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About Forms Equal to Zero at Each Vertex of a Cube

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Abstract—Conditions under which the form is equal to zero at each vertex of a cube are discussed. Codimensions of the spaces of these forms are calculated for certain values of the degree and dimension. Explicit expressions for the low-degree forms of this type are found.

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1. INTRODUCTION

A finite set of points with coordinates $\{a, b\}$ is called an (a, b)-cube over an arbitrary field $\mathbb{F}$, and the points themselves are called the vertices of the cube. The set of cube vertices lying on the hyperplane $x_k = a$ or $x_k = b$, is called a cube side.

All forms of degree d in n variables (together with the identically null form) create a liner space with the dimension

$$\frac{(n+d-1)!}{(n-1)!d!}.$$

The forms, equal to zero at each vertex of the cube, make up a linear space, which is determined by the linear relations for the coefficients of monomials corresponding to each vertex of the cube. However, calculation of the dimension of this subspace is complicated by the fact that the number of vertices of the cube rapidly increases with the dimension and corresponding relations contain many dependent ones.

2. HILBERT FUNCTIONS OF ±1-CUBES

Let the characteristic of the main field $\mathbb{F}$ be not equal to two. Let us identify the pairs of opposite vertices of the ±1-cube in the affine space $\mathbb{F}^n$ with 2^{n-1} points of the projective space $\mathbb{FP}^{n-1}$. These points make up the zero-dimensional set U_n, whose Hilbert polynomial is equal to constant 2^{n-1}. The codimension of the subspace of the forms of degree d in n variables, which are equal to zero at each vertex of the ±1-cube, is the value of the Hilbert function $\chi_n(d)$ of the set U_n. Note that the values of the Gilbert function coincide with the values of the Hilbert polynomial at all sufficiently large values of degree d [1, 2].

Proposition 1. For the sets U_n the following is fulfilled:

$$\chi_n(1) = n.$$
$$\chi_n(d+1) = \chi_n(d).$$
$$\chi_{n+1}(d+1) \geq 2\chi_n(d).$$
$$\text{If } d \geq n-1, \text{ then } \chi_n(d) = 2^{n-1}.$$

Proof. Linear forms can not become zero at each vertex of the cube. Hence, $\chi_n(1) = n$.

Let us consider linear operators on the spaces of linear forms $\theta_k: f \to x_k \cdot f$. If the linear subspace L consists of zero and some forms of degree d in n variables, which do not become zero simultaneously at all vertices of the cube, the subspace $\theta_1(L)$ consists of the forms of degree $(d + 1)$ in n variables, which do not become zero simultaneously at all vertices of the cube. Since dimensions of L and $\theta_1(L)$ coincide, $\chi_n(d + 1) \geq \chi_n(d)$.

Let us consider the direct sum of subspaces $\theta_1(L) \oplus \theta_{n+1}(L)$. This sum is direct, because the forms from the first subspace do are independent of x_{n+1} and each nonzero form of the second subspace depends on it. Let us show that each form from $\theta_1(L) \oplus \theta_{n+1}(L)$ is not equal to zero simultaneously at all vertices of the cube. Assume that $f \in \theta_1(L)$, $g \in \theta_{n+1}(L)$, and $f + g$ is equal to zero at each vertex. Let us associate each vertex of the cube **v** with the vertex **v** with opposite sign of the last coordinate: $v'_{n+1} = -v_{n+1}$. Since $f(\mathbf{v}) = f(\mathbf{v}')$ and $g(\mathbf{v}) = -g(\mathbf{v}')$, we obtain $g(\mathbf{v}) = 0$ for any **v**. The contradiction proves that, for some vertex **v**, the value of the form $f(\mathbf{v}) + g(v) \neq 0$. Hence, $\chi_{n+1}(d + 1) \geq 2\chi_n(d)$.

It is easy to show by induction on the number of variable n that $\chi_n(n - 1) = 2^{n-1}$. Since the Hilbert functions do not decrease and the values of the Hilbert function and the Hilbert polynomial coincide at large values of d, $\chi_n(d) = 2^{n-1}$ is fulfilled at all $d \geq n - 1$.

3. PROPERTIES OF THE FORMS EQUAL TO ZERO AT EACH VERTEX OF THE CUBE

Proposition 2. Let $f(x)$ be a polynomial in n variables of degree d over an arbitrary field $\mathbb{F}$. If a vertex of the cube in $\mathbb{F}^n$ does not lie on the hypersurface $f(\mathbf{x}) = 0$, the portion of vertices lying on the hypersurface does not exceed $1 - 2^{-\min(d, n)}$.

Proof. The upper boundary $1 - 2^{-n}$ is evident, because the cube has 2^n vertices. Let us show by the double induction on degree d and dimension n that the portion does not exceed $1 - 2^{-d}$.

Let $d = 1$. If $n = 1$, the cube has two vertices and the hypersurface is one point.

The step at $d = 1$. If each vertex of a cube side lies on the hyperplane, this hyperplane cannot pass through vertices on the opposite side. In this case, exactly a half of all vertices lie on the hyperplane. Otherwise, there are vertices on each of two opposite sides of the cube that do not lie on the hyperplane. Then, by the assumption of induction, no more than a half of vertices lie on each side on the hyperplane. Hence, the same is true for the whole cube.

The step at $d \geq 2$. If there are vertices on each of two opposite sides of the cube that do not lie on the hypersurface, then, by assumption of induction, the portions of vertices lying on the hypersurface do not exceed $1 - 2^{-d}$ on each side. Hence, the same is true for the whole cube.

Otherwise, without loss of generality, we can assume that all vertices with the coordinate $x_1 = a$ lie on the hypersurface and a vertex with the coordinate $x_1 = b$ does not lie on the hypersurface. The polynomial $f(\mathbf{x}) = g(x_2, ..., x_n) + (x_1 - a) \cdot h(\mathbf{x})$, where polynomial g is equal to zero at each vertex of the cube and the polynomial degree h does not exceed $d - 1$. The vertices of the cube lying at the intersection of the hyperplane $x_1 = b$ and the hypersurface $f(\mathbf{x}) = 0$ lie at the intersection of this hyperplane and hypersurface $h(\mathbf{x}) = 0$. By the assumption of induction, the portion of vertices lying on it does not exceed $1 - 2^{1-d}$. Hence, the portion of vertices of the whole cube does not exceed $(2 - 2^{1-d})/2 = 1 - 2^{-d}$.

Remark 1. It follows from the results obtained by Erdös [3] that boundary 1/2 is reachable over the field of characteristic zero only for linear functions depending on a small number of variables. If the function depends nontrivially on each of n variables, i.e., the corresponding hyperplane is not parallel to any coordinate axis, the portion of cube vertices lying on this hyperplane tends to zero as n increases. A similar result was obtained in [4] for second-degree polynomials having many monomials. On the contrary, over the field of characteristic two, the linear form $x_1 + ... + x_n$ becomes zero at a half of all vertices of the (0, 1)-cube. This means that boundary 1/2 is true in each dimension. For the same form, this portion is close to $1/p$ over the field of simple characteristic p, i.e., it does not tend to zero as the dimension increases.

Proposition 3. If a polynomial of degree d in n variables is equal to zero at each vertex of the n-dimensional (a, b)-cube, it does not have monomials in d different variables.

Proof. The proposition is evident if the number of variables n is strictly smaller than degree d. Let us consider the case of $n = d$. Any polynomial of degree d in d variables can be written as:

$$f(x_1, \cdots, x_d) = \alpha \prod_{i=1}^{d} (x_i - a) + g(x_1, \cdots, x_d),$$

where polynomial g does not have monomials in d variables. Note that, when $x = a$ and $x = b$, the values of the monomial x^m coincide with the values of the linear function

$$\frac{a^m}{a-b}(x-b) + \frac{b^m}{b-a}(x-a).$$

By replacing multiple occurrences of variables in monomials g by linear functions, we obtain a polynomial of degree no higher than $d - 1$ whose values coincide with the values of polynomial g at the vertices of the (a, b)-cube. According to Proposition 2, the portion of the cube vertices at which polynomial g becomes zero does not exceed $1 - 2^{1-d}$. On the other hand, the portion of the cube vertices at which the polynomial

$$\prod_{i=1}^{d} (x_i - a)$$

becomes zero is equal to $1 - 2^{-d}$. Hence, $\alpha = 0$.

It is easy to see that, when $n \geq d + 1$, if there is a polynomial in n variables of degree d with a monomial in d different variables $x_1, ..., x_d$, substitution of variable x_1 for variable x_i gives a polynomial in d variables with a monomial in d different variables for all indices $i \geq d + 1$. This is impossible.

4. EXPLICIT APPEARANCE OF SMALL-DEGREE FORMS EQUAL TO ZERO AT EACH VERTEX OF THE ±1-CUBE

Proposition 4. Quadratic forms equal to zero at each vertex of the ±1-cube can be written as:

$$\sum_{i=1}^{n} a_i x_i^2, \text{ were } \sum_{i=1}^{n} a_i = 0.$$

Moreover, for the sets U_n, the values of the Hilbert function

$$\chi_n(2) = \frac{n^2 - n}{2} + 1.$$

Proof. It follows from Proposition 3 that quadratic forms equal to zero at each vertex of the ±1-cube can be written as

$$\sum_{i=1}^{n} a_i x_i^2, \text{ were } \sum_{i=1}^{n} a_i = 0,$$

and make up a subset of dimension $(n - 1)$. The dimension of the space of all quadratic forms is

$$\frac{n^2 + n}{2}.$$

Hence,

$$\chi_n(2) = \frac{n^2+n}{2} - (n-1) = \frac{n^2-n}{2} + 1.$$

Proposition 5. If the cubic form $f(x_1, \cdots, x_n)$ is equal to zero at each vertex of the ± 1-cube, it can be written as

$$\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} x_j^2 \right), \tag{1}$$

where, for each index i, the condition

$$\sum_{j=1}^{n} a_{ij} = 0 \tag{2}$$

is met.

Moreover, for the sets U_n, the values of the Hilbert function

$$\chi_n(3) = \frac{n^3 - 3n^2 + 8n}{6}.$$

Proof. According to Proposition 3, if a cubic form is equal to zero at each vertex of the ± 1-cube, it has form (1). It is evident, that form f meeting condition (2) is equal to zero at each vertex of the cube. For each index i, the values of form f at two vertices of the cube that do not coincide only at the ith coordinate differ by

$$2\sum_{j=1}^{n} a_{ij}.$$

Hence, if form f is equal to zero at each vertex of the cube, condition (2) is met. All forms of type (1) meeting condition (2) make up a subspace of the dimension $n(n-1)$. The dimension of the linear space of all cubic forms is

$$\frac{n(n+1)(n+2)}{6}.$$

Since this space can be expanded into the direct sum of subspaces of dimensions $n(n-1)$ and $\chi_n(3)$, then:

$$\chi_n(3) = \frac{n(n+1)(n+2)}{6} - n(n-1)$$
$$= \frac{n^3 - (3n^2 + 8n)}{6}.$$

Proposition 6. If the forth-degree form $f(x_1, \cdots, x_n)$ is equal to zero at each vertex of the ± 1-cube, it can be written as

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_i x_j \left(\sum_{k=1}^{n} a_{ijk} x_k^2 \right) + \sum_{i=1}^{n} \sum_{j=1}^{n} (b_{ij} x_i^2 x_j^2), \tag{3}$$

where the condition

$$\sum_{k=1}^{n} a_{ijk} = 0 \tag{4}$$

is met for each pair of different indexes $i < j$, and, in addition, the condition

$$\sum_{i=1}^{b} \sum_{j=1}^{n} b_{ij} = 0 \tag{5}$$

is also met.

Moreover, for the sets U_n, the values of the Hilbert function

$$\chi_n(4) = \frac{n^4}{24} - \frac{n^3}{4} + \frac{23n^2}{24} - \frac{3n}{4} + 1.$$

Proof. According to Proposition 3, if the forth-degree form is equal to zero at each vertex of the ± 1-cube, it can be expressed as (3). It is evident that, when conditions (4) and (5) are met, form f is equal to zero at each vertex of the cube. Let us designate

$$A_{ij} = \sum_{k=1}^{n} a_{ijk}$$

for a pair of indexes $i < j$.

For each index $i < n$, the values of form f at two vertices of the cube that do not coincide only at the ith coordinate differ by

$$2\sum_{j=i+1}^{n} A_{ij}.$$

Hence, if form f is equal to zero at each vertex of the cube, the sum

$$\sum_{j=i+1}^{n} A_{ij} = 0.$$

Similarly, for each index $j \geq 2$, the sum

$$\sum_{i=1}^{j-1} A_{ij} = 0.$$

Thus, numbers A_{ij} are the solution to the system of homogeneous linearly independent equations in which the number of variables is equal to the number of equations. Hence, all $A_{ij} = 0$. Condition (4) is met. Now form f is equal to the sum of two summands, one of which is equal to zero at each vertex of the cube. Hence, the second summand, assuming a constant value at all vertices of the cube, is also equal to zero; i.e., the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} = 0.$$

Condition (5) is fulfilled. The forms of type (3) meeting conditions (4) and (5) make up a subspace of the dimension

$$\frac{n(n-1)^2}{2} + \left(\frac{n(n+1)}{2} - 1 \right),$$

where the first summand is equal to the number of linearly independent coefficients a_{ijk}, and the second summand is equal to b_{ij}. The dimension of the linear space of all forth-degree forms is

$$\frac{n(n+1)(n+2)(n+3)}{24}.$$

Hence,

$$\chi_n(4) = \frac{n(n+1)(n+2)(n+3)}{24} - \frac{n(n-1)^2}{2}$$

$$-\frac{n(n+1)}{2} + 1 = \frac{n^4}{24} - \frac{n^3}{4} + \frac{23n^2}{24} - \frac{3n}{4} + 1.$$

Remark 2. Propositions 1, 4, 5, and 6 allow one to completely describe the Hilbert functions for U_1, U_2, U_3, U_4, U_5, and U_6.

$$\chi_1(d) = 1$$

$$\chi_2(d) = 1$$

$$\chi_3(d) = \begin{cases} 3, & d = 1 \\ 4, & d \geq 2 \end{cases}$$

$$\chi_4(d) = \begin{cases} 4, & d = 1 \\ 7, & d = 2 \\ 8, & d \geq 3 \end{cases}$$

$$\chi_5(d) = \begin{cases} 5, & d = 1 \\ 11, & d = 2 \\ 15, & d = 3 \\ 16, & d \geq 4 \end{cases}$$

$$\chi_6(d) = \left\{\begin{cases} 6, & d = 1 \\ 16, & d = 2 \\ 26, & d = 3 \\ 31, & d = 4 \\ 32, & d \geq 5 \end{cases}\right\}.$$

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