On the infinitary pantachie of Du Bois Reymond*

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Abstract

Refining an earlier Du Bois Reymond's vague notion, Hausdorff defined a *pantachy* to be a maximal *chain* (a linearly ordered subset) in a partially ordered set of certain type, for instance, the set $\mathbb{N}^{\mathbb{N}}$ under eventual domination. The axiom of choice **AC** implies the existence of a pantachy in any partially ordered set. However the pantachy existence theorem fails in the absense of **AC**, and moreover, even if **AC** is assumed, hence pantachies do exist, one may not be able to come up with an individual, effectively defined example of a pantachy.

1 Introduction

The problem of infinity has been one of the most common topics of discussion in mathematics since the epoch of calculus of infinitesimals of XVII – XVIII centuries. Once of frequent use in the early era of calculus, infinite and infinitesimal quantities were condemned as mathematically inconsistent by both practicing mathematicians (see, e.g., Euler [5, Chapter III]) and philosophers, and eventually removed from rigorous mathematics by Cauchy, Weierstrass, Dedekind and their contemporaries.

Yet essentially in the middle of this period of general exorcism of the infinite and infinitesimal, Du Bois Reymond [3] came up with a rigorous notion leading to infinities and infinitesimals. Indeed define the *rate of growth* partial order $\leqslant_{\rm RG}$ on positive real functions so that $f \leqslant_{\rm RG} g$ iff the limit $\mathtt{lim}_{x\to+\infty} \frac{g(x)}{f(x)}$ exists and is > 0, and $f <_{\rm RG} g$ iff $\mathtt{lim}_{x\to+\infty} \frac{g(x)}{f(x)} = +\infty$. This ordering of functions was known long before Du Bois Reymond, but he was the first who considered $\leqslant_{\rm RG}$ and $<_{\rm RG}$ as relations on the whole totality of positive real functions, so that $f <_{\rm RG} g$ is understood that the quantity associated with g is essentially larger than one associated with f. Identifying ordinary real numbers with corresponding constant functions, we easily obtain "infinitely large" quantities as those associated with functions f such that $\mathtt{lim}_{x\to+\infty} f(x) = +\infty$, as well as "infinitesimals" associated with functions f such that $\mathtt{lim}_{x\to+\infty} f(x) = 0$.

It was also demonstrated in [3] that, unlike the order of the real line $\mathbb{R}$, the ordering $<_{\rm RG}$ is not countably cofinal: for any countable collection $\{f_n\}_{n\in\mathbb{N}}$ of positive real functions there is a function f satisfying $f_n <_{\rm RG} f$ strictly for all n.[1] Therefore we may think of a variety of "degrees of infinity". Later on, in a

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[1] Let $f(x) = x\ \mathtt{sup}_{n\le x} f_n(x)$. This was the first application of the diagonal method.

monograph [4], Du Bois Reymond stipulated that the totality of all real functions ordered by $<_{\rm RG}$, which he called *the infinitary pantachy*, might serve as an extension of the real line, where infinitesimal and infinitely large quantities coexist with usual reals (corresponding to constant functions), thus manifesting a sort of infinity which exceeds the infinity of the real line. This idea was met with mixed reception. In particular, Hausdorff [8, 9] noted that the obvious existence of $\leqslant_{\rm RG}$-incomparable functions makes *the infinitary pantachy* rather useless in the role of an extended analytic domain. (See more on controversies around Du Bois Reymond's approach in [6].) Instead, Hausdorff suggested to consider *maximal linearly ordered* sets of functions (or infinite real sequences, that can be ordered the same way), in the sense of $<_{\rm RG}$ or any other similar order based on the comparison of behaviour of functions or sequences at infinity. He called such maximal linearly ordered sets *pantachies*.

Hausdorff [8, 9] proved the existence of a pantachy in any partially ordered set. This result was one of the earliest explicit applications of the axiom of choice **AC** (or rather of *the maximality principle*, one of basic corollaries of **AC**). And, typically for the **AC**-based existence proofs, Hausdorff's argument did not produce anything near a concrete, individual, effectively defined example of a pantachy. Haudorff writes:

> Since the attempt to actually legitimately construct a pantachy seems completely hopeless, it would now be a matter of gathering information ... about the order type of any pantachy ... . ([8], p. 110.) [2]

Working in this direction, Hausdorff proved, in particular, that any pantachy is uncountably cofinal and uncountably coinitial — a type of infinity rather uncommon for mathematics of the early 1900s. Yet those studies left open the major problem of *effective existence* of pantachies. One may ask:

(A) can the pantachy existence be established not assuming **AC**, and

(B) even assuming **AC**, can one define an individual example of a pantachy.

Advances in modern set theory (in Russian see our monographs [1], [2] and survey papers [14], [15]) lead to **the negative answer** both for the $\leqslant_{\rm RG}$-ordering of positive functions and for a variety of similar partial orderings. This is the main result of this paper, and it supports Haudorff's observation cited above. The result is not unexpected. The unexpected feature is that we'll have to apply two difficult special results in set theory related to Solovay's models (propositions 6 and 7), since the basic technique of Solovay's models does not seem to be sufficient in this case.

2 Preliminaries

A *partial quasi-order*, PQO, is any *transitive* and *reflexive* binary relation $\leq$. An *associated equivalence relation* $x \equiv y$ iff $x \leq y \wedge y \leq x$ and an *associated strict partial order* $x < y$ iff $x \leq y \wedge y \not\leq x$ are defined on the same domain. If a PQO $\leq$ satisfies the *antisymmetry* condition $x \leq y \wedge y \leq x \Longrightarrow x = y$ then it is called a *partial order*, PO. A PQO is *linear*, LQO for brevity, if we have $x \leq y \vee y \leq x$ for all x, y in its domain. A *linear order*, or LO, is any antisymmetric LQO.

[2] English translation taken from [10].

An PQO $\langle X\,;\leq\rangle$ (meaning: X is the domain of $\leq$) is *of countable cofinality* iff there is a set $Y\subseteq X$, at most countable and *cofinal* in X, that is, if x belong to X then there exists an element $y\in Y$ such that $x\leq y$.

A *pantachy* in a PQO $\langle X\,;\leq\rangle$ is any set $P\subseteq X$ such that $\leq\restriction P$ is an LO and (the maximality!) if $x\in X\smallsetminus P$ then $\leq\restriction(P\cup\{x\})$ is **not** an LO.

If $\xi<\omega_1$ then 2^ξ is the set of all binary sequences of length ξ, and $2^{<\omega_1}=\bigcup_{\xi<\omega_1}2^\xi$. By $<_{\texttt{lex}}$ we denote the lexicographical order on $2^{<\omega_1}$, that is, if $s,t\in 2^{<\omega_1}$ then $s<_{\texttt{lex}}t$ iff $s\not\subset t$, $t\not\subset s$, and the least ordinal $\xi<\operatorname{\texttt{dom}}s,\operatorname{\texttt{dom}}t$ with $s(\xi)\neq t(\xi)$ satisfies $s(\xi)<t(\xi)$. Put $s\leqslant_{\texttt{lex}}t$ iff $s=t$ or $s<_{\texttt{lex}}t$.

Lemma 1. *If $\xi<\omega_1$ then any set $C\subseteq 2^\xi$ is countably $\leqslant_{\texttt{lex}}$-cofinal.*

Proof. Elementary transfinite induction on ξ. □

A PQO $\langle X\,;\leq\rangle$ is *Borel* iff the set X is a Borel set in a suitable Polish space $\mathbb{X}$, and the relation $\leq$ is a Borel subset of $\mathbb{X}\times\mathbb{X}$.

Corollary 2. *Every Borel LQO $\leq$ is countably cofinal, and moreover, there is no strictly increasing ω_1-sequences.*

Proof. It was established in [7] (see also [12]) that if $\langle X\,;\leq\rangle$ is a Borel LQO then there is an ordinal $\xi<\omega_1$ and a Borel map $\vartheta:X\to 2^\xi$ such that we have $x\leq y$ iff $\vartheta(x)\leqslant_{\texttt{lex}}\vartheta(y)$ for all $x,y\in X$. Now use Lemma 1. □

3 The main technical theorem

As usual, **ZFC** and **ZF** are Zermelo – Fraenkel set theories resp. with and without the axiom of choice **AC**. The *principle of dependent choices* **DC** allows countable sequences of choices even in the case when the set $X_n\neq\varnothing$, in which the next choice x_n is to be made, itself depends not only on the index $n\in\mathbb{N}$, but also on the results x_k, $k<n$, of all previous choices.

Let **WIC** be the the sentence “there is a weakly inaccessible cardinal”, that is, an uncountable regular limit cardinal number. **WIC** cannot be proved in **ZFC**. Newertheless **ZFC** + **WIC** is considered as a legitimate extension of **ZFC** itself, and accordingly consistency proofs carried out in the assumption of the consistency of **ZFC** + **WIC** are considered as legitimate consistency proofs.

Theorem 3. *Suppose that* **WIC** *is consistent with the axioms of* **ZFC**.

Then, first, the following sentence is consistent with **ZFC**:

(i) *if $\leq$ is a Borel PQO on a (Borel) set $D\subseteq\mathbb{N}^{\mathbb{N}}$, $X\subseteq D$ is a* **ROD** *set, and $\leq\restriction X$ is a LQO, then $\leq\restriction X$ is of countable cofinality.*

And second, the following sentence is consistent with **ZF** + **DC**:

(ii) *if $\leq$ is a Borel PQO on a (Borel) set $D\subseteq\mathbb{N}^{\mathbb{N}}$, $X\subseteq D$ is any set, and $\leq\restriction X$ is a LQO, then $\leq\restriction X$ is of countable cofinality.*

Recall that **ROD** is the class of *real-ordinal definable* sets, that is, those definable by a set theoretic formula with reals and ordinals as parameters — the class of all sets that can be considered as "effectively defined". Any nonexistence result for the **ROD** domain is usually treated in the sense that there is no individual, effectively defined examples of sets of the type considered.

Thus it is consistent with **ZFC** that all **ROD** linear suborders of Borel PQOs are countably cofinal, and it is consistent with **ZF** + **DC** that all in general linear suborders of Borel PQOs are countably cofinal. Now let's explain how Theorem 3 leads to the negative answers to questions (A) and (B) in the end of Section 1.

Definition 4. Let a *DBR-order* (from Du Bois Reymond) be any Borel PQO $\langle X\,;\leq\rangle$ such that for any countable set $Y \subseteq X$ there is an element $x \in X$ such that $y < x$ (that is, $y \leq x$ but $x \not\leq y$) for all $y \in Y$. □

A pantachy in a DBR-order cannot be countably cofinal, so we obtain

Corollary 5 (of Theorem 3)**.** *First, it is consistent with* **ZFC** *that no DBR-order contains a* **ROD** *pantachy. Second, it is consistent with* **ZF**+**DC** *that no DBR-order contains a pantachy of any kind.* □

There are many notable orders of this type, see, e.g., [11]. For instance let $X = \mathbb{N}^{\mathbb{N}}$ (sequences of natural numbers). For $x, y \in \mathbb{N}^{\mathbb{N}}$ let $x \leqslant_{\text{RG}} y$ iff the limit $\mathtt{lim}_{n\to\infty} \frac{y(n)}{x(n)}$ exists and is > 0. Easily $\langle \mathbb{N}^{\mathbb{N}}\,; \leqslant_{\text{RG}}\rangle$ is a DBR-order. (If $x_0, x_1, x_2, \ldots \in \mathbb{N}^{\mathbb{N}}$ then put $x(k) = k\,\mathtt{max}_{n\leq k}\, x_n(k)$ for all k; $x_n <_{\text{RG}} x$ for all n.) Thus by Corollary 5 it is consistent with **ZF** + **DC** that there is no pantachy in the structure $\langle \mathbb{N}^{\mathbb{N}}\,; \leqslant_{\text{RG}}\rangle$, and it is consistent with **ZFC** that there is no **ROD** pantachy in $\langle \mathbb{N}^{\mathbb{N}}\,; \leqslant_{\text{RG}}\rangle$. Thus *questions (A) and (B) in Section 1 answer in the negative* for the ordering $\langle \mathbb{N}^{\mathbb{N}}\,; \leqslant_{\text{RG}}\rangle$, and hence for $\langle (\mathbb{R}^+)^{\mathbb{N}}\,; \leqslant_{\text{RG}}\rangle$, in which $\mathbb{N}^{\mathbb{N}}$ is a cofinal subset.

The actual Du Bois Reymond's domain $\mathscr{F}^+$ of all real positive functions is **not** a set in a Polish space. Thus $\langle \mathscr{F}^+\,; \leqslant_{\text{RG}}\rangle$ is **not** a DBR-order. Nevertheless the negative result just obtained easily extends to $\langle \mathscr{F}^+\,; \leqslant_{\text{RG}}\rangle$. Indeed if $P \subseteq \mathscr{F}^+$ is a pantachy in $\langle \mathscr{F}^+\,; \leqslant_{\text{RG}}\rangle$ then $P \restriction \mathbb{N} = \{f \restriction \mathbb{N} : f \in P\}$ is a pantachy in $\langle (\mathbb{R}^+)^{\mathbb{N}}\,; \leqslant_{\text{RG}}\rangle$, and if P is **ROD** then so is $P \restriction \mathbb{N}$. Thus any pantachy-nonexistence result for $\langle (\mathbb{R}^+)^{\mathbb{N}}\,; \leqslant_{\text{RG}}\rangle$ implies a corresponding pantachy-nonexistence result for $\langle \mathscr{F}^+\,; \leqslant_{\text{RG}}\rangle$. It follows that *questions (A) and (B) in Section 1 answer in the negative* for Du Bois Reymond's ordered domain $\langle \mathscr{F}^+\,; \leqslant_{\text{RG}}\rangle$ as well.

4 The Solovay model

The proof of Theorem 3 involves *the Solovay model*, a model of set theory introduced in [16]. Basically, there are **two** Solovay models, that is,

(I) a model of **ZFC** in which all **ROD** sets of reals have some basic regularity properties, in particular, are Lebesgue measurable;

(II) a model of **ZF** + **DC** in which all sets of reals are Lebesgue measurable — it is equal to the class **HROD** of all hereditarily **ROD** sets [3] in (I).

[3] A set x is *hereditarily* **ROD** if x, all elements of x, all elements of elements of x, *et cetera*, are **ROD**. **HROD** is a transitive class containing all reals and all points of $\mathbb{N}^{\mathbb{N}}$.

The models are defined in the assumption that the sentence **WIC** ("there is a weakly inaccessible cardinal") is consistent with **ZFC**. Both models have the same reals and ordinals. Our applications of the models are based on the following two difficult results of modern set theory.

Proposition 6 (Stern [17]). *It holds in the Solovay model* (I) *that if* $\rho < \omega_1$ *then there is no* **ROD** ω_1*-sequence of pairwise different sets in the class* $\mathbf{\Sigma}^0_\rho$. □

Proposition 7 (Kanovei [13]). *It holds in the Solovay model* (I) *that if* $\le$ *is a* **ROD** *LQO on a set* $D \subseteq \mathbb{N}^{\mathbb{N}}$ *then there are an antichain* $A \subseteq 2^{<\omega_1}$ *and a* **ROD** *map* $\vartheta : D \stackrel{\text{onto}}{\longrightarrow} A$ *such that* $x \le y \iff \vartheta(x) \leqslant_{\texttt{lex}} \vartheta(y)$ *for all* $x, y \in D$. □

A set $A \subseteq 2^{<\omega_1}$ is an *antichain* if $s \not\subset t$ holds for every pair of $s \ne t$ in A. The lexicographic order $\leqslant_{\texttt{lex}}$ linearly orders any antichain $A \subseteq 2^{<\omega_1}$.

Using propositions 6 and 7, we'll prove the following result below:

Proposition 8. *Sentence* (i) *of Theorem 3 is true in the Solovay model* (I). *Therefore sentence* (ii) *of Theorem 3 is true in the Solovay model* (II).

The "therefore" claim here is an easy consequence of the first claim. Proposition 8 implies Theorem 3 since a sentence true in a model is consistent.

5 The proof

Here we prove Proposition 8. *We argue in the Solovay model* (I).

Accordingly to (i) of Theorem 3, suppose that $\le$ is a Borel PQO on a Borel set $D \subseteq \mathbb{N}^{\mathbb{N}}$, while $\equiv$ and $<$ are resp. the associated equivalence relation and the associated strict order, and in addition $X \subseteq D$ is a **ROD** set, and $\le \restriction X$ is a LQO. Our goal will be to show that X is countably $\le$-cofinal.

The restricted order $\le \restriction X$ is **ROD**, and hence, by Proposition 7, there is an antichain $A \subseteq 2^{<\omega_1}$ and a **ROD** map $\vartheta : X \stackrel{\text{onto}}{\longrightarrow} A$ such that $x \le y \iff \vartheta(x) \leqslant_{\texttt{lex}} \vartheta(y)$. If $\xi < \omega_1$ then let $A_\xi = A \cap 2^\xi$ and $X_\xi = \{x \in X : \vartheta(x) \in A_\xi\}$.

Case 1: there is an ordinal $\eta < \omega_1$ such that the set A_η is $\leqslant_{\texttt{lex}}$-cofinal in A. However, by Lemma 1, there is a set $A' \subseteq A_\eta$, countable and $\leqslant_{\texttt{lex}}$-cofinal in A_η, and hence $\leqslant_{\texttt{lex}}$-cofinal in A by the choice of η. If $s \in A'$ then pick an element $x_s \in X$ such that $\vartheta(x_s) = s$. Then $Y = \{x_s : s \in A'\}$ is a countable subset of X, $\le$-cofinal in X. This ends the proof of (i) of Theorem 3.

Case 2: not Case 1. That is, for any $\eta < \omega_1$ there is an ordinal $\xi < \omega_1$ and an element $s \in A_\xi$ such that $\eta < \xi$ and $t <_{\texttt{lex}} s$ for all $t \in A_\eta$.

Lemma 9. *The sequence of sets* $D_\xi = \{z \in D : \exists\, x \in X_\xi\, (z \le x)\}$ $(\xi < \omega_1)$ *has uncountably many different terms.*

Proof. As the sequence is $\subseteq$-increasing, it suffices to prove that for any $\eta < \omega_1$ there is an ordinal ξ, $\eta < \xi < \omega_1$, such that $D_\eta \subsetneqq D_\xi$. Let $\eta < \omega_1$. Then there exist: an ordinal ξ, $\eta < \xi < \omega_1$ and some $s \in A_\xi$ such that $t <_{\texttt{lex}} s$ for all $t \in A_\eta$. Take an element $z \in X_\xi$ such that $\vartheta(z) = s$. It remains to prove that $x \notin D_\eta$. Indeed otherwise we have $z \le x$ for some $x \in X_\eta$. By definition $t = \vartheta(x) \in A_\eta$, therefore $t <_{\mathtt{lex}} s$ by the choice of s. But on the other hand $s = \vartheta(z) \leqslant_{\mathtt{lex}} \vartheta(x) = t$ by the choice of ϑ, and this is a contradiction. $\square$

Recall that $\leq$ is a Borel relation, hence there is an ordinal $1 \leq \rho < \omega_1$ such that $\leq$ (as a set of pairs) belongs to the Borel class $\mathbf{\Sigma}^0_\rho$.

Lemma 10. *If $\xi < \omega_1$ then the set D_ξ belong to $\mathbf{\Sigma}^0_\rho$.*

Proof. By Lemma 1 there exists a countable set $A' = \{s_n : n < \omega\} \subseteq A_\xi$, $\leqslant_{\mathtt{lex}}$-cofinal in A_ξ. If $n < \omega$ then pick an element $x_n \in X_\xi$ such that $\vartheta(x_n) = s_n$. Then by the choice of ϑ any element $x \in X$ with $\vartheta(x) = s_n$ satisfies $x \equiv x_n$, where $\equiv$ is the equivalence relation on D associated with $\leq$. It follows that

$$D_\xi = \bigcup\nolimits_n Z_n \,, \quad \text{where} \quad Z_n = \{z \in D : z \leq x_n\}\,,$$

so each Z_n is a $\mathbf{\Sigma}^0_\rho$ set together with $\leq$. We conclude that D_ξ is a $\mathbf{\Sigma}^0_\rho$ set as a countable union of sets in $\mathbf{\Sigma}^0_\rho$. $\square$

The two lemmas contradict to Proposition 6, and the contradiction accomplishes the proof of Proposition 8 and Theorem 3.

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