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# On Hausdorff ordered structures

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**Abstract.** We suggest a classification of problems on the existence of structures such as limits, gaps, towers and scales in Hausdorff partially ordered sets of infinite sequences, including sequences with real terms and various partial order relations.

**Keywords:** Hausdorff ordered structure, Hausdorff gap, tower, limit.

## Introduction

Suppose that $\langle P; \leqslant\rangle$ is a partially ordered set whose domain $P$ consists either of real functions defined on the real half-line $[0, +\infty)$ or of infinite sequences of real numbers, where $f \leqslant g$ means that the function (or sequence) $g$ grows faster than $f$. Hausdorff regarded the use of any such partially ordered structure $\langle P; \leqslant\rangle$ as a *method of classifying* functions or sequences (*Graduierungsmethod*) in accordance with their rate of growth. Some examples of such structures (here called *Hausdorff structures*) are given in § 1.

The history of the study of such ordered structures goes back to the papers of du Bois-Reymond (see [1] and elsewhere) and was extended by Hadamard [2], Borel [3], Hardy [4] and others. Hausdorff proposed another approach to these structures in [5], [6] based on the study of certain *totally* and even *well* ordered substructures in a given *partial* order. These substructures are (usually transfinite) increasing or decreasing sequences, or can be reduced to such sequences. In particular, he considered *scales*, *towers*, *limits* and *gaps*, which are widely studied in modern set theory and set-theoretic topology (see, for example, [7]–[10]). We define these substructures in § 2 below.

It turns out that, except for a few simple non-existence theorems based on the diagonal construction and a rather complicated theorem of the existence of an $(\omega_1, \omega_1^*)$-gap, questions of the existence of these substructures in Hausdorff ordered structures lead to undecidable problems. For instance, it is impossible to either prove or refute the assertion of the existence of $(\omega_1, \omega^*)$-gaps, $\omega_1$-limits and others on the basis of the axioms of modern axiomatic set theory **ZFC**. We discuss this in more detail in § 6.

Yet not all of these problems are independent of each other. Indeed, Rothberger [11], [12] proved, for the order $\leqslant^*$ of eventual domination (see § 1), that the existence of an $\omega_1$-tower in $\mathbb{N}^{\mathbb{N}}$ is equivalent to that of an $(\omega_1, \omega^*)$-gap in $2^{\mathbb{N}}$, and that either of these two existence assertions implies the existence of $\omega_1$-limits

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in $2^{\mathbb{N}}$. (Each of these three existence assertions is undecidable in **ZFC**.) Since then, other results on the mutual reducibility and equivalence of such problems have been obtained (see, for instance, [10], [13]). However, these results did not exhaust the multitude of possible problems of this sort even in the most elementary case of the cardinal $\omega_1$. Indeed for $\omega_1$, almost any combination of one of the three domains $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$, one of the four Hausdorff order relations considered in § 1, and one of the four types of substructures (that is, scales and so on) is non-trivial.

The goal of this paper is to classify these problems. Besides several well-studied combinations of the domains $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$ and orders $\preccurlyeq$, $\leqslant^*$, we consider some more complicated problems connected with the domain $\mathbb{R}^{\mathbb{N}}$ of sequences of arbitrary real numbers and the less-studied orders $\trianglelefteq$, $\leqslant_{\mathrm{fro}}$. Our main result is Theorem 5 in § 5 (proved in § 7). In the case of the cardinal $\omega_1$, it says that all the problems considered, except for those few connected with gaps and limits for the order $\trianglelefteq$, are classified into the three types already known (scales, towers and gaps, and limits). In the case of cardinals $\kappa > \omega_1$, Theorem 5 yields results that are somewhat less complete.

In § 3 we study some questions related to the connection of the ordered structures considered with their *continual* forms, that is, structures consisting of continuous real functions defined on $[0, +\infty)$ rather than of infinite sequences.

## § 1. Hausdorff ordered structures

By a non-strict (partial) order we understand any transitive (that is, $x \leqslant y$ and $y \leqslant z$ implies $x \leqslant z$) and reflexive ($x \leqslant x$ for all $x$) binary relation $\leqslant$ on a given set $X$ called the *domain of* $\leqslant$. It is not assumed that $x \leqslant y \wedge y \leqslant x$ necessarily implies $x = y$. Nor do we assume linearity, that is, two elements $x, y \in X$ are not necessarily comparable under $\leqslant$. (Relations of this sort are sometimes called *pre-orders* or *quasi-orders*.) Given such an order $\leqslant$, we can define an equivalence relation ($x \equiv y$ if and only if $x \leqslant y$ and $y \leqslant x$) and a strict order ($x < y$ if and only if $x \leqslant y$ but $y \nleqslant x$) on the same domain. Conversely, given an equivalence relation $\equiv$ and an ($\equiv$)-invariant strict order $<$, we can define a non-strict order: $x \leqslant y$ if and only if $x < y$ or $x \equiv y$.

The domain of the following partially ordered sets is the set $\mathbb{R}^{\mathbb{N}}$ of all infinite sequences $a = \{a(n)\}_{n\in\mathbb{N}}$ of real numbers $a(n)$. We understand elements $a \in \mathbb{R}^{\mathbb{N}}$ as *functions* (from $\mathbb{N}$ to $\mathbb{R}$), reserving the word 'sequence' for transfinite sequences of elements of $\mathbb{R}^{\mathbb{N}}$. We also consider the subsets $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ of $\mathbb{R}^{\mathbb{N}}$. They consist of infinite sequences whose terms are positive integers (for the domain $\mathbb{N}^{\mathbb{N}}$) and numbers 0, 1 (for the *dyadic* domain $2^{\mathbb{N}}$).

The rate of growth order on $\mathbb{R}^{\mathbb{N}}$ is defined in [5] by putting

$$a \preccurlyeq b \quad \text{if and only if} \quad \exists \lim_{n\to\infty} \bigl(a(n) - b(n)\bigr) < +\infty.$$

It is different from du Bois-Reymond's original rate of growth order for real functions:

$$f \preccurlyeq g \quad \text{if and only if} \quad \lim_{x\to+\infty} \frac{f(x)}{g(x)} < +\infty.$$

However, the logarithm of the last fraction is equal to a difference of logarithms, and this induces an isomorphism between the version of the last definition for $a \in \mathbb{R}^{\mathbb{N}}$ with positive terms and the first definition. On the other hand, the definition by means of differences is technically somewhat more convenient, and that is why it is more often used in modern studies.

Simple examples show that the limit in the definition of $\preccurlyeq$ does not necessarily exist, and hence the domain $\mathbb{R}^{\mathbb{N}}$ contains ($\preccurlyeq$)-incomparable elements. To circumvent problems related to the non-existence of the limit, Hausdorff [5] suggested replacing the limit by the upper limit (which always exists). This leads to the following modified rate of growth order:

$$a \trianglelefteq b \quad \text{if and only if} \quad \limsup_{n\to\infty}\bigl(a(n) - b(n)\bigr) < +\infty.$$

However, incomparable elements still exist, such as the constant 1 and the function $a \in \mathbb{R}^{\mathbb{N}}$ defined by $a(n) = n$ for even $n$ and $a(n) = n^{-1}$ for odd $n$. In fact, no reasonable order on $\mathbb{R}^{\mathbb{N}}$ can compare any pair of elements of $\mathbb{R}^{\mathbb{N}}$ and be minimally compatible with $\preccurlyeq$ (see below).

The following ordering of $\mathbb{R}^{\mathbb{N}}$ was called the *final Rangordnung* in [5]:

$$\begin{aligned} a \leqslant_{\mathrm{fro}} b \quad \text{if and only if} \quad \exists\, n_0: \quad &\text{either} \quad a(n) < b(n) \quad \forall\, n \geqslant n_0,\\ &\text{or} \quad a(n) = b(n) \quad \forall\, n \geqslant n_0. \end{aligned}$$

For $a \preccurlyeq b$ to hold, it is clearly necessary and sufficient that $c + a \leqslant_{\mathrm{fro}} b$ for every constant $c$ (here $c + a$ denotes the function $a'(n) = a(n) + c$).

The *eventual domination* order $\leqslant^*$ was introduced in [14]:

$$a \leqslant^* b \quad \text{if and only if} \quad \exists\, n_0: \quad \forall\, n \geqslant n_0 \quad (a(n) \leqslant b(n)).$$

**Definition 1.** A *Hausdorff ordered structure* (HOS for brevity) is any partially ordered set $\langle D; \leqslant\rangle$ whose domain $D$ is one of the sets $\mathbb{R}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, $2^{\mathbb{N}}$ and whose order relation $\leqslant$ belongs to the list $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\mathrm{fro}}$, $\leqslant^*$, except for the uninteresting trivial structures $\langle 2^{\mathbb{N}}; \preccurlyeq\rangle$, $\langle 2^{\mathbb{N}}; \trianglelefteq\rangle$ and $\langle 2^{\mathbb{N}}; \leqslant_{\mathrm{fro}}\rangle$. Thus there are nine (non-trivial) HOS, of which one is the *dyadic* structure $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$, four others can be characterized as structures of $\mathbb{N}$-*type* (that is, those with domain $\mathbb{N}^{\mathbb{N}}$), and the last four are structures of $\mathbb{R}$-*type* (with domain $\mathbb{R}^{\mathbb{N}}$).

For each of the four order relations $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\mathrm{fro}}$, $\leqslant^*$ we naturally define (see above) the equivalence relations

$$\sim, \quad \bowtie, \quad \equiv_{\mathrm{fro}}, \quad \equiv^*$$

respectively and the strict order relations

$$\prec, \quad \lhd, \quad <_{\mathrm{fro}}, \quad <^*$$

respectively. For instance, $x \sim y$ if and only if $x \preccurlyeq y$ and $y \preccurlyeq x$, while $x \prec y$ if and only if $x \preccurlyeq y$ but $y \not\preccurlyeq x$.

Note that the order relations $\leqslant_{\mathrm{fro}}$ and $\leqslant^*$ are obviously different, yet they induce the same equivalence relation $\equiv_{\mathrm{fro}} = \equiv^*$, that is,

$$a \equiv_{\mathrm{fro}} b \Leftrightarrow a \equiv^* b \Leftrightarrow a(n) = b(n) \text{ for all but finitely many } n.$$

But the corresponding strict relations are different: $<_{\mathrm{fro}} \subsetneq <^*$.

## § 2. Gaps, limits, towers and scales

Several important types of linearly ordered substructures of HOS are defined here. We fix a partially ordered set $P = \langle P; \leqslant \rangle$ and let $<$ be the corresponding strict order. Let $\kappa$, $\lambda$ be a pair of arbitrary cardinals, each of which is assumed to be either infinite and regular, or finite and then equal to 0 or 1. (The other finite values are trivially reducible to these two in the context of this discussion.)

We introduce the following definitions.

1) A $(\kappa, \lambda^*)$-*pregap* is a pair that consists of a $(<)$-increasing sequence $X = \{x_\alpha\}_{\alpha<\kappa}$ and a $(<)$-decreasing sequence $Y = \{y_\beta\}_{\beta<\lambda}$ of elements $x_\alpha, y_\beta \in P$ such that $X < Y$ (that is, $x_\alpha < y_\beta$ for all $\alpha < \kappa$, $\beta < \lambda$).

2) Any element $z \in P$ satisfying $X < z < Y$ is said to *fill* the pregap $\langle X, Y \rangle$. If there are no such elements $z$, then the given $(\kappa, \lambda^*)$-pregap is called a $(\kappa, \lambda^*)$-*gap*.[1]

3) A $\kappa$-*limit* is any $(\kappa, 1^*)$-gap,[2] that is, a pair consisting of a $(<)$-increasing sequence $\{x_\alpha\}_{\alpha<\kappa}$ and an element $x \in P$ such that $x_\alpha < x$ for all $\alpha$ and there is no other element $y$ satisfying $x_\alpha < y < x$ for all $\alpha$. In this case we write $x = \lim_{\alpha\to\kappa} x_\alpha$.

4) A $\kappa$-*tower* is any $(\kappa, 0^*)$-gap, that is, a $(<)$-increasing $\kappa$-sequence that is unbounded above.[3]

5) A $\kappa$-*scale* is any $(<)$-increasing sequence $\{x_\alpha\}_{\alpha<\kappa}$ in $P$ such that for every $x \in P$ we have $x < x_\alpha$ for some $\alpha$.

Towers and scales are particular types of the much wider categories of unbounded and dominating sets (respectively).

An *unbounded set* is any set $X \subseteq P$ such that there is no $x \in P$ satisfying $X \leqslant x$ (that is, $x' \leqslant x$ for all $x' \in X$).

A *dominating set* is any set $X \subseteq P$ such that for every $x' \in P$ there is an $x \in X$ satisfying $x' \leqslant x$.

In this terminology, a tower in the structure $P = \langle P; \leqslant \rangle$ is any $(<)$-well-ordered unbounded set, while a scale is any $(<)$-well-ordered dominating set. Every dominating set is unbounded provided that there is no largest element.

## § 3. Continual structures

Each of the partially ordered structures defined in § 1 on the domain $\mathbb{R}^{\mathbb{N}}$ of all infinite sequences of real numbers has an obvious *continual* modification defined on the domain $C$ of all continuous functions[4] $f\colon [0, +\infty) \to \mathbb{R}$. One may consider even wider families of functions, for instance, those that are piecewise continuous but bounded on every bounded interval in $[0, +\infty)$.

[1] And frequently a $(\kappa, \lambda)$-*gap* as well, where the second cardinal $\lambda$ is understood as the order type of a decreasing sequence.

[2] In most of the cases considered here, the partially ordered sets will be symmetric enough to prove that the existence of $(\kappa, 1^*)$-gaps is equivalent to that of $(1, \kappa^*)$-gaps, and the latter type will be called *decreasing* limits.

[3] We also consider *decreasing* towers, that is, decreasing $\kappa$-sequences that are unbounded below.

[4] Thus the word 'continual' here reflects the nature of the domain rather than that of the functions considered. In fact, the continual versions of the orderings, say $\preccurlyeq$ and $\trianglelefteq$, historically precede the *discrete* forms (those defined on $\mathbb{R}^{\mathbb{N}}$). The latter were first defined and systematically studied by Hausdorff in his paper [5].

The following theorem contains several assertions connecting discrete and continual structures (ordered by the relations $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\text{fro}}$, $\leqslant^*$) regarding the existence of gaps and scales.

**Theorem 2.** *Suppose that $\leqslant$ is one of the order relations $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\text{fro}}$, $\leqslant^*$, and $\kappa$ is an infinite regular cardinal.*

(i) *Then the existence of a $\kappa$-scale in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant\rangle$ implies the existence of a $\kappa$-scale in $\langle C; \leqslant\rangle$.*

(ii) *Conversely, the existence of a $\kappa$-scale in $\langle C; \leqslant\rangle$ implies the existence of a $\kappa$-scale in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant\rangle$.*

*If, in addition, either $\lambda$ is an infinite regular cardinal, or $\lambda = 0$, or $\lambda = 1$ and $\leqslant$ is one of the relations $\preccurlyeq$, $\leqslant_{\text{fro}}$, then the following assertion holds.*

(iii) *The existence of a $(\kappa, \lambda^*)$-gap in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant\rangle$ implies that of a $(\kappa, \lambda^*)$-gap in $\langle C; \leqslant\rangle$.*

Recall that $(\kappa, 0^*)$-gaps are the same as $\kappa$-towers, while $(\kappa, 1^*)$-gaps are the same as $\kappa$-limits.

*Proof.* For any function $f\colon [0, +\infty) \to \mathbb{R}$, let $f \restriction \mathbb{N}$ denote the sequence $\{f(n)\}_{n\in\mathbb{N}}$ of the values of $f$ on positive integers. For the duration of the proof, $\leqslant$ will be any order relation in the list $\{\preccurlyeq, \trianglelefteq, \leqslant_{\text{fro}}, \leqslant^*\}$.

(i) Suppose that $\{a_\xi\}_{\xi<\kappa}$ is a scale in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant\rangle$. For any index $\xi$, define a function $f_\xi \in C$ in such a way that $a_\xi = f_\xi \restriction \mathbb{N}$ and $f_\xi$ is linear on every interval $[n, n+1]$. Clearly, the sequence of functions $\{f_\xi\}_{\xi<\kappa}$ is $(<)$-increasing when the given sequence $\{a_\xi\}_{\xi<\kappa}$ is, where $<$ is the strict order associated with the given non-strict order $\leqslant$. To see that $\{f_\xi\}$ is a scale, consider any function $f \in C$. As $f$ is continuous, the expression $a(n) = n \max_{0\leqslant x\leqslant n+1} f(x)$ is finite for any $n$, and hence $a = \{a(n)\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. Note that $a \leqslant a_\xi$ for some $\xi$, and then $f \leqslant f_\xi$.

(ii) Now suppose that $\{f_\xi\}_{\xi<\kappa}$ is a scale in $\langle C; \leqslant\rangle$. Define $a_\xi = f_\xi \restriction \mathbb{N}$ for all $\xi$. Let $a \in \mathbb{R}^{\mathbb{N}}$. There is a continuous function $f \in C$ such that $a = f \restriction \mathbb{N}$. Then $f \preccurlyeq f_\xi$ for some $\xi < \kappa$, and hence $a \preccurlyeq a_\xi$, as required. Further, the sequence $\{a_\xi\}_{\xi<\kappa}$ is obviously $(\leqslant)$-increasing, but not necessarily $(<)$-increasing in the case when $\leqslant$ is $\trianglelefteq$ or $\leqslant^*$ since in this case $f < g$ does not necessarily imply $f \restriction \mathbb{N} < g \restriction \mathbb{N}$. Therefore, removing appropriate terms, we can convert the sequence $\{a_\xi\}$ into a $(<)$-increasing sequence, that is, into a scale whose length does not exceed $\kappa$. This length cannot be strictly less than $\kappa$ since in that case we would get a shorter scale in $\langle C; \leqslant\rangle$ by (i), and this is impossible since two scales of different (transfinite regular) lengths cannot exist.

(iii) Suppose that $\langle \{a_\xi\}_{\xi<\kappa}, \{b_\eta\}_{\eta<\lambda}\rangle$ is a $(\kappa, \lambda^*)$-gap in the structure $\langle \mathbb{R}^{\mathbb{N}}; \leqslant\rangle$. There are functions $f_\xi, g_\eta \in C$ that are linear on every interval $[n, n+1]$ and satisfy $a_\xi = f_\xi \restriction \mathbb{N}$ and $b_\eta = g_\eta \restriction \mathbb{N}$. Then $f_\xi < f_{\xi'} < g_{\eta'} < g_\eta$ for all $\xi < \xi' < \kappa$ and $\eta < \eta' < \lambda$. We claim that the pair $\langle \{f_\xi\}_{\xi<\kappa}, \{g_\eta\}_{\eta<\lambda}\rangle$ is a gap. Indeed, otherwise there is a function $h \in C$ such that $f_\xi < h < g_\eta$ for all $\xi$, $\eta$. Then $c = h \restriction \mathbb{N}$ satisfies $a_\xi \leqslant c \leqslant b_\eta$ for all $\xi$, $\eta$. We consider several cases.

If $\kappa$ and $\lambda$ are limit ordinals, then $a_\xi < a_{\xi+1} \leqslant c \leqslant b_{\eta+1} < b_\eta$ and hence $a_\xi < c < b_\eta$ holds strictly, a contradiction.

When $\leqslant$ is one of the relations $\preccurlyeq$, $\leqslant_{\rm fro}$, it is clear that $f < g$ implies $f \restriction \mathbb{N} < g \restriction \mathbb{N}$, and hence the strict inequalities $a_\xi < c < b_\eta$ hold, a contradiction.

Finally, suppose that $\lambda = 0$, that is, $\{a_\xi\}_{\xi<\kappa}$ is a tower, and we wish to prove that so is the sequence $\{f_\xi\}$. Then we have $a_\xi \leqslant c$ (see above), a contradiction.

This completes the proof.

We do not know whether the converse of assertion (iii) of Theorem 2 is true. To demonstrate the difficulties here, consider any $(\kappa, \lambda^*)$-gap $\langle \{f_\xi\}_{\xi<\kappa}, \{g_\eta\}_{\eta<\lambda}\rangle$ in $\langle C; \preccurlyeq\rangle$. Put $a_\xi = f_\xi \restriction \mathbb{N}$ and $b_\eta = g_\eta \restriction \mathbb{N}$. Then $a_\xi \prec a_{\xi'} \prec b_{\eta'} \prec b_\eta$ for all $\xi < \xi' < \kappa$ and $\eta < \eta' < \lambda$. Now if $c \in \mathbb{R}^{\mathbb{N}}$ satisfies $a_\xi \preccurlyeq c \preccurlyeq b_\eta$ for all $\xi$, $\eta$, then it is not clear how to define a function $h \in C$ satisfying the equality $c = h \restriction \mathbb{N}$ and filling the gap $\langle \{f_\xi\}, \{g_\eta\}\rangle$.

Nor is it clear whether (iii) holds for $\lambda = 1$ (the case of limits) when the order $\leqslant$ is $\trianglelefteq$ or $\leqslant^*$. Indeed, suppose that $\langle \{a_\xi\}_{\xi<\kappa}, b_0\rangle$ is a $\kappa$-limit in, say, $\langle \mathbb{R}^{\mathbb{N}}; \trianglelefteq\rangle$ with the following property. When $\xi$ is an even ordinal, we have $a_\xi(n) = b_0(n)$ for even $n$, $a_\xi(n) < b_0(n)$ for odd $n$, and $b_0(n) - a_\xi(n) \to +\infty$, and when $\xi$ is odd, the same holds but with 'even' and 'odd' interchanged. Then the pair $\langle \{f_\xi\}_{\xi<\omega_1}, g_0\rangle$, defined as in the proof of assertion (iii) of Theorem 2, is not necessarily a $\kappa$-limit in $\langle C; \trianglelefteq\rangle$.

## § 4. Hausdorff's gap theorem

It is easy to prove (as Hausdorff does in [5]) that $(\omega, \omega^*)$-gaps and $\omega$-limits do not exist in structures of the type considered. The proof uses the diagonal construction of du Bois-Reymond [1]. For instance, suppose that

$$a_0 <^* a_1 <^* a_2 <^* \cdots <^* b_2 <^* b_1 <^* b_0, \qquad a_i, b_j \in \mathbb{N}^{\mathbb{N}}.$$

There is a sequence of positive integers $n_0 < n_1 < \cdots$ such that $a_i(n) < b_j(n)$ for all $k$ and $i, j \leqslant k$ whenever $n_k \leqslant n < n_{k+1}$. Put $c(n) = \max_{i\leqslant k} a_i(n)$ for all $n$ satisfying $n_k \leqslant n < n_{k+1}$. Then $a_n <^* c <^* b_n$ holds for all $n$.

The following theorem is much more difficult. We sketch its proof for the reader's convenience since it would be difficult to find a proof published in Russian.

**Theorem 3** (Hausdorff's gap theorem). *$(\omega_1, \omega_1^*)$-gaps exist in every Hausdorff ordered structure.*

The result for the structure $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\rm fro}\rangle$ appeared in [6]. A version for the dyadic structure $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$ appeared in [14], and this is a standard reference in modern literature. The proofs in [6] and [14] follow the same scheme, which is also applicable to any of the nine HOS with suitable modifications. On the other hand, such a generalization can also be established as a formal consequence of some rather transparent reductions established in § 7 below.

*Proof of Theorem* 3 (a sketch for the structure $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$). When $a, b \in 2^{\mathbb{N}}$ and $a \leqslant^* b$, let $(a; b)$ denote the least number $n_0$ satisfying $n \geqslant n_0 \Rightarrow a(n) \leqslant b(n)$. We define a $(<^*)$-increasing sequence $A = \{a_\xi\}_{\xi<\omega_1}$ and $(<^*)$-decreasing sequence $B = \{b_\xi\}_{\xi<\omega_1}$ of elements $a_\xi, b_\xi \in 2^{\mathbb{N}}$ satisfying the inequality $a_\eta <^* b_\xi$ for all $\xi$, $\eta$ (that is, $\langle A, B\rangle$ is a pre-gap) and the following key condition:

($*$) *for all $n \in \mathbb{N}$ and $\xi < \omega_1$, the set $\{\eta < \xi \colon (a_\eta; b_\xi) = n\}$ is finite.*

We can understand $(*)$ in the sense that although $b_\xi$ is strictly $(<^*)$-higher than all the $a_\eta$, there is still a certain degree of $(<^*)$-proximity of $b_\xi$ to the set $\{a_\eta : \eta < \xi\}$.

If such a construction is accomplished, then the pair $\langle A, B\rangle$ is the desired $(\omega_1, \omega_1^*)$-gap. Indeed, assume the opposite and let $c \in 2^{\mathbb{N}}$ be such that $a_\xi <^* c <^* b_\xi$ for all $\xi$. As $\omega_1$ is uncountable, there is an ordinal $\xi$ and a number $n$ such that $(a_\eta; c) = n$ for infinitely many $\eta < \xi$. But this clearly contradicts $(*)$ as $c <^* b_\xi$.

We now describe an inductive construction of terms of sequences satisfying $(*)$. The successor steps are rather trivial: if terms $a_\xi <^* b_\xi$ have been defined, choose $a_{\xi+1}$ and $b_{\xi+1}$ to be any pair $a, b \in 2^{\mathbb{N}}$ satisfying $a_\xi <^* a <^* b <^* b_\xi$. The limit steps need more effort. Suppose that $\lambda < \omega_1$ is a limit ordinal and $a_\xi$, $b_\xi$ have been defined for $\xi < \lambda$ in such a way that $(*)$ holds. The same argument as in the above proof of the absence of $(\omega, \omega^*)$-gaps enables us to define a $c \in 2^{\mathbb{N}}$ such that $a_\xi <^* c <^* b_\xi$ for all $\xi < \lambda$. It follows from the inductive assumption of $(*)$ that the set $\{\eta < \xi \colon (a_\eta; c) = n\}$ is finite for any number $n$ and any ordinal $\xi < \lambda$. In this case, another version of the same argument enables us to define a $b \in 2^{\mathbb{N}}$ such that $b <^* c$ and again $a_\xi <^* b$ for all $\xi < \lambda$, and in addition the set $\{\eta < \lambda \colon (a_\eta; b) = n\}$ is finite for any $n$. Put $b_\lambda = b$ and define $a_\lambda$ to be any $a \in 2^{\mathbb{N}}$ such that $a_\xi <^* a <^* b$ for all $\xi$.

## § 5. The principal problem and the main theorem

This paper is largely devoted to the following general problem concerning the partial orderings called HOS in § 1.

**Problem 4** (the principal problem). What are the structure, properties and spectra of the cardinals of the gaps, limits, towers and scales of any given partially ordered set $P = \langle P; \leqslant\rangle$? For instance, if $\kappa$, $\lambda$ are regular cardinals, does the set $P$ have $(\kappa, \lambda^*)$-gaps or $\kappa$-scales?

This problem includes a variety of more special questions on the existence of gaps (including limits and towers) and scales with certain cardinal characteristics. In particular, in the simplest (but most interesting) case $\kappa = \omega_1$, problems on the existence of $\omega_1$-limits, $\omega_1$-towers and $(\omega_1, \omega^*)$-gaps for various HOS were considered in the earliest of Hausdorff's studies, such as [6]. He regarded these problems as relevant to the continuum hypothesis **CH** (that is, the equality $\mathfrak{c} = \aleph_1$), which was still open at that time. Besides Theorem 3, Hausdorff's main results in his early studies [5], [6] relating to these particular problems amount to the following.

(I) The problems of the existence of $\omega_1$-limits, $\omega_1$-towers and $(\omega_1, \omega^*)$-gaps in the structure $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathrm{fro}}\rangle$ are equivalent to each other: the existence of any of them implies that of the other two.

(II) The continuum hypothesis **CH** implies the existence of $\omega_1$-limits, $\omega_1$-towers and $(\omega_1, \omega^*)$-gaps, as well as $\omega_1$-scales, in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathrm{fro}}\rangle$.

Further studies of the relations between these problems were undertaken by Rothberger in [11], [12], where it was established that, for the partially ordered set $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$, the existence of an $(\omega_1, \omega^*)$-gap is equivalent to the existence of an $\omega_1$-tower and implies the existence of an $\omega_1$-limit. To compare this with Hausdorff's result (I), one has to bear in mind that limits in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle$ and in the dyadic structure $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ are of a somewhat different nature.

At around the same time, another field of applications of these ideas was discovered by Luzin [15], [16]. Consider the set $\mathscr{P}(\mathbb{N}) = \{x \colon x \subseteq \mathbb{N}\}$ of all subsets of the set $\mathbb{N}$ of all positive integers, ordered by the relation of *almost-inclusion*: $x \subseteq^* y$ if and only if the difference $x \setminus y$ is finite. The structure $\langle \mathscr{P}(\mathbb{N}); \subseteq^* \rangle$ is isomorphic to the structure $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$, of course. Luzin defined a pair of strictly $(\subset^*)$-increasing sequences $\{x_\xi\}_{\xi<\omega_1}$ and $\{y_\xi\}_{\xi<\omega_1}$ of sets, $x_\xi, y_\xi \subseteq \mathbb{N}$, that are *orthogonal* (that is, all intersections $x_\xi \cap y_\xi$ are finite) but *inseparable* (that is, there is no set $z$ such that $x_\xi \subseteq^* z$ but $y_\xi \cap z$ is finite for all $\xi$). This is equivalent to the existence of an $(\omega_1, \omega_1^*)$-gap in $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$. He also posed problems on the existence of $\omega_1$-limits and $(\omega_1, \omega^*)$-gaps in $\langle \mathscr{P}(\mathbb{N}); \subseteq^* \rangle$ (in terms of the existence of a pair of orthogonal and unseparable sequences one of which has length $\omega_1$ and the other has length $\omega$, now called a *Luzin pair*). See [17] for more details.

Some other results in this direction, relating mainly to the orderings $\leqslant^*$ and $\preccurlyeq$ and the domains $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$, as well as their applications in set theory and topology, are to be found in [8]–[10], [18]. See § 6 below for some metamathematical independence results.

The following theorem (our main result) implies that for any given cardinal $\kappa \geqslant \omega_1$, the problems of the existence of $\kappa$-scales, $\kappa$-towers, $\kappa$-limits and $(\kappa, \omega^*)$-gaps in Hausdorff structures (see Definition 1) are reducible to a much shorter list of genuinely different problems, except for questions on the existence of gaps and limits in the $(\trianglelefteq)$-structures

$$\langle \mathbb{R}^{\mathbb{N}}; \trianglelefteq \rangle, \qquad \langle \mathbb{N}^{\mathbb{N}}; \trianglelefteq \rangle, \tag{1}$$

whose nature remains not fully clear. For all the other Hausdorff structures in Definition 1, that is, for

$$\langle \mathbb{R}^{\mathbb{N}}; \preccurlyeq \rangle, \quad \langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle, \quad \langle \mathbb{R}^{\mathbb{N}}; \leqslant^* \rangle, \quad \langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle, \quad \langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle, \quad \langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle, \quad \langle 2^{\mathbb{N}}; \leqslant^* \rangle, \tag{2}$$

all these questions are reducible to only three genuinely different problems in the most interesting case $\kappa = \omega_1$ (see Remark 7 below).

The content of the main theorem is illustrated by the diagrams displayed in Figs. 1 and 2. In these diagrams, the relation $X \Rightarrow Y$ (including the cases with vertical arrows) means that the existence of a $(\kappa, \lambda^*)$-gap (or a $\kappa$-limit) in $X$ implies that of the same gap (resp. limit) in $Y$, and the relation $\Leftrightarrow$ is understood accordingly. In Fig. 2, the relation $X \to Y$ means that the existence of a $\kappa$-limit in $X$ implies

$$\begin{tikzcd}
\langle 2^{\mathbb{N}}; \leqslant^* \rangle \arrow[r, Leftrightarrow] & \langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \langle \mathbb{N}^{\mathbb{N}}; \trianglelefteq \rangle \arrow[d, Leftrightarrow] \\
 & \langle \mathbb{R}^{\mathbb{N}}; \leqslant^* \rangle & \langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle \arrow[l, Rightarrow] & \langle \mathbb{R}^{\mathbb{N}}; \preccurlyeq \rangle \arrow[l, Rightarrow] & \langle \mathbb{R}^{\mathbb{N}}; \trianglelefteq \rangle
\end{tikzcd}$$

Figure 1. Relations between HOS regarding the existence of a $(\kappa, \lambda^*)$-gap

$$\begin{tikzcd}
 & \langle\mathbb{N}^\mathbb{N};\leqslant^*\rangle \arrow[r, Leftrightarrow] \arrow[dd, shift right, leftrightarrow] & \langle 2^\mathbb{N};\leqslant^*\rangle \\
\langle\mathbb{R}^\mathbb{N};\leqslant^*\rangle & \arrow[l, Rightarrow] & \\
 & \boxed{\text{towers}} \arrow[uu, shift left] \arrow[r, Leftrightarrow] \arrow[d, shift right] & \langle\mathbb{R}^\mathbb{N};\leqslant_{\mathrm{fro}}\rangle \\
 & \langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle \arrow[u, shift right, Rightarrow] \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \langle\mathbb{R}^\mathbb{N};\preccurlyeq\rangle \\
 & \langle\mathbb{N}^\mathbb{N};\trianglelefteq\rangle \arrow[r, Leftrightarrow] & \langle\mathbb{R}^\mathbb{N};\trianglelefteq\rangle
\end{tikzcd}$$

Figure 2. Relations between HOS regarding the existence of a $\kappa$-limit

the existence of a $\kappa'$-limit in $Y$ for a regular cardinal $\kappa' \leqslant \kappa$, the relation $\Leftarrow\!\updownarrow$ is understood in the sense of assertion 4), (vi) of Theorem 5, while the box $\boxed{\text{towers}}$ means the existence of $\kappa$-towers in the non-dyadic structures in the list (2).

**Theorem 5** (the main theorem). *Let $\kappa \geqslant \omega_1$ be a regular cardinal.*

1) *All the* HOS *in Definition 1, except for the dyadic structure $\langle 2^\mathbb{N};\leqslant^*\rangle$, are equivalent to each other with respect to the existence of $\kappa$-scales.*[5]

2) *The following assertions hold.*

(i) *All the* HOS *in the list (2), except for the dyadic structure $\langle 2^\mathbb{N};\leqslant^*\rangle$, are equivalent to each other with respect to the existence of $\kappa$-towers.*

(ii) *The existence of $\kappa$-towers in $\langle\mathbb{N}^\mathbb{N};\trianglelefteq\rangle$ follows from the existence of $\kappa$-towers in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$ and implies the existence of $\kappa'$-towers in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$ for some regular uncountable cardinal $\kappa' \leqslant \kappa$.*

3) *The following assertions hold.*

(i) *Let $\lambda \geqslant \omega$ be a regular cardinal. Then Fig. 1 displays the relations between Hausdorff structures with respect to the existence of a $(\kappa, \lambda^*)$-gap.*

(ii) *When $\lambda = \omega$, all occurrences of the implication sign $\Rightarrow$ in Fig. 1, except possibly for the implication $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle \Rightarrow \langle\mathbb{N}^\mathbb{N};\trianglelefteq\rangle$, can be replaced by $\Leftrightarrow$, and hence all seven of the* HOS *in (2) are equivalent with respect to the existence of a $(\kappa, \omega^*)$-gap.*

(iii) *The existence of a $(\kappa, \omega^*)$-gap in any of the* HOS *in (2) is equivalent to the existence of a $\kappa$-tower in any of the six non-dyadic* HOS *in (2).*

4) *Fig. 2 displays the relations between Hausdorff structures with respect to the existence of a $\kappa$-limit. More precisely, the content of the diagram in Fig. 2 is as follows.*

(i) *The structure $\langle\mathbb{N}^\mathbb{N};\leqslant_{\mathrm{fro}}\rangle$ has no $\kappa$-limits.*

[5] The questions of the existence of scales and towers are vacuous for the dyadic structure $\langle 2^\mathbb{N};\leqslant^*\rangle$ since it contains no scales or towers whose length is a limit ordinal and $2^\mathbb{N}$ contains $(\leqslant^*)$-largest elements. For instance, any $a \in 2^\mathbb{N}$ satisfying $a(n) = 1$ for almost all (except for finitely many) $n$, is such an element. Let us call any such $a$ an *almost constant* 1. Even if one removes the almost-1 sequences from $2^\mathbb{N}$, there will be no scales at all while any $\kappa$-towers, if they exist, will be the same as $\kappa$-limits in the given structure $\langle\mathbb{N}^\mathbb{N};\leqslant^*\rangle$. It follows that, without any loss of generality, we can eliminate $\langle 2^\mathbb{N};\leqslant^*\rangle$ whenever existence problems relating to towers and scales are involved.

(ii) *The structure* $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle$ *has* $\kappa$*-limits if and only if the structure* $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ *does.*

(iii) *The structure* $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle$ *has* $\kappa$*-limits if and only if the structure* $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle$ *has* $\kappa$*-towers.*

(iv) $\kappa$*-limits exist in* $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle$ *if and only if they exist in* $\langle \mathbb{R}^{\mathbb{N}}; \preccurlyeq \rangle$*. Either of these two existence claims implies the existence of* $\kappa$*-towers in* $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle$*.*

(v) *If the structure* $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle$ *has* $\kappa$*-towers, then the structure* $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle$ *has* $\kappa'$*-limits for some cardinal* $\kappa' \leqslant \kappa$*.*

(vi) *The structure* $\langle \mathbb{R}^{\mathbb{N}}; \leqslant^* \rangle$ *has* $\kappa$*-limits if and only if either* $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ *has* $\kappa$*-limits or* $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle$ *has* $\kappa$*-towers.*

(vii) $\kappa$*-limits exist in* $\langle \mathbb{N}^{\mathbb{N}}; \trianglelefteq \rangle$ *if and only if they exist in* $\langle \mathbb{R}^{\mathbb{N}}; \trianglelefteq \rangle$*. Each of these two existence claims follows from the existence of* $\kappa$*-limits in* $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle$*.*

5) *The following assertions hold.*

(i) *The existence of a* $\kappa$*-tower in any of the non-dyadic* HOS *in* (2) *follows from the existence of a* $\kappa$*-scale, is equivalent to the existence of a* $(\kappa, \omega^*)$*-gap, and implies the existence of a* $\kappa'$*-limit in* $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ *for some cardinal* $\kappa' \leqslant \kappa$*.*

(ii) *If* $\kappa$*-towers exist* (*in non-dyadic* HOS) *but the structure* $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ *has no* $\kappa$*-limits, then* $\kappa$*-scales exist.*

*Remark* 6. In the particular case $\lambda = \omega$, Theorem 5 reduces the multitude of existence problems for $\kappa$-scales, $\kappa$-towers, $\kappa$-limits and $(\kappa, \omega^*)$-gaps in the Hausdorff structures in Definition 1 to the following groups of mutually equivalent (within each group) problems.

A. The existence of a $\kappa$-limit in any (or, equivalently, in each) of the two structures $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ and $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle$.

B. The existence of a $\kappa$-tower in any non-dyadic HOS, the existence of a $(\kappa, \omega^*)$-gap in any HOS in (2), the existence of a $\kappa$-limit in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle$.

B′. The existence of a $\kappa$-limit in either of the structures $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle$, $\langle \mathbb{R}^{\mathbb{N}}; \preccurlyeq \rangle$.

C. The existence of a $\kappa$-scale in any non-dyadic HOS.

Here is another version.

A′. The existence of a $\kappa$-limit in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant^* \rangle$, which is equivalent to $\mathrm{A} \vee \mathrm{B}$.

The following problems are excluded from this scheme.

$\mathrm{A}^{\triangleleft}$. The existence of $\kappa$-limits in the structures $\langle \mathbb{R}^{\mathbb{N}}; \trianglelefteq \rangle$ and $\langle \mathbb{N}^{\mathbb{N}}; \trianglelefteq \rangle$.

$\mathrm{B}^{\triangleleft}$. The existence of $(\kappa, \omega^*)$-gaps in the structures $\langle \mathbb{R}^{\mathbb{N}}; \trianglelefteq \rangle$ and $\langle \mathbb{N}^{\mathbb{N}}; \trianglelefteq \rangle$.

Also the existence of a $\kappa$-limit in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant_{\mathrm{fro}} \rangle$, which is impossible.

Note that for any of the three HOS of $\mathbb{N}$-type in (2), the existence of a $(\kappa, \lambda^*)$-gap implies the existence of a $(\lambda, \kappa^*)$-gap since this is true for the structure $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ because of its obvious symmetry. (The existence of a suitable symmetry for HOS of $\mathbb{R}$-type is also fairly clear.)

*Remark* 7. The relations between the problems become even simpler in the most interesting case $\kappa = \omega_1$. Indeed, then we necessarily have $\kappa' = \kappa$ in assertion 5), (i) of Theorem 5 since there are no $\omega$-limits. Therefore problem B′ joins B, and hence B implies A. It follows that problem A′, that is, $\omega_1$-limits in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant^* \rangle$, also joins A.

Finally, C implies B. Thus we have

$$\begin{array}{ccccccl} A^{\triangleleft} & & B^{\triangleleft} & & & & \\ \Uparrow & & \Uparrow & & & & \\ A & \Leftarrow & B & \Leftarrow & C & : & \text{the case when } \kappa=\omega_1 \text{ and } \lambda=\omega. \\ \Updownarrow & & \Updownarrow & & & & \\ A' & & B' & & & & \end{array} \tag{3}$$

Note that the problems in the list (I) belong to type B.

Except for problems $A^{\triangleleft}$ and $B^{\triangleleft}$, the diagram in Fig. 2 is complete in the sense that nothing more can be proved on the mutual reducibility of the problems considered (see § 6).

The nature of problems $A^{\triangleleft}$ and $B^{\triangleleft}$ ($\kappa$-limits and $(\kappa,\omega^*)$-gaps in $\trianglelefteq$-structures) remains not completely clear. This is one of the most interesting problems here. For instance, one might want to prove the equivalences

$$A^{\triangleleft} \Leftrightarrow A, \qquad B^{\triangleleft} \Leftrightarrow B.$$

The key obstacle is that the relation $x \trianglelefteq y$ (where $x,y \in \mathbb{N}^{\mathbb{N}}$) is consistent, by definition, with the assumption that in fact $x(n) > y(n)$ for the vast majority of indices $n$. Hence there seems to be no reasonable way to convert $(\trianglelefteq)$-gaps and $(\trianglelefteq)$-limits into the corresponding structures in other Hausdorff orders.

**Problem 8.** Can one strengthen the implications in Fig. 1 to equivalences in the general case (that is, without assuming that $\kappa=\omega_1$ and $\lambda=\omega$)? It would be interesting to prove that the existence of a $(\kappa,\lambda^*)$-gap in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant^* \rangle$ implies the existence of the same gap in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle$. What are the relations between problems A′, B′, B in the case when $\kappa > \omega_1$?

The proof of Theorem 5 follows in § 7.

## § 6. Some metamathematical questions

Returning to diagram (3), let us discuss the question of whether the classification of the problems of the existence of $\omega_1$-scales, $\omega_1$-towers, $\omega_1$-limits and $(\omega_1,\omega^*)$-gaps in the Hausdorff structures given in the diagram is best possible. For instance, can some of the implications be strengthened to equivalences? As far as the implications A⇒$A^{\triangleleft}$ and B⇒$B^{\triangleleft}$ are concerned, the question is still open. On the other hand, the definitive nature of the rest diagram (3) has been established by a series of studies, presented here in brief for the reader's convenience.

Hausdorff himself demonstrated in [5], [6] that Cantor's continuum hypothesis **CH**, that is, $2^{\aleph_0}=\omega_1$, implies C, and then A and B as well, for $\kappa=\omega_1$. Yet, in the absence of **CH**, the state and interrelations of these problems were finally understood no earlier than the 1970s and 1980s, when the method of forcing had been successfully applied to show that there are no connections (provable in **ZFC**+¬**CH**) between these problems except for the double implication C⇒B⇒A and the equivalences B⇔B′ and A⇔A′ mentioned in Remark 7. These results are summarized in the next theorem.

**Theorem 9.** *Each of the following statements is consistent with* $\mathbf{ZFC} + \neg\mathbf{CH}$.

(i) *Problems* C, B, A *hold for* $\kappa = \omega_1$.

(ii) *Problem* C *fails*, *but problems* B, A *hold*, *for* $\kappa = \omega_1$.

(iii) *Problems* C, B *fail*, *but problem* A *holds*, *for* $\kappa = \omega_1$.

(iv) *Problems* C, B, A *fail for* $\kappa = \omega_1$.

Thus problems A, B, C are *undecidable* in the theory $\mathbf{ZFC} + \neg\mathbf{CH}$ for $\kappa = \omega_1$, and the implications C⇒B⇒A are irreversible in that theory.

It has become rather common in modern set theory to associate a certain *cardinal invariant* with each interesting type of transfinite object under consideration. This can be the cardinal $\kappa$ (usually in the interval $\omega_1 \leqslant \kappa \leqslant \mathfrak{c} = 2^{\aleph_0}$) equal to the least cardinality of a set of this type.

Among the multitude of cardinal invariants (see [7]), the following four are of principal interest here:

1) $\mathfrak{t}$ is the least cardinal $\kappa$ such that $\kappa$-limits exist in $2^{\mathbb{N}}$;

2) $\mathfrak{b}$ is the least cardinality of a $(\leqslant^*)$-unbounded subset of $\mathbb{N}^{\mathbb{N}}$ or, equivalently, the least length of a $(\leqslant^*)$-tower in $\mathbb{N}^{\mathbb{N}}$;

3) $\mathfrak{b}_6$ is the least cardinal $\kappa$ such that $(\kappa, \omega^*)$-gaps exist in $2^{\mathbb{N}}$;

4) $\mathfrak{d}$ is the least cardinal $\kappa$ such that $\kappa$-scales exist in $\mathbb{N}^{\mathbb{N}}$.

Then $\omega_1 \leqslant \mathfrak{t} \leqslant \mathfrak{b} = \mathfrak{b}_6 \leqslant \mathfrak{d} \leqslant \mathfrak{c}$ (see Remark 7 and also [7], §§ 3.1, 3.3). In these terms, the hypotheses A, B, C ($\kappa = \omega_1$) can be compactly written as the equalities $\mathfrak{t} = \omega_1$, $\mathfrak{b} = \omega_1$, $\mathfrak{d} = \omega_1$.

The theory of cardinal invariants involves a universal tool that enables one to make all these cardinals equal to the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum, independently of the relations between $\mathfrak{c}$ and $\omega_1$. This is *Martin's axiom* or **MA** (see [9], [19], [20]). It is known that **MA** is consistent with $\mathbf{ZFC} + \neg\mathbf{CH}$ (the negation of the continuum hypothesis), and hence any consequence of **MA** is consistent with $\neg\mathbf{CH}$. In particular, since **MA** implies[6] that $\mathfrak{t} = \mathfrak{c}$, it implies the absence of $\omega_1$-limits, $(\omega_1, \omega^*)$-gaps and $\omega_1$-scales. This proves assertion (iv) of Theorem 9.

The consistency of the combinations $\omega_1 = \mathfrak{t} = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ and $\omega_1 = \mathfrak{t} = \mathfrak{b} = \mathfrak{d} < \mathfrak{c}$ was established in [22], [23]. This proves assertions (i) and (ii) of Theorem 9 (for a more up-to-date proof, see [24]). Part (iii) of Theorem 9 was established in [25] (see also Theorem 5.3 in [7], which proves the consistency of the stronger combination $\omega_1 = \mathfrak{t} < \mathfrak{b} = \mathfrak{d} = \mathfrak{c}$).

We finish this section with an old but still unsolved problem in this field, first formulated by Hausdorff [5] and recently re-introduced by Solovay [26].

**Problem 10.** Does there exist (in a given Hausdorff structure) a maximal totally ordered subset that does not have $(\omega_1, \omega_1^*)$-gaps? (Compare with Theorem 3.)

There is little doubt that this problem has the same solution for all Hausdorff structures.

[6] See, for instance, Corollary 8 in [20], first proved perhaps in [21].

## § 7. Proof of Theorem 5

For the duration of the proof of the main theorem, $\kappa$ will denote a regular cardinal with $\kappa \geqslant \omega_1$.

**7.1. Towers and scales.** Here we prove assertions 1) and 2) of Theorem 5.

To begin with, let us eliminate $\mathbb{R}^{\mathbb{N}}$-structures.

**Lemma 11.** *If $\leqslant$ is one of the Hausdorff orders $\preccurlyeq$, $\trianglelefteq$, $\leqslant_{\mathrm{fro}}$, $\leqslant^*$, then the structures $\langle\mathbb{N}^{\mathbb{N}};\leqslant\rangle$ and $\langle\mathbb{R}^{\mathbb{N}};\leqslant\rangle$ are equivalent with respect to the existence of $\kappa$-scales. The same is true for $\kappa$-towers.*

*Proof.* First, any scale $\{a_\xi\}_{\xi<\kappa}$ in the structure $\langle\mathbb{N}^{\mathbb{N}};\leqslant\rangle$ remains a scale in $\langle\mathbb{R}^{\mathbb{N}};\leqslant\rangle$. Indeed, consider, for instance, the order $\leqslant^*$. Assume the opposite: there is an $x\in\mathbb{R}^{\mathbb{N}}$ such that no $\xi$ satisfies $x\leqslant^* a_\xi$. Define $x'\in\mathbb{N}^{\mathbb{N}}$ by letting, for every $n$, $x'(n)$ be the least positive integer bigger than $x(n)$. Clearly, $x\leqslant^* a$. Hence $a\leqslant^* a_\xi$ fails for every $\xi$, a contradiction.

Conversely, suppose that $\{x_\xi\}_{\xi<\kappa}$ is a scale in $\langle\mathbb{R}^{\mathbb{N}};\leqslant\rangle$. For each $\xi$, replacing negative terms $x_\xi(n)$ by zeros and positive terms by the integers nearest from above, we get $x'_\xi\in\mathbb{N}^{\mathbb{N}}$. Then $x_\xi\leqslant x'_\xi$ holds by definition, and hence the new sequence remains $(\leqslant)$-dominating. Finally, the sequence $\{a_\xi\}_{\xi<\kappa}$ is $(\leqslant)$-increasing (possibly non-strictly) and, as $\kappa$ is regular, it contains a strictly increasing subsequence.

With rather obvious changes, both parts of this argument also work for towers.

Thus it remains to consider towers and scales in the $(\mathbb{N}^{\mathbb{N}})$-structures

$$\langle\mathbb{N}^{\mathbb{N}};\preccurlyeq\rangle\,,\quad \langle\mathbb{N}^{\mathbb{N}};\trianglelefteq\rangle\,,\quad \langle\mathbb{N}^{\mathbb{N}};\leqslant_{\mathrm{fro}}\rangle\,,\quad \langle\mathbb{N}^{\mathbb{N}};\leqslant^*\rangle.$$

Note that any $(\preccurlyeq)$-tower $\{x_\xi\}_{\xi<\kappa}$ in $\mathbb{N}^{\mathbb{N}}$ (of any length) is also a $(\leqslant_{\mathrm{fro}})$-tower. Indeed, suppose that $x\in\mathbb{N}^{\mathbb{N}}$ and $x_\xi\leqslant_{\mathrm{fro}} x$ for all $\xi$. Then $x_\xi\preccurlyeq x$ for all $\xi$ since $x_\xi\leqslant_{\mathrm{fro}} x_{\xi+1}\preccurlyeq x$ implies that $x_\xi\preccurlyeq x$, a contradiction. The same argument shows that any $(\leqslant_{\mathrm{fro}})$-tower is a $(\leqslant^*)$-tower, and any $(\preccurlyeq)$-tower is a $(\trianglelefteq)$-tower. The proof for scales is similar.

Conversely, the map that sends every $x\in\mathbb{N}^{\mathbb{N}}$ to $x'(n)=\sum_{i=0}^{n}x(i)$ clearly transforms any $(\leqslant^*)$-tower (resp. scale) $\{x_\xi\}_{\xi<\kappa}$ of elements of $\mathbb{N}^{\mathbb{N}}$ into a $(\preccurlyeq)$-tower (resp. scale) $\{x'_\xi\}_{\xi<\kappa}$. Indeed, in the case of towers, assume the opposite and let $x\in\mathbb{N}^{\mathbb{N}}$ be such that $x'_\xi\preccurlyeq x$ for all $\xi$. Then, by definition, $x_\xi\preccurlyeq x'_\xi$ for all $\xi$. Therefore, $x_\xi\preccurlyeq x$ and hence $x_\xi\leqslant^* x$ for all $\xi$, a contradiction.

It remains to prove the converse for the order $\trianglelefteq$. Suppose that $\{x_\alpha\}_{\alpha<\kappa}$ is a $\kappa$-scale in $\langle\mathbb{N}^{\mathbb{N}};\trianglelefteq\rangle$. We claim that there is a $\kappa$-scale in $\langle\mathbb{N}^{\mathbb{N}};\preccurlyeq\rangle$. Indeed, by definition, for every set $X\subseteq\mathbb{N}^{\mathbb{N}}$ of cardinality $\operatorname{card}X<\kappa$, there is a function $y\in\mathbb{N}^{\mathbb{N}}$ such that $x\preccurlyeq y$ for all $x\in X$. (As $\{x_\alpha\}$ is a scale, there is an index $\alpha<\kappa$ satisfying $x\trianglelefteq x_\alpha$ for all $x\in X$. Put $y=x_\alpha$.) This enables us to define a $(\prec)$-increasing $\kappa$-sequence $\{y_\alpha\}_{\alpha<\kappa}$ of functions $y_\alpha\in\mathbb{N}^{\mathbb{N}}$ such that $x_\alpha\preccurlyeq y_\alpha$ for each $\alpha$. Then $\{y_\alpha\}$ is obviously a $\kappa$-scale in $\langle\mathbb{N}^{\mathbb{N}};\preccurlyeq\rangle$.

For towers, the converse holds in a weaker form, as in 2), (ii): if $\langle\mathbb{N}^{\mathbb{N}};\trianglelefteq\rangle$ has a $\kappa$-tower $\{x_\alpha\}_{\alpha<\kappa}$, then the structure $\langle\mathbb{N}^{\mathbb{N}};\preccurlyeq\rangle$ has a $\kappa'$-tower for some $\kappa'\leqslant\kappa$. Indeed, $\{x_\alpha\}$ remains an unbounded family in $\langle\mathbb{N}^{\mathbb{N}};\preccurlyeq\rangle$, but it is not necessarily $(\prec)$-increasing. Consider an arbitrary maximal $(\prec)$-increasing sequence $\{y_\alpha\}_{\alpha<\kappa'}$

such that $x_\alpha \preccurlyeq y_\alpha$ for all $\alpha < \kappa'$. Clearly, $\kappa' \leqslant \kappa$ (for otherwise the sequence $\{x_\alpha\}$ would not be a tower) while the maximality implies that the sequence $\{y_\alpha\}$ is unbounded and, therefore, is a tower in $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle$.

**7.2. Gaps.** Here we prove assertion 3), (i) of Theorem 5. The proof involves several lemmas of different levels of complexity. $\kappa \geqslant \omega_1$ is still an arbitrary regular cardinal, as in the theorem. The second parameter $\lambda$ may take any value $\lambda \geqslant \omega$ as in Theorem 5, and we admit the value $\lambda = 1$ in some of the lemmas, just to incorporate the case of limits. This is indicated in the preambles to Lemmas 12–16.

The following result belongs to Rothberger [11], [12].

**Lemma 12** ($\lambda \geqslant \omega$ or $\lambda = 1$). *The structures $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ and $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle$ are equivalent with respect to the existence of $(\kappa, \lambda^*)$-gaps.*

*Proof.* On the one hand, any gap in $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ remains a gap in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle$. Indeed, assume that $x \in \mathbb{N}^{\mathbb{N}}$ fills this gap in $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$. Then, changing every value $x(n) \neq 0$ to 1, we get an element $x \in 2^{\mathbb{N}}$ filling the same gap, a contradiction.

Conversely, any gap in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle$ can be transformed into a gap in the structure $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$ with both sequences having the same length. Indeed, replace any element $a \in \mathbb{N}^{\mathbb{N}}$ that occurs in a given gap, first by the set $X_a = \{\langle i, n \rangle \colon i < a(n)\} \subseteq \mathbb{N}^2$, then by the image $Y_a = \{f(i, n) \colon \langle i, n \rangle \in X_a\}$ of this set under any fixed bijection $f \colon \mathbb{N}^2 \overset{\text{onto}}{\to} \mathbb{N}$, and finally by the characteristic function of $Y_a$. This construction yields the required gap in the structure $\langle 2^{\mathbb{N}}; \leqslant^* \rangle$.

**Lemma 13** ($\lambda \geqslant \omega$ or $\lambda = 1$). *If $\leqslant$ is any of the relations $\leqslant^*$, $\leqslant_{\mathrm{fro}}$, $\preccurlyeq$, $\trianglelefteq$, then every $(\kappa, \lambda^*)$-gap in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant \rangle$ remains a gap in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant \rangle$.*

*Proof.* Indeed, consider for instance a $\kappa$-limit, that is, a $(\kappa, 1^*)$-gap $\langle \{a_\xi\}_{\xi<\kappa}, a \rangle$, in the structure $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^* \rangle$. Assume the opposite: an element $x \in \mathbb{R}^{\mathbb{N}}$ fills this gap, that is, satisfies the strict inequalities $a_\xi <^* x <^* a$ for all $\xi$. Without any loss of generality, we can suppose that $0 \leqslant x(n) < a(n)$ for all $n$. Given any $n$, let $x'(n)$ be the largest integer satisfying $x'(n) \leqslant x(n)$. Then $x' \in \mathbb{N}^{\mathbb{N}}$, $x' <^* a$ (since $x'(n) \leqslant x(n)$ for any $n$) and, obviously, $a_\xi <^* x'$ for every $\xi$ because all the $a_\xi$ belong to $\mathbb{N}$.

The next lemma concerns two order relations.

**Lemma 14** ($\lambda \geqslant \omega$ or $\lambda = 1$). *If $\leqslant$ is one of the relations $\preccurlyeq$, $\trianglelefteq$, then the structures $\langle \mathbb{N}^{\mathbb{N}}; \leqslant \rangle$ and $\langle \mathbb{R}^{\mathbb{N}}; \leqslant \rangle$ are equivalent with respect to the existence of $(\kappa, \lambda^*)$-gaps.*

*Proof.* To pass from $\langle \mathbb{R}^{\mathbb{N}}; \preccurlyeq \rangle$ to $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq \rangle$, we simply replace all the terms of a given gap in $\mathbb{R}^{\mathbb{N}}$ by the nearest integers from above. The orders $\preccurlyeq$ and $\trianglelefteq$ are obviously preserved under such a change. (This argument does not work for the orders $\leqslant^*$ and $\leqslant_{\mathrm{fro}}$.)

The following lemma obtains 'weaker' gaps from 'stronger' ones.

**Lemma 15** ($\lambda \geqslant \omega$). *If $D$ is one of the sets $\mathbb{N}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$, then*

(i) *any $(\kappa, \lambda^*)$-gap in the structure $\langle D; \preccurlyeq \rangle$ remains a gap in $\langle D; \trianglelefteq \rangle$;*

(ii) *any $(\kappa, \lambda^*)$-gap in $\langle D; \leqslant_{\mathrm{fro}} \rangle$ remains a gap in $\langle D; \leqslant^* \rangle$;*

(iii) *any $(\kappa, \lambda^*)$-gap in the structure $\langle D; \preccurlyeq \rangle$ remains a gap in $\langle D; \leqslant_{\mathrm{fro}} \rangle$.*

*Proof.* (i) Assume the opposite: the pair $\langle \{a_\xi\}_{\xi<\kappa}, \{b_\eta\}_{\eta<\lambda}\rangle$ is a $(\preccurlyeq)$-gap in $\mathbb{N}^\mathbb{N}$ but $x \in \mathbb{N}^\mathbb{N}$ satisfies $a_\xi \lhd x \lhd b_\eta$ for all $\xi < \kappa$ and $\eta < \lambda$. As $\kappa$ and $\lambda$ are limit ordinals, it follows that $a_{\xi+1} \lhd x \lhd b_{\eta+1}$. But $f \lhd g \prec h$ implies that $f \prec h$, and hence $a_\xi \lhd x \lhd b_\eta$, a contradiction.[7]

Claims (ii) and (iii) are proved similarly.

The next lemma completes the cycle of structures of $\mathbb{N}$-type with orders $\preccurlyeq$, $\leqslant_{\mathrm{fro}}$, $\leqslant^*$ with respect to the existence of gaps (except for limits and towers).

**Lemma 16** $(\lambda \geqslant \omega)$. *If the structure $\langle 2^\mathbb{N}; \leqslant^*\rangle$ has a $(\kappa, \lambda^*)$-gap, then such a gap exists in $\langle \mathbb{N}^\mathbb{N}; \preccurlyeq\rangle$ as well.*

*Proof.* Consider a gap $\langle \{a_\xi\}_{\xi<\kappa}, \{b_\eta\}_{\eta<\lambda}\rangle$ in the structure $\langle 2^\mathbb{N}; \leqslant^*\rangle$. For any $a \in 2^\mathbb{N}$ we define $\widetilde{a} \in \mathbb{N}^\mathbb{N}$ by $\widetilde{a}(n) = \sum_{i=0}^{n} 2^i a(i)$. Then the sequence $\{\widetilde{a}_\xi\}_{\xi<\kappa}$ is $(\prec)$-increasing, while the sequence $\{\widetilde{b}_\eta\}_{\eta<\lambda}$ is, accordingly, $(\prec)$-decreasing, and we have $\widetilde{a}_\xi \prec \widetilde{b}_\eta$ for all $\xi$, $\eta$. To prove that this is a $(\preccurlyeq)$-gap, assume the opposite and let $\widetilde{c} \in \mathbb{N}^\mathbb{N}$ be such that $\widetilde{a}_\xi \preccurlyeq \widetilde{c} \preccurlyeq \widetilde{b}_\eta$ for all $\xi$, $\eta$. Define $c \in 2^\mathbb{N}$ in such a way that $c(n) = 1$ if and only if $\widetilde{c}(n) \geqslant 2^n$. Then we easily see that $a_\xi \leqslant^* c \leqslant^* b_\eta$ for all $\xi$, $\eta$, a contradiction.

**7.3. Gaps and towers.** Here we prove assertions 3), (ii) (the case $\lambda = \omega$) and 3), (iii) of Theorem 5. According to assertion 3), (i) already established, to prove 3), (ii) it suffices to check that the structures $\langle \mathbb{R}^\mathbb{N}; \leqslant^*\rangle$ and $\langle \mathbb{N}^\mathbb{N}; \leqslant^*\rangle$ are equivalent with respect to the existence of $(\kappa, \omega^*)$-gaps. We shall prove this existence claim in such a way that assertion 3), (iii) will be proved simultaneously. Our strategy will be to obtain a $\kappa$-tower in $\langle \mathbb{N}^\mathbb{N}; \leqslant^*\rangle$ from the 'weakest' gap and then obtain the 'strongest' gap from such a tower. The plan is realized by the following two lemmas, first established by Hausdorff [6] for gaps and towers in $\langle \mathbb{R}^\mathbb{N}; \leqslant_{\mathrm{fro}}\rangle$ (see the results (I) in § 5) and then by Rothberger [12] for gaps ant towers in $\langle 2^\mathbb{N}; \leqslant^*\rangle$. We consider this question here in a more general context.

**Lemma 17.** *If the structure $\langle \mathbb{R}^\mathbb{N}; \leqslant^*\rangle$ has a $(\kappa, \omega^*)$-gap, then $\kappa$-gaps exist in the structure $\langle \mathbb{N}^\mathbb{N}; \leqslant^*\rangle$ and hence (see § 7.1) in any other non-dyadic* HOS *in the list* (2), *in particular in $\langle \mathbb{N}^\mathbb{N}; \leqslant_{\mathrm{fro}}\rangle$.*

*Proof.* Let $\langle \{a_\xi\}_{\xi<\kappa}, \{b_n\}_{n\in\mathbb{N}}\rangle$ be a $(\kappa, \omega^*)$-gap in $\langle \mathbb{R}^\mathbb{N}; \leqslant^*\rangle$. We can assume that $b_{n+1}(k) \leqslant b_n(k)$ for all $n$, $k$. If $a \in \mathbb{R}^\mathbb{N}$ satisfies $a \leqslant^* b_n$ for all $n$, then, for any $n$, let $\widetilde{a}(n)$ be the least positive integer such that $a(k) \leqslant b_n(k)$ for all $k \geqslant \widetilde{a}(n)$. The sequence $\{\widetilde{a}_\xi\}_{\xi<\kappa}$ is obviously $(\leqslant^*)$-increasing. Thus it suffices to show that it is unbounded in $\langle \mathbb{N}^\mathbb{N}; \leqslant^*\rangle$ (for then it contains a strictly $(<^*)$-increasing cofinal subsequence). Assume the opposite and let $c \in \mathbb{N}^\mathbb{N}$ be such that $\widetilde{a}_\xi \leqslant^* c$ for all $\xi < \kappa$.

Define $k_{-1} = 0$ and then, by induction, $k_n = \max\{c(n) + 1, k_{n-1}\}$. Put $a(k) = b_n(k)$ whenever $k$ satisfies $k_n \leqslant k < k_{n+1}$. (We also put $a(k) = b_0(k)$ for $k < k_0$.) It follows from our assumptions that then $a(k) \leqslant b_n(k)$ for all $k \geqslant k_n$, and hence

[7] We cannot prove (i) in the case of limits. Indeed, if a pair $\langle \{a_\xi\}_{\xi<\kappa}, b\rangle$ is a limit in $\langle \mathbb{N}^\mathbb{N}; \preccurlyeq\rangle$ and (assuming the opposite) $a_\xi \lhd x \lhd b$, then $a_\xi \prec x$ still holds for all $\xi$. However it is impossible to deduce that $x \prec b$.

$a \leqslant^* b_n$ for every $n$. It now suffices to prove that $a_\xi \leqslant^* a$ for all $\xi$. For then $a$ fills the given gap, a contradiction.

Recall that $\widetilde{a}_\xi \leqslant^* c$. Hence there is an index $N$ such that $\widetilde{a}_\xi(n) \leqslant c(n) \leqslant k_n$ for all $n \geqslant N$. Take any half-open interval of the form $I_n = (k_n, k_{n+1}]$, $n \geqslant N$. Then $a_\xi(k) \leqslant b_n(k) = a(k)$ for all $k \in I_n$ because $\widetilde{a}_\xi(n) \leqslant k_n$. Thus $a_\xi(k) \leqslant a(k)$ for all $k > k_N$. Therefore $a_\xi \leqslant^* a$, as required.

**Lemma 18.** *If a $\kappa$-tower exists in the structure $\langle \mathbb{N}^\mathbb{N}; \leqslant_{\mathrm{fro}} \rangle$, then a $(\kappa, \omega^*)$-gap exists in $\langle \mathbb{N}^\mathbb{N}; \leqslant^* \rangle$.*

*Proof.* Let $\{c_\xi\}_{\xi<\kappa}$ be a $(\leqslant_{\mathrm{fro}})$-tower in $\mathbb{N}^\mathbb{N}$. We can assume that each $c_\xi$ is a strictly increasing sequence as an element of $\mathbb{N}^\mathbb{N}$ (otherwise put $c'_\xi(n) = n + \sum_{k\leqslant n} c_\xi(k)$). Thus $c_\xi(n) \geqslant n$. Define $a_\xi \in \mathbb{N}^\mathbb{N}$ for any $\xi$ in such a way that $a_\xi(k) = n$ whenever $c_\xi(n) \leqslant k < c_\xi(n+1)$. Then $a_\xi$, as a map $\mathbb{N} \to \mathbb{N}$, is in some sense an inverse of $c_\xi$. Clearly $a_\eta \leqslant^* a_\xi$ for all $\xi < \eta < \kappa$. We claim that $a_\eta <^* a_\xi$ strictly for all $\xi < \eta < \kappa$. Indeed, if $c_\xi(n) < c_\eta(n)$ (and this happens for infinitely many $n$ since $c_\xi \leqslant^* c_\eta$), then by definition $n - 1 = a_\eta(c_\eta(n) - 1) < a_\xi(c_\eta(n) - 1) = n$.

Thus $\{a_\xi\}_{\xi<\kappa}$ is a strictly $(<^*)$-decreasing sequence in $\mathbb{N}^\mathbb{N}$. Note that every $a_\xi$ is an increasing function (as a map $\mathbb{N} \to \mathbb{N}$), possibly non-strictly increasing. It is also unbounded, that is, $\mathbf{0} \prec a_\xi$, where $\mathbf{0} \in 2^\mathbb{N}$ is the constant 0, but $a_\xi(k) \leqslant k$ for all $k$. We claim that

$(*)$ there are no elements $a \in \mathbb{N}^\mathbb{N}$ such that $\mathbf{0} \prec a$ and $a \leqslant^* a_\xi$ for all $\xi$.

Indeed, assume the opposite and let $a \in \mathbb{N}^\mathbb{N}$ be a counterexample. Without any loss of generality, we can assume that $a$ is an increasing function (non-strictly), and $a(n+1) \leqslant a(n) + 1$ for all $n$. Then there is a unique strictly increasing function $c \in \mathbb{N}^\mathbb{N}$ such that $a(k) = n$ for all $n$ and $k$ satisfying $c(n) \leqslant k < c(n+1)$. Since $a \leqslant^* a_\xi$, we have $c_\xi \leqslant^* c$ for all $\xi$. But this contradicts the choice of a tower.

It follows that the pair of sequences $\langle \{b_n\}_{n\in\mathbb{N}}, \{a_\xi\}_{\xi<\kappa} \rangle$ is a $(\omega, \omega_1^*)$-gap in $\langle \mathbb{N}^\mathbb{N}; \leqslant^* \rangle$ if we put $b_n = \omega \times \{n\}$ (where $n$ is a constant). Now to obtain a $(\omega_1, \omega^*)$-gap, put $a'_\xi(k) = k - a_\xi(n)$ (recall that $a_\xi(k) \leqslant k$) and $b'_n(k) = \max\{0, k - n\}$ for all $\xi$, $k$, $n$.

This completes the proof of assertion 3) of Theorem 5.

**7.4. Limits.** We start with the proof of assertion 4) of Theorem 5. To prove 4), (i), that is, the absence of $\kappa$-limits in $\langle \mathbb{N}^\mathbb{N}; \leqslant_{\mathrm{fro}} \rangle$, note that every $a \in \mathbb{N}^\mathbb{N}$ has an exact $(\leqslant_{\mathrm{fro}})$-predecessor $a_- \in \mathbb{N}^\mathbb{N}$ defined by $a_-(n) = \max\{a(n) - 1, 0\}$ for all $n$.

Furthermore, assertion 4), (ii) follows from Lemma 12.

The remaining parts of assertion 4) need some effort.

4), (iii). Suppose that $\{a_\xi\}_{\xi<\kappa}$ is a tower in $\langle \mathbb{N}^\mathbb{N}; \leqslant_{\mathrm{fro}} \rangle$. We get a limit $\langle 0, 0, 0, \dots \rangle = \lim_{\xi\to\kappa} c_\xi$ in $\langle \mathbb{R}^\mathbb{N}; \leqslant_{\mathrm{fro}} \rangle$, where $c_\xi(n) = \frac{1}{a_\xi(n)}$. (For any $\xi$, there may be finitely many cases of division by 0 in this formula. The results of these can be set equal to, say, 1.) The converse is proved similarly.

4), (iv). The equivalence follows from Lemma 14. The construction of a tower resembles the construction in the final part of the proof of Lemma 18. Consider a $\kappa$-limit $a = \lim_{\xi\to\kappa} a_\xi$ in the structure $\langle \mathbb{N}^\mathbb{N}; \preccurlyeq \rangle$, where $a_\xi \prec a_\eta$ for all $\xi < \eta < \kappa$. Put $b_n(k) = \max\{0, a(k) - n\}$. Then $\{b_n\}_{n\in\mathbb{N}}$ is a $(\leqslant_{\mathrm{fro}})$-descending sequence, and the pair $\langle\{a_\xi\}_{\xi<\kappa},\{b_n\}_{n\in\mathbb{N}}\rangle$ is a $(\kappa,\omega^*)$-gap in $\langle\mathbb{N}^\mathbb{N};\leqslant_{\mathrm{fro}}\rangle$. To derive a $\kappa$-tower in $\langle\mathbb{N}^\mathbb{N};\leqslant_{\mathrm{fro}}\rangle$, apply Lemma 17.

4), (v). We say that a tower $\{c_\xi\}_{\xi<\kappa}$ (in any Hausdorff structure) is *regular* if and only if it satisfies the following condition.

$(**)$ For every $\xi<\kappa$ there is an ordinal $\eta$, $\xi<\eta<\kappa$, and a number $n_0$ such that $c_\eta(n)\geqslant c_\xi(n+1)$ for all $n\geqslant n_0$. In other words, it is required that for every $\xi$ there is an ordinal $\eta>\xi$ satisfying $c_\xi^+\leqslant^* c_\eta$, where $c_\xi^+(n)=c_\xi(n+1)$ for all $n$.

The regularity in this sense hardly follows from the definition of a tower. Accordingly, it seems to us that the proof of $2\Rightarrow 3$ in [10], Theorem 14 (the existence of $\kappa$-limits in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$ follows from the existence of $\kappa$-towers for the same $\kappa$) contains a gap in its key part, Claim 5 on p. 454. On the other hand, we know of no example of a non-regular tower. Note also that the regularity holds in the case when $\{c_\xi\}$ is a scale. Hence $\kappa$-scales do indeed generate $\kappa$-limits in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$.

**Lemma 19.** *If the structure $\langle\mathbb{N}^\mathbb{N};\leqslant_{\mathrm{fro}}\rangle$ has a $\kappa$-tower, then there is a regular cardinal $\kappa'<\kappa$ such that the structure $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$ has $\kappa'$-limits.*

*Proof.* It follows from assertion 2), (i) of Theorem 5 that there is a $\kappa$-tower $\{c_\xi\}_{\xi<\kappa}$ in the structure $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$. Then there is a regular tower $\{c'_\xi\}_{\xi<\kappa'}$ of length $\kappa'\leqslant\kappa$ in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$ satisfying $c_\xi\preccurlyeq c'_\xi$ for all $\xi<\kappa'$. (One can even achieve a stronger condition: $c'_{\xi+1}(n)\geqslant c'_\xi(n+1)$ for all $\xi$ and $n$. Define $c'_\xi\in\mathbb{N}^\mathbb{N}$ by induction on $\xi$ in such a way that $c_{\xi+1}\preccurlyeq c'_{\xi+1}$ and $c'_{\xi+1}(n)\geqslant c'_\xi(n+1)$ for all $n$ in the step $\xi\mapsto\xi+1$ and, in the limit steps $\lambda<\kappa$, if $\{c'_\xi\}_{\xi<\lambda}$ is not yet a tower in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$, we take an element $c'_\lambda\in\mathbb{N}^\mathbb{N}$ such that $c_\lambda\preccurlyeq c'_\lambda$ and $c'_\xi\preccurlyeq c'_\lambda$ for every $\xi<\lambda$.)

Following the proof of Lemma 18, we define $a_\xi\in\mathbb{N}^\mathbb{N}$ for $\xi<\kappa'$ using the regular tower $\{c'_\xi\}_{\xi<\kappa'}$ already defined in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$. Now, if $\xi<\eta<\kappa'$ and the inequality $c'_\eta(n)\geqslant c'_\xi(n+1)$ holds for all $n\geqslant n_0$, we obtain $a_\eta<_{\mathrm{fro}}a_\xi$ (not just $a_\eta<^*a_\xi$) in Lemma 18 and hence $\{a_\xi\}$ has a cofinal strictly $(<_{\mathrm{fro}})$-decreasing subsequence. Moreover, the limit terms of such a subsequence form a cofinal and now $(\prec)$-decreasing subsequence. Therefore, by $(*)$ there is a $\kappa'$-limit in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$.

4), (vi). Any $\kappa$-limit in $\langle 2^\mathbb{N};\leqslant^*\rangle$ remains a $\kappa$-limit in $\langle\mathbb{R}^\mathbb{N};\leqslant^*\rangle$ by Lemma 13, while any $\kappa$-tower in $\langle\mathbb{N}^\mathbb{N};\leqslant_{\mathrm{fro}}\rangle$ can be transformed into a $\kappa$-limit in $\langle\mathbb{R}^\mathbb{N};\leqslant^*\rangle$ as follows. First we convert the tower into a $\kappa$-tower $\{a_\xi\}_{\xi<\kappa}$ in $\langle\mathbb{N}^\mathbb{N};\preccurlyeq\rangle$ that consists only of increasing elements (sequences) $a_\xi\in\mathbb{N}^\mathbb{N}$. Then, following the proof of assertion 4), (iv), we put $c_\xi(n)=\frac{1}{a_\xi(n)}$. We claim that $\{c_\xi\}_{\xi<\kappa}$ is a $\kappa$-limit in $\langle\mathbb{R}^\mathbb{N};\leqslant^*\rangle$ (not only in $\langle\mathbb{R}^\mathbb{N};\leqslant_{\mathrm{fro}}\rangle$, as in 4), (iv)) with the limit value $\lim_{\xi\to\kappa}c_\xi=\mathbf{0}$. Indeed, assume the opposite and let $x\in\mathbb{R}^\mathbb{N}$ be such that $\mathbf{0}<^*x\leqslant^*c_\xi$ for all $\xi$. The set $D=\{k\colon x(k)\neq 0\}$ is infinite as $\mathbf{0}<^*x$ strictly. We write $D=j_0<j_1<j_2<\cdots$ and put $a(k)=\frac{1}{x(k)}$ for $k\in D$. Clearly, $x\restriction D\leqslant^*c_\xi\restriction D$ and hence $a_\xi\restriction D\leqslant^*a$ for all $\xi$. Now take any strictly increasing element $b\in\mathbb{N}^\mathbb{N}$ satisfying $b(k)\geqslant a(j_{n+1})$ whenever $j_n\leqslant k<j_{n+1}$. Then $a_\xi\leqslant^*b$ because $a_\xi$ is also increasing. Therefore the transfinite sequence $\{a_\xi\}_{\xi<\kappa}$ is bounded. But this sequence is a tower. This is a contradiction.

Let us prove the converse. Consider an arbitrary $(\leqslant^*)$-limit $\{c_\xi\}_{\xi<\kappa}$ in $\mathbb{R}^\mathbb{N}$. To simplify the argument, we assume that the sequence $\{c_\xi\}_{\xi<\kappa}$ is $(\leqslant^*)$-decreasing, the limit value $\lim_{\xi\to\kappa} c_\xi$ is $\mathbf{0}$ (the constant 0), and all the values $c_\xi(n)$ are non-negative. Put $D_\xi = \{n\colon c_\xi(n) = 0\}$ and let $h_\xi$ be the characteristic function of $D_\xi$. The sequence of functions $h_\xi$ is $(\leqslant^*)$-increasing. Hence the proof will be complete if we can show that $\lim_{\xi\to\omega_1} h_\xi = \mathbf{1}$ (the constant 1) in $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$.

Suppose that this is not the case: there is an $h \in 2^{\mathbb{N}}$ such that $h_\xi \leqslant^* h <^* \mathbf{1}$ for all $\xi$. Then the set $D = \{n\colon h(n) = 1\}$ is co-infinite in $\mathbb{N}$ and we have $D_\xi \subseteq^* D$ for all $\xi$ since $h_\xi \leqslant^* h$. It follows that the infinite set $Z = \mathbb{N} \setminus D$ has finite intersection with each of the $D_\xi$. This enables us to define $a_\xi(k) = \frac{1}{c_\xi(k)}$ for all $k \in Z$ and all $\xi$. (For each $\xi$, finitely many divisions by 0 can be treated as above.) The sequence of functions $a_\xi\colon Z \to \mathbb{N}$ is $(\leqslant^*)$-increasing (at least non-strictly) because the sequence $\{c_\xi\}$ is $(\leqslant^*)$-decreasing. Moreover, the sequence $\{a_\xi\}$ is $(\leqslant^*)$-unbounded in the family $\mathbb{N}^Z$ of all functions $a\colon Z \to \mathbb{N}$ because the given sequence $\{c_\xi\}$ is a limit (and remains a limit if we restrict all its terms to $Z$). It follows that $\{a_\xi\}$ has a strictly $(<^*)$-increasing subsequence. Thus we have a tower in $\langle Z; \leqslant^*\rangle$. To transform it into a tower in $\mathbb{N}^{\mathbb{N}}$, just use any bijection of $D$ onto $\mathbb{N}$.

Note that by 4), (vi), $(\leqslant^*)$-limits in $\mathbb{R}^{\mathbb{N}}$ are of at least two different types: those homological to towers in $\mathbb{N}^{\mathbb{N}}$, and those homological to $(\leqslant^*)$-limits in $2^{\mathbb{N}}$ (or, equivalently, in $\mathbb{N}^{\mathbb{N}}$).

4), (vii). The equivalence of the structures $\langle \mathbb{N}^{\mathbb{N}}; \trianglelefteq\rangle$ and $\langle \mathbb{R}^{\mathbb{N}}; \trianglelefteq\rangle$ with respect to the existence of $\kappa$-limits follows from Lemma 14. Furthermore, any $\kappa$-limit in $\langle \mathbb{N}^{\mathbb{N}}; \preccurlyeq\rangle$ can easily be transformed into a decreasing limit $\{x_\xi\}_{\xi<\kappa}$ with limit value $\mathbf{0}$ (the constant 0). Thus we have $\mathbf{0} \prec x_\eta \prec x_\xi$ whenever $\xi < \eta < \kappa$, and there is no $x \in \mathbb{N}^{\mathbb{N}}$ such that $\mathbf{0} \prec x \prec x_\xi$ for all $\xi$. We can assume that every element $x_\xi$ is increasing, that is, $x_\xi(n) < x_\xi(n+1)$ for all $n$, as otherwise each $x_\xi$ can be replaced by $x'_\xi$, where $x'_\xi(n) = n + \sum_{k\leqslant n} x_\xi(k)$ for all $n$. We claim that the sequence $\{x_\xi\}_{\xi<\kappa}$ is a limit in $\langle \mathbb{N}^{\mathbb{N}}; \trianglelefteq\rangle$.

Indeed, assume the opposite and let $x \in \mathbb{N}^{\mathbb{N}}$ be such that $\mathbf{0} \lhd x \lhd x_\xi$ for all $\xi < \kappa$. Then $x \prec x_\xi$ for all $\xi$ (see the proof of Lemma 15). Put $y(n) = \max\{x(k)\colon k \leqslant n\}$, so that $y \in \mathbb{N}^{\mathbb{N}}$ is an increasing function (possibly non-strict), and hence we have not only $\mathbf{0} \lhd y$ but also $\mathbf{0} \prec y$. It remains to check that we still have $y \prec x_\xi$ for all $\xi$: this yields the desired contradiction. Basically, it is enough to prove that $y \leqslant^* x_\xi$ for all $\xi$.

We claim that $y \leqslant^* x_\xi$. Since $x \prec x_\xi$, there is a number $n_0$ such that $x(n) < x_\xi(n)$ for all $n \geqslant n_0$. Furthermore, as $x_\xi$ is an increasing function, there is an $n_1 \geqslant n_0$ such that $\max_{k<n_0} x(k) < x_\xi(n)$ for all $n \geqslant n_1$. Thus, we have $x(k) < x_\xi(n)$ whenever $n \geqslant n_1$ and $k \leqslant n$. It follows from the construction that $y(n) < x_\xi(n)$ for all $n \geqslant n_1$, as required.

**7.5. Gaps and limits.** The proof of the last part of Theorem 5 is based on the following lemma (see [12] for the case $\kappa = \omega_1$).

**Lemma 20.** *If the structure $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^*\rangle$ has a $\kappa$-tower, then $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$ has a $\kappa'$-limit for some uncountable cardinal $\kappa' \leqslant \kappa$. In particular, since $\omega$-limits do not exist, the existence of an $\omega_1$-tower in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^*\rangle$ implies the existence of a $\omega_1$-limit in $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$. In addition, if there are no $\kappa$-limits in $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$, then every $\kappa$-tower in the structure $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^*\rangle$ is a $\kappa$-scale.*

*Proof.* Given an infinite set $x \subseteq \mathbb{N}$, we write $\varphi_x$ for the unique increasing bijection $\mathbb{N} \overset{\text{onto}}{\to} x$. Suppose that $\{f_\alpha\}_{\alpha<\kappa}$ is a $\kappa$-tower in $\langle \mathbb{N}^{\mathbb{N}}; \leqslant^*\rangle$. We can assume that all the $f_\alpha$ are strictly increasing functions (otherwise replace $f_\alpha$ by $g_\alpha(k) = k + \sum_{n=0}^{k} f_\alpha(n)$). We shall define a $(\subset^*)$-decreasing sequence $\{x_\alpha\}_{\alpha<\kappa'}$ of infinite sets $x_\alpha \subseteq \mathbb{N}$ such that $f_\alpha \leqslant^* \varphi_{x_\alpha}$ for all $\alpha < \kappa'$. The ordinal $\kappa' \leqslant \kappa$ will be determined in the course of the construction.

Suppose that $\lambda \leqslant \kappa$ and all the $x_\alpha$, $\alpha < \lambda$, have been defined.

*Case* 1. There is an infinite set $x \subseteq \mathbb{N}$ such that $x \subseteq^* x_\alpha$ for all $\alpha < \lambda$. Then $f_\alpha \leqslant^* \varphi_{x_\alpha} \leqslant^* \varphi_x$ for all $\alpha$. Therefore $\lambda < \kappa$. Clearly, there is an infinite set $y \subset^* x$ satisfying $f_\alpha \leqslant^* \varphi_y$. Put $x_\lambda = y$.

*Case* 2. There is no such set $x$. Then the sequence $\{x_\alpha\}_{\alpha<\lambda}$ can easily be converted into a $\lambda$-limit in the structure $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$, and hence we can take $\kappa' = \lambda$.

We now prove the additional claim in the lemma. Consider an arbitrary function $f \in \mathbb{N}^{\mathbb{N}}$ and suppose on the contrary that $f \nleqslant^* f_\alpha$ for some $\alpha < \kappa$. It can be assumed that $f$ is strictly increasing, and so are all the $f_n$. Then every set $x_\alpha = \{n \colon f_\alpha(n) < f(n)\}$ is infinite, and we have $x_\beta \subseteq^* x_\alpha$ whenever $\alpha < \beta < \kappa$ since $f_\alpha \leqslant^* f_\beta$. We claim that there is an infinite set $x \subseteq \mathbb{N}$ satisfying $x \subseteq^* x_\alpha$ for all $\alpha$. Indeed, if the sequence $\{x_\alpha\}_{\alpha<\kappa}$ contains a cofinal strictly decreasing subsequence, then such a set $x$ does exist since otherwise the subsequence would give us a $\kappa$-limit in $\langle 2^{\mathbb{N}}; \leqslant^*\rangle$. If cofinal strictly decreasing subsequences do not exist, we have $\forall\, \xi > \gamma (x_\xi \equiv^* x_\gamma)$ for some $\gamma < \kappa$, and hence $x = x_\gamma$ is the required set.

Thus let $x$ be a set as indicated. Then $f_\alpha \restriction x \leqslant^* f \restriction x$ (in the sense that the set $\{n \in x \colon f(n) < f_\alpha(n)\}$ is finite) for all $\alpha$. Let

$$x = \{0 = i_0 < i_1 < \cdots < i_n < \cdots\}.$$

We define $g(k) = f(i_{n+1})$ whenever $i_n \leqslant k < i_{n+1}$. Since $f$ and all the $f_\alpha$ are increasing functions, we have $f_\alpha \leqslant^* g$ for all $\alpha$. But this contradicts the assumption that the sequence of all $f_\alpha$ is a tower.

The proof of Theorem 5 is complete.

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