

# Cauchy's Infinitesimals, His Sum Theorem, and Foundational Paradigms

Tiziana Bascelli<sup>1</sup> · Piotr Błaszczyk<sup>2</sup> · Alexandre Borovik<sup>3</sup> · Vladimir Kanovei<sup>4</sup> · Karin U. Katz<sup>5</sup> · Mikhail G. Katz<sup>5</sup> · Semen S. Kutateladze<sup>6</sup> · Thomas McGaffey<sup>7</sup> · David M. Schaps<sup>8</sup> · David Sherry<sup>9</sup>

Published online: 24 June 2017  
© Springer Science+Business Media B.V. 2017

**Abstract** Cauchy's *sum theorem* is a prototype of what is today a basic result on the convergence of a series of functions in undergraduate analysis. We seek to interpret Cauchy's proof, and discuss the related epistemological questions involved in comparing distinct interpretive paradigms. Cauchy's proof is often interpreted in the modern framework of a Weierstrassian paradigm. We analyze Cauchy's proof closely and show that it finds closer proxies in a different modern framework.

**Keywords** Cauchy's infinitesimal · Sum theorem · Quantifier alternation · Uniform convergence · Foundational paradigms

---

✉ Mikhail G. Katz  
katzmik@macs.biu.ac.il

Tiziana Bascelli  
tiziana.bascelli@liceocorradini.gov.it

Piotr Błaszczyk  
pb@up.krakow.pl

Alexandre Borovik  
alexandre.borovik@manchester.ac.uk

Vladimir Kanovei  
kanovei@googlemail.com

Karin U. Katz  
katzmik@math.biu.ac.il

Semen S. Kutateladze  
sskut@math.nsc.ru

Thomas McGaffey  
Thomas.McGaffey@sjcd.edu

David M. Schaps  
dschaps@gmail.com

David Sherry  
David.Sherry@nau.edu

## 1 Introduction

Infinitesimals were to disappear from mathematical practice  
in the face of Weierstrass'  $\varepsilon$  and  $\delta$  notation.

(Bottazzini 1986, p. 208)

Nearly two centuries after Cauchy first published his controversial *sum theorem*, historians are still arguing over the nature of its hypotheses and Cauchy's modification thereof in a subsequent paper.

### 1.1 The Sum Theorem

In 1821 Cauchy presented the sum theorem as follows:

When the various terms of series  $[u_0 + u_1 + u_2 + \cdots + u_n + u_{n+1} \dots]$  are functions of the same variable  $x$ , continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum  $s$  of the series is also a continuous function of  $x$  in the neighborhood of this particular value.

This definition is found in his textbook *Cours d'Analyse*, Theorem I in Sect. 6.1; the translation above is from Bradley and Sandifer (2009, p. 90). Cauchy returned to the sum theorem in an article Cauchy (1853), and the matter has been debated by mathematicians and historians alike ever since.

Cauchy is often claimed to have modified<sup>1</sup> the hypothesis of his sum theorem in 1853 to a stronger hypothesis of uniform convergence. What did that modification consist of precisely? Writes G. Arsac:

Assez curieusement, Robinson montre que si l'on interprète de façon non standard les hypothèses de l'énoncé de Cauchy de 1853 que nous venons d'étudier, alors cet énoncé implique la convergence uniforme. (Arsac 2013, p. 134)

Our text can be viewed as an extended commentary on Arsac's *rather curious* observation.

The received scholarship on this issue contains a tension between a pair of contradictory contentions. On the one hand, it holds that Cauchy worked with an Archimedean

<sup>1</sup> Lyceum Gymnasium "F. Corradini", Thiene, Italy

<sup>2</sup> Institute of Mathematics, Pedagogical University of Cracow, Kraków, Poland

<sup>3</sup> School of Mathematics, University of Manchester, Oxford Street, Manchester M13 9PL, United Kingdom

<sup>4</sup> IPPI, Moscow, and MIIT, Moscow, Russia

<sup>5</sup> Department of Mathematics, Bar Ilan University, Ramat Gan 5290002, Israel

<sup>6</sup> Sobolev Institute of Mathematics, Novosibirsk State University, Novosibirsk, Russia

<sup>7</sup> Rice University, Houston, USA

<sup>8</sup> Department of Classical Studies, Bar Ilan University, Ramat Gan 5290002, Israel

<sup>9</sup> Department of Philosophy, Northern Arizona University, Flagstaff AZ 86011, US

<sup>1</sup> A debate of long standing concerns the issue of whether Cauchy *modified* or *clarified* the hypothesis of the sum theorem of 1853 as compared to 1821. Our analysis of the 1853 text is independent of this debate and our main conclusions are compatible with either view.

continuum (A-track for short) exclusively; on the other, it holds that Cauchy introduced the condition of uniform convergence in 1853. These contentions are contradictory because Cauchy's 1853 condition was not formulated in an A-track fashion and on the contrary involved infinitesimals.

In more detail, uniform convergence is traditionally formulated over an Archimedean continuum in terms of alternating quantifiers which are conspicuously absent from Cauchy's text, whereas infinitesimals are conspicuously present there. From the A-track viewpoint, the modification required for uniform convergence focuses on how the remainder term  $r_n(x)$  behaves both (1) as  $n$  increases and (2) as a function of  $x$ . This is, however, not what Cauchy does in his 1853 paper. Namely, Cauchy declares  $n$  to be infinite, and requires  $r_n$  to be infinitesimal for all  $x$  (*toujours*).

An infinitesimal-enriched continuum (B-track for short) enables a characterisation of uniform convergence by means of enlargement of the domain, along the lines of what Cauchy suggests by means of his term *toujours* (*always*) as widely discussed in the literature; see e.g., Laugwitz (1987), Cutland et al. (1988). We argue that Cauchy's procedures as developed in his 1853 text admit better proxies in modern infinitesimal frameworks<sup>2</sup> than in modern Weierstrassian frameworks limited to an Archimedean continuum.

Thus, a B-framework allows us to follow Cauchy in presenting a statement of the sum theorem without quantifier alternations, and also to be faithful to Cauchy in retaining his infinitesimals rather than providing *paraphrases* for them in terms of either a variable subordinate to a universal quantifier, or a sequence tending to zero.

## 1.2 Wartofski's Challenge

Wartofsky launched a challenge to the historians of science. Wartofsky (1976) called for a philosophical analysis of the ontology of a field like mathematics (as practiced during a particular period) as a prior condition for a historiography that possesses vision. We propose a minor step in this direction by introducing a distinction between, on the one hand, the *procedures* exploited in the mathematical practice of that period, and, on the other, the *ontology* (in the narrow sense) of the mathematical entities like points, numbers, or functions, as used during that period. We use Cauchy's sum theorem as a case study.

We seek an approach to Cauchy that interprets his sum theorem in a way faithful to his own *procedures*. We argue that, leaving aside issues of the *ontology of mathematical entities* and their justification in terms of one or another *foundational* framework, a B-track (infinitesimal-enriched) framework provides better proxies for interpreting Cauchy's *procedures* and inferential moves than does an A-track (Archimedean) framework. More specifically, we will provide a proxy for Cauchy's proof of the sum theorem and show that our A-track opponents, who criticize Cauchy's proof, have to ignore some of Cauchy's procedures.

As I. Grattan-Guinness pointed out, Cauchy's procedures in 1853 are *difficult to interpret* in an A-framework (see Sect. 7.1 for more details). Indeed, nowhere does Cauchy rely on quantifier alternations; furthermore, he exploits the term *always* to refer to an extension of the possible inputs to the function, which must now include infinitesimals in addition to ordinary values. Meanwhile, an A-framework can't express uniform

<sup>2</sup> Some historians are fond of recycling the claim that Abraham Robinson used *model theory* to develop his system with infinitesimals. What they tend to overlook is not merely the fact that an alternative construction of the hyperreals via an ultrapower requires nothing more than a serious undergraduate course in algebra (covering the existence of a maximal ideal), but more significantly the distinction between *procedures* and *foundations* (see Sect. 1.2) which highlights the point that whether one uses Weierstrass's foundations or Robinson's is of little import, procedurally speaking.

convergence without quantifier alternations, nor countenance any extension beyond the real numbers. We document strawmanship and crude mathematical errors by A-track scholars in response to B-track interpretations of Cauchy.

The goal of a Cauchy scholar is not to translate Cauchy's procedures into contemporary mathematics so as to obtain a correct result, but rather to understand them on his (Cauchy's) own terms, or as close as possible to his own terms. We argue that Cauchy's procedures are best understood in the sense of infinitesimal mathematics, rather than paraphrased to fit the *Epsilon* mode. Recent articles deals with Cauchy's foundational stance include Sørensen (2005), Ramis (2012), and Nakane (2014).

### 1.3 A Re-evaluation

This article is part of a re-evaluation of the history of infinitesimal mathematics. Such studies have recently appeared in journals as varied as *Erkenntnis* (Katz and Sherry 2013), *Foundations of Science* (Bascelli et al. 2017), *HOPOS* (Bascelli et al. 2016), *JGPS* (Bair et al. 2017), *Notices of the American Mathematical Society* (Bair et al. 2013), *Perspectives on Science* (Katz et al. 2013), *Studia Leibnitiana* (Sherry and Katz 2014).<sup>3</sup> Abraham Robinson's framework has recently become more visible due to the activity of some high-profile advocates like Terry Tao; see e.g., Tao (2014), Tao and Vu (2016). The field has also had its share of high-profile detractors like Bishop (1977) and Connes et al. (2001). Their critiques were analyzed in Katz and Katz (2011a), Katz and Leichtnam (2013), Kanovei et al. (2013), and Sanders (2017a, b). Additional criticisms were voiced by Earman (1975), Easwaran (2014), Edwards (2007), Ferraro (2004), Grabiner (1981), Gray (2015), Halmos (1985), Ishiguro (1990), Schubring (2005), Sergeev (2015), and Spalt (2002). These were dealt with respectively in Katz and Sherry (2013), Bascelli et al. (2014), Kanovei et al. (2015), Bair et al. (2017), Borovik and Katz (2012), Błaszczyk et al. (2016, 2017a, b), Bascelli et al. (2016), Gutman et al. (2017), and Katz and Katz (2011b).

## 2 The Revised Sum Theorem

Cauchy denotes the  $n$ th partial sum of the series

$$s(x) = u_0(x) + u_1(x) + u_2(x) + \dots$$

by  $s_n(x)$ . He also denotes by  $r_n(x) = s(x) - s_n(x)$  the  $n$ th remainder of the series. Cauchy's revised sum theorem from Cauchy (1853, pp. 456–457) as reprinted in Cauchy (1900), states the following.<sup>4</sup>

**Théorème 1.** Si les différents termes de la série

$$u_0, u_1, u_2, \dots, u_n, u_{n+1}, \dots \quad (1)$$

sont des fonctions de la variable réelle  $x$ , continues, par rapport à cette variable, entre des limites données; si, d'ailleurs, la somme

$$u_n + u_{n+1} + \dots + u_{n'-1} \quad (3)$$

devient *toujours* infiniment petite pour des valeurs infiniment grandes des nombres entiers  $n$  et  $n' > n$ , la série (1) sera convergente et la somme  $s$  de la série sera, entre

<sup>3</sup> See <http://u.cs.biu.ac.il/~katzmik/infinitesimals.html> for a more detailed list.

<sup>4</sup> Here the equation numbers (1) and (3) are in Cauchy's text as reprinted in Cauchy (1900).

les limites données, fonction continue de la variable  $x$ . (Cauchy 1900, p. 33) (emphasis added)

We will highlight several significant aspects of the procedures exploited by Cauchy in connection with the sum theorem.

### 2.1 Infinite Numbers

Cauchy considers two instantiations (which he refers to as *valeurs*) of integer numbers, denoted  $n$  and  $n'$ . Note that he refers to them as infinitely large *numbers*, rather than either *quantities* or *sequences* increasing without bound.

### 2.2 Cauchy’s $\varepsilon$

Cauchy points out that in order to prove the sum theorem, one needs to show that “le module de  $r_n$  [soit] inférieur à un nombre  $\varepsilon$  aussi petit que l’on voudra’.” This comment can be formalized in two different ways, [A] and [B]:

$$[A] \quad (\forall \varepsilon \in \mathbb{R})(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})[(\varepsilon > 0) \wedge (n > m) \Rightarrow (|r'_n| < \varepsilon)].$$

$$[B] \quad (\forall \varepsilon \in \mathbb{R})[(\varepsilon > 0) \rightarrow (|r'_n| < \varepsilon)].$$

In neither interpretation does  $\varepsilon$  assume infinitesimal values.

Here  $n$  is a bound variable in formula [A] in the sense of being subordinate to the universal quantifier. Meanwhile,  $n$  is interpreted as a specific value in [B], i.e., it is a free variable. For formula [B] to be true,  $n$  must be infinite (and  $r_n$  infinitesimal). Interpretations [A] and [B] correspond to the two tracks compared in Sect. 3.4.

Clarifying the formulas completely would require adding a quantifier over the variable  $x$  of  $r_n$ , as well as specifying explicitly in [B] that  $n$  is infinite.

### 2.3 When is $n' = \infty$ ?

Cauchy examines the series<sup>5</sup>

$$\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \tag{2}$$

closely related to Abel’s purported counterexample to the sum theorem, namely to the sum theorem as stated in 1821. The series sums to a (discontinuous) sawtooth waveform. Cauchy argues that the series does not in fact satisfy the hypotheses of his 1853 theorem, as follows. He considers the difference  $s_{n'} - s_n = u_n + u_{n+1} + \dots + u_{n'-1}$  which translates into the following sum in the case of the series (2)<sup>6</sup>:

$$\frac{\sin(n+1)x}{n+1} + \frac{\sin(n+2)x}{n+2} + \dots + \frac{\sin(n')x}{n'}. \tag{4}$$

Cauchy proceeds to assign the value  $\infty$  to the index  $n'$ , by writing  $n' = \infty$ , and points out that the sum (4) is then precisely the remainder term  $r_n$ .

<sup>5</sup> The equation number (2) is in Cauchy’s original text.

<sup>6</sup> The equation number (4) is in Cauchy’s original text.

## 2.4 Evaluating at $\frac{1}{n}$

Cauchy goes on to evaluate the remainder  $r_n(x)$  at  $x = \frac{1}{n}$  and points out in Cauchy (1853) that the result

se réduira sensiblement à l'intégrale

$$\int_1^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} - 1 + \frac{1}{1.2.3} \frac{1}{3} - \frac{1}{1.2.3.4.5} \frac{1}{5} + \dots = 0.6244\dots$$

Here one can evaluate the integral of  $\frac{\sin x}{x}$  by viewing the series as an infinite Riemann sum; see Laugwitz (1989, p. 212). Cauchy concludes that the remainder is not infinitesimal and that therefore the purported counterexample (2) in fact does not satisfy the hypotheses of the sum theorem as formulated in 1853. For a discussion of Cauchy's proof of the theorem see Sect. 8.4.

## 3 Methodological Issues

### 3.1 Issues Raised by Cauchy's Article

Cauchy's 1853 article raises several thorny issues:

- (1) If one is to interpret Cauchy's new hypothesis as uniform convergence, how can one explain the absence of any quantifier alternations in the article (cf. Sect. 7.1, claim GG4)?
- (2) If the term *always* is supposed to signal a transition from the hypothesis of convergence in 1821 to a hypothesis of uniform convergence in 1853, what amplification does the term provide, given that the convergence was already required for every  $x$  in the domain  $[x_0, X]$ , and there are no further real inputs at which one could impose additional conditions?
- (3) The traditional reading of convergence, whether ordinary or uniform, leaves no room for indices given by *infinite* numbers  $n$  and  $n'$  referred to by Cauchy (see Sect. 2.1).
- (4) How can  $|r_n|$  be smaller than every real  $\varepsilon > 0$  if the option of interpreting this inequality in terms of alternating quantifiers is not available (see Sect. 2.2)?
- (5) How can  $r_n(x)$  evaluated at  $x = \frac{1}{n}$  produce the definite integral  $\int_1^{\infty} \frac{\sin x}{x} dx$  when the answer should patently be dependent on  $n$  (see Sect. 2.4)?

Before seeking to answer the questions posed in Sect. 3.1 (see Sect. 5), we first address some methodological concerns and point out possible pitfalls.

### 3.2 An Ontological Disclaimer

Before answering such questions, one needs to clarify what kind of answer can be expected when dealing with the work of a mathematician with no access to modern conceptualisations of mathematical entities like numbers and their set-theoretic ontology prevalent today.

Such a historiographic concern is ubiquitous. Thus, what could a modern historian mean, without falling into a trap of presentism, when he asserts that Lagrange was the first

to define the notion of the derivative  $f'$  of a real function  $f$ ? Such an attribution is apparently problematic since Lagrange shared neither our set-theoretic ontology of a punctiform continuum<sup>7</sup> given by the set of real numbers, nor our notion of a function as a relation of a certain type, again in a set-theoretic context.

Some helpful insights in this area were provided by Benacerraf and Quine, in terms of a dichotomy of *mathematical procedures and practice* versus *ontology of mathematical entities* like numbers. Thus, Quine wrote:

Arithmetic is, in this sense, all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic. (Quine 1968, p. 198)

In a related development, Benacerraf (1965) pointed out that if observer E learned that the natural numbers “are” the Zermelo ordinals  $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots$ , while observer J learned that they are the von Neumann ordinals  $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots$  then, strictly speaking, they are dealing with different things. Nevertheless, observer E’s actual mathematical practice (and the mathematical structures and procedures he may be interested in) is practically the same as observer J’s. Hence, different ontologies may underwrite one and the same practice.

The distinction between practice and ontology can be seen as a partial response to the challenge to the historians of science launched in Wartofsky (1976), as discussed in Sect. 1.2.

When dealing with Lagrange, Cauchy, or for that matter any mathematician before 1872, we need to keep in mind that their ontology of mathematical entities like numbers is distinct from ours. However, the *procedures* Lagrange uses can often be interpreted and clarified in terms of their modern proxies, i.e., procedures used by modern mathematicians, as in the case of the concept of the derivative.

The dichotomy of practice, procedures, and inferential moves, on the one hand, versus foundational issues of ontology of mathematical entities, on the other, is distinct from the familiar opposition between intuition and rigor. In a mathematical framework rigorous up to modern standards, the syntactic procedures and inferential moves are just as rigorous as the semantic foundational aspects, whether set-theoretic or category-theoretic.<sup>8</sup>

### 3.3 Felix Klein on Two Tracks

In addition to an ontological disclaimer as in Sect. 3.2, one needs to bear in mind a point made by Felix Klein in 1908, involving the non-uniqueness of a conceptual framework for interpreting pre-1872 procedures.

Felix Klein described a rivalry of dual approaches in the following terms. Having outlined the developments in real analysis associated with Weierstrass and his followers, Klein pointed out that

<sup>7</sup> Historians often use the term *punctiform continuum* to refer to a continuum made out of points (as for example the traditional set-theoretic  $\mathbb{R}$ ). Earlier notions of continuum are generally thought to be non-punctiform (i.e., not made out of points). The term *punctiform* also has a technical meaning in topology unrelated to the above distinction.

<sup>8</sup> Nelson’s syntactic recasting of Robinson’s framework is a good illustration, in that the logical procedures in Nelson’s framework are certainly up to modern standards of rigor and are expressed in the classical Zermelo–Fraenkel set theory (ZFC). With respect to an enriched set-theoretic language, infinitesimals in Nelson’s Internal Set Theory (IST) can be found within the real number system  $\mathbb{R}$  itself. The semantic/ontological issues are handled in an appendix to Nelson (1977), showing Nelson’s IST to be a conservative extension of ZFC.

The scientific mathematics of today is built upon the series of developments which we have been outlining. But an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries (Klein 1908, p. 214).

Such a different conception, according to Klein,

harks back to old metaphysical speculations concerning the structure of the continuum according to which this was made up of [...] infinitely small parts (ibid.).

Klein went on to formulate a criterion,<sup>9</sup> in terms of the mean value theorem, for what would qualify as a successful theory of infinitesimals, and concluded:

I will not say that progress in this direction is impossible, but it is true that none of the investigators have achieved anything positive (Klein 1908, p. 219).

Thus, the approach based on notions of infinitesimals is not limited to “the work of Fermat, Newton, Leibniz and many others in the seventeenth and eighteenth centuries,” as implied by Katz (2014). Rather, it was very much a current research topic in Felix Klein’s mind.

Of course, Klein in 1908 had no idea at all of modern infinitesimal frameworks, such as those of Robinson (see Robinson 1961), Lawvere–Moerdijk–Reyes–Kock (see Kock 2006 or Bell 2006). What Klein was referring to is the *procedural* issue of how analysis is to be presented, rather than the *ontological* issue of a specific realisation of an infinitesimal-enriched number system in the context of a traditional set theory.

### 3.4 Track A and Track B in Leibniz

The dichotomy mentioned by Klein (see Sect. 3.3) can be reformulated in the terminology of dual methodology as follows. One finds both A-track (i.e., Archimedean) and B-track (Bernoullian, i.e., involving infinitesimals) methodologies in historical authors like Leibniz, Euler, and Cauchy. Note that scholars attribute the first systematic use of infinitesimals as a foundational concept to Johann Bernoulli. While Leibniz exploited both infinitesimal methods and “exhaustion” methods usually interpreted in the context of an Archimedean continuum, Bernoulli never wavered from the infinitesimal methodology. To note the fact of such systematic use by Bernoulli is not to say that Bernoulli’s foundation is adequate, or that it could distinguish between manipulations with infinitesimals that produce only true results and those manipulations that can produce false results.

In addition to ratios of differentials like  $\frac{dy}{dx}$ , Leibniz also considered ratios of ordinary values which he denoted  $(d)y$  and  $(d)x$ , so that the ratio  $\frac{(d)y}{(d)x}$  would be what we call today the derivative. Here  $dx$  and  $(d)x$  were distinct entities since Leibniz described them as respectively *inassignable* and *assignable* in his text *Cum Prodiisset Leibniz* (1701):

...although we may be content with the assignable quantities  $(d)y$ ,  $(d)v$ ,  $(d)z$ ,  $(d)x$ , etc., ... yet it is plain from what I have said that, at least in our minds, the unassignables [*inassignables* in the original Latin]  $dx$  and  $dy$  may be substituted for them by a method of supposition even in the case when they are evanescent; ... (as translated in Child 1920, p. 153).

<sup>9</sup> A similar criterion was formulated in Fraenkel (1928, pp. 116–117). For a discussion of the Klein–Fraenkel criterion see Kanovei et al. (2013, Section 6.1).



Echoes of Leibnizian terminology are found in Cauchy when he speaks of *assignable numbers* while proving various convergence tests, see e.g., Bradley and Sandifer (2009, pp. 37, 40, 87, 369). An alternative term used by Cauchy and others is *a given quantity*. Fisher (1978) shares the view that historical evaluations of mathematical analysis in general and Cauchy's work in particular can profitably be made in the light of both A-track and B-track approaches.

Leibniz repeatedly asserted that his infinitesimals, when compared to 1, violate what is known today as the Archimedean property, viz., Euclid's *Elements*, Definition V.4; see e.g., Leibniz (1695, p. 322), as cited in Bos (1974, p. 14).

## 4 From Indivisibles to Infinitesimals: Reception

### 4.1 Indivisibles Banned by the Jesuits

Indivisibles and infinitesimals were perceived as a theological threat and opposed on doctrinal grounds in the seventeenth century. The opposition was spearheaded by clerics and more specifically by the jesuits. Tracts opposing indivisibles were composed by jesuits Paul Guldin, Mario Bettini, and André Tacquet (Redondi 1987, p. 291). P. Mancosu writes:

Guldin is taking Cavalieri to be composing the continuum out of indivisibles, a position rejected by the Aristotelian orthodoxy as having atomistic implications. ... Against Cavalieri's proposition that "all the lines" and "all the planes" are magnitudes – they admit of ratios – Guldin argues that "all the lines ... of both figures are infinite; but an infinite has no proportion or ratio to another infinite." (Mancosu 1996, p. 54)

Tacquet for his part declared that the method of indivisibles "makes war upon geometry to such an extent, that if it is not to destroy it, it must itself be destroyed." (Festa 1992, p. 205; Alexander 2014, p. 119).

In 1632 (the year Galileo was summoned to stand trial over heliocentrism) the Society's Revisors General led by Jacob Bidermann banned teaching indivisibles in their colleges (Festa 1990, 1992, p. 198). Indivisibles were placed on the Society's list of permanently banned doctrines in 1651 (Hellyer 1996). The effects of anti-indivisible bans were still felt in the 18th century, when most jesuit mathematicians adhered to the methods of Euclidean geometry, to the exclusion of the new infinitesimal methods:

...le grand nombre des mathématiciens de [l'Ordre] resta jusqu'à la fin du XVIII<sup>e</sup> siècle profondément attaché aux méthodes euclidiennes. (Bosmans 1927, p. 77)

Echoes of such bans were still heard in the nineteenth century when jesuit Moigno, who considered himself a student of Cauchy, wrote:

In effect, either these magnitudes, smaller than any *given* magnitude, still have substance and are divisible, or they are simple and indivisible: in the first case their existence is a chimera, since, necessarily greater than their half, their quarter, etc., they are not actually less than any *given* magnitude; in the second hypothesis, they are no longer mathematical magnitudes, but take on this quality, this would renounce the idea of the continuum divisible to infinity, a necessary and fundamental point of

departure of all the mathematical sciences (ibid., xxiv f.) (as quoted in Schubring 2005, p. 456) (emphasis added)

Moigno saw a contradiction where there is none. Indeed, an infinitesimal is smaller than any *assignable*, or *given* magnitude; and has been so at least since Leibniz, as documented in Sect. 3.4. Thus, infinitesimals need not be less than “their half, their quarter, etc.,” because the latter are not assignable.

In fact, the dichotomy of Moigno’s critique is reminiscent of the dichotomy of a critique of Galileo’s indivisibles penned by Moigno’s fellow jesuit Orazio Grassi some three centuries earlier. P. Redondi summarizes it as follows:

As for light – composed according to Galileo of indivisible atoms, more mathematical than physical – in this case, logical contradictions arise. Such indivisible atoms must be finite or infinite. If they are finite, mathematical difficulties would arise. If they are infinite, one runs into all the paradoxes of the separation to infinity which had already caused Aristotle to discard the atomist theory ... (Redondi 1987, p. 196).

This criticism appeared in the first edition of Grassi’s book *Ratio ponderum librae et simbellae*, published in Paris in 1626. In fact, this criticism of Grassi’s

exhumed a discounted argument, copied word-for-word from almost any scholastic philosophy textbook. ... The Jesuit mathematician [Paul] Guldin, great opponent of the geometry of indivisibles, and an excellent Roman friend of [Orazio] Grassi, must have dissuaded him from repeating such obvious objections. Thus the second edition of the *Ratio*, the Neapolitan edition of 1627, omitted as superfluous the whole section on indivisibles. (Redondi 1987, p. 197).

Alas, unlike father Grassi, father Moigno had no Paul Guldin to dissuade him.

## 4.2 Unguru’s Admonition

There are several approaches to uniform convergence. These approaches are equivalent from the viewpoint of modern mathematics (based on classical logic):

- [A] the epsilon-delta approach in the A-track context;
- [B] the approach involving a Bernoullian continuum;
- [C] the sequential approach in the A-track context.

The approach usually found in textbooks is the approach [A] via the epsilon-delta formulation involving quantifier alternations as detailed in Sect. 6.5. The B-track approach is detailed in Sect. 5. An alternative approach [C] via sequences runs as follows. A sequence of functions  $r_n$  as  $n$  runs through  $\mathbb{N}$  converges uniformly to 0 on a domain  $D$  if for each sequence of inputs  $x_n$  in  $D$ , the sequence of outputs  $r_n(x_n)$  tends to zero. Giusti pursues approaches [A] and [C] to Cauchy respectively at Giusti (1984, pp. 38, 50) (see Sect. 8). Meanwhile, Unguru admonished:

It is ... a historically unforgiveable sin ... to assume wrongly that mathematical equivalence is tantamount to historical equivalence. (Unguru 1976, p. 783)

When seeking a mathematical interpretation of a historical text we must consider, in addition to questions of mathematical coherence, also questions of coherence with the actual procedures adopted by the historical author.

### 4.3 Bottazzini: What Kept Cauchy From Seeing?

The points made in Sects. 3.2–3.4 about ontology vs procedures, as well as multiple possibilities for modern interpretive frameworks, seem straightforward enough. However, the consequences of paying insufficient attention to such distinctions can be grave. Thus, Bottazzini opines that

the techniques that Abel used were those of Cauchy; his definitions of continuity and of convergence are the same, as are his use of infinitesimals in demonstrations. It is precisely this latter fact that kept both men from finding the weak point in Cauchy's demonstration and from seeing that there is another form of convergence, what is today called the *uniform convergence* of a series of functions. (Bottazzini 1986, pp. 115–116) (emphasis in the original)

Bottazzini's claim that 'it is precisely' 'the use of infinitesimals in demonstrations' that 'kept [Cauchy] from finding ... and from seeing, etc.' only makes sense in the context of an interpretive paradigm exclusively limited to track A (see Sect. 3.4). Such an approach views the evolution of 19th century analysis as inevitable progress toward the foundations of analysis purged of infinitesimals as established by "the great triumvirate" (Boyer 1949, p. 298) of Cantor, Dedekind, and Weierstrass. In a similar vein, J. Dauben notes:

There is nothing in the language or thought of Leibniz, Euler, or Cauchy ... that would make them early Robinsonians. (Dauben 1988, p. 180)

"Early Robinsonians" perhaps not if this means familiarity with ultrafilters,<sup>10</sup> but the extent to which the procedures as found in Leibniz, Euler, and Cauchy admit straightforward proxies in Robinson's framework does not emerge clearly from the above comment.

It seems almost as if, in an effort to save Cauchy as a herald for Weierstrassian arithmetical analysis, some A-track enforcers need carefully to purge the possibility that Cauchy might also be something quite different, something generally seen *nowadays* as contrary to the rational structure of his iconage; e.g., their image seems to be that this *other* was surely a remnant of the irrational in his methodology.

### 4.4 Fraser Versus Laugwitz

In the abstract of his 1987 article in *Historia Mathematica*, Laugwitz is careful to note that he interprets Cauchy's sum theorem "with his [i.e., Cauchy's] own concepts":

It is shown that the famous so-called errors of Cauchy are correct theorems when interpreted with his own concepts. (Laugwitz 1987, p. 258)

In the same abstract, Laugwitz goes on to emphasize: "No assumptions on uniformity or on nonstandard numbers are needed." Indeed, in Sect. 7 on pages 264–266, Laugwitz gives a lucid discussion of the sum theorem in terms of Cauchy's infinitesimals, with not a whiff of modern number systems. In particular this section does not mention the article Schmieden and Laugwitz (1958). In a final section 15 entitled "Attempts toward theories of infinitesimals," Laugwitz presents a rather general discussion, with no specific reference to the sum theorem, of how one might formalize Cauchyan infinitesimals in modern set-theoretic terms. A reference to Schmieden and Laugwitz (1958) appears in this final section

<sup>10</sup> For the role of these in a possible construction of the hyperreals see Sect. 5.2.

only. Thus, Laugwitz carefully distinguishes between his analysis of Cauchy's procedures, on the one hand, and the ontological issues of possible implementations of infinitesimals in a set-theoretic context, on the other.

Alas, all of Laugwitz's precautions went for naught. In 2008, he became a target of damaging innuendo in the updated version of *The Dictionary of Scientific Biography*. Here C. Fraser writes as follows in his article on Cauchy:

Laugwitz's thesis is that certain of Cauchy's results that were criticized by later mathematicians are in fact valid *if one is willing to accept certain assumptions about Cauchy's understanding and use of infinitesimals. These assumptions reflect a theory of analysis and infinitesimals that was worked out by Laugwitz and Curt Schmieden during the 1950s.* (Fraser 2008, p. 76) (emphasis added)

What is particularly striking about Fraser's misrepresentation of Laugwitz's scholarship is Fraser's failure to recognize the dichotomy of procedure versus ontology. Fraser repeats the performance in 2015 when he writes:

Laugwitz, ... some two decades following the publication by Schmieden and him of the  $\Omega$ -calculus commenced to publish a series of articles arguing that their non-Archimedean formulation of analysis is well suited to interpret Cauchy's results on series and integrals. (Fraser 2015, p. 27)

What Fraser fails to mention is that Laugwitz specifically separated his analysis of Cauchy's *procedures* from attempts to account *ontologically* for Cauchy's infinitesimals in modern terms.

#### 4.5 The Rhetoric of Reaction

Schubring's disagreement with Laugwitz's interpretation of Cauchy found expression in the following comment:

[Giusti's article] spurred Laugwitz to even more detailed attempts to banish the error and *confirm that Cauchy had used hyper-real numbers*. On this basis, he claims, the errors vanish and the theorems become correct, or, rather, they always were correct (see Laugwitz 1990, p. 21). (Schubring 2005, p. 432) (emphasis added)

Here Schubring is *referring to* the article Laugwitz (1990) though he is most decidedly not *quoting* it. In fact, there is no mention of hyperreals on page 21 in Laugwitz (1990), contrary to Schubring's claim. What one does find here is the following comment:

The "mistakes" show rather, as *experimenta crucis* that one must understand Cauchy's terms/definitions [*Begriffe*], in the spirit of the motto,<sup>11</sup> in an infinitesimal-mathematical sense. (Laugwitz 1990, p. 21) (translation ours)

We fully endorse Laugwitz's comment to the effect that Cauchy's procedures must be understood in the sense of infinitesimal mathematics, rather than paraphrased to fit the *Epsilon* mode. Note that we are dealing with an author, namely Laugwitz, who published Cauchy studies in the leading periodicals *Historia Mathematica* (Laugwitz 1987) and

<sup>11</sup> Here Laugwitz is referring to Cauchy's motto to the effect that "Mon but principal a été de concilier la rigueur, dont je m'étais fait une loi dans mon *Cours d'analyse* avec la simplicité que produit la considération directe des quantités infiniment petites."

*Archive for History of Exact Sciences* (Laugwitz 1989).<sup>12</sup> The idea that Laugwitz would countenance the claim that Cauchy “had used hyper-real numbers” whereas both the term *hyper-real* and the relevant construction were not introduced until Hewitt (1948, p. 74), strikes us as far-fetched. Meanwhile, in a colorful us-against-“them” passage, J. Grabiner opines that

[Schubring] effectively rebuts the partisans of nonstandard analysis who wish to make Cauchy one of *them*, using the work of Cauchy’s disciple the Abbé Moigno to argue for Cauchy’s own intentions. (Grabiner 2006, p. 415). (emphasis added)

Schubring indeed uses the work of Moigno in an apparent attempt to refute infinitesimals. Jesuit Moigno’s confusion on the issue of infinitesimals was detailed in Sect. 4.1.

Contrary to Schubring’s claim, Laugwitz did not attribute 20th century number systems to Cauchy, but rather argued that modern (B-track in the terminology of Sect. 3.4) infinitesimal frameworks provide better proxies for Cauchy’s *procedures* than modern Weierstrassian (A-track in the terminology of Sect. 3.4) frameworks. Laugwitz also sought to understand Cauchy’s inferential moves in terms of their modern proxies.

But Schubring’s insights don’t even reach a quarter of his student K. Viertel’s, who writes:

...one can find assertions of Cauchy that contradict a consequent NSA-interpretation. With respect to the function  $\frac{-m}{x}$  and  $x^{-m}$  and their continuity he remarks: Both become infinite and, as a consequence, discontinuous when  $x = 0$ . (Bradley and Sandifer 2009, p. 28). ... In non-standard analysis, however, one has  $x^{-1} \simeq \infty$  for all  $x \simeq 0$  which entails the continuity of the function  $\frac{1}{x}$  for the point zero. Thus Cauchy’s concept of continuity is not always consistent (or is not always compatible) with the concept of continuity of nonstandard analysis. (Viertel 2014, Section 4.3.1.1, pp. 56–57) (translation ours)

Viertel’s apparent intuition that all infinite numbers  $x^{-1}$  are infinitely close to each other makes one wonder how such intuitions could be squared with Cauchy’s apparently distinct infinite integers  $n$  and  $n'$  (see Sects. 2.3, 8.3). Viertel’s claims concerning the *reciprocal function* in Robinson’s framework are of course incorrect.

Such crude mathematical errors are symptomatic of questionable priority scales. Viertel’s attempt to “prove” that the *reciprocal function*  $\frac{1}{x}$  is continuous at the origin in nonstandard analysis is doomed from the start since the extension  $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$  preserves all the first-order properties under transfer.<sup>13</sup> Anti-infinitesimal rhetoric by A-track scholars appears to be *reciprocal* to mathematical competence in matters infinitesimal.

<sup>12</sup> The fact that Laugwitz had published articles in leading periodicals does not mean that he couldn’t have said something wrong. However, it does suggest the existence of a strawman aspect of Schubring’s sweeping claims against him.

<sup>13</sup> The *transfer principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system (or more general mathematical structure), still apply (i.e., are *transferred*) to an extended number system (or more general mathematical structure). Thus, the familiar extension  $\mathbb{Q} \hookrightarrow \mathbb{R}$  preserves the properties of an ordered field. To give a negative example, the extension  $\mathbb{R} \hookrightarrow \mathbb{R} \cup \{\pm\infty\}$  of the real numbers to the so-called *extended reals* does not preserve the properties of an ordered field. The hyperreal extension  $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$  preserves *all* first-order properties. For example, the identity  $\sin^2 x + \cos^2 x = 1$  remains valid for all hyperreal  $x$ , including infinitesimal and infinite values of  $x \in {}^*\mathbb{R}$ . In particular, the properties of the reciprocal function remain the same after it is extended to the hyperreal domain.

Arsac tries to explain nonstandard analysis but he seems to be as unaware of the transfer principle as Viertel: “Une fonction continue au sens habituel est une fonction continue aux points standards, mais elle ne

## 4.6 Gestalt Switch

The debate over Cauchy's understanding of infinitesimals bears some of the marks of a paradigm shift in the sense of Thomas Kuhn. A change is taking place in the historiography of mathematics, involving the entry of professional mathematicians into the field. Mathematicians bring with them the understanding that the same mathematical object or structure can be defined and described in a variety of mathematical languages. Similarly, the same theorem can be proved (or the same theory developed) in a multitude of languages and conceptual frameworks.

When a mathematician of the past used (and in some cases, invented) several distinct mathematical languages (as was the case for Cauchy), it is natural for a modern mathematician to exploit modern formalisations of these languages in the analysis of his predecessor's work. This adds depth and added dimension to our vision of the past, amounting to a *gestalt switch* as formulated in Kuhn (1996, pp. 150–151):

[T]he transition between competing paradigms cannot be made a step at a time, forced by logic and neutral experience. Like a gestalt switch, it must occur all at once (though not necessarily in an instant) or not at all. How, then, are scientists brought to make this transition? Part of the answer is that they are very often not. Copernicanism made few converts for almost a century after Copernicus' death. Newton's work was not generally accepted, particularly on the Continent, for more than half a century after the *Principia* appeared. Priestley never accepted the oxygen theory, nor Lord Kelvin the electromagnetic theory, and so on. The difficulties of conversion have often been noted by scientists themselves. ... Max Planck, surveying his own career in his *Scientific Autobiography*, sadly remarked that "a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it."

Here Kuhn is quoting (Planck 1950, pp. 33–34). In short, science (including its historiography) makes progress one funeral at a time.

## 5 Interpreting Cauchy's Infinitesimal Mathematics

Cutland et al. agree with Grattan-Guinness (see Sect. 7.1) concerning the *difficulty* of interpreting Cauchy's procedures in an Archimedean framework:

[Cauchy's] modification of his theorem is anything but clear if we interpret his conception of the continuum as identical with the 'Weierstrassian' concept. Abraham Robinson [1966] first discusses Cauchy's original 'theorem' ... (Cutland et al. 1988, p. 376)

---

Footnote 13 continued

l'est pas obligatoirement aux points non standard" (Arsac 2013, p. 133). Contrary to his claim, the natural extension of a continuous function  $f$  will be continuous at *all* hyperreal points  $c$  in the sense of the standard definition  $\forall \epsilon > 0 \exists \delta > 0 : |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$ , by the transfer principle. Indeed, Arsac confused S-continuity and continuity...

## 5.1 Robinson's Reading

The first B-track interpretation of Cauchy's sum theorem was developed in (Robinson 1966, pp. 271–273).

Modern versions of the sum theorem require an additional hypothesis to obtain a mathematically correct result, either (1) in terms of uniform convergence, or (2) in terms of equicontinuity of the family of functions.

Robinson considers both possibilities in the context of Cauchy's (1821) formulation of the sum theorem, which is sufficiently ambiguous to accommodate various readings. The last mention of the term *equicontinuous* occurs in Robinson (1966) on page 272, line 10 from bottom.

Meanwhile, starting on page 272, line 2 from bottom, Robinson turns to an analysis of Cauchy's (1853) formulation of the sum theorem and its proof. Robinson writes: "Over thirty years after the publication of [the sum theorem], Cauchy returned to the problem [in] (Cauchy 1853)." (ibid.) Robinson concludes:

If we interpret this theorem in the sense of Non-standard Analysis, so that 'infiniment petite' is taken to mean 'infinitesimal' and translate 'toujours' by 'for all  $x$ ' (and not only 'for all standard  $x$ '), then the condition introduced by Cauchy ... amounts precisely to uniform convergence in accordance with (1) above." (Robinson 1966, p. 273)

Thus, in his analysis of Cauchy's (1853) article, Robinson only envisions the possibility (1) of uniform convergence. The reason is that Cauchy's proof contains hints pointing toward such an interpretation, while it contains no hints in the direction of (2) equicontinuity.

Robinson's interpretation is based on a characterisation of uniform convergence exploiting a hyperreal extension  $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ ; cf. (Davis 1977, p. 62, Theorem 6.3). Down-to-earth treatments of the hypereals can be found e.g., in Lindstrøm (1988) and Prestel (1990); see Sect. 5.2.

Robinson wrote:

**Theorem 4.6.1** The sequence of standard functions  $\{f_n(x)\}$  converges to the standard function  $f(x)$  uniformly on  $B \subset T$  [where  $T$  is a metric space] if and only if  $f(p) \approx f_n(p)$  for all  $p \in {}^*B$  and for all infinite  $n$ . (Robinson 1966, p. 116)

Here the relation of infinite proximity  $f_n(x) \approx f(x)$  signifies that the difference  $f_n(x) - f(x)$  is infinitesimal.

## 5.2 Constructing a Hyperreal Field

We outline a construction of a hyperreal field  ${}^*\mathbb{R}$ . Let  $\mathbb{R}^{\mathbb{N}}$  denote the ring of sequences of real numbers, with arithmetic operations defined termwise. Then we have

$${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}} / \text{MAX}$$

where MAX is the maximal ideal consisting of all "negligible" sequences  $(u_n)$ , i.e., sequences which vanish for a set of indices of full measure  $\xi$ , namely,  $\xi(\{n \in \mathbb{N} : u_n = 0\}) = 1$ .

Here  $\xi : \mathcal{P}(\mathbb{N}) \rightarrow \{0, 1\}$  (thus  $\xi$  takes only two values, 0 and 1) is a finitely additive measure taking the value 1 on each cofinite set, where  $\mathcal{P}(\mathbb{N})$  is the set of subsets of  $\mathbb{N}$ .

The subset  $\mathcal{F}_\varepsilon \subseteq \mathcal{P}(\mathbb{N})$  consisting of sets of full measure is called a *free ultrafilter*. These originate with Tarski (1930). The construction of a Bernoullian continuum outlined here was therefore not available prior to that date. Further details on the hyperreals can be found in Fletcher et al. (2017). The educational advantages of the infinitesimal approach are discussed in Katz and Polev (2017).

### 5.3 An Example

To illustrate the mathematical issue involved, we let  $B = [0, 1]$  and consider the sequence of real functions  $f_n(x) = (1 - x)^n$  on the interval  $B$ . Then the limiting function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

is discontinuous. The condition that  $f(x) \approx f_n(x)$  for an infinite  $n$  is satisfied at all real inputs  $x$ , but fails at the infinitesimal input  $x = \frac{1}{n}$  since  $f_n(\frac{1}{n}) = (1 - \frac{1}{n})^n \approx \frac{1}{e}$  and  $\frac{1}{e}$  is appreciable:  $\frac{1}{e} \not\approx 0$ . The limit of the sequence can of course be re-written as the sum of a series, producing a series of continuous real functions with discontinuous sum.

A similar phenomenon can be illustrated by means of the family  $f_n(x) = \arctan nx$ .

### 5.4 Cauchy’s B-Track Procedures

We will now interpret Cauchy’s *procedures* (as opposed to *ontology*; see Sect. 3.2) in such a framework. The functions  $f_n$  in Robinson’s Theorem 4.6.1 correspond to Cauchy’s partial sums  $s_n$ . Our main thesis, in line with the comment by Grattan-Guinness cited in Sect. 6.2, is the following.

**Main Thesis.** Cauchy’s term *toujours* suggests *extending* the possible inputs to the function.

We interpret this *extension* procedure in terms of the following hyperreal proxy. We extend the domain  $B$  of a real function  $f : B \rightarrow \mathbb{R}$  to its hyperreal extension  ${}^*B$ , obtaining a hyperreal function  $f : {}^*B \rightarrow {}^*\mathbb{R}$ . Thus, if  $B$  is Cauchy’s interval  $[x_0, X]$  with, say,  $x_0 = 0$  [as in the example of the series (4) in Sect. 2.3], then the hyperreal extension  ${}^*B = {}^*[0, X]$  will contain in particular positive infinitesimals of the form  $\alpha = \frac{1}{n}$  where  $n$  is an infinite hyperinteger.

The condition  $f_n(x) \approx f(x)$  is equivalent to the remainder term  $r_n = f(x) - f_n(x)$  being infinitesimal. This is required to hold *at all hyperreal inputs*  $x$ , serving as a proxy for Cauchy’s 1853 modified hypothesis with its implied extension of the domain.

Note that Robinson’s Theorem 4.6.1 cited in Sect. 5.1 contains no quantifier alternations. This characterisation of uniform convergence closely parallels the condition found in Cauchy’s text.

The condition  $|r_n| < \varepsilon$  for all real positive  $\varepsilon$  amounts to requiring  $r_n$  (more precisely,  $r_n(x)$ ) to be infinitesimal, as per item [B] in Sect. 2.2. As Cauchy pointed out, the condition is not satisfied by the series  $\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots$

In the hyperreal framework, the quantity  $r_n(\frac{1}{n})$  for all infinite  $n$  will possess nonzero standard part (shadow) given precisely by the integral  $\int_1^\infty \frac{\sin x}{x} dx$ , in line with Cauchy’s comment cited in Sect. 2.4 (a more detailed discussion can be found in Cleave 1971). This answers all the queries formulated in Sect. 3.1.



## 5.5 An Issue of Adequacy and Refinement

Bottazzini opines that

the language of infinites and infinitesimals that Cauchy used here seemed ever more *inadequate* to treat the sophisticated and complex questions then being posed by analysis... The problems posed by the study of nature, such as those Fourier had faced, now reappeared everywhere in the most delicate questions of “pure” analysis and necessarily led to the elaboration of techniques of inquiry considerably more *refined* than those that had served French mathematicians at the beginning of the century. Infinitesimals were to disappear from mathematical practice in the face of Weierstrass’  $\varepsilon$  and  $\delta$  notation, ... (Bottazzini 1986, p. 208). (emphasis added)

Are infinitesimal techniques “inadequate,” or less “refined” than epsilonic ones, as Bottazzini appears to suggest? We have argued that, *procedurally* speaking, the infinitesimal techniques of the classical masters ranging from Leibniz to Euler to Cauchy are more robust than is generally recognized by historians whose training is limited to a Weierstrassian framework.

## 6 Grabiner’s Reading

In this section we will analyze the comments by J. Grabiner related to Cauchy’s alleged confusion in the context of the sum theorem. Two subtle mathematical distinctions emerged in the middle of the nineteenth century: (1) continuity versus uniform continuity, and (2) convergence versus uniform convergence. We will show that Grabiner herself is confused about whether Cauchy’s purported error consisted in his failure to distinguish continuity from uniform continuity, or his failure to distinguish convergence from uniform convergence.

### 6.1 Whose Confusion?

Grabiner alleges that Cauchy’s (1821) theorem failed to distinguish pointwise from uniform convergence, and as a result he left his sum theorem open to some obvious counterexamples. Grabiner presents her case for Cauchy’s alleged “confusion” in the following terms:

... in treating series of functions, Cauchy did not distinguish between pointwise and uniform convergence. The verbal formulations like “for all” that are involved in choosing deltas did not distinguish between “for any epsilon and for all  $x$ ” and “for any  $x$ , given any epsilon” [19]. Nor was it at all clear in the 1820’s how much depended on this distinction, since proofs about continuity and convergence were in themselves so novel. We shall see the same *confusion* between uniform and pointwise convergence as we turn now to Cauchy’s theory of the derivative. (Grabiner 1983, p. 191) (emphasis added)

Here the bracketed number [19] refers to her footnote 19 that reads as follows:

19. I. Grattan-Guinness, *Development of the Foundations of Mathematical Analysis from Euler to Riemann*, M. I. T. Press, Cambridge and London, 1970, p. 123, puts

it well: “Uniform convergence was tucked away in the word ‘always,’ with no reference to the variable at all.” (Grabiner 1983, p. 194)

The views of Grattan-Guinness will be examined in Sect. 7. We now summarize Grabiner’s position as follows:

- G1. Cauchy did not distinguish between pointwise convergence and uniform convergence.
- G2. Cauchy did not distinguish between “for any epsilon and for all  $x$ ” and “for any  $x$ , given any epsilon.”
- G3. The failure to distinguish between the quantified clauses in item (2) is indicative of a “confusion between uniform and pointwise convergence” in Cauchy’s procedures.
- G4. Cauchy’s term *always* is somehow related to uniform convergence.
- G5. Cauchy’s use of the term *always* made no reference to the variable of the function.
- G6. No mention of infinitesimals whatsoever is made by Grabiner in connection with Cauchy’s treatment of the sum theorem.

## 6.2 Quantifier Order

What does Cauchy’s mysterious term *always* signify? A reader sufficiently curious about it to trace the sources will discover that Grattan-Guinness (as cited by Grabiner) is not referring to the 1821 *Cours d’Analyse*, but rather to the article Cauchy (1853), where Cauchy

stated a new theorem which embodied the *extension* of the necessary and sufficient conditions for convergence, and a revised theorem 4.4 to cover uniform convergence ... (Grattan-Guinness 1970, p. 122). (emphasis added)

We will get back to Cauchy’s *always* in due time. We can now analyze Grabiner’s claims G1, G2, and G3 as stated in Sect. 6.1. Her claim G1 is directly contradicted by Grattan-Guinness in the passage just cited. Here it turns out that in 1853 Cauchy *did modify* the original 1821 condition by *extending* it.

Grabiner’s claim G2 involves terms “for any”, “for all”, and “given any”, each of which expresses the universal quantifier. Thus claim G2 postulates a distinction between a pair of quantifier clauses, namely,  $(\forall x)(\forall \epsilon)$  and  $(\forall \epsilon)(\forall x)$ . However, these two clauses are logically equivalent. Indeed, the order of the quantifiers is significant only when one has alternating existential and universal quantifiers, contrary to her claim G2. Quantifier alternation is irrelevant when only universal quantifiers are involved.

Grabiner’s claim G3 postulates a “confusion” on Cauchy’s part but is based on incorrect premises, and is therefore similarly incorrect. Grabiner’s indictment of Cauchy in Grabiner (1983, p. 191) does not stand up to scrutiny, at least if we are to believe her source, namely Grattan-Guinness. Since her claims G5 and G6 are based on Grattan-Guinness, we will postpone their analysis until Sect. 7.

## 6.3 What Does It Take to Elucidate a Distinction?

Turning now to Grabiner’s (1981) book, we find the following:

The elucidation of the difference between convergence and uniform convergence by men like Stokes, Weierstrass, and Cauchy himself was still more than a decade away. (Grabiner 1981, p. 12)

Apparently Cauchy did provide an “elucidation of the difference between convergence and uniform convergence,” after all. The passage immediately following is particularly revealing:

The verbal formulations of limits and *continuity* used by Cauchy and Bolzano obscured the distinction between “for any epsilon, there is a delta that works for all  $x$ ” and “for any epsilon and for all  $x$ , there is a delta.” The only tools for handling such distinctions were words, and the usual formulation with the word “always” suggested “for all  $x$ ” as well as “for any epsilon.” (ibid., pp. 12–13) (emphasis added)

This passage immediately follows the passage on p. 12 on uniform *convergence*, but it appears to deal instead with uniform *continuity*. Indeed, the quantifier clause  $\forall\epsilon\exists\delta\forall x$  that Grabiner refers to customarily appears in the definition of uniform continuity of a function rather than uniform convergence of a series of functions.<sup>14</sup> Grabiner's position as expressed in the 1981 book involves the following claims (cf. those of her 1983 article enumerated in Sect. 6.1):

- G7. More than a decade later, Cauchy elucidated the difference between convergence and uniform convergence;
- G8. To handle the distinction between continuity and uniform continuity, Cauchy needed to handle the quantifier-order distinction between  $\forall\epsilon\exists\delta\forall x$ , on the one hand, and  $\forall\epsilon\forall x\exists\delta$ , on the other.
- G9. Cauchy only had the term *always* at his disposal.

We will analyze these in Sect. 6.5.

## 6.4 Cauchy's Definition of Continuity

Having mentioned both continuity and convergence in Sect. 6.3, we should point out that, while Cauchy's definition of continuity in 1821 was ambiguous to the extent that it is not clear whether he was defining ordinary or uniform continuity, what is clear is that he exploited the term *toujours* to refer to the possible inputs to the function. He formulated the definition as follows in 1853:

... une fonction  $u$  de la variable réelle  $x$  sera *continue*, entre deux limites données de  $x$ , si, cette fonction admettant pour chaque valeur intermédiaire de  $x$  une valeur unique et finie, un accroissement infiniment petit attribué à la variable produit *toujours*, entre les limites dont il s'agit, un accroissement infiniment petit de la fonction elle-même. (Cauchy 1853, pp. 455–456) (emphasis on *continue* in the original; emphasis on *toujours* ours)

The phrase “*toujours*, entre les limites dont il s'agit” suggests that the term *toujours* refers to the possible inputs to the function. While Cauchy apparently did not distinguish between distinct notions of continuity (such as modern notions of pointwise continuity as opposed to uniform continuity), he did use the term *toujours* in reference to the possible inputs.

<sup>14</sup> The symbol  $\delta$  is not used in reference to an integer tending to infinity.

One can surmise that when he uses the term *toujours* in the definition of convergence in the same article, he is also referring to the possible inputs.

Cauchy's definition of convergence in 1821 cited in Sect. 1.1 did not use the term *toujours*. Thus Cauchy was possibly referring to ordinary convergence. By contrast, in 1853 the definition did use the term *toujours* in defining convergence, which arguably corresponds to uniform convergence (see Sect. 5 for a more detailed discussion).

## 6.5 Uniform Convergence Today

To fix notation, recall that today a series  $\sum_{n=0}^{\infty} u_n(x)$  is said to converge uniformly to a function  $s(x)$  on a domain  $I$  if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall x \in I)(\forall m \in \mathbb{N}) \left[ (m > N) \Rightarrow \left| s(x) - \sum_{n=0}^m u_n \right| < \epsilon \right]$$

where all variables and functions are real. Grabiner's claim G8 to the effect that Cauchy failed to distinguish  $\forall \epsilon \exists \delta \forall x$  from  $\forall \epsilon \forall x \exists \delta$  (see Sect. 6.3 above) is puzzling coming as it does in the midst of a discussion of the dichotomy of convergence *vs* uniform convergence of series of functions, rather than continuity *vs* uniform continuity. Furthermore, we are not familiar with any text where Cauchy dealt explicitly with a distinction related to continuity *vs* uniform continuity. In fact, Grabiner herself writes:

Weierstrass and his school distinguished—as Cauchy had not—between pointwise and uniform continuity. (Grabiner 1981, p. 97)

We will therefore assume that what Grabiner meant on page 13 of her book was actually the following modification of her claim G8:

G8'. To handle the distinction between *convergence* and *uniform convergence*, Cauchy needed to handle the quantifier-order distinction between  $\forall \epsilon \exists N \forall x$ , on the one hand, and  $\forall \epsilon \forall x \exists N$ , on the other.

By page 140 Grabiner seems to have changed her mind once again about the status of convergence in Cauchy, as she writes:

Actually, [Cauchy's] proof implicitly assumed the function to be uniformly continuous, though he did not distinguish between continuity and uniform continuity, *just as he had not distinguished between convergence and uniform convergence.* (Grabiner 1981, p. 140) (emphasis added)

We will assume that she is referring to Cauchy (1821) here rather than Cauchy (1853). Grabiner does not explain Cauchy's use of the term *always*, but she assumes in claims G7–G9 that the use of this term somehow corresponds to requiring the stronger *uniform convergence*. She also assumes in G8' that successfully dealing with the dichotomy of convergence *vs* uniform convergence necessarily requires a quantifier-order distinction, indicative of an A-track *parti pris*.

## 6.6 A-Tracking Cauchy's Infinitesimals

Grabiner wrote in 1983:

Now we come to the last stage in our chronological list: definition. In 1823, Cauchy defined the derivative of  $f(x)$  as the limit, when it exists, of the quotient of differences  $\frac{f(x+h)-f(x)}{h}$  as  $h$  goes to zero [4, pp. 22–23]. (Grabiner 1883, p. 204)

Her reference “4, pp. 22–23” is to Cauchy (1823). Here is Cauchy's definition that Grabiner claims to report on:

Lorsque la fonction  $y = f(x)$  reste continue entre deux limites données de la variable  $x$ , et que l'on assigne à cette variable une valeur comprise entre les deux limites dont il s'agit, un accroissement *infinitement petit*, attribué à la variable, produit un accroissement *infinitement petit* de la fonction elle-même. Par conséquent, si l'on pose alors  $\Delta x = i$ , les deux termes du rapport aux différences

$$\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}$$

seront des quantités *infinitement petites*. Cauchy (1823) (emphasis added)

Notice that the infinitely small have been mentioned *three times*. Cauchy continues:

Mais, tandis que ces deux termes s'approcheront indéfiniment et simultanément de la limite zéro, le rapport lui-même pourra converger vers une autre limite, soit positive, soit négative. Cette limite, lorsqu'elle existe, a une valeur déterminée, pour chaque valeur particulière de  $x$  ... on donne à la nouvelle fonction le nom de fonction dérivée, ... (ibid)

Grabiner's paraphrase of Cauchy's definition is a blatant example of failing to see what is right before one's eyes; for Cauchy's *infinitely small*  $i$  have been systematically purged by Grabiner, only to be replaced by a post-Weierstrassian imposter “as  $h$  goes to zero.” The difference between the Cauchy and Weierstrassian notions of limit was recently analyzed in Nakane (2014).

## 7 Interpretation by Grattan-Guinness

As we saw in Sect. 6, Grabiner restricts her discussion of the sum theorem to mentioning that in 1853, Cauchy introduced uniform convergence (which is a sufficient condition for the validity of the sum theorem as stated today). Grattan-Guinness treats the sum theorem in more detail. He notes that

Cauchy lived through the appearance of modes of convergence and returned to his [sum theorem] in a short paper of 1853... [Cauchy] admitted that

$$\frac{\pi - x}{2} = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots$$

... was a counterexample with its discontinuity of magnitude  $\pi$  when  $x$  equals a multiple of  $\pi$ . (Grattan-Guinness 1970, p. 122).

### 7.1 Difficult to Interpret

Recall that Cauchy denotes by  $s_n(x)$  the  $n$ th partial sum of the series  $s(x) = u_0(x) + u_1(x) + \dots$ . He also denotes by  $r_n(x) = s(x) - s_n(x)$  the  $n$ th remainder of the series. Grattan-Guinness writes:

So a repair was needed, precisely at the point in his original proof of [the sum theorem] where he had said that  $[r_n(x_0 + \alpha) - r_n(x_0)]$  “becomes insensible at the same time” as  $r_n(x_0)$ . (ibid.)

Here an understanding of the term  $r_n(x_0 + \alpha)$  is crucial to interpreting Cauchy’s proof, and we will return to it in Sect. 7.2. Grattan-Guinness proceeds to make the following key comment:

This remark [of Cauchy’s] is *difficult to interpret* against [i.e., in the context of] the classification of modes of uniform convergence given here ... since  $\alpha$  is an infinitesimally small increment of  $x$ . (ibid.) (emphasis added)

The “modes of convergence” he is referring to here appear in his book Grattan-Guinness (1970, pp. 119–120) (rather than in Cauchy’s article). They are all formulated in terms of quantifier alternations, while the variable  $x$  ranges through the real domain of the function. Grattan-Guinness then paraphrases the revised version of Cauchy’s theorem from Cauchy (1853, p. 33), asserts that the proof is based on “the combined smallness of  $r_n(x_0)$  and  $r_n(x_0 + \alpha)$ ” (Grattan-Guinness 1970, p. 123) among other ingredients, and concludes that “uniform convergence itself was tucked away in the word ‘always’ with no reference to the variable at all.” (ibid.) We summarize his position as follows:

- GG1. Cauchy exploits infinitesimals in his attempted proof of the sum theorem.
- GG2. More specifically, Cauchy evaluates the remainder term  $r_n$  at an input  $x_0 + \alpha$  where  $\alpha$  is an infinitesimal.
- GG3. Cauchy’s proof is based on the smallness of  $r_n(x_0 + \alpha)$ .
- GG4. The procedures in Cauchy (1821) and Cauchy (1853) involve no quantifier alternations.<sup>15</sup>
- GG5. Cauchy’s arguments relying on infinitesimals are difficult to interpret in the context of the traditional characterisations of uniform convergence which rely on quantifier alternations.
- GG6. Cauchy’s term *always* alludes to uniform convergence.
- GG7. Cauchy’s term *always* makes no reference to the variable  $x$ .

## 7.2 Shortcomings

The analysis of Cauchy’s procedures by Grattan-Guinness contains three shortcomings. First, claim GG3 concerning the expression  $r_n(x_0 + \alpha)$  does not mention the crucial relation

$$\alpha = \frac{1}{n}$$

between  $n$  and the infinitesimal. This relation is crucial to Cauchy’s analysis of series (2) namely  $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots$  already discussed in Sect. 2.3. In fact in 1853 Cauchy did not use the symbol  $\alpha$  at all but worked directly with the infinitesimal  $\frac{1}{n}$ .

As far as claim GG6 is concerned, Grattan-Guinness makes no attempt to explain the relation between uniform convergence and the term *always* exploited in the formulation of Cauchy’s theorem.

<sup>15</sup> Grattan-Guinness acknowledges this point GG4 implicitly in his summary of Cauchy’s proof in (Grattan-Guinness 1970, p. 123). Meanwhile, an  $\varepsilon$  does occur in Cauchy on p. 32; see Sect. 2.2.

Finally, his claim GG7 is misleading. The term *always* is indeed meant to refer to the variable  $x$  (as it did in Cauchy's definition of continuity; see Sect. 6.4). Neither Grabiner nor Grattan-Guinness properly address this point. But in order to explain precisely in what way the term refers to  $x$ , one needs to abandon the limitations of the particular interpretive paradigm they are both laboring under, and avail oneself of a modern framework where Cauchy's procedures are *not* "difficult to interpret"; see Sect. 3.

## 8 Giusti on the Meaning of *toujours*

Giusti proposed two different interpretations of Cauchy's sum theorem, which we explore in Sects. 8.1 and 8.2.

### 8.1 Giusti's Epsilontic Track

In a 1984 study of Cauchy's sum theorem, Giusti wrote:

The comparison between the statement and the proof *clarifies immediately the meaning* of the word "toujours" in this context. The sentence: "the sum (3)<sup>16</sup> always becomes infinitesimal for infinite  $n$  and  $n' > n$ " turns out to mean nothing but that "it is possible to assign to  $n$  a value sufficiently large so that it follows that  $|s_{n'} - s_n| < \varepsilon$  for every  $n' > n$  and every  $x$ "; (Giusti 1984, p. 38) (emphasis added)

Giusti's comment indeed "clarifies immediately the meaning" of the sum theorem but only in the sense of providing a mathematically correct result loosely corresponding to Cauchy's theorem. Giusti continues:

in other words, it indicates the *uniform* convergence of the series. As support for this, Cauchy shows how in the case of Abel's series the condition is not satisfied, and the sum (3) does *not* become infinitely small for infinitely large values of  $n$  and  $n'$ . (ibid.) (emphasis in the original; translation ours)

To summarize, Giusti's formulation relies on thinly veiled quantifier alternations, requires interpreting what Cauchy calls an "infinite number  $n$ " as a variable subordinate to a universal quantifier, and sheds little light on Cauchy's term *always*. Cauchy's own *procedures* are therefore not fully clarified.

### 8.2 Giusti's Sequential Track

A few pages later, Giusti offers an alternative interpretation in the following terms:

[In Cauchy's sum theorem] the magnitude which plays a crucial role is the remainder term  $r_n(x)$ , depending on two variables. The convergence of that series implies that  $r_n(x)$  always tends to 0 for infinite values of  $x$  [sic], which means, from the standpoint of our interpretation, that for each sequence  $x_n$  the variable  $r_n(x_n)$  is infinitesimal. Once again, this [procedure] is equivalent to [proving] uniform convergence. ... [In Cauchy's analysis of the series of Abel], as we have already explained, Cauchy proves that if the sum  $u_n + u_{n+1} + \dots + u_{n'-1}$  becomes always infinitely small for some infinitely big values of  $n$  and  $n'$ , then the series  $u_1 + u_2 +$

<sup>16</sup> This is a reference to Cauchy's sum (3) namely  $u_n + u_{n+1} \dots u_{n'-1}$  discussed in Sect. 2.

$u_3 + \dots$  converges and its sum is a continuous function. (Giusti 1984, p. 50)  
[translation ours]

Thus, Giusti provides a sequential interpretation of Cauchy's argument, still in an Archimedean context (see Sect. 4.2, item [C]). However, as Laugwitz pointed out,

Giusti gives a correct translation of the example into the language of sequences, ... [Giusti 1984, p. 50]. But he fails to translate the general theorem and its proof. Actually both  $x$  (or  $x + \alpha$ ) and  $n$  will have to be replaced by sequences which becomes troublesome as soon as they are not connected as in the example where  $x = 1/n$ . Moreover, the theorem shows the power of Cauchy's "direct consideration of infinitesimals". (Laugwitz 1987, p. 266)

A paraphrase of Cauchy's proof along the lines of Giusti's sequential reading may be harder to implement than the analysis of Abel's example, and may depart significantly from Cauchy's own proof as summarized in Grattan-Guinness (1970, formula (6.30), p. 123), which we present in Sect. 8.4. An additional difficulty (with Giusti's [C]-track reading; see Sect. 4.2) not mentioned by Laugwitz is detailed in Sect. 8.3.

### 8.3 Order $n' > n$

If  $n$  and  $n'$  are (infinite) *numbers* then one naturally expects as Cauchy does that they are comparable; as Cauchy writes, we have  $n' > n$ . However if they are *sequences*  $n = (n_k)$  and  $n' = (n'_k)$  tending to infinity, then there is no reason to assume that they are comparable. For example, if  $n_k = k$  whereas  $n'_k = k + (-1)^k$  then who is to say which sequence is bigger,  $n'$  or  $n$ ? Meanwhile in any ordered Bernoullian continuum (see Sect. 3.4) comparability of numbers is automatic. This is not to say that Cauchy *constructed* an ordered Bernoullian continuum, but rather that his *procedures* presume that the infinite numbers are ordered, and this idea finds more faithful proxies in modern infinitesimal theories than in Giusti's [C]-track reading.

### 8.4 Cauchy's Track

To show that  $s(x_0 + \alpha) - s(x_0)$  is infinitesimal, Cauchy writes  $r_n = s - s_n$ , so that

$$s(x_0 + \alpha) - s(x_0) = [s'_n(x_0 + \alpha) - s_n(x_0)] + [r'_n(x_0 + \alpha) - r_n(x_0)]$$

(see Sect. 5.4). The terms  $r'_n(x_0 + \alpha)$  and  $r_n(x_0)$  are infinitesimal for *each* infinite  $n$  by Cauchy's *always* hypothesis. To account for the smallness of the first summand, Grattan-Guinness appeals to the "continuity of  $s_n(x)$ ." However, this is immediate only for finite  $n$ , whereas Cauchy's proof of continuity requires  $\Delta s_n = s'_n(x_0 + \alpha) - s_n(x_0)$  to be infinitesimal for *some* infinite  $n$ .

Can one guarantee that if  $\Delta s_n$  is infinitesimal for each finite  $n$ , this property will persist for some infinite  $n$ ? In the context of hyperreal proxies for Cauchy's procedures, the answer is affirmative (every internal sequence which is infinitesimal for all finite  $n$  will remain infinitesimal for *some* infinite  $n$ ). What is perhaps even more remarkable than the validity of this type of *permanence principle* is the fact that a related principle can indeed be found in Cauchy (1829), as reported in Robinson (1966, p. 274). The relevant technical result in a hyperreal framework is Robinson (1966, Theorem 3.3.20, p. 65). McKinzie and Tuckey (1997, p. 49) follow Laugwitz (1990) in interpreting Euler in terms of *hidden*



lemmas using a related result called the *overspill theorem*; see also the *sequential theorem* in McKinzie and Tuckey (McKinzie and Tuckey 2001, p. 361).

## 9 Conclusion

Interpretation of texts written in the nineteenth century, and the meaning we give to technical terms, procedures, theories, and the like are closely related to what we already know as well as our expectations and assumptions. This paper provides evidence that a change in the cultural-technical framework of a historian provides new explanations, which are arguably more natural, and new insights into Cauchy's work.

**Acknowledgements** V. Kanovei was supported in part by the RFBR Grant Number 17-01-00705. M. Katz was partially funded by the Israel Science Foundation Grant Number 1517/12. We are grateful to Dave L. Renfro for helpful suggestions.

## References

- Alexander, A. (2014). *Infinitesimal: How a dangerous mathematical theory shaped the modern world*. Straus and Giroux: Farrar.
- Arsac, G. (2013). *Cauchy, Abel, Seidel, Stokes et la convergence uniforme. De la difficulté historique du raisonnement sur les limites*. Paris: Hermann.
- Bair, J., Błaszczyk, P., Ely, R., Henry, V., Kanovei, V., Katz, K., Katz, M., Kutateladze, S., McGaffey, T., Schaps, D., Sherry, D., & Shnider, S. (2013). Is mathematical history written by the victors? *Notices of the American Mathematical Society*, 60(7), 886–904. See <http://www.ams.org/notices/201307/rnoti-p886.pdf> and [arXiv:1306.5973](https://arxiv.org/abs/1306.5973)
- Bair, J., Błaszczyk, P., Ely, R., Henry, V., Kanovei, V., Katz, K., et al. (2017). Interpreting the infinitesimal mathematics of Leibniz and Euler. *Journal for General Philosophy of Science*, 48(2), 195–238. See doi:10.1007/s10838-016-9334-z and [arXiv:1605.00455](https://arxiv.org/abs/1605.00455)
- Bascelli, T., Bottazzi, E., Herzberg, F., Kanovei, V., Katz, K., Katz, M., Nowik, T., Sherry, D., & Shnider, S. (2014). Fermat, Leibniz, Euler, and the gang: The true history of the concepts of limit and shadow. *Notices of the American Mathematical Society*, 61(8), 848–864. See <http://www.ams.org/notices/201408/rnoti-p848.pdf> and [arXiv:1407.0233](https://arxiv.org/abs/1407.0233)
- Bascelli, T., Błaszczyk, P., Kanovei, V., Katz, K., Katz, M., Schaps, D., et al. (2016). Leibniz versus Ishiguro: Closing a quarter-century of syncategoremania. HOPOS. *The Journal of the International Society for the History of Philosophy of Science*, 6(1), 117–147. doi:10.1086/685645. [arXiv:1603.07209](https://arxiv.org/abs/1603.07209).
- Bascelli, T., Błaszczyk, P., Kanovei, V., Katz, K., Katz, M., Kutateladze, S., Nowik, T., Schaps, D., & Sherry, D. (2017). Gregory's sixth operation. *Foundations of Science*. See doi:10.1007/s10699-016-9512-9 and [arXiv:1612.05944](https://arxiv.org/abs/1612.05944)
- Bell, J. (2006). *The continuous and the infinitesimal in mathematics and philosophy*. Milan: Polymetrica.
- Benacerraf, P. (1965). What numbers could not be. *Philosophical Review*, 74, 47–73.
- Bishop, E. (1977). Book review: Elementary calculus. *Bulletin of the American Mathematical Society* 83, 205–208 (review of the first edition of Keisler 1986)
- Borovik, A., & Katz, M. (2012). Who gave you the Cauchy–Weierstrass tale? The dual history of rigorous calculus. *Foundations of Science*, 17(3), 245–276. doi:10.1007/s10699-011-9235-x. [arXiv:1108.2885](https://arxiv.org/abs/1108.2885).
- Bos, H. (1974). Differentials, higher-order differentials and the derivative in the Leibnizian calculus. *Archive for History of Exact Sciences*, 14, 1–90.
- Bosmans, H. (1927). André Tacquet (S. J.) et son traité d' 'Arithmétique théorique et pratique'. *Isis*, 9(1), 66–82.
- Bottazzini, U. (1986). *The higher calculus: A history of real and complex analysis from Euler to Weierstrass*. Translated from the Italian by Warren Van Egmond: Springer-Verlag, New York.
- Boyer, C. (1949). *The concepts of the calculus*. New York, NY: Hafner Publishing.
- Bradley, R., & Sandifer, C. (2009). *Cauchy's Cours d'analyse. An annotated translation. Sources and studies in the history of mathematics and physical sciences*. New York, NY: Springer.

- Błaszczyk, P., Borovik, A., Kanovei, V., Katz, M., Kudryk, T., Kutateladze, S., et al. (2016). A non-standard analysis of a cultural icon: The case of Paul Halmos. *Logica Universalis*, 10(4), 393–405. doi:[10.1007/s11787-016-0153-0](https://doi.org/10.1007/s11787-016-0153-0). arXiv:[1607.00149](https://arxiv.org/abs/1607.00149).
- Błaszczyk, P., Kanovei, V., Katz, K., Katz, M., Kutateladze, S., & Sherry, D. (2017a). Toward a history of mathematics focused on procedures. *Foundations of Science*. See doi:[10.1007/s10699-016-9498-3](https://doi.org/10.1007/s10699-016-9498-3) and arXiv:[1609.04531](https://arxiv.org/abs/1609.04531)
- Błaszczyk, P., Kanovei, V., Katz, M., & Sherry, D. (2017b). Controversies in the foundations of analysis: Comments on Schubring's *Conflicts*. *Foundations of Science*. See doi:[10.1007/s10699-015-9473-4](https://doi.org/10.1007/s10699-015-9473-4) and arXiv:[1601.00059](https://arxiv.org/abs/1601.00059)
- Cauchy, A. L. (1821). *Cours d'Analyse de L'Ecole Royale Polytechnique Première Partie. Analyse algébrique*. Paris: Imprimerie Royale.
- Cauchy, A. L. (1823). *Résumé des Leçons données à l'Ecole Royale Polytechnique sur le Calcul Infinitésimal*. Paris: Imprimerie Royale.
- Cauchy, A. L. (1829). Leçons sur le calcul différentiel. *Oeuvres Complètes, Series, 2*, 4.
- Cauchy, A. L. (1853). Note sur les séries convergentes dont les divers termes sont des fonctions continues d'une variable réelle ou imaginaire, entre des limites données. *Comptes Rendus de l'Académie Royale des Sciences*, 36, 454–459. Reprinted as (Cauchy 1900).
- Cauchy, A. L. (1900). Note sur les séries convergentes dont les divers termes sont des fonctions continues d'une variable réelle ou imaginaire, entre des limites données. *Oeuvres complètes, Series 1, Vol. 12*, pp. 30–36. Paris, Gauthier-Villars.
- Child, J., ed. (1920). *The early mathematical manuscripts of Leibniz*. Translated from the Latin texts published by Carl Immanuel Gerhardt with critical and historical notes by J. M. Child. The Open Court Publishing, Chicago-London. Reprinted by Dover in 2005.
- Cleave, J. (1971). Cauchy, convergence and continuity. *The British Journal for the Philosophy of Science*, 22, 27–37.
- Connes, A., Lichnerowicz, A., & Schützenberger, M. (2001). *Triangle of thoughts*. Translated from the 2000 French original by Jennifer Gage. American Mathematical Society, Providence, RI.
- Cutland, N., Kessler, C., Kopp, E., & Ross, D. (1988). On Cauchy's notion of infinitesimal. *The British Journal for the Philosophy of Science*, 39(3), 375–378.
- Dauben, J. (1988). Abraham Robinson and nonstandard analysis: history, philosophy, and foundations of mathematics. *History and philosophy of modern mathematics (Minneapolis, MN, 1985)*, 177–200, Minnesota Stud. Philos. Sci., XI, Univ. Minnesota Press, Minneapolis, MN.
- Davis, M. (1977). *Applied nonstandard analysis. Pure and applied mathematics*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney. Reprinted by Dover, NY, 2005. See <http://store.doverpublications.com/0486442292.html>
- Earman, J. (1975). Infinities, infinitesimals, and indivisibles: The Leibnizian labyrinth. *Studia Leibnitiana*, 7(2), 236–251.
- Easwaran, K. (2014). Regularity and hyperreal credences. *Philosophical Review*, 123(1), 1–41.
- Ebbinghaus, H.-D., Hermes, H., Hirzebruch, F., Koecher, M., Mainzer, K., Neukirch, J., Prestel, A., & Remmert, R. (1990). *Numbers*. With an introduction by K. Lamotke. Translated from the second German edition by H. L. S. Orde. Translation edited and with a preface by J. H. Ewing. Graduate Texts in Mathematics, 123. Readings in Mathematics. Springer-Verlag, New York.
- Edwards, H. (2007). Euler's definition of the derivative. *Bulletin of the American Mathematical Society (N.S.)*, 44(4), 575–580.
- Ferraro, G. (2004). Differentials and differential coefficients in the Eulerian foundations of the calculus. *Historia Mathematica*, 31(1), 34–61.
- Festa, E. (1990). La querelle de l'atomisme: Galilée, Cavalieri et les Jésuites. *La Recherche* (sept. 1990), 1038–1047.
- Festa, E. (1992). Quelques aspects de la controverse sur les indivisibles. *Geometry and Atomism in the Galilean School*, 193–207, Bibl. Nuncius Studi Testi, X, Olschki, Florence.
- Fisher, G. (1978). Cauchy and the infinitely small. *Historia Mathematica*, 5(3), 313–331.
- Fletcher, P., Hrbacek, K., Kanovei, V., Katz, M., Lobry, C., & Sanders, S. (2017). Approaches to analysis with infinitesimals following Robinson, Nelson, and others. *Real Analysis Exchange*, 42(2). See arXiv:[1703.00425](https://arxiv.org/abs/1703.00425)
- Fraenkel, A. (1928). *Einleitung in die Mengenlehre*. Dover Publications, New York, NY, 1946 (originally published by Springer, Berlin, 1928).
- Fraser, C. (2008). *Cauchy. The new dictionary of scientific biography* (Vol. 2). New York, NY: Scribners and Sons.

- Fraser, C. (2015). Nonstandard analysis, infinitesimals, and the history of calculus. In D. Row & W. Horng (Eds.), *A delicate balance: Global perspectives on innovation and tradition in the history of mathematics* (pp. 25–49). Heidelberg: Springer: Birkhäuser.
- Gerhardt, C. (Ed.). (1846). *Historia et Origo calculi differentialis a G. Hannover: G. Leibnitio conscripta*.
- Gerhardt, C. (Ed.). (1850–1863). *Leibnizens mathematische Schriften*. Berlin and Halle, Eidmann.
- Gilain, C. (1989). Cauchy et le cours d'analyse de l'École polytechnique. With an editorial preface by Emmanuel Grison. *Bulletin de la Sabix. Société des amis de la Bibliothèque et de l'Histoire de l'École Polytechnique*, no. 5.
- Giusti, E. (1984). Cauchy's 'errors' and the foundations of analysis. (*Italian*) *Bollettino di Storia delle Scienze Matematiche*, 4(2), 24–54.
- Grabiner, J. (1883). The Changing concept of change: The derivative from Fermat to Weierstrass. *Mathematics Magazine*, 56(4), 195–206.
- Grabiner, J. (1981). *The origins of Cauchy's rigorous calculus*. Cambridge, MA: MIT Press.
- Grabiner, J. (1983). Who gave you the epsilon? Cauchy and the origins of rigorous calculus. *The American Mathematical Monthly*, 90(3), 185–194.
- Grabiner, J. (2006). Review of Schubring (2005). *SIAM Review*, 48(2), 413–416 (jun., 2006).
- Grattan-Guinness, I. (1970). *The development of the foundations of mathematical analysis from Euler to Riemann*. Cambridge, MA: The MIT Press.
- Gray, J. (2015). *The real and the complex: A history of analysis in the nineteenth century.*, Springer Undergraduate Mathematics Series Cham: Springer.
- Gutman, A., Katz, M., Kudryk, T., & Kutateladze, S. (2017). The mathematical intelligencer flunks the olympics. *Foundations of Science*. See doi:[10.1007/s10699-016-9485-8](https://doi.org/10.1007/s10699-016-9485-8) and [arXiv:1606.00160](https://arxiv.org/abs/1606.00160)
- Halmos, P. (1985). *I want to be a mathematician. An automathography*. New York, NY: Springer.
- Hellyer, M. (1996). 'Because the authority of my superiors commands': Censorship, physics and the German Jesuits. *Early Science and Medicine*, 3, 319–354.
- Hewitt, E. (1948). Rings of real-valued continuous functions. I. *Transactions of the American Mathematical Society*, 64, 45–99.
- Ishiguro, H. (1990). *Leibniz's philosophy of logic and language* (2nd ed.). Cambridge: Cambridge University Press.
- Kanovei, V., Katz, M., & Mormann, T. (2013). Tools, objects, and chimeras: Connes on the role of hyperreals in mathematics. *Foundations of Science*, 18(2), 259–296. doi:[10.1007/s10699-012-9316-5](https://doi.org/10.1007/s10699-012-9316-5). [arXiv:1211.0244](https://arxiv.org/abs/1211.0244).
- Kanovei, V., Katz, K., Katz, M., & Sherry, D. (2015). Euler's lute and Edwards' oud. *The Mathematical Intelligencer*, 37(4), 48–51. doi:[10.1007/s00283-015-9565-6](https://doi.org/10.1007/s00283-015-9565-6). [arXiv:1506.02586](https://arxiv.org/abs/1506.02586).
- Katz, V. (2014). Review of "Bair et al., Is mathematical history written by the victors? *Notices of the American Mathematical Society* (2013) 607, 886–904." See <http://www.ams.org/mathscinet-getitem?mr=3086638>
- Katz, K., & Katz, M. (2011a). Meaning in classical mathematics: Is it at odds with intuitionism? *Intellectica*, 56(2), 223–302. [arXiv:1110.5456](https://arxiv.org/abs/1110.5456).
- Katz, K., & Katz, M. (2011b). Cauchy's continuum. *Perspectives on Science*, 19(4), 426–452. doi:[10.1162/POSC\\_a\\_00047](https://doi.org/10.1162/POSC_a_00047). [arXiv:1108.4201](https://arxiv.org/abs/1108.4201).
- Katz, M., & Leichtnam, E. (2013). Commuting and noncommuting infinitesimals. *American Mathematical Monthly*, 120(7), 631–641. doi:[10.4169/amer.math.monthly.120.07.631](https://doi.org/10.4169/amer.math.monthly.120.07.631). [arXiv:1304.0583](https://arxiv.org/abs/1304.0583).
- Katz, M., & Polev, L. (2017). From Pythagoreans and Weierstrassians to true infinitesimal calculus. *Journal of Humanistic Mathematics*, 7(1), 87–104. doi:[10.5642/jhummath.201701.07](https://doi.org/10.5642/jhummath.201701.07). [arXiv:1701.05187](https://arxiv.org/abs/1701.05187).
- Katz, M., Schaps, D., & Shnider, S. (2013). Almost equal: The method of adequality from Diophantus to Fermat and beyond. *Perspectives on Science*, 21(3), 283–324. doi:[10.1162/POSC\\_a\\_00101](https://doi.org/10.1162/POSC_a_00101). [arXiv:1210.7750](https://arxiv.org/abs/1210.7750).
- Katz, M., & Sherry, D. (2013). Leibniz's infinitesimals: Their fictionality, their modern implementations, and their foes from Berkeley to Russell and beyond. *Erkenntnis*, 78(3), 571–625. doi:[10.1007/s10670-012-9370-y](https://doi.org/10.1007/s10670-012-9370-y). [arXiv:1205.0174](https://arxiv.org/abs/1205.0174).
- Keisler, H. J. (1986). *Elementary calculus: An infinitesimal approach*, 2nd edn. Boston, MA: Prindle, Weber and Schmidt. See <http://www.math.wisc.edu/~keisler/calc.html>
- Klein, F. (1908). *Elementary mathematics from an advanced standpoint. Vol. I. Arithmetic, algebra, analysis*. Translation by E. R. Hedrick and C. A. Noble [Macmillan, New York, 1932] from the third German edition [Springer, Berlin, 1924]. Originally published as Elementarmathematik vom höheren Standpunkte aus (Leipzig, 1908).
- Kock, A. (2006). *Synthetic differential geometry* (2nd ed.), London Mathematical Society Lecture Note Series, 333 Cambridge: Cambridge University Press.
- Kuhn, T. (1996). *The structure of scientific revolutions* (3rd ed.). Chicago, IL: University of Chicago.

- Laugwitz, D. (1987). Infinitely small quantities in Cauchy's textbooks. *Historia Mathematica*, 14, 258–274.
- Laugwitz, D. (1989). Definite values of infinite sums: Aspects of the foundations of infinitesimal analysis around 1820. *Archive for the History of Exact Sciences*, 39, 195–245.
- Laugwitz, D. (1990). Das mathematisch Unendliche bei Cauchy und bei Euler. ed. Gert König, *Konzepte des mathematisch Unendlichen im 19. Jahrhundert* (Göttingen, Vandenhoeck u. Ruprecht, 1990), 9–33.
- Leibniz, G. (1695). Responsio ad nonnullas difficultates a Dn. Bernardo Niewentii circa methodum differentialem seu infinitesimalem motas. *Act. Erudit. Lips.* In Gerhardt (1850–1863), vol. V, 320–328. A French translation is in (Leibniz 1989, pp. 316–334).
- Leibniz, G. (1701). Cum Prodiisset... mss "Cum prodiisset atque increbuisset Analysis mea infinitesimalis ..." in Gerhardt (1846, pp. 39–50). Online at [http://books.google.co.il/books?id=UOM3AAAAMAAJ&source=gb\\_s\\_navlinks\\_s](http://books.google.co.il/books?id=UOM3AAAAMAAJ&source=gb_s_navlinks_s)
- Leibniz, G. (1989). *La naissance du calcul différentiel*. 26 articles des *Acta Eruditorum*. Translated from the Latin and with an introduction and notes by Marc Parmentier. With a preface by Michel Serres. Mathesis. Librairie Philosophique J. Vrin, Paris.
- Lindström, T. (1988). An invitation to nonstandard analysis. *Nonstandard analysis and its applications* (Hull, 1986), 1–105, London Math. Soc. Stud. Texts, 10, Cambridge University Press, Cambridge.
- Mancosu, P. (1996). *Philosophy of mathematics and mathematical practice in the seventeenth century*. New York, NY: The Clarendon Press, Oxford University Press.
- McKinzie, M., & Tuckey, C. (1997). Hidden lemmas in Euler's summation of the reciprocals of the squares. *Archive for History of Exact Sciences*, 51, 29–57.
- McKinzie, M., & Tuckey, C. (2001). Higher trigonometry, hyperreal numbers, and Euler's analysis of infinities. *Mathematics Magazine*, 74(5), 339–368.
- Nakane, M. (2014). Did Weierstrass's differential calculus have a limit-avoiding character? His definition of a limit in  $\epsilon$ - $\delta$  style. *BSHM Bulletin: Journal of the British Society for the History of Mathematics*, 29(1), 51–59.
- Nelson, E. (1977). Internal set theory: A new approach to nonstandard analysis. *Bulletin of the American Mathematical Society*, 83(6), 1165–1198.
- Planck, M. (1950). *Scientific autobiography and other papers* (F. Gaynor, Trans.). London.
- Prestel, A. (1990). Chapter 12. Nonstandard analysis. Pp. 305–327 in Ebbinghaus et al. (1990).
- Quine, W. (1968). Ontological relativity. *The Journal of Philosophy*, 65(7), 185–212.
- Ramis, J.-P. (2012). Poincaré et les développements asymptotiques (première partie). *Gazette Mathématique No.*, 133, 33–72.
- Redondi, P. (1987). *Galileo: heretic*. Translated from the Italian by Raymond Rosenthal: Princeton University Press, Princeton, NJ.
- Robinson, A. (1961). Non-standard analysis. *Nederl. Akad. Wetensch. Proc. Ser. A*, 64 = *Indag. Math.*, 23 (1961), 432–440. Reprinted in Selected Papers (Robinson 1979), pp. 3–11.
- Robinson, A. (1966). *Non-standard analysis*. Amsterdam: North-Holland Publishing.
- Robinson, A. (1979). *Selected papers of Abraham Robinson. Vol. II. Nonstandard analysis and philosophy*. Edited and with introductions by W. A. J. Luxemburg and S. Körner. Yale University Press, New Haven, Conn.
- Sanders, S. (2017a). Reverse Formalism 16. *Synthese*, Special issue for SOTFOMIII. See doi:[10.1007/s11229-017-1322-2](https://doi.org/10.1007/s11229-017-1322-2) and [arXiv:1701.05066](https://arxiv.org/abs/1701.05066)
- Sanders, S. (2017b). To be or not to be constructive. *Indagationes Mathematicae* and the Brouwer volume *L.E.J. Brouwer, fifty years later*, 2017–2018, p. 68. See [arXiv:1704.00462](https://arxiv.org/abs/1704.00462)
- Schmieden, C., & Laugwitz, D. (1958). Eine Erweiterung der infinitesimalrechnung. *Mathematische Zeitschrift*, 69, 1–39.
- Schubring, G. (2005). *Conflicts between generalization, rigor, and intuition. Number concepts underlying the development of analysis in 17–19th century France and Germany*. Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag, New York.
- Sergeyev, Y. (2015). The olympic medals ranks, lexicographic ordering, and numerical infinities. *The Mathematical Intelligencer*, 37(2), 4–8.
- Sherry, D., & Katz, M. (2014). Infinitesimals, imaginaries, ideals, and fictions. *Studia Leibnitiana*, 44(2) (2012), 166–192. See <http://www.jstor.org/stable/43695539> and [arXiv:1304.2137](https://arxiv.org/abs/1304.2137) (Article was published in 2014 even though the journal issue lists the year as 2012)
- Spalt, D. (2002). Cauchys kontinuum. Eine historiografische Annäherung via Cauchys summensatz. *Archive for History of Exact Sciences*, 56(4), 285–338.
- Sørensen, H. (2005). Exceptions and counterexamples: Understanding Abel's comment on Cauchy's theorem. *Historia Mathematica*, 32(4), 453–480.
- Tao, T. (2014). *Hilbert's fifth problem and related topics*. Graduate Studies in Mathematics Providence, RI: American Mathematical Society.

- Tao, T. & Vu, V. (2016). Sum-avoiding sets in groups. *Discrete Analysis* 2016:15, 31 pp, see doi:[10.19086/da.887](https://doi.org/10.19086/da.887) and [arXiv:1603.03068](https://arxiv.org/abs/1603.03068)
- Tarski, A. (1930). Une contribution à la théorie de la mesure. *Fundamenta Mathematicae*, 15, 42–50.
- Unguru, S. (1976). Fermat revived, explained, and regained. *Francia*, 4, 774–789.
- Viertel, K. (2014). *Geschichte der gleichmäßigen Konvergenz. Ursprünge und Entwicklungen des Begriffs in der Analysis des 19. Jahrhunderts*. UK: Springer Spektrum.
- Wartofsky, M. (1976). The relation between philosophy of science and history of science. In R. S. Cohen, P. K. Feyerabend, & M. W. Wartofsky (Eds.), *Essays in memory of Imre Lakatos* (Vol. XXXIX, pp. 717–737). Boston Studies in the Philosophy of Science Dordrecht: D. Reidel Publishing.

**Tiziana Bascelli** graduated in Mathematics (1993) and philosophy (2006), and obtained a Ph.D. in theoretical philosophy (2010) from University of Padua, Italy. She co-authored *Campano da Novara's Equatorium Planetarium. Transcription, Italian translation and commentary* (Padua 2007) and *Galileo's 'Sidereus Nuncius' or 'A Sidereal Message'*, translated from the Latin by William R. Shea, introduction and notes by William R. Shea and Tiziana Bascelli. Sagamore Beach, MA (USA): Science History Publications, 2009. She is an independent researcher in integrated history and philosophy of science, in the field of early modern mechanics and mathematics. Her research interests are in the development of infinitesimal objects and procedures in Seventeenth-century mathematics.

**Piotr Błaszczyk** is Professor at the Institute of Mathematics, Pedagogical University (Cracow, Poland). He obtained degrees in mathematics (1986) and philosophy (1994) from Jagiellonian University (Cracow, Poland), and a Ph.D. in ontology (2002) from Jagiellonian University. He authored *Philosophical Analysis of Richard Dedekind's memoir Stetigkeit und irrationale Zahlen* (2008, Habilitationsschrift). He co-authored *Euclid, Elements, Books V–VI. Translation and commentary*, 2013; and *Descartes, Geometry. Translation and commentary* (Cracow 2015). His research interest is in the idea of continuum and continuity from Euclid to modern times.

**Alexandre Borovik** is Professor of Pure Mathematics at the University of Manchester, United Kingdom. His principal research interests are to algebra, model theory, and combinatorics. He co-authored monographs *Groups of Finite Morley Rank*, 1994, *Coxeter Matroids*, 2003, and *Simple Groups of Finite Morley Rank*, 2008. His latest work, on probabilistic recognition of so-called black box groups, focuses on the boundary between finite and infinite in mathematics.

**Vladimir Kanovei** graduated in 1973 from Moscow State University, and obtained a Ph.D. in physics and mathematics from Moscow State University in 1976. In 1986, he became Doctor of Science in physics and mathematics at Moscow Steklov Mathematical Institute (MIAN). He is currently Principal Researcher at the Institute for Information Transmission Problems (IITP) and Professor at Moscow State University of Railway Engineering (MIIT), Moscow, Russia. Among his publications is the book *Borel equivalence relations. Structure and classification*. University Lecture Series 44. American Mathematical Society, Providence, RI, 2008.

**Karin U. Katz** (B.A. Bryn Mawr College, 1982); Ph.D. Indiana University, 1991) teaches Mathematics at Bar Ilan University, Ramat Gan, Israel. Among her publications is the joint article “Proofs and retributions, or: why Sarah can't take limits” published in *Foundations of Science*.

**Mikhail G. Katz** (BA Harvard 1980; PhD Columbia 1984) is Professor of Mathematics at Bar Ilan University, Ramat Gan, Israel. He is interested in Riemannian geometry, infinitesimals, debunking mathematical history written by the victors, as well as in true infinitesimal differential geometry; see *Journal of Logic and Analysis* 7:5 (2015), 1–44 at <http://www.logicandanalysis.com/index.php/jla/article/view/237>

**Semen S. Kutateladze** was born in 1945 in Leningrad (now St. Petersburg). He is a senior principal officer of the Sobolev Institute of Mathematics in Novosibirsk and professor at Novosibirsk State University. He authored more than 20 books and 200 papers in functional analysis, convex geometry, optimization, and nonstandard and Boolean valued analysis. He is a member of the editorial boards of *Siberian Mathematical Journal*, *Journal of Applied and Industrial Mathematics*, *Positivity*, *Mathematical Notes*, etc.

**Thomas McGaffey** teaches Mathematics at San Jacinto College in Houston, US.

**David M. Schaps** is Professor Emeritus of Classical Studies at Bar Ilan University, Israel. His books include *Economic Rights of Women in Ancient Greece*, *The Invention of Coinage and the Monetization of Ancient Greece*, and *Handbook for Classical Research*; among his articles are “The Woman Least Mentioned: Etiquette and Women’s Names”, “What was Free about a Free Athenian Woman?”, and “Zeus the Wife-Beater”.

**David Sherry** is Professor of Philosophy at Northern Arizona University, in the tall, cool pines of the Colorado Plateau. He has research interests in philosophy of mathematics, especially applied mathematics and non-standard analysis. Recent publications include “Fields and the Intelligibility of Contact Action,” *Philosophy* 90 (2015), 457–478. “Leibniz’s Infinitesimals: Their Fictionality, their Modern Implementations, and their Foes from Berkeley to Russell and Beyond,” with Mikhail Katz, *Erkenntnis* 78 (2013), 571–625. “Infinitesimals, Imaginaries, Ideals, and Fictions,” with Mikhail Katz, *Studia Leibnitiana* 44 (2012), 166–192. “Thermoscopes, Thermometers, and the Foundations of Measurement,” *Studies in History and Philosophy of Science* 24 (2011), 509–524. “Reason, Habit, and Applied Mathematics,” *Hume Studies* 35 (2009), 57–85.