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THEORY AND METHODS OF INFORMATION PROCESSING

A Novel Algorithm for Solution of a Combinatory Set Partitioning Problem

V. A. Lyubetsky and A. V. Seliverstov

Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute), Bol'shoi Karetnyi per. 19, str. 1, Moscow, 127051 Russia

e-mail: slvstv@iitp.ru

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Abstract—A novel efficient algorithm for solution of the problem of equal partitioning of a set with predefined weights of elements is proposed. The algorithm is based on calculation of a linear group preserving an invariant: the set of zeros of a cubic form. Algorithms for solution of related problems, including the problem of the search for the second solution if the first solution is known, are discussed.

Keywords: partitioning algorithm, cubic form, computational complexity

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1. SET PARTITIONING PROBLEM AND PRELIMINARY INFORMATION

It appears that many purely combinatory problems are efficiently reduced to the search for singular points on the projective hypersurface, and this is the new approach of the authors to solution of these problems. As an example of such combinatory problems, we consider the well-known problem of partitioning of a given set with weights. Below, we everywhere consider multidimensional complex spaces, mainly the $(n + 1)$-dimensional space $\mathbb{C}^{n+1}$, where $\mathbb{C}$ is the field of complex numbers.

Set partitioning problem. Let us have a set of $n + 1$ elements in which weight α_j (where α_j is a positive integer) is assigned to the jth element. Does a partitioning of this set into two parts with equal sum weights of the parts exist? This problem has a natural parameter: number k of its solutions. A problem with no more than k solutions will be referred to as a *k-problem*. The set together with fixed weights will be referred to as the *data* of the original problem. This problem is equivalent to the search for a cube vertex with coordinates −1 or +1 lying in a hyperplane with equation $\alpha_0 x_0 + \ldots + \alpha_n x_n = 0$. Thus, it is required to find the cube vertex lying in the hyperplane.

A widespread opinion is that this problem is algorithmically difficult [1, 2]. The analysis of related problems confirms its high computational complexity [3–5]. On the other hand, an efficient algorithm can be found for particular cases of algorithmically difficult problems that are close in their sense to the problem under study [6–8]. Relations between upper levels of the polynomial hierarchy are still poorly investigated and one may think that its structure differs substantially from the structure of the arithmetical hierarchy or similarly defined hierarchies of the descriptive theory of sets, while methods for the analysis of these hierarchies are based on similar approaches [9–13].

It has been shown in [14] that the set partitioning problem can be reduced to the analysis of hypersurface $\alpha_0 x_0^d + \ldots + \alpha_n x_n^d = 0$ having not only cubic but also a higher odd degree d. Namely, cube vertices in the hyperplane $\alpha_0 x_0 + \ldots + \alpha_n x_n = 0$ are its points of contact with the hypersurface.

Another approach is to consider a hyperplane section of the hypersurface that contains all vertices of the cube and each of these vertices is singular. For example, this is the hypersurface defined by quartic equation $g((x_0^2 - 1), \ldots, (x_n^2 - 1)) = 0$, where g is any nondegenerate quadratic form in $n + 1$ variables. Arbitrariness in selection of form g allows us to avoid the appearance of additional singular points in the hyperplane section. An estimate of the dimensionality of the space of hypersurfaces of fixed dimensionality containing of cube vertices was given in [15]. By considering forms of higher degrees, it is possible to investigate belonging to the hyperplane of not only the cube vertices but also vertices of other polytopes embedded into the complex space.

Transition from discrete problems to the analysis of hypersurfaces allows us to obtain information about the original problem via the study of hypersurface points with special analytic properties. Here, an analogy with error-correcting codes can be seen [16–18]. On the other hand, there are easily computable invariants allowing efficient discrimination of hypersurfaces in applied object description and recognition problems [19].

By definition, the form gradient is zero at each point of the *singular straight line*.

Lemma 1. *Let us have cubic form f in n + 1 variables. If the cone f = 0 is invariant with respect to the action of the linear group* $G = \{A, A^2\}$ *in* $\mathbb{C}^{n+1}$ *and the number of singular straight lines on the cone f = 0 is odd, then* $\mathbb{C}^{n+1}$ *is decomposed into a direct sum one-dimensional and n-dimensional G-invariant subspaces, one of which containing an odd number of singular straight lines of this cone.*

We can consider the restriction of the form $\alpha_0 x_0^3 + \ldots + \alpha_n x_n^3$ to the hyperplane $\alpha_0 x_0 + \ldots + \alpha_n x_n = 0$ as the cubic form.

Proof. The set of singular straight lines that do not remain fixed under involution A is separated into pairs of straight lines that can be transformed into each other. Since the total number of singular straight lines is odd, the number of fixed singular straight lines is also odd. The linear representation of Abelian group G over $\mathbb{C}$ is decomposed into a direct sum of one-dimensional G-invariant subspaces. Two cases are possible.

1. If action G is nontrivial, there is such a subspace among one-dimensional G-invariant subspaces whose points are not G-invariant. Let us denote it by M. If M does not coincide with a singular straight line of the cone $f = 0$, then all fixed singular straight lines lie in the direct adjunct of M, which is also G-invariant.

2. If action G is trivial, then each singular straight line is G-invariant.

The following lemma strengthens the result from [14] in the case of cubic forms.

Lemma 2. *Let* $h = \alpha_0 x_0 + \ldots + \alpha_n x_n$ *be a linear form of no less than four variables with nonzero integer coefficients. Opposite pairs of cube vertices belonging to the hyperplane h = 0 bijectively correspond to singular straight lines along which this hyperplane touches the cone* $\alpha_0 x_0^3 + \ldots + \alpha_n x_n^3 = 0$. *Moreover, they are singular straight lines of the hyperplane section of the cone.*

The proof consists in direct calculation.

2. NEW HYPOTHESES

Below, we will discuss formulae in the language of the theory of fields with additional constants for coefficients of preset cubic form f. The equivalence of the formulae is considered in the theory of algebraically closed fields of zero characteristic.

Lemma 3. *There is an algorithm of polynomial time with respect to n constructing a closed E-formula that is true in* $\mathbb{C}$ *if the cone f = 0 does not contain singular straight lines and false if it contains an odd number of singular straight lines.*

The existence of the A-formula, which expresses smoothness of the hypersurface, is the definition of a singular point. It is unknown whether an E-formula equivalent to an arbitrary A-formula of this language can be found in a polynomial time. If the computational complexity is not limited, the existence of such a formula is evident. The computational complexity of the check for the provability of closed formulae as a function of the number of quantifier changes was considered in detail in [20].

Proof (outline). A cubic form defining a cone without singular straight lines is reduced to form $y^2 z + g$, where cubic form g is independent of variable y, by means of a nondegenerate linear transformation. In the case of three variables, this is a Weierstrass normal form. This form is invariant under the involution changing the sign of coordinate y. Applying Lemma 1, we find the singular straight line or restrict the search for the singular straight line by the plane section $y = 0$. If the initial cone does not contain singular straight lines, its section also does not contain them. Therefore, each section has its own involution. This fact allows us to reduce the dimensionality until three variables remain. The cubic form in three variables determining a cone without singular straight lines can be reduced to the structure $y^2 z = x^3 + pxz^2 + qz^3$, where polynomial $x^3 + px + q$ does not have multiple roots, i.e., its discriminant $-4p^3 - 27q^2$ is not equal to zero, by means of a linear transformation of coordinates. It can be readily checked that the transformation reduces the cubic form to the required structure and is nondegenerate. And these conditions are expressible in the language of the field theory.

In the case of good approximations of corresponding complex numbers, an algorithm appears that allows one to discriminate a smooth cubic surface and surface with odd number of singular points by performing the number of operations over these approximations that is polynomially bounded by the record length of the initial form.

Assumption 1. *There is a nondeterministic algorithm of polynomial time with respect to n that receives data for each 1-problem if and only if no one solution exists.*

If we discard the constraint on the number of solutions, this assumption is equivalent to the equality $NP = coNP$, which is usually accepted to be false.

The outline of the proof. Let us consider subspace H in $\mathbb{C}^{n+1}$ that is defined by the equation $\alpha_0 x_0 + \ldots + \alpha_n x_n = 0$ and form F in this subspace that is obtained by means of restriction of the form $\alpha_0 x_0^3 + \ldots + \alpha_n x_n^3$. According to Lemma 2, it is necessary to find out whether or not the cone $F = 0$ in H contains a singular straight line. Using Lemma 3, we find involution A and set $G = \{A, A^2\}$. For this purpose, it is necessary to nondeterministically determine values of variables related by the existential quantifier. As a result, numbers allowing short writing and approximating corresponding complex numbers must appear. While Lemma 1 says about a group of order 2, in order to reduce the influence of errors of the above approximations, we can consider a larger group G of linear transformations with respect to which the cone $F = 0$ is invariant. Let us build a G-invariant eigensubspace L in H that includes the cube vertex in H, if it exists. If group G includes a finite subgroup not generated by a small number of elements, then the dimensionality of subspace L is small. Indeed, according to Lemma 1, such an L exists for the subgroup of order 2. If G contains two involutions corresponding to subspaces L' and L'', we set $L = L' \cap L''$. As the number of involutions in G increases, the dimensionality of intersection L of these subspaces decreases. Elements of a higher order can also be used. As a result, the search for the cube vertex from H is reduced to solution of a similar problem in the subspace of lesser dimensionality. In this case, the aforementioned process of descent runs faster. For the dimensionality equal to three, this problem is solved trivially.

For hypersurfaces of degrees higher than three, there are upper bounds on the group order [21]. Thorough analysis of the groups of linear transformations with invariant cubic cones leads to Assumption 2, which is a natural strengthening of Lemma 3. For example, for the surface $x^2 + y^2 = z^3$, such a group is infinite.

Assumption 2. *Let f be a cubic form. There is an algorithm of polynomial time with respect to n constructing a closed E-formula individual for each n that expresses in* $\mathbb{C}$ *evenness of the number of singular straight lines of the cubic cone if this set is finite.*

In some cases, the cubic form is reduced by a nondegenerate linear transformation of coordinates to the form that allows determination of singular points of the hypersurface, if they exist [22, 23]. Moreover, there is an iterative algorithm for reduction to this form under additional conditions [24].

Probably, the analysis of hypersurfaces will allow one to refine the results on the complexity of finding the second solution of the NP-complete problem [25]. Indeed, if one singular point and the automorphism of the hypersurface are known, the image of a singular point is also a singular point. Thus, the search for the second singular point is reduced to the search for an automorphism not leaving the first point stationary.

While the proposed algorithms are nondeterministic, the development of such algorithms will allow certification of the results of calculations on supercomputers [26]. An increase in the performance of computers leads to difficulties in the tests of computations that cannot be executed on commonly available computers. The development of fast nondeterministic algorithms that require small memory volumes and short computation times allows one to efficiently check the results of operation of multiprocessor computers in the case when they present a certificate containing all nondeterministic computation steps in the course of execution of the algorithm to the customer. In particular, this may be of great importance for decision making in transport [27], medicine [28], and image processing [29].

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