 \cup \ldots \cup P_n = P = Q{\downarrow}_{\subseteq \boldsymbol{i}}$, as required.

□ (Lemma 56.2)

To proceed with another lemma, we fix a $[\subseteq \boldsymbol{i}]$-admissible function $\phi \in \mathfrak{D}$, $\phi : \omega \to [\subseteq \boldsymbol{i}]$ (meaning that if $\boldsymbol{j} \subseteq \boldsymbol{i}$ then $\phi(k) = \boldsymbol{j}$ for infinitely many k).

Lemma 56.3. *Let $n < \omega$, and $\langle Y_s \rangle_{s \in 2^n}$ be a system of sets $Y_s \in \mathscr{Q}$, satisfying* S1 *of Definition 15.1 with $\zeta = [\subseteq \boldsymbol{i}]$. Let $\mathscr{Y} \in \mathfrak{D}$, $\mathscr{Y} \subseteq \mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $\mathscr{Y}$ be dense in $\mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$. Then there is a system $\langle Q_s \rangle_{s \in 2^n}$ of sets $Q_s \in \mathscr{Q}$, $Q_s \subseteq Y_s$, satisfying* S1 *and $Q_s{\downarrow}_{\subset \boldsymbol{i}} = Y_s{\downarrow}_{\subset \boldsymbol{i}}$, $Q_s \subseteq^{\mathtt{fin}} \bigcup \mathscr{Y}$ for all $s \in 2^n$.*

Proof. If $s \in 2^n$ then, by Lemma 56.2, pick a set $Q_s \in \mathscr{Q}$, $Q_s \subseteq Y_s$, such that $Q_s{\downarrow}_{\subset \boldsymbol{i}} = Y_s{\downarrow}_{\subset \boldsymbol{i}}$ and $Q_s \subseteq^{\mathtt{fin}} \bigcup \mathscr{Y}$. The system $\langle Q_s \rangle_{s \in 2^n}$ still satisfies S1 (with $\zeta = [\subseteq \boldsymbol{i}]$) because if $s \neq t$ belong to 2^n then $\zeta_\phi[s,t] \subseteq [\subset \boldsymbol{i}]$, hence $Q_s{\downarrow}\zeta_\phi[s,t] = Y_s{\downarrow}\zeta_\phi[s,t] = Y_t{\downarrow}\zeta_\phi[s,t] = Q_t{\downarrow}\zeta_\phi[s,t]$. □ (Lemma 56.3)

Finalization. Now we are able to accomplish the proof of Theorem 56.1. We define a ϕ-fusion sequence $\langle Q_u \rangle_{u \in 2^{<\omega}}$ of sets $Q_u \in \mathscr{Q}$ (still with $\zeta = [\subseteq \boldsymbol{i}]$ in Definition 16.1) satisfying

(1) $Q_\Lambda = U \cap (X{\uparrow}^{\subseteq \boldsymbol{i}})$ — this is even an atom by the choice of U, X in Theorem 56.1;

(2) if $\mathscr{Y} \in \mathfrak{D}$, $\mathscr{Y} \subseteq \mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $\mathscr{Y}$ is dense in $\mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, then there is $m < \omega$ such that $Q_u \subseteq^{\mathtt{fin}} \bigcup \mathscr{Y}$ for all $u \in 2^m$;

(3) if $m < \omega$ then $\bigcup_{u \in 2^m} Q_u{\downarrow}_{\subset \boldsymbol{i}} = Q_\Lambda{\downarrow}_{\subset \boldsymbol{i}} = X$.

Namely suppose that a layer $\langle Q_u \rangle_{u \in 2^m}$ has been defined so that both S1,S2 of Definition 15.1, and (3), hold for this m. Let $Y_{u^\frown e} = (Q_u)_{\to \boldsymbol{i}, e}$ for all $u \in 2^m$ and $e = 0, 1$, where $\boldsymbol{i} = \phi(m)$, so that $\langle Y_s \rangle_{s \in 2^{m+1}}$ is a clopen expansion of $\langle Q_u \rangle_{u \in 2^m}$ by Lemma 15.6. Each Y_s belongs to $\mathscr{Q}$ by (C) above. Lemma 56.3 yields a system $\langle Q_s \rangle_{s \in 2^{m+1}}$ of sets $Q_s \in \mathscr{Q}$, $Q_s \subseteq Y_s$, satisfying S1 and $Q_s{\downarrow}_{\subset \boldsymbol{i}} = Y_s{\downarrow}_{\subset \boldsymbol{i}}$, $Q_s \subseteq^{\mathtt{fin}} \bigcup \mathscr{Y}$ for all $s \in 2^{m+1}$, as required.

Having an (1)-(2)-(3) fusion sequence in $\mathscr{Q}$, we define $X' = \bigcap_m \bigcup_{u \in 2^m} Q_u$. Then $X' \in \mathbf{IPS}_{\subseteq \boldsymbol{i}}$ by Theorem 16.2, $X' \subseteq Q_\Lambda = U$ by construction, $X' {\downarrow}_{\subset \boldsymbol{i}} = X$ by (3), X' is $\mathfrak{D}$-generic by (2).

Further, $\mathbb{F}_\lambda$ avoids $X' {\Downarrow} \boldsymbol{i}$ on $\overline{Y}$ by (∗) and because $X' \subseteq U$. Moreover, if $\boldsymbol{j} \subset \boldsymbol{i}$ and $\boldsymbol{j} \in \mathfrak{G}$ then $\mathbb{F}_\lambda$ avoids $X' {\Downarrow} \boldsymbol{j}$ on $\overline{Y}$ since $X' {\downarrow}_{\subset \boldsymbol{i}} = X$ and X is a $\mathfrak{G}$-set. Thus overall $\mathbb{F}_\lambda$ avoids $X' {\Downarrow} \boldsymbol{j}$ on $\overline{Y}$ for every $\boldsymbol{j} \in [\subseteq \boldsymbol{i}] \cap \mathfrak{G}$, and hence X' is a $\mathfrak{G}$-set, as required. □ (Theorem 56.1)

57 Consequences of the lifting theorem

Consider the system $\mathcal{K} = \langle \mathcal{K}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \daleth}$ of sets

$$\mathcal{K}_{\boldsymbol{i}} = \{X \in \mathbf{IPS}_{\subseteq \boldsymbol{i}} : X \text{ is a } \mathfrak{D}\text{-generic } \mathfrak{G}\text{-set}\}.$$

Corollary 57.1. (i) *Let $\boldsymbol{j} \subset \boldsymbol{i}$ belong to $\daleth$, $U \in \mathcal{U}_\lambda {\downarrow}_{\subseteq \boldsymbol{i}}$, $X \in \mathcal{K}_{\boldsymbol{j}}$, $X \subseteq U {\downarrow}_{\subseteq \boldsymbol{j}}$. Then there is a set $X' \in \mathcal{K}_{\boldsymbol{i}}$, $X' \subseteq U$, such that $X' {\downarrow}_{\subseteq \boldsymbol{j}} = X$;*

(ii) *in particular, with $U = \mathscr{D}^{\boldsymbol{I}[<\boldsymbol{i}]}$, if $X \in \mathcal{K}_{\boldsymbol{j}}$ then there is a set $X' \in \mathcal{K}_{\boldsymbol{i}}$ such that $X' {\downarrow}_{\subseteq \boldsymbol{j}} = X$;*

(iii) *the system $\mathcal{K} = \langle \mathcal{K}_{\boldsymbol{i}} \rangle_{\boldsymbol{i} \in \daleth}$ is a $\daleth$-kernel.*

Proof. (i) is an immediate corollary of Theorem 56.1 (applied by induction on $\mathtt{lh}(\boldsymbol{i}) - \mathtt{lh}(\boldsymbol{j})$), with (ii) being a particular case of (i).

To prove (iii), note that (ii) implies 1∗ of Section 22 for $\mathcal{K}$. Condition 2∗ in Section 22 is obvious, whereas 3∗, 4∗ hold because the property of being a $\mathfrak{D}$-generic $\mathfrak{G}$-set is transferred to all smaller sets still in $\mathbf{IPS}$. (Note that Z in 3∗ and Y in 4∗ belong to $\mathbf{IPS}_{\subseteq \boldsymbol{i}}$ by Lemma 10.5, resp., Lemma 11.3.) Finally 5∗ holds because all notions related to the property of being a $\mathfrak{D}$-generic $\mathfrak{G}$-set are invariand under the action of $\pi_{\boldsymbol{ij}}$ because $\pi_{\boldsymbol{ij}} \in \mathfrak{D}$. □

Following Section 40, we consider the rudiment

$$\mathcal{P} = \mathcal{P}(\mathcal{K}) := \{X \in \daleth = \boldsymbol{I}[<\lambda] : \forall\, \boldsymbol{i} \in \daleth\, (X {\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}})\} \in \mathbf{Rud}_\lambda .$$

Corollary 57.2. (i) *$\mathcal{P} \in \mathbf{Rud}_\lambda$ and $\mathcal{P} {\downarrow}_{\subseteq \boldsymbol{i}} = \mathcal{K}_{\boldsymbol{i}}$ for all $\boldsymbol{i} \in \daleth$;*

(ii) *$\mathcal{P}$ is a refinement of $\mathcal{U}_\lambda$: $\mathcal{U}_\lambda \sqsubset \mathcal{P}$ in the sense of Section 42;*

(iii) *$\overline{Y} \in \mathcal{P}$.*

Proof. (i) holds by Lemma 40.2.

(ii) We have to check $5^\dagger$, $6^\dagger$, $7^\dagger$ of Section 42.

Of them, $5^\dagger$ (*i.e.*, $\mathscr{D}^{\mathbb{T}} \in \mathcal{U}_\lambda$) holds by Theorem 43.2(i).

To prove $6^\dagger$, assume that $\eta \in \mathbf{FT}(\xi)$, $U \in \mathcal{U}_\lambda$, $Y \in \mathcal{P}$, $Y{\downarrow}\eta \subsetneq U{\downarrow}\eta$, and the goal is to find $Z \in \mathcal{P}$ satisfying $Z \subseteq U$ and $Z{\downarrow}\eta = Y{\downarrow}\eta$. For that purpose, we define a system of sets $X_{\boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{T}$, such that

(a) $X_{\boldsymbol{i}} = Y{\downarrow}_{\subset \boldsymbol{i}}$ for all $\boldsymbol{i} \in \eta$;

(b) $X_{\boldsymbol{i}} \subseteq U{\downarrow}_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i}$;

(c) if $\boldsymbol{j} \subset \boldsymbol{i}$, $\mathtt{lh}(\boldsymbol{i}) = \mathtt{lh}(\boldsymbol{j}) + 1$, then $X_{\boldsymbol{i}}{\downarrow}_{\subset \boldsymbol{i}} = X_{\boldsymbol{j}}$.

The construction goes on as follows. Assume that $\boldsymbol{j} \subset \boldsymbol{i}$ in $\mathbb{T}$, $\mathtt{lh}(\boldsymbol{i}) = \mathtt{lh}(\boldsymbol{j}) + 1$, $\boldsymbol{i} \notin \eta$, and a set $X_{\boldsymbol{j}} \in \mathcal{K}_{\boldsymbol{j}}$, $X_{\boldsymbol{j}} \subseteq U{\downarrow}_{\subseteq \boldsymbol{j}} = U{\downarrow}_{\subset \boldsymbol{i}}$ has been defined. Use Corollary 57.1(i) to get a set $X_{\boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}$, $X_{\boldsymbol{i}} \subseteq U{\downarrow}_{\subseteq \boldsymbol{i}}$, with $X_{\boldsymbol{i}}{\downarrow}_{\subset \boldsymbol{i}} = X_{\boldsymbol{j}}$.

After the construction of sets $X_{\boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}$ satyisfying (a),(b),(c) is accomplished, the set $Z = \{x \in \mathscr{D}^{\mathbb{T}} : \forall\, \boldsymbol{i} \in \mathbb{T}\, (x{\downarrow}\boldsymbol{i} \in X_{\boldsymbol{i}})\}$ is as required for $6^\dagger$.

To prove $7^\dagger$, assume that $\boldsymbol{i} \in \mathbb{T}$, $U \in \mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$, $Y \in \mathcal{P}{\downarrow}_{\subseteq \boldsymbol{i}}$. Then $U \cap Y$ is clopen in Y by Theorem 54.4(iv), as required.

(iii) As $\overline{Y}$ is $\mathfrak{D}$-generic by Lemma 55.4, we conclude that each $\overline{Y}{\downarrow}_{\subseteq \boldsymbol{i}}$ is $\mathfrak{D}$-generic as well by Theorem 54.4(i). And $\overline{Y}{\downarrow}_{\subseteq \boldsymbol{i}}$ is a $\mathfrak{S}$-set since such is $\overline{Y}$ itself still by Lemma 55.4. □

58 The construction of a sub-rudiment

We know that the set $\overline{Y}$ chosen in Section 55 belongs to $\mathcal{P}$ by Corollary 57.2(iii). Here we define another special set $\overline{Y_1} \in \mathcal{P}$, related rather to condition (B) of Definition 52.1, and then define a set $\mathcal{P}'$ required, in the form of a countable sub-rudiment of $\mathcal{P}$ containing both $\overline{Y}, \overline{Y_1}$. In some similarity to (1) of Section 53, we first define $\overline{X_1}$ as follows:

$1^\vartriangle$: if $\mathbf{M}^{\#}_{\lambda} \in \mathbf{NH}(\varphi{\restriction}\lambda)$ and $\|\mathbf{M}^{\#}_{\lambda}\| \subseteq \mathbb{T} = \boldsymbol{I}[{<}\lambda]$ then let $\overline{X_1} = \mathbf{M}^{\#}_{\lambda}{\uparrow}\mathbb{T}$, otherwise let $\overline{X_1} = \mathscr{D}^{\mathbb{T}}$, so $\overline{X_1} \in \mathbf{NH}(\varphi{\restriction}\lambda)$, $\|\overline{X_1}\| = \mathbb{T}$ in both cases.

Corollary 58.1 (of Lemma 40.2)**.** *If $\boldsymbol{i} \in \mathbb{T}$ then $\overline{X_1}{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{U}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}}$.* □

Whereas we do not assume that $\overline{X_1}$ as a whole belongs to $\mathcal{U}_\lambda$!

Corollary 58.2 (of Corollaries 58.1 and 57.1(i))**.** *There is a system of sets $Y_{\boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}, \boldsymbol{i} \in \mathbb{T}$ such that $Y_{\boldsymbol{i}} \subseteq \overline{X_1}{\downarrow}_{\subseteq \boldsymbol{i}}$ and if $\boldsymbol{j} \subset \boldsymbol{i}$ then $Y_{\boldsymbol{j}} = Y_i{\downarrow}_{\subset \boldsymbol{j}}$.* □

Recall that $\mathcal{K}_{\boldsymbol{i}}$ and $\mathcal{P}$ were defined in Section 57.

Corollary 58.3 (of Corollary 12.3)**.** *There is a set $\overline{Y_1} \in \mathbf{IPS}_{\mathbb{T}}$ such that $\overline{Y_1}{\downarrow}_{\subset \boldsymbol{i}} = Y_{\boldsymbol{i}}$ for all $\boldsymbol{i} \in \mathbb{T}$. Note that then $\overline{Y_1} \in \mathcal{P}$ as $Y_{\boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}, \forall\, \boldsymbol{i}$.* □

To conclude, we have got a set $\overline{Y}_1 \in \mathcal{P}$ satisfying $\overline{Y}_1 \subseteq \overline{X}_1$ (because $\overline{Y}_1{\downarrow}_{\subset \boldsymbol{i}} = Y_{\boldsymbol{i}} \subseteq \overline{X}_1{\downarrow}_{\subseteq \boldsymbol{i}}$). Recall that $\overline{Y} \in \mathcal{P}$ and $\mathcal{U}_\lambda \sqsubseteq \mathcal{P}$, by Corollary 57.2.

Lemma 58.4. *There is a **countable** sub-rudiment $\mathcal{P}' \subseteq \mathcal{P}$ still containing $\overline{Y}, \overline{Y}_1$ and satisfying $\mathcal{U}_\lambda \sqsubseteq \mathcal{P}'$.*

Proof. A routine "elementary substructure" argument. □

Lemma 58.5. *$\mathcal{P}'$ is a 1-5-$\Bbbk$ extension of $\boldsymbol{\varphi} = \langle \mathcal{Q}_\alpha \rangle_{\alpha<\lambda}$.*

Proof. Basically we have to check (A), (B), and (C) (including (C2)—-(C5)) of Definition 52.1 for $\mathcal{Q}_\lambda := \mathcal{P}'$.

(B) Suppose that $\mathbf{M}^{\#}_\lambda \in \mathbf{NH}(\boldsymbol{\varphi}{\restriction}\lambda)$ and $\|\mathbf{M}^{\#}_\lambda\| \subseteq \Bbbk = \boldsymbol{I}[{<}\lambda]$. Thus $\overline{X}_1 = \mathbf{M}^{\#}_\lambda{\uparrow}\Bbbk$ by $1^\vartriangle$ above. However $\overline{Y}_1 \in \mathcal{P}'$ and $\overline{Y}_1 \subseteq \overline{X}_1$ by construction, and this completes the proof of (B).

(C2) –(C5). In accordance to Definition 52.1, we assume that $\mathbf{M}^{\#}_\lambda \in \mathcal{Q}_{<\lambda} := \bigcup_{\gamma<\lambda} \mathcal{Q}_\gamma$ — then $\overline{X} = \mathbf{M}^{\#}_\lambda{\uparrow}\Bbbk$ by (1) of Section 53 — and the goal is to find a set $Y \in \mathcal{Q}_\lambda$ satisfying both $Y \downarrow\subseteq \mathbf{M}^{\#}_\lambda$ and each of (C2),(C3),(C4),(C5). Let's chech that the set $\overline{Y}$ defined in Section 55 is as required. First of all, note that $\overline{Y} \in \mathcal{P}'$ and $\overline{Y} \subseteq \overline{X} \downarrow\subseteq \mathbf{M}^{\#}_\lambda$ by construction. It remains to check (C2) –(C5) of Definition 52.1 for $\overline{Y}$.

(C2) Suppose that if λ is limit, $k < \omega$, $\mathbf{B}_{\lambda k}{}^\sharp \subseteq \mathcal{Q}_{<\lambda}$, and $\mathbf{B}_{\lambda k}{}^\sharp$ is dense in $\mathcal{Q}_{<\lambda}$. Then $\overline{Y} \subseteq^{\mathtt{fd}} \bigcup \mathbf{B}_{\lambda k}{}^\sharp$ holds by $2^\vartriangle$, as required.

(C3) and (C4) are immediate corollaries of $3^\vartriangle$, $4^\vartriangle$.

(C5) This is not so straightforward. First of all we claim that

(∗) if $Z \in \mathcal{P}$ and $\boldsymbol{i} \in \Bbbk$ then $\mathbb{F}_\lambda$ avoids $Z{\Downarrow}\boldsymbol{i}$ on $\overline{Y}$.

Indeed $Z' = Z{\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{K}_{\boldsymbol{i}}$ by Corollary 57.2(i), meaning that Z' is a $\Bbbk$-set. It follows that $\mathbb{F}_\lambda$ avoids $Z{\Downarrow}\boldsymbol{i} = Z'{\Downarrow}\boldsymbol{i}$ on $\overline{Y}$ because $\boldsymbol{i} \in \Bbbk$, as required.

Case 1: (a) of $5^\vartriangle$ in Section 55. Then $\Bbbk = \Bbbk$, and hence $\mathbb{F}_\lambda$ avoids $Z{\Downarrow}\boldsymbol{i}$ on $\overline{Y}$ for all $\boldsymbol{i} \in \Bbbk = \boldsymbol{I}[{<}\lambda]$ by (∗). Thus we have (C5)(a) of Definition 52.1.

Case 2: (b) of $5^\vartriangle$ in Section 55. Then accordingly $\Bbbk = \{\boldsymbol{i} \in \Bbbk : \boldsymbol{i} \not\approx_{\mathtt{par}} \mathbb{j}\}$ (see Remark 55.3) for some $\mathbb{j} \in \Bbbk$ as in $5^\vartriangle$(b). In other words, $\mathbb{F}_\lambda$ is a $\mathbb{j}$-axis map on $\overline{Y}$, and $\mathbb{F}_\lambda$ avoids $Z{\Downarrow}\boldsymbol{i}$ on $\overline{Y}$ for all $\boldsymbol{i} \in \Bbbk$, $\boldsymbol{i} \not\approx_{\mathtt{par}} \mathbb{j}$, by (∗). Thus we have (C5)(b) of Definition 52.1, as required. □ (Lemma 58.5)

□ (Theorem 53.1)

X The final forcing construction

Theorem 53.1 obviously allows to define, in $\mathbf{L}$, a **Rud** sequence $\boldsymbol{\varphi} = \langle \mathcal{Q}_\lambda \rangle_{\lambda<\omega_1}$ of length ω_1, such that each term $\mathcal{Q}_\lambda$ is a 1-5-$\mathbb{n}$ extension of the subsequence $\langle \mathcal{Q}_\alpha \rangle_{\alpha<\lambda}$, for a given $\mathbb{n} \geq 1$ of Theorem 52.2. Our next and the final step in the proof of Theorems 52.2– 34.1–1.1 will be to maintain such a construction so that the global definability condition $\mathfrak{P}_6^{\mathbb{n}}$ also holds. This will be the content of this Chapter.

We argue in L in this Chapter.

59 Some simple definability claims

We continue to argue in L. As usual, $\mathscr{P}_{\texttt{fin}}(X) = \{Y \subseteq X : Y \text{ is finite}\}$. To countably code the topology of spaces $\mathscr{D}^\xi$, put $U^\xi(\boldsymbol{i}, k, e) = \{x \in \mathscr{D}^\xi : x(\boldsymbol{i})(k) = e\}$ for all $\boldsymbol{i} \in \xi \in \boldsymbol{\Xi}$, $k < \omega$, $e = 0, 1$. If $u \subseteq \xi \times \omega \times 2$ is finite and *consistent* (that is, for no $\boldsymbol{i}, k$ both $\langle \boldsymbol{i}, k, 0\rangle$ and $\langle \boldsymbol{i}, k, 1\rangle$ belong to u) then put $U^\xi_u = \bigcap_{\langle \xi, k, e\rangle \in u} U^\xi(\boldsymbol{i}, k, e)$ (a basic clopen cube in $\mathscr{D}^\xi$). Finally, if

$$b \in \mathbf{cCO}_\xi := \mathscr{P}_{\texttt{fin}}(\mathscr{P}_{\texttt{fin}}(\xi \times \omega \times 2))$$

is *consistent*, in the sense that each $u \in b$ is such, then put $\mathbf{CO}^\xi_b = \bigcup_{u\in b} U^\xi_u$, an arbitrary clopen subset of $\mathscr{D}^\xi$. (**cCO** from *codes of ClOpen (sets)*.)

If $\xi \in \boldsymbol{\Xi}$ then let $\mathbf{Ctbl}_\xi = \{X \subseteq \mathscr{D}^\xi : X \text{ is finite or countable}\}$.

If $X, Y \in \mathbf{Ctbl}_\xi$ then let $X \cap^* Y = (X \cap Y^\#) \cup (Y \cap X^\#)$; then clearly $X \cap^* Y \in \mathbf{Ctbl}_\xi$ and $(X \cap^* Y)^\# = X^\# \cap Y^\#$.

If $\eta \subseteq \xi$ belong to $\boldsymbol{\Xi}$ and $Y \in \mathbf{Ctbl}_\eta$ then let $Y {\uparrow}^* \xi$ consist of all points $x \in \mathscr{D}^\xi$ such that $y = x{\downarrow}\eta \in Y$ and the set $\{\langle \boldsymbol{i}, k\rangle : \boldsymbol{i} \in \xi \smallsetminus \eta \wedge x(\boldsymbol{i})(k) = 1\}$ is finite. Thus $Y {\uparrow}^* \xi \in \mathbf{Ctbl}_\xi$ provided $Y \in \mathbf{Ctbl}_\eta$, whereas $Y {\uparrow} \xi$ is not necessarily countable, of course, but still $(Y {\uparrow}^* \xi)^\# = Y {\uparrow} \xi$.

1$^\blacktriangle$: the sets ω_1, ω, $\boldsymbol{I}$, $\boldsymbol{\Xi}$, $\mathbf{Ctbl} = \bigcup_{\xi\in\boldsymbol{\Xi}} \mathbf{Ctbl}_\xi$, $\{\langle \xi, X\rangle : \xi \in \boldsymbol{\Xi} \wedge X \in \mathbf{Ctbl}_\xi\}$ are $\Delta^{\mathbf{HC}}_1$ (as subsets of $\mathbf{HC}$);

2$^\blacktriangle$: the maps $\xi \mapsto \mathbf{cCO}_\xi$ and $\xi, X, b \mapsto X \cap \mathbf{CO}^\xi_b$ belong to $\Delta^{\mathbf{HC}}_1$;

3$^\blacktriangle$: the set $\{\langle X, Y\rangle : X, Y \in \mathbf{Ctbl} \wedge X^\# \subseteq Y^\#\}$ is $\Delta^{\mathbf{HC}}_1$;

4$^\blacktriangle$: the map $\langle \boldsymbol{i}, \boldsymbol{j}, X\rangle \mapsto \pi_{\boldsymbol{ij}} {\bullet} X$ (Section 13) belongs to $\Delta^{\mathbf{HC}}_1$;

5$^\blacktriangle$: the maps $\langle X, Y\rangle \mapsto X \cap^* Y$ and $\langle \xi, Y\rangle \mapsto Y {\uparrow}^* \xi$ belong to $\Delta^{\mathbf{HC}}_1$;

6$^\blacktriangle$: the maps $\xi \in \boldsymbol{\Xi} \mapsto \mathbf{FT}(\xi)$ (subsets of finite type, Section 40) and $\alpha \mapsto \boldsymbol{I}[{<}\alpha] := \alpha^{<\omega} \smallsetminus \{\Lambda\}$ (Section 8) belong to $\Delta^{\mathbf{HC}}_1$.

The proof of 1▲–6▲ is based on one common principle. Let $\mathbf{Z}\aleph_0$ be the theory of Zermelo $\mathbf{Z}$ sans the Power Set axiom, plus the axiom saying that every set x is at most countable. An $\in$-formula $\varphi(x,y,\dots)$ is $\mathbf{Z}\aleph_0$-*absolute*, if for any transitive model $\mathfrak{M}\in\mathbf{HC}$, $\mathfrak{M}\models\mathbf{Z}\aleph_0$, and any $x,y,\dots\in\mathfrak{M}$, the equivalence $(\mathbf{HC}\models\varphi(x,y,\dots)) \iff (\mathfrak{M}\models\varphi(x,y,\dots))$ holds.

Theorem 59.1. *If $\varphi(x,y,\dots)$ is a $\mathbf{Z}\aleph_0$-absolute $\in$-formula then the set $X=\{\langle x,y,\dots\rangle : \mathbf{HC}\models\varphi(x,y,\dots)\}$ is of the definability class $\varDelta^{\mathbf{HC}}_1$.*

Proof. The relation $\langle x,y,\dots\rangle\in X$ is equivalent to each of the two formulas

$$\exists\,\mathfrak{M}\in\mathbf{HC}\,\big(\mathfrak{M}\models\mathbf{Z}\aleph_0\wedge\mathfrak{M}\text{ is transitive}\wedge\mathfrak{M}\models\varphi(x,y,\dots)\big),$$
$$\forall\,\mathfrak{M}\in\mathbf{HC}\,\big(\mathfrak{M}\models\mathbf{Z}\aleph_0\wedge\mathfrak{M}\text{ is transitive}\implies\mathfrak{M}\models\varphi(x,y,\dots)\big).$$

The first formula provides $X\in\varSigma^{\mathbf{HC}}_1$, the second one gives $X\in\varPi^{\mathbf{HC}}_1$. □

Now to prove 1▲–6▲ it suffices to check that some natural formulas, which define the sets and relations mentioned in 1▲–6▲, are $\mathbf{Z}\aleph_0$-absolute. This is entirely routine, except perhaps for the relation $X^\#\subseteq Y^\#$, which we have to rewrite as follows. If $X,Y\in\mathbf{Ctbl}_\xi$ for one and the same $\xi\in\mathbf{\Xi}$ then we let $\xi(X,Y)=\xi$, otherwise keep $\xi(X,Y)$ undefined. Now, $X^\#\subseteq Y^\#$ is equivalent to the following formula, easily shown to be $\mathbf{Z}\aleph_0$-absolute:

$$\xi=\xi(X,Y)\text{ is defined and }\forall\,b\in\mathbf{cCO}_\xi(X\cap\mathbf{CO}^\xi_b\neq\varnothing\implies Y\cap\mathbf{CO}^\xi_b\neq\varnothing).$$

60 Definability of iterated perfect sets

Recall that $\mathbf{cIPS}_\xi=\{X\in\mathbf{Ctbl}_\xi : X^\#\in\mathbf{IPS}_\xi\}$ and $\mathbf{cIPS}=\bigcup_{\xi\in\mathbf{\Xi}}\mathbf{cIPS}_\xi$.

Theorem 60.1. *$\mathbf{cIPS}$ and $\{\langle\xi,A\rangle:\xi\in\mathbf{\Xi}\wedge A\in\mathbf{cIPS}_\xi\}$ are $\varDelta^{\mathbf{HC}}_1$ sets.*

Proof. We use the notation of Section 59. Let $\Psi(\xi,A)$ say the following:

(1) $\xi\in\mathbf{\Xi}$ and $A\subseteq\mathscr{D}^\xi$, and

(2) there is a set $C\subseteq\mathscr{D}^\xi$ and a bijection $h:C\xrightarrow{\text{onto}}A$ such that:

   (a) C is topologically dense in $\mathscr{D}^\xi$;

   (b) if $b_1\in\mathbf{cCO}_\xi$ and $\mathbf{CO}^\xi_{b_1}\cap A\neq\varnothing$ then there is $b\in\mathbf{cCO}_\xi$ such that the image $h``(C\cap\mathbf{CO}^\xi_b)$ is equal to $\mathbf{CO}^\xi_{b_1}\cap A$;

   (c) if $\boldsymbol{i}\in\xi$ and $x,y\in Z$ then $x{\downarrow}_{\subseteq\boldsymbol{i}}=y{\downarrow}_{\subseteq\boldsymbol{i}}$ iff $h(x){\downarrow}_{\subseteq\boldsymbol{i}}=h(y){\downarrow}_{\subseteq\boldsymbol{i}}$.

We assert that (*) $\big(\mathbf{HC}\models\Psi(\xi,A)\big)$ iff $\big(\xi\in\boldsymbol{\Xi}$ and $A\in\mathbf{cIPS}_\xi\big)$.

In the nontrivial direction, assume that $\xi,A\in\mathbf{HC}$ and $\Psi(\xi,A)$ is true in $\mathbf{HC}$. Then $\xi\in\boldsymbol{\Xi}$ by (1), thus it remains to prove that $A^{\#}\in\mathbf{IPS}_\xi$.

Let, by (2), a set $C\subseteq\mathscr{D}^\xi$ and a bijection $h:C\xrightarrow{\text{onto}}A$ satisfy (2)a, (2)b, (2)c in $\mathbf{HC}$, so that in fact $C\in\mathbf{Ctbl}_\xi$ is dense in $\mathscr{D}^\xi$ by (2)a. In particular, $C^{\#}=\mathscr{D}^\xi$. Let $H=h^{\#}$ be the topological closure of H in $\mathscr{D}^\xi\times\mathscr{D}^\xi$.

It easily follows from (2)b (and the compactness of the spaces considered) that H is a homeomorphism from $C^{\#}=\mathscr{D}^\xi$ onto $A^{\#}$. Finally, (2)c implies that H is projection–keeping, hence $A^{\#}\in\mathbf{IPS}_\xi$, as required. This completes the proof of (*).

It remains to prove that Ψ defines a $\varDelta^{\mathbf{HC}}_1$ relation. This looks somewhat doubtful (in spite of the rather obvious $\mathbf{Z}\aleph_0$-absoluteness of (1), (2)a, (2)b, (2)c and Theorem 59.1), because the $\exists$ quantifier in (2) does not seem to be replaceable by a $\forall$ quantifier. Yet we can apply the following trick.

Recall that $\boldsymbol{I}[{<}\omega]=\omega^{<\omega}\smallsetminus\{\Lambda\}\in\boldsymbol{\Xi}$. Clearly each $\xi\in\boldsymbol{\Xi}$ can be embedded in $\boldsymbol{I}[{<}\omega]$ via a map $\pi\in\Gamma_\xi$, where Γ_ξ consists of all $\subset$-preserving and length-preserving injections $\pi:\xi\to\boldsymbol{I}[{<}\omega]$. Thus

$$\begin{aligned}\Psi(\xi,A)\quad&\Longleftrightarrow\quad\exists\,\pi\in\Gamma_\xi\,\exists\,\xi'\,\exists\,A'\big(\xi'=\pi{\bullet}\xi\wedge A'=\pi{\bullet}A\wedge\Psi(\xi',A')\big)\\&\Longleftrightarrow\quad\forall\,\pi\in\Gamma_\xi\,\forall\,\xi'\,\forall\,A'\big(\xi'=\pi{\bullet}\xi\wedge A'=\pi{\bullet}A\Longrightarrow\Psi(\xi',A')\big).\end{aligned}$$

On the other hand, if it is assumed that $\xi'\subseteq\boldsymbol{I}[{<}\omega]$ and $A'\in\mathbf{Ctbl}_{\xi'}$, then the formula $\Psi(\xi',A')$ is convertible to an equivalent $\varSigma^1_1$ form by a suitable coding of ξ',A' by reals, and hence Ψ defines a $\varDelta^{\mathbf{HC}}_1$ relation in this particular domain by Proposition 7.1. It follows that the first line of the double equivalence above provides a $\varSigma^{\mathbf{HC}}_1$ definition of the relation defined by Ψ, whereas the second line provides its $\varPi^{\mathbf{HC}}_1$ definition, as required. □

61 Definability of rudiments

We come back to Definition 51.1.

Given any set $\mathcal{B}\subseteq\mathbf{cIPS}$ (so that $\mathcal{B}$ consists of codes of sets in $\mathbf{IPS}$), we let $\mathcal{B}^\sharp:=\{A^{\#}:A\in\mathcal{B}\}$; thus $\mathcal{B}^\sharp\subseteq\mathbf{IPS}$. Let $\alpha<\omega_1$, $\xi=\boldsymbol{I}[{<}\alpha]$. Say that $\mathcal{B}\subseteq\mathbf{cIPS}_\xi$ is *a coded rudiment of width* α, in symbol $\mathcal{B}\in\mathbf{cRud}_\alpha$, if $\mathcal{B}^\sharp\in\mathbf{Rud}_\alpha$. To evaluate the complexity of $\mathbf{cRud}_\alpha$ in the next theorem, we define several related notions. If $\alpha<\omega_1$, $\xi=\boldsymbol{I}[{<}\alpha]$, $\mathcal{B}\subseteq\mathbf{cIPS}_\xi$ then let $\mathcal{B}^+=\mathcal{B}^+_1\cup\mathcal{B}^+_2\cup\mathcal{B}^+_3$ be the union of the three following sets:

$$\begin{aligned}\mathcal{B}^+_1&=\{X\cap^*((Y{\downarrow}\eta){\uparrow}^*\xi):X,Y\in\mathcal{B}\wedge\eta\in\mathbf{FT}(\xi)\wedge(Y{\downarrow}\eta)^{\#}\subseteq(X{\downarrow}\eta)^{\#}\};\\\mathcal{B}^+_2&=\{X\cap\mathbf{CO}^\xi_b:X\in\mathcal{B}\wedge b\in\mathbf{cCO}_\xi\wedge X\cap\mathbf{CO}^\xi_b\in\mathbf{cIPS}_\xi\};\\\mathcal{B}^+_3&=\{\pi_{\boldsymbol{ij}}{\bullet}X:X\in\mathcal{B}\wedge\boldsymbol{i},\boldsymbol{j}\in\xi\wedge\boldsymbol{i}\approx_{\text{par}}\boldsymbol{j}\}.\end{aligned}$$

We also define $\mathbf{cRH}(\mathcal{B}) = \bigcup_n \mathcal{B}_n$ (the *coded* rudimentary hull), where $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_{n+1} = (\mathcal{B}_n)^+$, $\forall n$. Then: (1) $\mathbf{cRH}(\mathcal{B}) \in \mathbf{cRud}_\alpha$,

(2) $(\mathbf{cRH}(\mathcal{B}))^\sharp = \mathbf{RH}(\mathcal{B}^\sharp)$ (rudimentary hull, Section 41),

(3) $\mathcal{B} \in \mathbf{cRud}_\alpha$ iff $\mathcal{B}^\sharp = (\mathbf{cRH}(\mathcal{B}))^\sharp$.

Theorem 61.1. *The following sets belong to* $\Delta_1^{\mathbf{HC}}$:

(i) $W_1 = \{\langle \mathcal{B}, \mathbf{cRH}(\mathcal{B})\rangle : \exists\, \alpha < \omega_1\, (\mathcal{B} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}) \wedge \mathcal{B}$ is countable$\}$;

(ii) $W_2 = \{\langle \alpha, \mathcal{B}\rangle : \alpha < \omega_1 \wedge \mathcal{B} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\alpha]} \wedge \mathcal{B} \in \mathbf{cRud}_\alpha\}$.

Proof. (i) For any $\mathcal{B}$, if there is an ordinal α such that $\mathcal{B} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}$ then let $\alpha(\mathcal{B}) := \alpha$. Then $\langle \mathcal{B}, \mathcal{B}'\rangle \in W_1$ iff $\Phi_1(\mathcal{B}, \mathcal{B}')$ holds in $\mathbf{HC}$, where

$$\Phi_1(\mathcal{B}, \mathcal{B}') \ := \ \big(\mathcal{B}' = \mathbf{cRH}(\mathcal{B}) \wedge \alpha(\mathcal{B}) = \alpha \text{ exists} \wedge \mathcal{B}, \mathcal{B}' \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\alpha]}\big).$$

In this formula, the two first summands are $\mathbf{Z}\aleph_0$-absolute, hence $\Delta_1^{\mathbf{HC}}$ by Theorem 59.1, whereas the rightmost summand is $\Delta_1^{\mathbf{HC}}$ by Theorem 60.1.

(ii) Quite similarly, $\langle \alpha, \mathcal{B}\rangle \in W_2$ iff $\Phi_2(\mathcal{B}, \mathcal{B}')$ holds in $\mathbf{HC}$, where

$$\Phi_2(\alpha, \mathcal{B}) \ := \ \big(\alpha(\mathcal{B}) = \alpha \wedge \mathcal{B} \subseteq \mathbf{IPS}_{\boldsymbol{I}[<\alpha]} \wedge \mathcal{B} = \mathbf{cRH}(\mathcal{B})\big),$$

and then replace $\mathcal{B} = \mathbf{cRH}(\mathcal{B})$ by $\langle \mathcal{B}, \mathcal{B}\rangle \in W_1$ and refer to (i). □

62 Definability of rudimentary sequences

Recall that a sequence $\beta = \langle \mathcal{B}_\alpha\rangle_{\alpha<\lambda}$ is a *coded* **Rud** *sequence of length* λ, or a **cRud** *sequence*, if each $\mathcal{B}_\alpha \in \mathbf{cRud}_\alpha$ is countable and the sets $\mathcal{Q}_\alpha = \mathcal{B}_\alpha^\sharp := \{A^\# : A \in \mathcal{B}_\alpha\} \in \mathbf{Rud}_\alpha$ form a **Rud** sequence $\varphi = \beta^\sharp := \langle \mathcal{Q}_\alpha\rangle_{\alpha<\lambda}$.

Theorem 62.1. *The following set belongs to* $\Delta_1^{\mathbf{HC}}$:

$$W = \{\langle \alpha, \beta\rangle : \alpha < \omega_1 \wedge \beta \text{ is a coded } \mathbf{Rud} \text{ sequence of length } \alpha\}.$$

Proof. Conditions (A), (B), (C) of Definition 43.1 find their $\Delta_1^{\mathbf{HC}}$ forms by different results above. In particular, as far as (C) is concerned, make use of $3^\blacktriangle$ in Section 59. Recall the remaining condition (D):

(D) if $3 \le \nu < \lambda$ then $\mathbf{RH}(\varphi \restriction \nu) \sqsubset \mathcal{Q}_\nu$ in the sense of Definition 42.1; here $\mathbf{RH}(\varphi \restriction \nu) = \mathbf{RH}(\bigsqcup(\varphi \restriction \nu)) = \mathbf{RH}(\bigcup_{\alpha<\nu}(\mathcal{Q}_\alpha \uparrow \boldsymbol{I}[<\nu]))$ and $\mathcal{Q}_\alpha = \varphi(\alpha)$.

In terms of a coded **Rud** sequence $\beta = \langle \mathcal{B}_\alpha\rangle_{\alpha<\lambda}$, it takes the form:

(cD) if $3 \le \nu < \lambda$ then $\mathcal{B}_{<\nu} \sqsubset^\sharp \mathcal{B}_\nu$ — where $\mathcal{B} \sqsubset^\sharp \mathcal{B}'$ means $\mathcal{B}^\sharp \sqsubseteq \mathcal{B}'^\sharp$ provided $\mathcal{B}, \mathcal{B}' \subseteq \mathbf{cIPS}_\nu$, $\mathcal{B}_{<\nu} = \mathbf{cRH}(\{A \uparrow^* \boldsymbol{I}[<\nu] : A \in \bigcup_{\alpha<\nu} \mathcal{B}_\alpha\})$, whereas $\mathbf{cRH}(\mathcal{B})$ and $\uparrow^*$ are defined in Sections 61, resp., 59.

Thus it remains to prove that $\mathcal{B} \sqsubset^{\sharp} \mathcal{B}'$ is a $\mathbf{Z}\aleph_0$-absolute, hence a $\varDelta_1^{\mathbf{HC}}$ relation by Theorem 59.1. To check this, we return to Definition 42.1. In terms of $\mathcal{P} = \mathcal{B}^{\sharp}$ and $\mathcal{Q} = \mathcal{B}'^{\sharp}$, conditions $5^{\dagger}$, $6^{\dagger}$, $7^{\dagger}$ there take the form:

c$5^{\dagger}$. There is $A \in \mathcal{B}$ dense in $\mathscr{D}^{\xi}$, so that $A^{\#} = \mathscr{D}^{\xi}$.

c$6^{\dagger}$. If $\eta \in \mathbf{FT}(\xi)$, $A \in \mathcal{B}$, $B \in \mathcal{B}'$, $(B{\downarrow}\eta)^{\#} \subseteq (A{\downarrow}\eta)^{\#}$, then there is $C \in \mathcal{B}'$ such that $C^{\#} \subseteq A^{\#}$ and $(C{\downarrow}\eta)^{\#} = (B{\downarrow}\eta)^{\#}$.

c$7^{\dagger}$. If $\boldsymbol{i} \in \xi$, $A \in \mathcal{B}{\downarrow}_{\subseteq \boldsymbol{i}}$, $B \in \mathcal{B}'{\downarrow}_{\subseteq \boldsymbol{i}}$, then $A^{\#} \cap B^{\#}$ is *clopen* in $B^{\#}$.

That c$5^{\dagger}$ is $\mathbf{Z}\aleph_0$-absolute, is pretty clear.

See the end of Section 59 regarding the conversion of formulas like $C^{\#} \subseteq A^{\#}$ in c$6^{\dagger}$ to a $\mathbf{Z}\aleph_0$-absolute form.

Finally, $A^{\#} \cap B^{\#} = A \cap^{*} B^{\#}$. Then the clopenness of $A \cap^{*} B^{\#}$ in $B^{\#}$ is equivalent to the following $\mathbf{Z}\aleph_0$-absolute formula:

$$\exists\, b \in \mathbf{cCO}_{\xi}\, (A \cap^{*} B^{\#} \cap \mathbf{CO}_b^{\xi} = B^{\#}).$$

Thus c$5^{\dagger}$+c$6^{\dagger}$+c$7^{\dagger}$, as a whole, is $\mathbf{Z}\aleph_0$-absolute, and $\varDelta_1^{\mathbf{HC}}$, as required. □

63 Definability claims related to continuous functions

Recall the notions $\mathbf{Rat}_{\xi}$, $\mathbf{cCF}_{\xi}$, $\mathbf{cCF}^{*}_{\xi}$, $\mathbf{cCF} = \bigcup_{\xi \in \boldsymbol{\Xi}} \mathbf{cCF}_{\xi}$, and $\mathbf{cCF}^{*} = \bigcup_{\xi \in \boldsymbol{\Xi}} \mathbf{cCF}^{*}_{\xi}$, related to codes of continuous functions $\mathscr{D}^{\xi} \to \omega^{\omega}$ and $\mathscr{D}^{\xi} \to 2^{\omega} = \mathscr{D}$, $\xi \in \boldsymbol{\Xi}$, and defined in $\mathbf{L}$ in Section 35.

See Sections 18,19,20 on axis maps and avoidance.

Theorem 63.1 (in $\mathbf{L}$)**.** *The following sets belong to* $\varDelta_1^{\mathbf{HC}}$:

(i) $\{\langle \xi, f\rangle : \xi \in \boldsymbol{\Xi} \wedge f \in \mathbf{cCF}_{\xi}\}$ *and* $\{\langle \xi, f\rangle : \xi \in \boldsymbol{\Xi} \wedge f \in \mathbf{cCF}^{*}_{\xi}\}$;

(ii) $\{\langle \xi, A, f, \boldsymbol{i}\rangle : \xi \in \boldsymbol{\Xi} \wedge f \in \mathbf{cCF}^{*}_{\xi} \wedge A \in \mathbf{cIPS}_{\xi} \wedge$
$\wedge\, f^{\#}$ is an $\boldsymbol{i}$-axis map on $A^{\#}\}$;

(iii) $\{\langle \xi, A, f, \mathcal{U}\rangle : \xi \in \boldsymbol{\Xi} \wedge f \in \mathbf{cCF}^{*}_{\xi} \wedge A \in \mathbf{cIPS}_{\xi} \wedge \mathcal{U} \in \mathbf{HC}$ consists of countable subsets of $\mathscr{D} \wedge f^{\#}$ avoids $E^{\#}$ on $A^{\#}$ for any $E \in \mathcal{U}\}$;

(iv) $\{\langle \xi, A\rangle : \xi \in \boldsymbol{\Xi} \wedge A \in \mathbf{cIPS}_{\xi} \wedge A^{\#}$ is uniform as in Section 17$\}$.

Proof. (i) Let $f : \mathbf{Rat}_{\xi} \to \omega^{\omega}$. Then $f \in \mathbf{cCF}_{\xi}$ iff for any $m, k < \omega$ there exists $b \in \mathbf{cCO}_{\xi}$ (a code of a clopen set in $\mathscr{D}^{\xi}$) such that for all $x \in \mathbf{Rat}_{\xi}$ the equivalence $x \in \mathbf{CO}_b^{\xi} \iff f(x)(m) = k$ holds. This yields a $\mathbf{Z}\aleph_0$-absolute definition, and hence the class $\varDelta_1^{\mathbf{HC}}$, for the first set.

(ii) Let $f \in \mathbf{cCF}^*_\xi$. Then $f^\#$ is an $\boldsymbol{i}$-axis map on $A^\#$ iff for all $b \in \mathbf{cCO}_\xi$, $k < \omega$, and $e = 0, 1$ the following holds:

$$\forall\, x \in \mathbf{CO}^\xi_b \cap \mathbf{Rat}_\xi \big(x(\boldsymbol{i})(k) = e \wedge f(x)(k) = 1 - e\big) \implies A \cap \mathbf{CO}^\xi_b = \varnothing,$$

and this is a $\mathbf{Z}\aleph_0$-absolute formula.

(iii) According to the compactness of the spaces considered, if a continuous map $f^\#$ avoids $E^\#$ on $A^\#$ then there exist clopen supersets $X \supseteq A^\#$ and $Y \supseteq E^\#$ such that $f^\#$ avoids Y on X. We conclude that the relation "$f^\#$ avoids $E^\#$ on $A^\#$" is equivalent to the following $\mathbf{Z}\aleph_0$-absolute formula:

$$\exists\, b, c \in \mathbf{cCO}_\xi \left(A \subseteq \mathbf{CO}^\xi_b \wedge E \subseteq \mathbf{CO}^\xi_c \wedge \forall\, x \in \mathbf{CO}^\xi_b \cap \mathbf{Rat}_\xi \left(f(x) \notin \mathbf{CO}^\xi_c\right)\right).$$

(iv) For $A^\#$ to be uniform it's necessary that A itself is uniform, *i.e.*, if $\boldsymbol{i} \subset \boldsymbol{j}$ belong to $\xi = \|A\|$ and $x, y \in A$ satisfy $x(\boldsymbol{j}) = y(\boldsymbol{j})$ then $x(\boldsymbol{i}) = y(\boldsymbol{i})$ as well. In other words, there is a map $h_{\boldsymbol{ij}} : A{\Downarrow}\boldsymbol{j} \to A{\Downarrow}\boldsymbol{i}$ satisfying $x(\boldsymbol{i}) = h_{\boldsymbol{ij}}(x(\boldsymbol{j}))$ for all $x \in A$. Thus the condition that (*) every closure $h^\#_{\boldsymbol{ij}}$ in the according space $\mathscr{D} \times \mathscr{D}$ remains a map, is necessary and sufficient for $A^\#$ to be uniform. On the other hand, (*) is $\mathbf{Z}\aleph_0$-absolute by an argument similar to used in the proof of (i). We leave the details to the reader. □

64 Definability of the forcing approximation

Still arguing in L, now we come back to the notion of forcing approximation `forc` introduced by Definition 36.1. The goal of the next theorem is to evaluate the complexity of the sets

$$\begin{aligned}
\mathbf{Forc}(\Sigma^1_n) &= \{\langle X, \varphi\rangle : X \in \mathbf{cIPS} \wedge \varphi \text{ a closed } \mathfrak{L}\Sigma^1_n \text{ formula} \wedge X^\# \,\texttt{forc}\, \varphi\};\\
\mathbf{Forc}(\Pi^1_n) &= \{\langle X, \varphi\rangle : X \in \mathbf{cIPS} \wedge \varphi \text{ a closed } \mathfrak{L}\Pi^1_n \text{ formula} \wedge X^\# \,\texttt{forc}\, \varphi\}.
\end{aligned}$$

Theorem 64.1. *The set* $\mathbf{Forc}(\Pi^1_1)$ *belongs to* $\varDelta^{\mathbf{HC}}_1$.
The set $\mathbf{Forc}(\Sigma^1_1)$ *belongs to* $\varPi^{\mathbf{HC}}_1$.
If $n \geq 1$ *then* $\mathbf{Forc}(\Sigma^1_{n+1})$ *belongs to* $\varSigma^{\mathbf{HC}}_n$, $\mathbf{Forc}(\Pi^1_{n+1})$ *belongs to* $\varPi^{\mathbf{HC}}_n$.

Proof. *Case* $\varPi^1_1$. Assume that $X \in \mathbf{cIPS}$, φ is a closed $\mathfrak{L}\Pi^1_1$ formula, $\xi = \|X\| \cup \|\varphi\| \in \mathbf{\Xi}$. Using the same trick as in the end of Section 60, note that ξ can be embedded in $\boldsymbol{I}[{<}\omega]$ via a map $\pi \in \Gamma_\xi$, where Γ_ξ consists of all $\subset$-preserving and length-preserving injections $\pi : \xi \to \boldsymbol{I}[{<}\omega]$. Then $X^\# \,\texttt{forc}\, \varphi$ is equivalent to each of the two formulas:

$$\exists\, \pi \in \Gamma_\xi\, \exists\, \xi'\, \exists\, A'\, \exists\, \varphi' \left(\xi' = \pi{\bullet}\xi \wedge X' = \pi{\bullet}X \wedge \varphi' = \pi{\bullet}\varphi \wedge X'^\# \,\texttt{forc}\, \varphi'\right),$$
$$\forall\, \pi \in \Gamma_\xi\, \forall\, \xi'\, \forall\, A'\, \forall\, \varphi' \left(\xi' = \pi{\bullet}\xi \wedge X' = \pi{\bullet}X \wedge \varphi' = \pi{\bullet}\varphi \implies X'^\# \,\texttt{forc}\, \varphi'\right).$$

On the other hand, if it is assumed that $\xi' \subseteq \boldsymbol{I}[<\omega]$ then "$X'^{\#}$ $\texttt{forc}\,\varphi'$" is essentially a $\mathit{\Pi}^1_1$ relation via a suitable coding of φ', X' by reals, by 1° of Definition 36.1, and hence we have a $\mathit{\Delta}^{\mathbf{HC}}_1$ relation in this particular domain by Proposition 7.1. It follows that the first line of the double equivalence above provides a $\mathit{\Sigma}^{\mathbf{HC}}_1$ definition of the relation "$X'^{\#}$ $\texttt{forc}\,\varphi'$", whereas the second line provides its $\mathit{\Pi}^{\mathbf{HC}}_1$ definition, as required.

Case $\mathit{\Sigma}^1_1$*.* Essentially the same argument, but if φ is a Σ^1_1 formula then 1° of Definition 36.1 yields a $\mathit{\Pi}^1_2$ relation, hence $\mathit{\Pi}^{\mathbf{HC}}_1$ relation.

Inductive step $\mathit{\Pi}^1_n \to \mathit{\Sigma}^1_{n+1}$*,* $n \ge 1$*.* By 2° of Definition 36.1, $\mathbf{Forc}(\Sigma^1_{n+1})$ consists of all pairs $\langle X, \exists\, x\, \varphi(x)\rangle$, where φ is a $\mathfrak{L}\Pi^1_n$ formula and there is $f \in \mathbf{cCF}$ satisfying $\langle X, \varphi(f)\rangle \in \mathbf{Forc}(\Pi^1_n)$. Thus if $\mathbf{Forc}(\Pi^1_n)$ belongs to $\mathit{\Pi}^{\mathbf{HC}}_{n-1}$ or at worst $\mathit{\Delta}^{\mathbf{HC}}_n$ then $\mathbf{Forc}(\Sigma^1_{n+1})$ belongs to $\mathit{\Sigma}^{\mathbf{HC}}_n$.

Inductive step $\mathit{\Sigma}^1_{n+1} \to \mathit{\Pi}^1_{n+1}$*,* $n \ge 1$*.* By 3° of Definition 36.1, $\mathbf{Forc}(\Pi^1_{n+1})$ consists of all pairs $\langle X, \varphi^-\rangle$, where $X \in \mathbf{cIPS}$, φ is a closed $\mathfrak{L}\Sigma^1_{n+1}$ formula, and there is no $Y \in \mathbf{cIPS}$ satisfying $Y^{\#} \downarrow\!\subseteq X^{\#}$ and $\langle Y, \varphi\rangle \in \mathbf{Forc}(\Sigma^1_{n+1})$. Thus if $\mathbf{Forc}(\Sigma^1_{n+1})$ belongs to $\mathit{\Pi}^{\mathbf{HC}}_n$ then $\mathbf{Forc}(\Pi^1_{n+1})$ belongs to $\mathit{\Pi}^{\mathbf{HC}}_n$. □

65 Definability of being an 1-5-$\mathbb{n}$ extension

Here we collect all the previous results of this chapter to prove the following main definability theorem. If $\mathbb{n} \ge 1$ then let $\mathrm{EXT}_{\mathbb{n}}$ be the set of all pairs $\langle \mathbb{\beta}, \mathcal{B}_\lambda\rangle$, where $\mathbb{\beta} = \langle \mathcal{B}_\alpha\rangle_{\alpha<\lambda}$ is a coded **Rud** sequence of length some $\lambda < \omega_1$, $\mathcal{B}_\lambda \in \mathbf{cRud}_\lambda$, and the set $\mathcal{Q}_\lambda = \mathcal{B}^\sharp_\lambda := \{A^{\#} : A \in \mathcal{B}_\lambda\}$ is an 1-5-$\mathbb{n}$ extension of the **Rud** sequence

Theorem 65.1 (in **L**)**.** *Let* $\mathbb{n} \ge 1$*. Let* $\mathrm{EXT}_{\mathbb{n}}$ *be the set of all pairs* $\langle \mathbb{\beta}, \mathcal{B}_\lambda\rangle$*, where* $\mathbb{\beta} = \langle \mathcal{B}_\alpha\rangle_{\alpha<\lambda}$ *is a coded* **Rud** *sequence of length some* $\lambda < \omega_1$*,* $\mathcal{B}_\lambda \in$ $\mathbf{cRud}_\lambda$*, and the set* $\mathcal{Q}_\lambda = \mathcal{B}^\sharp_\lambda := \{A^{\#} : A \in \mathcal{B}_\lambda\}$ *is an 1-5-*$\mathbb{n}$ *extension of the* **Rud** *sequence* $\boldsymbol{\varphi} = \mathbb{\beta}^\sharp := \langle \mathcal{Q}_\alpha\rangle_{\alpha<\lambda}$*, where* $\mathcal{Q}_\alpha = \mathcal{B}^\sharp_\alpha := \{A^{\#} : A \in \mathcal{B}_\alpha\}$*,* $\forall\, \alpha$*.*

Then $\mathrm{EXT}_{\mathbb{n}}$ *belongs to* $\mathit{\Delta}^{\mathbf{HC}}_{\mathbb{n}}$*.*

Proof. We have to evaluate coded forms of conditions (A), (B), (C) (including (C2)–(C5) in the last one) as in Definition 52.1.

(A) *The extended sequence* $\mathbb{\beta}^\frown\mathcal{B}_\lambda$ *is a* **cRud** *sequence* (of length $\lambda + 1$).

This condition is $\mathit{\Delta}^{\mathbf{HC}}_1$ by Theorem 62.1.

(B) *If* $\mathbf{M}^{\#}_\lambda \in \mathbf{NH}(\mathbb{\beta}^\sharp)$, $\|\mathbf{M}^{\#}_\lambda\| \subseteq \boldsymbol{I}[<\lambda]$ *then* $\exists\, A \in \mathcal{B}_\lambda\, (A^{\#} \downarrow\!\subseteq \mathbf{M}^{\#}_\lambda)$.

This needs some bit of work. Recall that the map $\alpha \mapsto \mathbf{M}_\alpha$ is $\Delta_1^{\mathbf{HC}}$ by Lemma 45.1. The relation $A^\# \subseteq B^\#$ is $\Delta_1^{\mathbf{HC}}$ by $3^\blacktriangle$ in Section 59. Thus the 2nd and 3rd subformulas in (B) define $\Delta_1^{\mathbf{HC}}$ relations. Let's focus on the 1st subformula $\mathbf{M}_\lambda^\# \in \mathbf{NH}(\beta^\sharp)$. Here $\mathbf{NH}(\beta^\sharp) = \mathbf{NH}(\varphi) := \mathbf{NH}(\mathcal{Q}_{<\lambda})$, where

$$\mathcal{Q}_{<\lambda} = \bigcup_{\alpha<\lambda} \mathcal{Q}_\alpha = \mathcal{B}_{<\lambda}{}^\sharp \quad \text{and} \quad \mathcal{B}_{<\lambda} = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha\,,$$

and $\mathbf{NH}(\cdot)$ is the normal hull, Definition 21.2.

To eliminate the operation $\mathbf{NH}(\cdot)$ of indefinite complexity, we define $\mathcal{U} = \mathbf{RH}(\mathcal{Q}_{<\lambda}{\upharpoonright}\boldsymbol{I}[<\lambda])$ (the rudimentary hull, Section 40), so that $\mathcal{U} \in \mathbf{Rud}_\lambda$ is countable. At the level of codes, we put $\mathcal{A} = \mathcal{B}_{<\lambda}{\upharpoonright}^*\boldsymbol{I}[<\lambda]$ (see Section 59 on ${\upharpoonright}^*$), so that $\mathcal{A} \subseteq \mathbf{cIPS}_\lambda$ is countable and $\mathcal{A}^\sharp = \mathcal{Q}{\upharpoonright}\boldsymbol{I}[<\lambda]$.

We further define $\mathscr{C} = \mathbf{cRH}(\mathcal{A})$ (the coded rudimentary hull, Section 61), hence $\mathscr{C} \in \mathbf{cRud}_\lambda$ and $\mathcal{U} = \mathscr{C}^\sharp := \{C^\# : C \in \mathscr{C}\}$.

Now suppose that $\mathbf{M}_\lambda \in \mathbf{cIPS}$ and $\xi = \|\mathbf{M}_\lambda\| \subseteq \boldsymbol{I}[<\lambda]$. We are going to define the relation $\mathbf{M}_\lambda^\# \in \mathscr{X}$, where $\mathscr{X} = \mathbf{NH}(\beta^\sharp)$, in terms of the above notation, so that it becomes $\Delta_1^{\mathbf{HC}}$. First of all, $\mathbf{M}_\lambda^\# \in \mathscr{X}$ iff $\mathbf{M}_\lambda^\# {\downarrow}_{\subseteq \boldsymbol{i}} \in \mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \xi$, by 6° of Section 21. On the other hand, $\mathscr{X}{\downarrow}_{\subseteq \boldsymbol{i}} = \mathcal{U}{\downarrow}_{\subseteq \boldsymbol{i}}$ by Lemma 40.2. Thus (*) $\mathbf{M}_\lambda^\# \in \mathscr{X}$ iff $\mathbf{M}_\lambda^\# {\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{U}{\downarrow}_{\subseteq \boldsymbol{i}}$ for all $\boldsymbol{i} \in \xi = \|\mathbf{M}_\lambda\|$.

On the other hand, the relation $\mathbf{M}_\lambda^\# {\downarrow}_{\subseteq \boldsymbol{i}} \in \mathcal{U}{\downarrow}_{\subseteq \boldsymbol{i}}$ is equivalent to

$$\exists\, C \in \mathscr{C}\, \big((\mathbf{M}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}})^\# = (C{\downarrow}_{\subseteq \boldsymbol{i}})^\#\big).$$

This allows to rewrite (*) as follows:

$$\mathbf{M}_\lambda^\# \in \mathbf{NH}(\beta^\sharp) \iff \forall\, \boldsymbol{i} \in \|\mathbf{M}_\lambda\|\, \exists\, C \in \mathscr{C}\, \big((\mathbf{M}_\lambda{\downarrow}_{\subseteq \boldsymbol{i}})^\# = (C{\downarrow}_{\subseteq \boldsymbol{i}})^\#\big), \quad (\dagger)$$

where $\mathscr{C} = \mathbf{cRH}(\mathcal{A}) = \mathbf{cRH}((\mathcal{B}_{<\lambda}{\upharpoonright}^*\boldsymbol{I}[<\lambda]))$. Finally note that the right-hand side of (†) contains only $\Delta_1^{\mathbf{HC}}$ relations and operations by $3^\blacktriangle$ and $5^\blacktriangle$ in Section 59 and Theorem 61.1. We conclude that "$\mathbf{M}_\lambda^\# \in \mathbf{NH}(\beta^\sharp)$" is a $\Delta_1^{\mathbf{HC}}$ relation, and hence so is (B) as a whole (with λ, $\mathcal{B}_\lambda$, β as arguments).

(C) $\mathbf{M}_\lambda^\# \in \mathcal{Q}_{<\lambda} \Longrightarrow \exists\, Y \in \mathcal{Q}_\lambda\, (Y \downarrow\!\subseteq \mathbf{M}_\lambda^\# \wedge \text{(C2)–(C5)})$, or equivalently,

$\exists\, B \in \mathcal{B}_{<\lambda}\, (\mathbf{M}_\lambda^\# = B^\#) \Longrightarrow \exists\, A \in \mathcal{B}_\lambda\, (A^\# \downarrow\!\subseteq \mathbf{M}_\lambda^\# \wedge \text{(C2)–(C5)})$.

Temporarily leaving (C2)–(C5) aside in the 2nd line of (B) here, note that the subrelations $\exists\, B \in \mathcal{B}_{<\lambda}\, (\mathbf{M}_\lambda^\# = B^\#)$ and $\exists\, A \in \mathcal{B}_\lambda\, (A^\# \downarrow\!\subseteq \mathbf{M}_\lambda^\#)$ are $\Delta_1^{\mathbf{HC}}$ by $3^\blacktriangle$ in Section 59. Now consider (C2)–(C5) one by one, assuming that $\mathbf{M}_\lambda^\# \in \mathcal{Q}_{<\lambda}$, or equivalently, that some $B \in \mathcal{B}_{<\lambda}$ satisfies $\mathbf{M}_\lambda^\# = B^\#$.

(C2) If λ is limit, $k < \omega$, $\mathbf{B}_{\lambda k}{}^\sharp \subseteq \mathcal{Q}_{<\lambda}$, and $\mathbf{B}_{\lambda k}{}^\sharp$ is dense in $\mathcal{Q}_{<\lambda}$, then $A^\# \subseteq^{\mathtt{fd}} \bigcup \mathbf{B}_{\lambda k}{}^\sharp$.

Here we recall that $\lambda, k \mapsto \mathbf{B}_{\lambda k}{}^{\sharp}$ is a $\Delta_1^{\mathbf{HC}}$ map by Lemma 45.1. Then replace the subformula $\mathbf{B}_{\lambda k}{}^{\sharp} \subseteq \mathcal{Q}_{<\lambda}$ by $\forall\, A \in \mathbf{B}_{\lambda k}{}^{\sharp}\, \exists\, B \in \mathcal{B}_{<\lambda}\, (A \in \mathbf{cIPS} \wedge A^{\#} = B^{\#})$ — which defines a $\Delta_1^{\mathbf{HC}}$ relation by 3$^\blacktriangle$ in Section 59. Similar routine $\Delta_1^{\mathbf{HC}}$ replacements apply also for the subformulas "$\mathbf{B}_{\lambda k}{}^{\sharp}$ is dense in $\mathcal{Q}_{<\lambda}$" and $A^{\#} \subseteq^{\mathtt{fd}} \bigcup \mathbf{B}_{\lambda k}{}^{\sharp}$, with an extra reference to 5$^\blacktriangle$ in Section 59. After that, we conclude that (C2) is a $\Delta_1^{\mathbf{HC}}$ relation.

(C3) If $\mathtt{n} \geq 2$ and $\mathbf{M}'_\lambda$ is a closed formula φ in $\bigcup_{k \leq \mathtt{n}} \mathfrak{L}\Sigma^1_k$ then $A^{\#}\, \mathtt{forc}\, \varphi$ or $A^{\#}\, \mathtt{forc}\, \varphi^-$ — *this is void in case* $\mathtt{n} = 1$.

Use Theorem 64.1 to see that (C3) is a $\Delta_{\mathtt{n}}^{\mathbf{HC}}$ condition.

(C4) $A^{\#}$ is a uniform set. — *Still a* $\Delta_1^{\mathbf{HC}}$ *condition by Theorem 63.1*(iv).

(C5) Either (a) $\mathbb{F}_\lambda := \mathbb{f}^{\#}_\lambda$ avoids $E^{\#}$ on $A^{\#}$ for all $\boldsymbol{i} \in \boldsymbol{I}[<\lambda]$ and $E \in \mathcal{B}_\lambda \Downarrow \boldsymbol{i}$, or (b) there is $\boldsymbol{j} \in \boldsymbol{I}[<\lambda]$ such that $\mathbb{f}^{\#}_\lambda$ is an $\boldsymbol{j}$-axis map on $A^{\#}$ but $\mathbb{f}^{\#}_\lambda$ avoids $E^{\#}$ on $A^{\#}$ for all $E \in \mathcal{B}_\lambda \Downarrow \boldsymbol{i}$ and $\boldsymbol{i} \in \boldsymbol{I}[<\lambda]$, $\boldsymbol{i} \not\approx_{\mathtt{par}} \boldsymbol{j}$.

Theorem 63.1 (different items) implies that (C5) is $\Delta_1^{\mathbf{HC}}$, too.

This completes the proof of Theorem 65.1: all components of the definition of $\mathrm{EXT}_{\mathtt{n}}$ are $\Delta_1^{\mathbf{HC}}$ except for (C3) which is $\Delta_{\mathtt{n}}^{\mathbf{HC}}$. □

66 The final forcing construction

Proof (Theorem 52.2, finalization, in $\mathbf{L}$)**.** Let $\mathtt{n} \geq 1$. Theorem 53.1 implies that for any coded **Rud** sequence $\mathbb{\beta}'$ of length $\lambda = \mathtt{dom}\,\mathbb{\beta}' < \omega_1$ there exists a coded rudiment $\mathcal{B}_\lambda \in \mathbf{cRud}_\lambda$ satisfying $\langle \mathbb{\beta}', \mathcal{B}_\lambda \rangle \in \mathrm{EXT}_{\mathtt{n}}$. Let $\mathcal{B}_\lambda(\mathbb{\beta}')$ be the $\leqslant_{\mathbf{L}}$-*minimal* of such coded rudiments $\mathcal{B}_\lambda \in \mathbf{cRud}_\lambda$.

Define a coded **Rud** sequence $\mathbb{\beta} = \langle \mathcal{B}_\lambda \rangle_{\lambda<\omega_1}$ so that $\mathcal{B}_\lambda = \mathcal{B}_\lambda(\mathbb{\beta} \restriction \lambda)$ for all $\lambda < \omega_1$. Then, by Theorem 65.1, $\mathbb{\beta}$ belongs to $\Delta_{\mathtt{n}}^{\mathbf{HC}}$, because it is known that iterated constructions, by taking the $\leqslant_{\mathbf{L}}$-minimal choice in the domain bounded by a $\Delta_{\mathtt{n}}^{\mathbf{HC}}$ relation, lead to $\Delta_{\mathtt{n}}^{\mathbf{HC}}$ final results (say by Proposition 7.2(iii)). It follows that the according **Rud** sequence $\boldsymbol{\varphi} = \langle \mathcal{Q}_\lambda \rangle_{\lambda<\omega_1}$, where $\mathcal{Q}_\lambda = (\mathcal{B}_\lambda)^{\sharp}$, $\forall\, \lambda$, satisfies the global definability condition $\mathfrak{P}_6^{\mathtt{n}}$ via $\mathbb{\beta}$.

On the other hand, each $\mathcal{Q}_\lambda$ is a a 1-5-$\mathtt{n}$ extension of $\boldsymbol{\varphi} \restriction \lambda$, because $\langle \mathbb{\beta} \restriction \lambda, \mathcal{B}_\lambda \rangle \in \mathrm{EXT}_{\mathtt{n}}$ by construction.

Thus the sequence $\boldsymbol{\varphi}$ witnesses Theorem 52.2. □

Proof (Theorem 1.1, finalization)**.** It remains to recall that Theorem 52.2 implies Theorem 1.1, see Section 52. □ (Theorem 1.1)

XI Proof of the second main theorem

Here we prove Theorem 1.2. The model $\mathfrak{M}[\boldsymbol{v}] \cap 2^\omega$ defined in Section 67 will be a set in an $\mathscr{X}$-generic extension $\mathbf{L}[\boldsymbol{v}]$, where $\mathscr{X}$ is given by Theorem 34.1.

67 The model

If $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ is an $\boldsymbol{I}$-array of reals then let $\boldsymbol{I}[\boldsymbol{v}]$ consist of all tuples $\boldsymbol{i} \in \boldsymbol{I}$ such that the ordinal $\alpha = \boldsymbol{i}(0)$ is odd (hence $\alpha{-}1$ is well-defined), and

($*$) for any $1 \le k < \mathtt{lh}(\boldsymbol{i})$, if $\boldsymbol{i}(k)$ is even then $\boldsymbol{v}(\langle \alpha{-}1 \rangle)(k) = 0$.

We put $\boldsymbol{\Omega}_5[\boldsymbol{v}] = \{\xi \in \boldsymbol{\Xi} : \xi \subseteq \boldsymbol{I}[\boldsymbol{v}]\}$ and $\mathfrak{M}[\boldsymbol{v}] = \bigcup_{\xi \in \boldsymbol{\Omega}_5[\boldsymbol{v}]} \mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$.

Lemma 67.1. (i) *If $\eta \subseteq_{\mathsf{odd}} \xi$ belong to $\boldsymbol{\Xi}$ then $\eta \in \boldsymbol{\Omega}_5[\boldsymbol{v}] \Longrightarrow \xi \in \boldsymbol{\Omega}_5[\boldsymbol{v}]$.*

(ii) *If $\alpha < \omega_1^{\mathbf{L}}$ is odd and $k \ge 1$ then TFAE: 1) there is an even tuple $\boldsymbol{i} \in \boldsymbol{I}[\boldsymbol{v}]$ with $\boldsymbol{i}(0) = \alpha$ and $\mathtt{lh}(\boldsymbol{i}) = k+1$, and 2) $\boldsymbol{v}(\langle \alpha{-}1 \rangle)(k) = 0$.*

(iii) *If $\boldsymbol{i} = \langle \alpha \rangle \in \boldsymbol{I}$ then $\boldsymbol{i} \in \boldsymbol{I}[\boldsymbol{v}]$ iff α is odd.* □

Lemma 67.2. *Let $\mathscr{X} \in \mathbf{NF}$ be a normal forcing in $\mathbf{L}$, and $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ be $\mathscr{X}$-generic over $\mathbf{L}$. Then $\boldsymbol{I}[\boldsymbol{v}], \boldsymbol{\Omega}_5[\boldsymbol{v}] \in \mathbf{L}[\boldsymbol{v}]$* (not necessarily $\in \mathbf{L}$) *and:*

(i) *if $\boldsymbol{i} \in \boldsymbol{I}$ then $\boldsymbol{v}(\boldsymbol{i}) \in \mathfrak{M}[\boldsymbol{v}]$ iff $\boldsymbol{i} \in \boldsymbol{I}[\boldsymbol{v}]$;*

(ii) *if $\boldsymbol{i} = \langle \alpha \rangle \in \boldsymbol{I}$ then $\boldsymbol{v}(\boldsymbol{i}) \in \mathfrak{M}[\boldsymbol{v}]$ iff α is odd.*

Proof. (i) If $\boldsymbol{i} \in \boldsymbol{I}[\boldsymbol{v}]$ then obviously $[\subseteq \boldsymbol{i}] \in \boldsymbol{\Omega}_5[\boldsymbol{v}]$ and we are done. To prove the converse suppose that $\boldsymbol{v}(\boldsymbol{i}) \in \mathfrak{M}[\boldsymbol{v}]$, hence $\boldsymbol{v}(\boldsymbol{i}) \in \mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$ for some $\xi \in \boldsymbol{\Omega}_5[\boldsymbol{v}]$. Then $\boldsymbol{i} \in \xi$ by Corollary 26.4, hence $\boldsymbol{i} \in \boldsymbol{I}[\boldsymbol{v}]$.

To prove (ii) use (i) and Lemma 67.1(iii). □

Theorem 67.3. *Assume that $\mathtt{n} \ge 1$ and $\mathscr{X} \in \mathbf{NF}$ is a normal forcing as in Theorem 34.1, i.e., $\mathscr{X}$ has the fusion, the $(\mathtt{n})$-odd expansion, and the $(\mathtt{n})$-definability properties in $\mathbf{L}$. Let $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ be $\mathscr{X}$-generic over $\mathbf{L}$. Then:*

(i) *$\mathbf{CA}(\boldsymbol{\Sigma}^1_{\mathtt{n}+2})$ (with parameters) fails in $\langle \omega ; \mathfrak{M}[\boldsymbol{v}] \cap 2^\omega \rangle$.*

(ii) *$\mathbf{CA}(\boldsymbol{\Sigma}^1_{\mathtt{n}+1})$ (with parameters) holds in $\langle \omega ; \mathfrak{M}[\boldsymbol{v}] \cap 2^\omega \rangle$.*

(iii) *$\mathbf{AC}_\omega(\Sigma^1_\infty)$ and $\mathbf{CA}(\Sigma^1_\infty)$ (parameter-free) hold in $\langle \omega ; \mathfrak{M}[\boldsymbol{v}] \cap 2^\omega \rangle$.*

Reals $x \in \mathfrak{M}[\boldsymbol{v}] \cap 2^\omega$ are identified with sets $\{k : x(k) = 0\}$, so that we view $\mathfrak{M}[\boldsymbol{v}] \cap 2^\omega$ as a subset of $\mathscr{P}(\omega)$ in the context of this theorem.

Quite obviously Theorem 67.3 implies Theorem 1.2.

The proof of Theorem 67.3 goes on below in this Chapter, each of the items taking a separate Section because of pretty different methods involved.

68 Item 1: violation of Comprehension at the level $\mathbb{n}+2$

Proof (item (i) of Thm 67.3)**.** By the ($\mathbb{n}$)-definability property of $\mathscr{X}$ as in Definition 32.1(II), the set $E = \mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) \cap \mathfrak{M}[\boldsymbol{v}]$ is $\Pi^1_{\mathbb{n}+1}$ over $\mathfrak{M}[\boldsymbol{v}]$, where

$$\mathbf{E}^{\mathtt{evn}}(\boldsymbol{v}) = \{\langle k, \boldsymbol{v}(\boldsymbol{i})\rangle : k \geq 1 \wedge \boldsymbol{i} \in \boldsymbol{I} \text{ is even } \wedge \mathtt{lh}(\boldsymbol{i}) = k\}.$$

Here it is not claimed that $E \in \mathfrak{M}[\boldsymbol{v}]$. What **is** asserted is that there is a parameter-free $\Pi^1_{\mathbb{n}+1}$ formula $\varphi(k,x)$ such that

$$E = \{\langle k, x\rangle : x \in \mathfrak{M}[\boldsymbol{v}] \wedge \mathfrak{M}[\boldsymbol{v}] \models \varphi(k,x)\}. \tag{1}$$

Now we claim that, for any $k \geq 1$,

$$\boldsymbol{v}(\langle 0\rangle)(k) = 0 \iff \exists\, x(\langle k{+}1, x\rangle \in E \wedge \boldsymbol{v}(\langle 1\rangle) \in \mathbf{L}[x]). \tag{2}$$

From left to right, let $\boldsymbol{v}(\langle 0\rangle)(k) = 0$. By Lemma 67.1(ii) ($\alpha = 1$), there is an even tuple $\boldsymbol{i} \in \boldsymbol{I}[\boldsymbol{v}]$ with $\boldsymbol{i}(0) = 1$ and $\mathtt{lh}(\boldsymbol{i}) = k+1$. Let $x = \boldsymbol{v}(\boldsymbol{i})$. By definition, $\langle k{+}1, x\rangle \in E$. Moreover $\boldsymbol{v}(\langle 1\rangle) \in \mathbf{L}[x]$ by Definition 32.1(I), since $\langle 1\rangle \subseteq \boldsymbol{i}$ by construction. Thus the right-hand side of (2) holds.

From left to right, suppose that the right-hand side of (2) holds, and this is witnessed by some x. Then $x = \boldsymbol{v}(\boldsymbol{i})$, where $\boldsymbol{i} \in \boldsymbol{I}$ is even and $\mathtt{lh}(\boldsymbol{i}) = k{+}1$, and, as $\langle k{+}1, x\rangle \in E \subseteq \mathfrak{M}[\boldsymbol{v}]$, we have $x \in \mathfrak{M}[\boldsymbol{v}]$, and hence $\boldsymbol{i} \in \boldsymbol{I}[\boldsymbol{v}]$ by (i). Moreover, as $\boldsymbol{v}(\langle 1\rangle) \in \mathbf{L}[x]$, we have $\langle 1\rangle \subseteq \boldsymbol{i}$ by Definition 32.1(I), therefore $\boldsymbol{i}(0) = 1$. To conclude, $\boldsymbol{i} \in \boldsymbol{I}[\boldsymbol{v}]$ is even, $\mathtt{lh}(\boldsymbol{i}) = k{+}1$, $\boldsymbol{i}(0) = 1$. This implies $\boldsymbol{v}(\langle 0\rangle)(k) = 0$ by Lemma 67.1(ii) ($\alpha = 1$), as required.

Combining (1) and (2), it is clear now that $\boldsymbol{v}(\langle 0\rangle)$ is definable over $\mathfrak{M}[\boldsymbol{v}]$ by a $\Sigma^1_{\mathbb{n}+2}$ formula (note the quantifier $\exists\, x$ in (2)!), with $\boldsymbol{v}(\langle 1\rangle) \in \mathfrak{M}[\boldsymbol{v}]$ as the only parameter. However $\boldsymbol{v}(\langle 0\rangle) \notin \mathfrak{M}[\boldsymbol{v}]$ by Lemma 67.2(ii). □

69 Item 2: verification of Comprehension at the level $\mathbb{n}+1$

Proof (item (ii) of Thm 67.3)**.** The first step is the following claim, motivated by the the ($\mathbb{n}$)-odd expansion property of $\mathscr{X}$ and Lemma 67.1(i):

(1) $\mathfrak{M}[\boldsymbol{v}]$ is an elementary submodel of $\mathbf{L}[\boldsymbol{v}]$ w.r.t. all $\Sigma^1_{\mathbb{n}+1}$ formulas with reals in $\mathfrak{M}[\boldsymbol{v}]$ as parameters.

Now let $\varphi(p,k)$ be a $\Sigma^1_{\mathbb{n}+1}$ formula with some $p \in \mathfrak{M}[\boldsymbol{v}] \cap 2^\omega$ as the only parameter. We are going to prove that the set $X = \{k : \mathfrak{M}[\boldsymbol{v}] \models \varphi(p,k)\}$ belongs to $\mathfrak{M}[\boldsymbol{v}]$. By definition, $p \in \mathbf{L}[v{\downarrow}\eta]$ for some $\eta \in \boldsymbol{\Omega}_5[\boldsymbol{v}]$. Let

$$\Omega_\eta = \{\xi \in \boldsymbol{\Xi} : \eta \subseteq_{\mathsf{odd}} \xi\}, \quad \text{all odd expansions of } \eta \text{ in } \boldsymbol{\Xi},$$

and $\mathfrak{M}_\eta[\boldsymbol{v}] = \bigcup_{\xi\in\Omega_\eta} \mathbf{L}[\boldsymbol{v}{\downarrow}\xi]$. Note that $\Omega_\eta \subseteq \boldsymbol{\Omega}_5[\boldsymbol{v}]$ by Lemma 67.1(i), and Ω_η obviously satisfies the same property, that is, if $\eta \subseteq_{\mathsf{odd}} \xi$ belong to $\boldsymbol{\Xi}$ then $\eta \in \Omega_\eta \Longrightarrow \xi \in \Omega_\eta$. Therefore, similarly to (1), we obtain:

(2) $\mathfrak{M}_\eta[\boldsymbol{v}]$ is an elementary submodel of $\mathbf{L}[\boldsymbol{v}]$ — and hence of $\mathfrak{M}[\boldsymbol{v}]$ as well by (1) — w.r.t. all $\Sigma^1_{\mathfrak{n}+1}$ formulas with reals in $\mathfrak{M}_\eta[\boldsymbol{v}]$ as parameters.

(3) Hence in particular $X = \{k : \mathfrak{M}_\eta[\boldsymbol{v}] \models \varphi(p,k)\}$.

Note finally that unlike $\boldsymbol{\Omega}_5[\boldsymbol{v}]$ the set Ω_η belongs to $\mathbf{L}$, and is closed under countable unions. It follows that $\mathfrak{M}_\eta[\boldsymbol{v}] \cap 2^\omega = \mathbf{L}[\boldsymbol{v}{\downarrow}\boldsymbol{I}[\boldsymbol{v}]] \cap 2^\omega$, hence the set $\mathfrak{M}_\eta[\boldsymbol{v}] \cap 2^\omega$ satisfies the full schema of **CA**. It follows that $X \in \mathfrak{M}_\eta[\boldsymbol{v}] \subseteq \mathfrak{M}[\boldsymbol{v}]$ by (3), as required. □

70 Item 3: verification of the parameter-free Choice

Proof (item (iii) of Thm 67.3)**.** This will be rather similar to the proof of Theorem 29.1 in the version of its last claim.

To begin with, consider the subgroup $\boldsymbol{\Gamma}_5 \in \mathbf{L}$ of the group $\boldsymbol{\Pi}$ of parity-preserving permutations π of $\boldsymbol{I}$ (Section 13) which consists of all $\pi \in \boldsymbol{\Pi}$ such that, for each odd α, if $\pi(\langle\alpha\rangle) = \langle\gamma\rangle$ (also odd!) then $\pi(\langle\alpha{-}1\rangle) = \langle\gamma{-}1\rangle$.

Lemma 70.1. *Let* $\boldsymbol{v} \in \mathscr{D}^{\boldsymbol{I}}$ *be* $\mathscr{X}$*-generic over* $\mathbf{L}$*, and* $\pi \in \boldsymbol{\Gamma}_5$*. Then*

(i) $\pi{\bullet}\boldsymbol{v}$ *is* $\mathscr{X}$*-generic over* $\mathbf{L}$*,*

(ii) $\boldsymbol{I}[\pi{\bullet}\boldsymbol{v}] = \pi{\bullet}\boldsymbol{I}[\boldsymbol{v}]$*,*

(iii) $\boldsymbol{\Omega}_5[\pi{\bullet}\boldsymbol{v}] = \pi{\bullet}\boldsymbol{\Omega}_5[\boldsymbol{v}]$*,*

(iv) $\mathfrak{M}[\boldsymbol{v}] = \mathfrak{M}[\pi{\bullet}\boldsymbol{v}]$*.*

Proof (lemma)**.** (ii) Let $\boldsymbol{v}' = \pi{\bullet}\boldsymbol{v}$, $\boldsymbol{i} \in \boldsymbol{I}$, $\alpha = \boldsymbol{i}(0)$, $\boldsymbol{j} = \pi{\bullet}\boldsymbol{i}$, $\alpha' = \boldsymbol{j}(0)$, so that $\langle\alpha'\rangle = \pi{\bullet}\langle\alpha\rangle$. If α is even then so is α' (as π is parity-preserving), and we have $\boldsymbol{i} \notin \boldsymbol{I}[\boldsymbol{v}]$, $\boldsymbol{j} \notin \boldsymbol{I}[\boldsymbol{v}']$. Thus suppose that α is odd.

Then α' is odd too, and the even ordinals $\gamma = \alpha{-}1$, $\gamma' = \alpha'{-}1$ are defined and satisfy $\gamma' = \pi{\bullet}\gamma$ since $\pi \in \boldsymbol{\Gamma}_5$, and moreover (I) $\boldsymbol{v}'(\gamma') = \boldsymbol{v}(\gamma)$. It remains to note that (II) if $1 \le k < \mathtt{lh}(\boldsymbol{i}) = \lg \boldsymbol{j}$ then the ordinals $\boldsymbol{i}(k)$ and $\boldsymbol{j}(k)$ are both even or both odd. We conclude from (I),(II) that condition $(*)$ of Section 67 holds for $\boldsymbol{i}, \boldsymbol{v}$ and $\boldsymbol{j}, \boldsymbol{v}'$ simultaneously, as required.

This completes the proof of (ii). The other two equalities (iii), (iv) are easy corollaries. □ (lemma)

To begin the proof of the theorem, fix a parameter-free Σ^1_∞ formula $\varphi(k,x)$, and assume that (*) $\mathfrak{M}[\boldsymbol{v}] \models \forall\, k\, \exists\, x\, \varphi(k,x)$. By necessity, the arguments somewhat change w.r.t. the proof of Theorem 29.1. First of all, for any $\alpha \in \mathbf{Ord}$ and suitable set z, $\mathfrak{F}_\alpha(z)$ will denote the αth element of

$\mathbf{L}[z]$ in the sense of the Gödel well-ordering of $\mathbf{L}[z]$. Then it follows from (*) that, in $\mathbf{L}$, there exist sequences of conditions $X_k \in \mathscr{X}$, ordinals α_k, and sets $\xi_k \in \boldsymbol{\Omega}_5[\boldsymbol{v}]$, satisfying

(1) $X_k \Vdash_{\mathscr{X}} \big(\mathfrak{M}[\underline{\boldsymbol{v}}] \models \varphi(k, \mathfrak{F}_{\alpha_k}(\underline{\boldsymbol{v}}{\downarrow}\xi_k))\big)$ — for all $k < \omega$.

Now assume to the contrary that $\mathfrak{M}[\boldsymbol{v}] \models \neg\,\exists\, f\, \forall\, k\, \varphi(k, f(k))$, and hence there exists a condition $X \in \mathscr{X}$, satisfying

(2) $X \Vdash_{\mathscr{X}} \big(\mathfrak{M}[\underline{\boldsymbol{v}}] \models \neg\,\exists\, f\, \forall\, k\, \varphi(k, f(k))\big)$.

Let $\tau = \|X\|$, $\tau_k = \|X_k\|$. Arguing in $\mathbf{L}$, we get a sequence of permutations $\pi_k \in \boldsymbol{\Gamma}_5$ by induction, satisfying $\vartheta_k \cap \vartheta_j = \vartheta_k \cap \tau = \varnothing$ whenever $k \neq j$, where $\vartheta_k = \pi_k{\bullet}\tau_k \in \boldsymbol{\Xi}$. Let $Y_k = \pi_k{\bullet}X_k$, thus $Y_k \in \mathscr{X}_{\vartheta_k}$. Let $\sigma_k = \pi_k{\bullet}\xi_k$; $\sigma_k \in \boldsymbol{\Omega}_5[\boldsymbol{v}]$ by Lemma 70.1. Then (1) implies by Theorem 25.2:

(3) $Y_k \Vdash_{\mathscr{X}} \big(\mathfrak{M}[\pi_k{\bullet}\underline{\boldsymbol{v}}] \models \varphi(k, \mathfrak{F}_{\alpha_k}((\pi_k{\bullet}\underline{\boldsymbol{v}}){\downarrow}\xi_k))\big)$, $\quad \forall\, k < \omega$,

Here $\mathfrak{M}[\pi_k{\bullet}\underline{\boldsymbol{v}}]$ can be replaced by just $\mathfrak{M}[\underline{\boldsymbol{v}}]$ by Lemma 70.1(iv), whereas $(\pi_k{\bullet}\underline{\boldsymbol{v}}){\downarrow}\xi_k$ can be replaced by $\pi_k{\bullet}(\underline{\boldsymbol{v}}{\downarrow}\sigma_k)$. This implies

(4) $Y_k \Vdash_{\mathscr{X}} \big(\mathfrak{M}[\underline{\boldsymbol{v}}] \models \varphi(k, \mathfrak{F}_{\alpha_k}(\pi_k{\bullet}(\underline{\boldsymbol{v}}{\downarrow}\sigma_k)))\big)$, $\quad \forall\, k$.

Now let $\vartheta = \bigcup_k \vartheta_k$. Then the set $Y = \bigcap_k (Y_k{\uparrow}\vartheta)$ belongs to $\mathscr{X}{\downarrow}\vartheta$ by Lemma 21.1 (w.r.t. Lemma 12.2). As obviously $Y \downarrow\subseteq Y_k$, (4) implies:

(5) $Y \Vdash_{\mathscr{X}} \big(\mathfrak{M}[\underline{\boldsymbol{v}}] \models \varphi(k, \mathfrak{F}_{\alpha_k}(\pi_k{\bullet}(\underline{\boldsymbol{v}}{\downarrow}\sigma_k)))\big)$.

Now follows the key step. The set $\sigma = \bigcup_k \sigma_k$ belongs to $\boldsymbol{\Omega}_5[\boldsymbol{v}]$ because so does each $\sigma_k = \pi_k{\bullet}\xi_k$. The term $\mathfrak{F}_{\alpha_k}(\pi_k{\bullet}(\underline{\boldsymbol{v}}{\downarrow}\sigma_k))$ in (5), as a function of k and $\underline{\boldsymbol{v}}{\downarrow}\sigma$, is defined in $\mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\sigma]$ by an absolute formula with parameters $k \mapsto \alpha_k$, $k \mapsto \pi_k$, $k \mapsto \sigma_k$ (all three maps belong to $\mathbf{L}$ by construction). Therfore the map $f(k) = \mathfrak{F}_{\alpha_k}(\pi_k{\bullet}(\underline{\boldsymbol{v}}{\downarrow}\sigma_k))$ is forced by Y to belong to $\mathbf{L}[\underline{\boldsymbol{v}}{\downarrow}\sigma]$. We conclude that

(6) $Y \Vdash_{\mathscr{X}} \exists\, f\, \forall\, k < \omega\, \big(\mathfrak{M}[\underline{\boldsymbol{v}}] \models \varphi(k, f(k))\big)$.

Thus conditions Y and X are incompatible in $\mathscr{X}$ by (2). However $\|Y\| \cap \|X\| = \sigma \cap \tau = \varnothing$ by construction, which implies that Y and X are indeed compatible. This is a contradiction. □ (item (iii) of Thm 67.3)

□ (Theorem 67.3 and Theorem 1.2)

XII Final remarks and questions

71 Working on the basis of the consistency of $\mathbf{PA}_2$

The main results of this paper, Theorems 1.1 and 1.2, can be naturally viewed as formal consistency results related to certain subsystems of 2nd order Peano arithmetic $\mathbf{PA}_2$ and obtained by means of forcing technique and other tools of **ZFC** which go way beyond $\mathbf{PA}_2$ itself. Therefore it is usually a tempting problem in such cases to reproduce the consistency results obtained on the basis of Consis $\mathbf{PA}_2$, the formal consistency of $\mathbf{PA}_2$.

Such a reproduction of another result, the consistency of the assertion $\mathbf{WO}_n \wedge \neg \mathbf{WO}_{n-1}$, based of the consistency of $\mathbf{PA}_2$, where

$\mathbf{WO}_n$**:** *there is a wellordering of the reals of class* Δ^1_n*,*

has been recently achieved, for any given $n \geq 3$, by adapting the proof of the consistency of $\mathbf{WO}_n \wedge \neg \mathbf{WO}_{n-1}$ with **ZFC** in an earlier paper [40].

The adaptation of this **ZFC**-based proof to $\mathbf{PA}_2$ was carried out in [38]. There we utilize $\mathbf{ZFC}^-$, a subtheory of **ZFC** obtained by removing the Power Set axiom and some changes in other axioms, as a proxy theory. (See *e.g.* [16] for a comprehensive account of $\mathbf{ZFC}^-$.) The advantage of $\mathbf{ZFC}^-$ is that this theory is equiconsistent with $\mathbf{PA}_2$, while it is still a rather forcing-friendly theory. The equiconsistency of $\mathbf{ZFC}^-$ and $\mathbf{PA}_2$ is considered to be a well-known result, although, as far as we know, no complete proof has ever been published. A sketch given in [38] involves some results of [5, 43] and other earlier papers.

On the other hand, $\mathbf{ZFC}^-$ allows to adapt many typical forcing notions related to reals, in the form of *pre-tame class forcings*, based on appropriate coding of the "continual" forcing conditions by real-like objects, and the general class forcing theory set up in [11, 3, 4]. Such an adaptation contains a lot of routine (but nevertheless time and space consuming) work. In addition, regarding the $\mathbf{ZFC}^-$-adapted proof in [38], there are two non-routine issues. Firstly, this is getting rid of countable transitive models, of theories similar to $\mathbf{ZFC}^-$, in evaluation of the definability level of some constructions, as in Theorem 59.1 above. Secondly, circumwenting the use of diamond, which is definitely not a $\mathbf{ZFC}^-$ result in its common formulation and proof. Note that the requirement of cardinal-preservation of the forcing notion considered in the **ZFC** setting is a *conditio sine qua non* for such an adaptation, because generic collapse of cardinals is definitely beyond the formal consistency of $\mathbf{PA}_2$.

Anyway, we were able to overcome these difficulties in [38] and prove the consistency of $\mathbf{WO}_n \wedge \neg \mathbf{WO}_{n-1}$ (for any given $n \geq 3$) with $\mathbf{PA}_2$, based on the consistency of $\mathbf{PA}_2$ itself (equivalently, of $\mathbf{ZFC}^-$). Metamathematically, this means that Consis $\mathbf{PA}_2$ implies Consis $(\mathbf{PA}_2 + \mathbf{WO}_n + \neg\mathbf{WO}_{n-1})$.

The methods developed in [38] (and in [41] with respect to some other problem) are also applicable to the main results of this article (Theorems 1.1 and 1.2). Adapting their proofs, we are able to establish the following form of our main results:

Theorem 71.1 (1st main theorem for $\mathbf{PA}_2$)**.** *Assume that* $\mathtt{n} \geq 1$. *Then* Consis $\mathbf{PA}_2$ *implies the consistency of the following theories*:

(1) $\mathbf{PA}_2 + \mathbf{DC}(\mathbf{\Pi}^1_{\mathtt{n}}) + \neg\mathbf{AC}_\omega(\mathit{\Pi}^1_{\mathtt{n}+1})$;

(2) $\mathbf{PA}_2 + \mathbf{AC}_\omega(\mathrm{OD}) + \mathbf{DC}(\mathit{\Pi}^1_{\mathtt{n}+1}) + \neg\mathbf{AC}_\omega(\mathbf{\Pi}^1_{\mathtt{n}+1})$;

(3) $\mathbf{PA}_2 + \mathbf{AC}_\omega + \mathbf{DC}(\mathbf{\Pi}^1_{\mathtt{n}}) + \neg\mathbf{DC}(\mathit{\Pi}^1_{\mathtt{n}+1})$;

(4) $\mathbf{PA}_2 + \mathbf{AC}_\omega + \mathbf{DC}(\mathit{\Pi}^1_{\mathtt{n}+1}) + \neg\mathbf{DC}(\mathbf{\Pi}^1_{\mathtt{n}+1})$. □

Theorem 71.2 (2nd main theorem for $\mathbf{PA}_2$)**.** *Assume that* $\mathtt{n} \geq 1$. *Then* Consis $\mathbf{PA}_2$ *implies* Consis $(\mathbf{PA}^0_2 + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty) + \mathbf{CA}(\mathbf{\Sigma}^1_{\mathtt{n}+1}) + \neg\mathbf{CA}(\mathbf{\Sigma}^1_{\mathtt{n}+2}))$. □

Corollary 71.3. *It follows from Theorem 71.2 that the full schema of* $\mathbf{CA}$ *is not finitely axiomatizable over* $\mathbf{PA}^0_2$ *and over* $\mathbf{PA}^0_2 + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty)$.

It follows from Theorem 71.1 that the full schema of $\mathbf{AC}_\omega$ *is not finitely axiomatizable over* $\mathbf{PA}_2$, *and the full schema of* $\mathbf{DC}$ *is not finitely axiomatizable over* $\mathbf{PA}_2 + \mathbf{AC}_\omega$. □

The details will appear elsewhere.

Identifying theories with their deductive closures, we may present the concluding statement of Theorem 71.2 as follows:

$$\mathbf{PA}^0_2 + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty) + \mathbf{CA}(\mathbf{\Sigma}^1_{\mathtt{n}+1}) \subsetneqq \mathbf{PA}^0_2 + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty) + \mathbf{CA}(\mathbf{\Sigma}^1_{\mathtt{n}+2}).$$

Studies on subsystems of $\mathbf{PA}_2$ have discovered many cases in which $S \subsetneqq S'$ holds for a given pair of subsystems S, S', see *e.g.* [57]. And it is a rather typical case that such a strict extension is established by demonstrating that S' proves the consistency of S. One may ask whether this is the case for the result in the displayed line above. The answer is in the negative: namely

the theories $\mathbf{PA}^0_2 + \mathbf{AC}_\omega(\mathit{\Sigma}^1_\infty)$ *and the full* $\mathbf{PA}_2$ *are equiconsistent*

by a result in [9, Lemma 3.1.7], also mentioned in [53]. This equiconsistency result also follows from a somewhat sharper theorem in [54, 1.5].

72 Remarks and questions

In this study, the technique of countable-support generalized iterations of Jensen forcing, combined with the method of definable generic forcing notions, was employed to the construction of models of **ZF** and $\mathbf{PA}_2$ with different effects related to the Choice and Comprehension axioms. The main results obtained show that the strength of a Choice or Comprehension principle naturally depends on the next three factors in essential way:

1) the type of the principle considered: **CA**, $\mathbf{AC}_\omega$, or **DC**;

2) the level considered in the projective hierarchy,

3) admission or non-admission of parameters.

These results (Theorems 1.1 and 1.2) are significant strengthenings of previously known results in this area, including our own earlier results in [42, 39], especially with regard to the transfer of ensuing independence results to an arbitrary level of the projective hierarchy. These are new results and valuable improvements upon much of known independence results in this area. The technique developed in this paper may lead to further progress in studies of different aspects of the projective hierarchy.

This theorem continues our series of resent research such as

– a $\Pi^1_{\mathbb{n}}$ real singleton $\{a\}$ such that a codes a cofinal map $f : \omega \to \omega_1^{\mathbf{L}}$, while every $\Sigma^1_{\mathbb{n}}$ set $X \subseteq \omega$ is constructible and hence cannot code a cofinal map $\omega \to \omega_1^{\mathbf{L}}$, [34],

– a non-ROD-uniformizable $\Pi^1_{\mathbb{n}}$ set with countable cross-sections, while all $\mathbf{\Sigma}^1_{\mathbb{n}}$ sets with countable cross-sections are $\mathbf{\Delta}^1_{\mathbb{n}+1}$-uniformizable [35],

– a model of **ZFC**, in which , is defined in [36];

– a model of **ZFC**, in which the full basis theorem holds in the absence of analytically definable well-orderings of the reals, is defined in [37].

These results also bring us closer to solving the following extremely important problem by S. D. Friedman [11, P. 209], [12, P. 602]: assuming the consistency of an inaccessible cardinal, find a model for a given n in which all $\mathbf{\Sigma}^1_n$ sets of reals are Lebesgue measurable and have the Baire and perfect set properties, but there is a $\mathbf{\Delta}^1_{n+1}$ well-ordering of the reals.

From our study, it is concluded that the technique of *definable generic* inductive constructions of forcing notions in $\mathbf{L}$, developed for Jensen-type generalized forcing iterations, succeeds to solve important descriptive set theoretic problems.

We present several questions related to possible extensions of the results achieved in this paper, that arise from our study.

Problem 72.1. Recall that $\mathbf{DC}(\mathrm{OD}) \Longleftrightarrow \mathbf{DC}(\mathrm{ROD})$ by Lemma 2.2(vi). Is $\mathbf{DC}(\mathrm{OD})$ equivalent to the full **DC** in **ZF**?

Problem 72.2. Still about the Dependent Choices principle. Three different forms of this axiom were introduced by Definition 2.1: $\mathbf{DC}(K)$, $\mathbf{DC}^{-}(K)$, $\mathbf{DC}^{*}(K)$. Lemma 2.2 contains several results on the relationship of these forms of **DC** to each other. But still many questions are unresolved. For instance, consider the implications $\mathbf{DC}^{*}(K) \Longrightarrow \mathbf{DC}(K) \Longrightarrow \mathbf{DC}^{-}(K)$ in Lemma 2.2(i). The first implication is actually an equivalence for appropriate classes K by Lemma 2.2(iv). What about $\mathbf{DC}(K) \Longrightarrow \mathbf{DC}^{-}(K)$, the second one? Can we split it by suitable models, provided $K = \varSigma^1_n$ or $\mathbf{\Sigma}^1_n$?

Problem 72.3. Does the implication $\mathbf{DC}^{-}(\varPi^1_{n+1}) \Longrightarrow \mathbf{DC}^{-}(\mathbf{\Pi}^1_n)$ hold, similarly to (v) of Lemma 2.2?

Problem 72.4 (Communicated by Ali Enayat)**.** A natural question is whether the main results of this paper also hold for second order set theory (the Kelley-Morse theory of classes). This may involve a generalization of the Sacks forcing to uncountable cardinals, as in [10, 28], as well as the new models of set theory recently defined by Fuchs [14], on the basis of further development of the methods of *class forcing* introduced by S. D. Friedman [11].

Now we return to the result on consistency of hypothesis $\mathbf{WO}_n \wedge \neg \mathbf{WO}_{n-1}$, discussed in Section 71. The generic model used to prove this consistency claim in [40] definitely satisfies the continuum hypothesis $2^{\aleph_0} = \aleph_1$. The problem of obtaining models of **ZFC** in which $2^{\aleph_0} > \aleph_1$ and there is a projective well-ordering of the real line, has been known since the early years of modern set theory. See, *e.g.*, problem 3214 in an early survey [46] by Mathias. Harrington [22] solved this problem by constructing a generic model of **ZFC**, in which $2^{\aleph_0} > \aleph_1$ and there is a $\varDelta^1_3$ well-ordering of the continuum. This model involves various forcing notions like the almost-disjoint forcing [24] and a forcing notion by Jensen and Johnsbråten [27].

Problem 72.5. Prove the consistency of $\mathbf{WO}_n \wedge \neg \mathbf{WO}_{n-1}$ by a model satisfying the additional requirement that the negation $2^{\aleph_0} > \aleph_1$ of the continuum hypothesis holds.

A very recent preprint [59] presents another study of interrelations between various forms of Choice from somewhat different point of view. In particular Theorem in [59, page 5] claims a model of

$$\mathbf{ZF} + \mathbf{DC}(\mathbb{R}, \mathbf{\Pi}^1_n) + \neg\mathbf{AC}_\omega(\mathbb{R}, \mathrm{unif}\, \mathit{\Pi}^1_{n+1}) + \neg\mathbf{AC}_\omega(\mathbb{R}, \mathbf{Ctbl})$$

for any $n \geq 1$, where:

$\mathbf{DC}(\mathbb{R}, \mathbf{\Pi}^1_n)$ asserts that if $\varnothing \neq X \subseteq \omega^\omega$ is a $\mathbf{\Pi}^1_n$ set and $P \subseteq X \times X$ is a $\mathbf{\Pi}^1_n$ relation with $\mathtt{dom}\, P = X$, then there is a chain $\langle x_k \rangle_{k<\omega}$ of reals $x_i \in X$ satisfying $x_k \; P \; x_{k+1}$ for all k — in fact this is equivalent to our $\mathbf{DC}(\mathbf{\Pi}^1_n)$ by Lemma 2.2(iv);

$\mathbf{AC}_\omega(\mathbb{R}, \mathrm{unif}\, \mathit{\Pi}^1_{n+1})$ asserts that if $\varnothing \neq X_k \subseteq \omega^\omega$ are sets in $\mathit{\Pi}^1_{n+1}$ and the set $\{k^\frown x : k < \omega \wedge x \in X_k\}$ belongs to $\mathit{\Pi}^1_{n+1}$ either — this is equivalent to our $\mathbf{AC}_\omega(\mathit{\Pi}^1_{n+1})$ as in Definition 2.1;

$\mathbf{AC}_\omega(\mathbb{R}, \mathbf{Ctbl})$ asserts that any family of countable or finite sets $\varnothing \neq X_k \subseteq \omega^\omega$ admits a choice function — note that in $\mathbf{ZF}$ the union $\bigcup_k X_k$ is not necessarily countable, and the set $\widehat{X} = \{\langle k, x \rangle : k < \omega \wedge x \in X_k\}$ is not necessarily even ROD, in this case under $\mathbf{ZF}$.

Problem 72.6. Find out whether axiom $\mathbf{AC}_\omega(\mathbb{R}, \mathbf{Ctbl})$ as above is fulfilled in the models that are built to prove our Theorem 1.1.

It should be noted that, when dealing with $\mathbf{AC}_\omega(\mathbb{R}, \mathbf{Ctbl})$ in the choiceless environment of $\mathbf{ZF}$, the behaviour of countable sets can be different from what a mathematician is assustomed with, see *e.g.* [48, 49].

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