ISSN 1064-2269, Journal of Communications Technology and Electronics, 2017, Vol. 62, No. 6, pp. 663–668. © Pleiades Publishing, Inc., 2017.
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MATHEMATICAL MODELS
AND COMPUTATIONAL METHODS

Projective-Invariant Description of a Meandering River

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Received September 29, 2016

Abstract—How can the projective invariant of the cubic curve approximating the river bed near its meander be calculated? A well-known approach uses the Weierstrass normal form. However, it is important to find this form by means of calculations tolerant to curve representation errors and, in particular, using calculations that do not require computation of tangent lines or inflection points. A new algorithm is proposed for calculation of the projective invariant of the cubic curve. This algorithm can be used to describe river meanders.

Keywords: cubic curve, Weierstrass normal form, projective invariant, image description, image recognition, machine vision, algorithm

DOI: 10.1134/S1064226917060201

1. INTRODUCTION

Detection of contours that are planar curves is often used in image analysis [1, 2]. In this case, it is important to recognize projectively equivalent curves, since they correspond to different projections on the plane of the same object in 3D space [3]. For example, significant distortions occur in wide-angle photo shooting and in the alignment of images taken from different aspect angles [4–7]. Note that real objects often have symmetry that can be difficult to recognize in the image [8, 9]. On the other hand, artificial distortion of image proportions is intentionally used to create caricatures [10]. Below, we will consider smooth curves on a real projective plane.

The problem of description of an oval with implicit symmetry that is invariant with respect to projective transformations of the plane provided that the oval intersects each line at no more than two points was considered in [11] and in earlier works cited therein. Similar methods are used to describe some nonconvex ovals [12]. These methods use only calculation of the tangent line. Other methods for calculation of invariants are based on computation of higher derivatives [3, 13], which leads to significant computational difficulties, because of the need in compensation for the inaccuracies in sampling and quantization of the initial image. Calculations are also reduced to solution of a system of algebraic equations. This can be done by means of algorithms based on the computation of Gröbner bases [14, 15]. A formula for solution in the form of a hypergeometric series of coefficients is known for the given systems of n algebraic equations in n unknowns [16]. However, these algorithms do not use the specific features of the problem and usually are time and memory consuming and unstable to initial data approximation errors.

2. PRELIMINARIES

Let us recall that the real projective plane is a nonorientable surface. It can be intuitively represented as a result of gluing of the disc and the Möbius band along the edge. When the projective plane is immersed in the 3D affine space, the surface self-intersection inevitably occurs. Another model is obtained by identifying antipodal points on a two-dimensional sphere. In this case, the path between antipodal points on the sphere corresponds to the nonorientable closed curve that is not a boundary. Coordinates on the sphere determine the coordinates on the projective plane. In particular, it becomes possible to define the concept of a random point uniformly distributed on the projective curve (a closed curve in the projective plane is assumed). On the other hand, a projective plane is obtained by adding a projective line at infinity to the affine plane. In turn, the projective line is obtained by adding a point at infinity to the affine line.

The class of projectively equivalent curves corresponds to a cone in the 3D space. The sections of this cone by the planes not passing through its apex are affine curves with equivalent projective closures. The corresponding affine curves can have a different number of connected components and, at first sight, can be completely different. For example, all smooth curves of the second degree (conics) are projectively equivalent. Cubic curves form a one-parameter family. A smooth irreducible projective cubic curve over the field of real

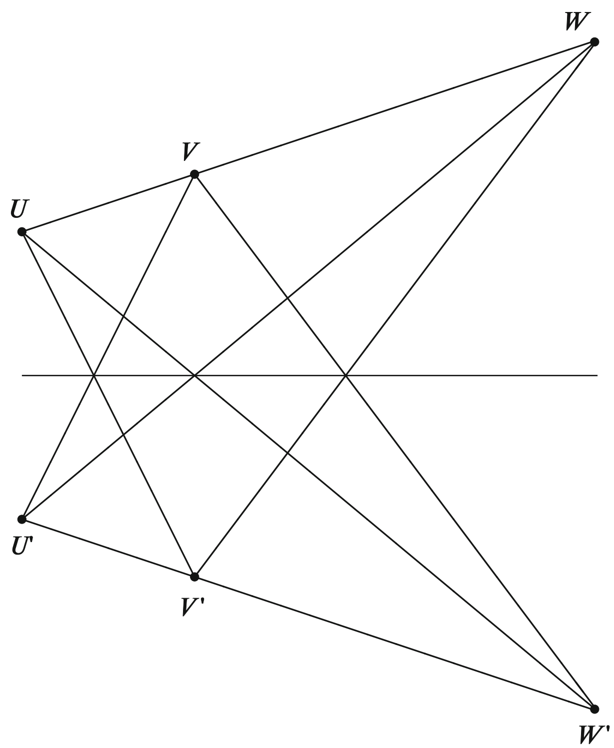

Fig. 1. Illustration for the Pappus theorem.

numbers can be reduced to the Weierstrass normal form by a projective transformation [17, p. 29]. In the affine space, it is given by the following equation:

$$y^2 = x^3 + px + q, \tag{1}$$

where the polynomial on the right-hand side does not have multiple roots, i.e., the following discriminant is nonzero:

$$-4p^3 - 27q^2. \tag{2}$$

Two curves in the Weierstrass normal form are projectively equivalent if and only if the following ratios are equal [17, p. 143]

$$\frac{q^2}{p^3}. \tag{3}$$

If discriminant (2) is negative, the curve is connected. If it is positive, the curve consists of two connected components, one of which is nonorientable and the other is an oval without inflection points.

The projective transformation of the plane that maps the considered curve onto itself will be called the curve symmetry. The symmetry coinciding with the reverse symmetry is called *involution*. Let us say that points on the curve are located symmetrically, if they change into each other in the case of the considered symmetry of the curve. The curve in Weierstrass normal form (1) is symmetrical with respect to the change of the sign of coordinate y.

It is important to select a convenient curve parameterization for successive scan of the points. In this case, it is convenient to consider the projective plane as a sphere with identified antipodal points. In particular, if the projective curve corresponds to the cubic form of three variables $f(x, y, z)$ that determines a cone, then the corresponding curve on the sphere consists of the points in which form f vanishes on three direction cosines. All cosines simultaneously change sign in the case of transition to the antipodal point. Therefore, at the antipodal points, values of cubic form f differ only in their signs. Hence, the resulting equation in the direction cosines correctly defines the curve in the projective plane.

A smooth projective cubic curve over the field of reals numbers has exactly three real inflection points. Moreover, if a straight line intersects this curve at three points two of which are inflection points, then the third intersection point is also the inflection point [17, p. 31]. The other six complex inflection points do not belong to the real plane. A smooth cubic curve in the Weierstrass form over the field of real numbers crosses the line placed at infinity at one point, which is an inflection point. This curve in another normal form crosses the line placed at infinity at three inflection points. Note that an iterative algorithm for reduction to this form was proposed in [18].

Any two triples of pairwise different points on the projective line change into each other under a certain projective transformation. However, this transformation does not always exist for four points. The *double ratio* $[T, U, V, W]$, which is equal to the following expression of the coordinates of points, remains invariant under projective transformations for four points on the affine line T, U, V, and W with coordinates t, u, v, and w:

$$[T,U,V,W] = \frac{v-t}{v-u} : \frac{w-t}{w-u}. \tag{4}$$

The proof is given in [19, pp. 64–65]. Triples of points resulting from such coincidences are projectively equivalent. For any finite set of points, it is possible to select a system of affine coordinates such that the abscissas of all points are different. In this case, double ratios (4) of the quadruples of points on each line can be calculated using the abscissas of the points as their coordinates on the line.

Theorem (Pappus of Alexandria). *Points U, V, and W on line L and points U', V', and W' on line L' are given. Three intersection points of three pairs of lines UV' and VU', UW' and WU', and VW' and WV' lie on one line.*

The proof is given in [17, p. 14] and [19, p. 71]. Figure 1 shows an example of location of points and lines of the theorem.

If the position of the inflection point on the cubic curve is known, then it is easy to reduce it to the Weierstrass form. The point of the cubic curve fixed in the case of involution is either an inflection point or a point at which the tangent line transversally intersects the curve at the inflection point. However, exact calculation of the inflection point, i.e., the point of the curve at which the Hessian of the corresponding cubic form is zero, is associated with computational difficulties. Indeed, in the case of the shift of the curve in the vicinity of the inflection point, the position of the tangent line remains almost the same.

Let us consider an easily verifiable condition of location of points on the cubic curve symmetrical with respect to the involution, which does not require exact calculations of tangent lines and inflection points. Note that, in the case of involution of the projective plane, points of a line and another point that does not lie on this line remain fixed. A similar construction was considered in [20].

Theorem 1 (Necessary condition for symmetrical location). *Let a smooth cubic curve be given. Assume that the points of the curve U, V, and W lie on a line intersecting straight line L, and the points U', V', and W', which are their images under the involution φ, lie on other line intersecting straight line L'. Let the straight lines L and L' intersect each other at point T. Then the values of the projective invariant coincide with each other for two quadruples of points* [*T*, *U*, *V*, *W*] *and* [*T*, *U'*, *V'*, *W'*]. *Moreover, point T and three intersection points of three pairs of lines UV' and VU', UW' and WU', and VW' and WV' lie on one line fixed under the involution* φ.

Proof. Under the involution φ, straight line *L* is mapped onto straight line *L'*. Therefore, the point of intersection of these lines *T* remains fixed. Consequently, double ratios are equal: [*T*, *U*, *V*, *W*] = [*T*, *U'*, *V'*, *W'*]. By virtue of the Pappus theorem, three points of intersection of three pairs of lines *UV'*and *VU'*, *UW'* and *WU'*, and *VW'*and *WV'* lie on one line. Because of the uniqueness of such a line, it is fixed under the involution φ. The theorem is proved.

Remark. If straight lines *L* and *L'* are located asymmetrically, point *T* cannot belong to the line constructed by the Pappus theorem. However, if point *T* belongs to it for some pair of secants, it is not sufficient for the existence of involution. It is necessary to consider the second pair of secants *M* and *M'*.

Theorem 2. *Points U, V, and W on straight line L and points U', V', and W' on straight line L' are given. The following three conditions are equivalent*:

1. Point T and three intersection points of three pairs of lines UV' and VU', UW'and WU', and VW' and WV' lie on one straight line.

2. There is a projective transformation φ, *for which* φ(*T*) = *T*, φ(*U*) = *U'*, φ(*V*) = *V'*, *and* φ(*W*) = *W'*.

3. In the coordinate system in which invariants are defined, double ratios are equal: [*T*, *U*, *V*, *W*] = [*T*, *U'*, *V'*, *W'*].

Proof. The second and third conditions are equivalent [19, pp. 64–65]. In order to prove the equivalence

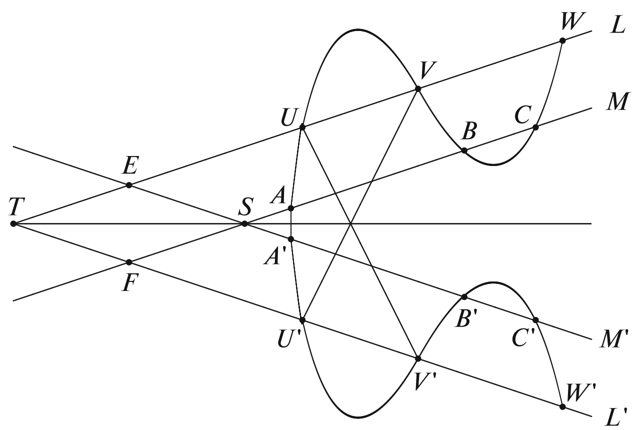

Fig. 2. Example of location of points and some lines in Theorem 3.

of the first two conditions, it can be assumed that point *T* is at infinity and straight lines *L* and *L'* are parallel in the affine plane. In this case, the straight line constructed in the Pappus theorem is parallel to them if and only if the quadrangles *VV'U'U* and *WW'U'U* are parallelograms. Then the corresponding points are superimposed by means of parallel translation. The theorem is proved.

Let us show that there is an efficiently verifiable sufficient condition for location of points on a cubic curve that are symmetric with respect to some involution (Fig. 2).

Theorem 3 (Sufficient condition for symmetric location). *Assume that we have a smooth cubic curve and four pairwise different intersecting straight lines L, L', M, and M', each intersecting the curve at three points. Straight lines L and L' intersect each other at point T that does not lie on the curve. Straight lines M and M' intersect each other at the point S that does not lie on the curve and is different from point T. Straight lines L and M' intersect each other at point E that does not lie on the curve. Straight lines M and L' intersect each other at point F that does not lie on the curve. Let U, V, and W be the intersection points of the curve with secant L*; *U'*, *V'*, *and W', with secant L'*; *A, B, and C, with secant M; and A', B', and C', with secant M'. Suppose that six intersection points of six pairs of straight lines lie on straight line ST: UV' and VU', UW' and WU', VW' and WV', AB' and BA', AC' and CA', BC' and CB'. Then there is an involution φ that preserves the curve at which* φ(*S*) = *S*, φ(*T*) = *T*, φ(*U*) = *U'*, φ(*V*) = *V'*, φ(*W*) = *W'*, φ(*A*) = *A'*, φ(*B*) = *B'*, *and* φ(*C*) = *C'*.

Proof. By virtue of Theorem 2, for a given set of points, there is an involution φ that leaves points *S* and *T* fixed and rearranges 12 points *U*, *V*, *W*, *U'*, *V'*, *W'*, *A*, *B*, *C*, *A'*, *B'*, and *C'* in accordance with the requirement. Let us show that, in the case of this mapping, points of the curve are changed into the points of the same curve. At least eight different considered points are on the curve. In fact, each two out of four straight lines L, L', M, and M' intersect each other at one point. There is a total of six intersection points. By condition, four of them, E, F, S, and T, do not lie on the curve. Even if the other two lie on the curve, 12 points U, V, W, U', V', W', A, B, C, A', B', and C' include ten different points. However, only one cubic curve (including the reducible one) passes through ten different points of the plane, since the total cubic form has ten monomials. Therefore, the initial curve coincides with the symmetrical curve, which is completed by half of the initial curve using involution φ. The theorem is proved.

Remark. According to the Pappus theorem, some part of the conditions in Theorem 3 is redundant. However, additional conditions can be useful in order to improve the accuracy. For a more accurate determination of the position of the straight line, it is more convenient to use the points located far from each other in order to avoid errors of approximate calculations. On the other hand, Theorem 2 makes it possible to replace some of the incidence tests for straight lines and points with the equality tests of projective invariants for points on secants, which can be more efficient than auxiliary geometric constructions of the Pappus theorem. Only if the numerical values coincide, it is reasonable to test all conditions.

3. REDUCTION TO THE NORMAL FORM

Knowing the line the reflection with respect to which leaves the cubic curve invariant, it is easy to calculate a linear transformation of coordinates for reduction to the Weierstrass normal form. Note that, in the general case, this transformation is not orthogonal. In the new coordinate system, each point on the x axis remains fixed under involution, and the y axis passes through two different points that change into each other under this involution.

Thus, reduction of a cubic curve to the Weierstrass normal form is based on the search for two pairs of symmetrically located secants. If the position of two inflection points is approximately known, the secants should be selected so as to intersect a curve at different inflection points. Let us use the notation from Theorem 3. In the case of fixed point V on the secant L we will select point V' of intersection by the second secant L' with another curve near the inflection point. Then, we will select point T of intersection of L and L', in such a way that it is located either inside the oriented component of the curve or inside the oval formed by the arc of the curve and segment VV'. Points T, V, and V' uniquely define secants L and L'. Note that we assume only a rough localization of inflection points that is insufficient for reduction to the normal form without additional constructions. It is also possible to select the second pair of secants M and M' that cross this curve at points V and V' but select point S different from point T. After arbitrary fixing of point V, it is necessary to vary three points: point V' on the arc of the curve and two points T and S in a bounded region of the plane.

Let us vary points V', S, and T until sufficient conditions of Theorem 3 are met. The constructed straight line ST becomes the x axis in the new coordinate system on the affine plane. The y axis passes through two symmetrically located points on the curve, for example, through points U and U'. At new coordinates, the affine curve is determined by the equation of the following type

$$y^2 = ax^3 + bx^2 + cx + d.$$

Without loss of generality it can be assumed that the higher coefficient $a = 1$. In order to calculate coefficients b, c, and d, it is sufficient to know the images under projective transformation of four arbitrary points of the curve in the general position, i.e., not passing into each other under involution. For example, it is possible to use points U, V, and W and the intersection point of straight line ST with the curve. Further, the linear change of variable x with the difference $(x–b/3)$ reduces it to form (1), which is convenient for calculation of the projective invariant of the curve equal to ratio (3). In particular, calculation of this projective invariant of the curve does not require calculation of the equation of the curve in initial coordinates and does not use calculation of tangent lines or the curvature radius of this curve.

On the other hand, the knowledge of the approximate position of the inflection points greatly simplifies the calculations. Looking at the figure, it is usually easy to specify approximate positions of symmetrically located points mentioned in Theorem 3. The test on the symmetry of positions of the guessed points and subsequent calculation of the projective invariant of the curve are carried out using only transversely intersecting straight lines and the curve.

Since the source data contain sampling errors, for the calculation of the projective invariant, it is appropriate to use a greater number of points selecting the average or the most probable of the calculated values. Moreover, it is possible to correct errors not only in the initial data but also in intermediate calculations [21]. The implementation of the described algorithms for multiprocessor computing systems is of particular interest [22]. Different versions of straight lines L', M, and M' can be considered independently from each other without the use of interprocessor exchange in the case of parallel processing of versions.

4. LOCAL DESCRIPTION OF THE CURVE SEGMENTS AND PRACTICAL APPLICATION

A natural source of initial data for the practical application of our method are half-tone or multispectral (including infrared) images of the river bed obtained by using wide-angle lens or obtained at different angles to the Earth's surface. Using known image processing methods (such as thresholding, edge linking, obtaining the skeleton of the region, i.e., skeletonizing, etc.), they are converted into binary images of the river central line. In the general case, it is multiply connected. The method results in description of the river bed by the set of invariants independent of projective transformations. Informally, if a curve in the Weierstrass normal form describes a symmetric meander, the projective invariant of the curve consisting of one connected component describes the relative length of the channel needed to straighten the river meander. Figure 2 shows an example of the connected smooth cubic curve in the Weierstrass normal form. The curve is symmetrical in the case of reflection with respect to the x axis.

Similarly, the invariant of a curve consisting of two connected components indicates the relative remoteness of the oxbow from the main stem of the river.

Since there are three inflection points on a real smooth cubic curve (inflection points can lie on the line placed at infinity), it cannot approximate a curve with a greater number of inflection points. In particular, if the curve corresponds to the riverbed, the cubic curve usually describes the river meander well. Thus, projective invariant (3) calculated by our method is the characteristic of a small segment. Description of the curve as a whole requires an implicit approximation by cubic splines and attributing the invariants to each of them separately. In this case, linear and quadratic splines are not suitable for description independently of projective transformations, since all of them are projectively equivalent. On the contrary, the use of cubic splines makes it possible to calculate such an invariant [13].

Although calculation of the invariant does not require explicit construction of the cubic spline in initial coordinates, it is necessary to select connected segments of the curve that knowingly contain two inflection points inside and quite long flanks at the edges. This requires rough localization of inflection points. It is important that it is sufficient to locate them with low accuracy that is insufficient for direct calculation of the Weierstrass normal form as such. Inflection points can be approximately localized by the Hough transformation. Indeed, the tangent line at the inflection point approximates the curve well, thus making a relatively large contribution during the voting procedure. Note that a similar method was used to recognize road signs in real time [23]. On the other hand, the considered calculations make it possible to test how well the curve is approximated by the cubic curve.

As was noted in the beginning, we do not require connectivity of the cubic curve. Conversely, the curves consisting of an oval and nonorientable connected component can be considered. The Weierstrass normal form of this curve has form (1), where the polynomial on the right-hand side has three real roots. This is equivalent to the positivity of discriminant (2).

When describing a river, this oval corresponds to the oxbow. Therefore, the use of curves with two components also appears to be natural for the application.

Note that, in [24], high sensitivity to errors in the initial data of geometric constructions based on the Pascal hexagon theorem, which is a special case of the Pappus theorem, is discussed. Namely, the the Pascal angle is highly sensitive to the deviation of the curve from a conic.

5. CONCLUSIONS

Despite significant advances in the field of differential geometry in the investigation of smooth curves, algorithms based on the methods of descriptive geometry but tolerant to errors in initial data are more efficient for the analysis of digital images. The image analysis method proposed by us has exactly this property. Application of our method involves combination with previously developed skeletonizing methods and linking of edges with discontinuities. However, the errors arising during this procedure are not critical for application of the method. Note also the apparent similarity of our approach with error-correcting codes and problems of discrete optimization.

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Translated by O. Pismenov