Journal of Mathematical Sciences, Vol. 158, No. 5, 2009

BOREL REDUCIBILITY AS AN ADDITIVE PROPERTY OF DOMAINS

V. G. Kanovei* and V. A. Lyubetsky* UDC 510.225

We prove that under certain requirements, if E *and* F *are Borel equivalence relations,* $X = \bigcup_n X_n$ *is a countable union of Borel sets, and* $\mathsf{E} \restriction X_n$ *is Borel reducible to* F *for all* n*, then* $\mathsf{E} \restriction X$ *is Borel reducible to* F*. Thus the property of Borel reducibility to* F *is countably additive as a property of domains. Bibliography: 19 titles.*

1. Introduction

Recall that if E and F are Borel equivalence relations on Borel sets X and Y, respectively, then $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ (the Borel reducibility of E to F) means that there exists a Borel map $\vartheta : X \to Y$ such that the equivalence

$$x \mathsf{E} x' \longleftrightarrow f(x) \mathsf{F} f(x')$$

holds for all $x, x' \in X$. Borel maps are those satisfying the requirement that the preimage of every Borel set is a Borel set. For Polish spaces this is equivalent to the condition that the graph of a map is a Borel set in the product space (for more detail on Borel reducibility, see [5] and [1, 6, 3]).

Assume that E is a Borel equivalence relation on a Borel set X in a Polish space. We consider restricted equivalence relations of the form $\mathsf{E} \restriction Y$, where Y is a Borel subset of X. These restricted equivalence relations can satisfy $\mathsf{E} \restriction Y \le_{\mathrm{B}} \mathsf{F}$, where F is another fixed Borel equivalence relation. It is quite clear that if $\mathsf{E} \restriction Y \le_{\mathrm{B}} \mathsf{F}$ and $Y' \subseteq Y$ is a Borel set, then we still have $\mathsf{E} \restriction Y' \le_{\mathrm{B}} \mathsf{F}$. In other words, the condition $\mathsf{E} \restriction Y \le_{\mathrm{B}} \mathsf{F}$ as a property of Y (and with E and F fixed) is a smallness-type property. In such a case, it is a typical problem to figure out to which extent this property is additive. Our main theorem proves that under certain conditions the property is countably additive.

2. The main theorem

We begin with several technical definitions. Assume that F is an equivalence relation on a set X. For any integer n, let $n\mathsf{F}$ denote the equivalence relation defined on the set $n \times X = \{\langle k, x\rangle : k < n \wedge x \in X\}$ so that $\langle k, x\rangle n\mathsf{F} \langle j, y\rangle$ if and only if $k = j$ and $x \mathsf{F} y$. Thus $n\mathsf{F}$ can be viewed as the union of n independent copies F_k, $k < n$, of the relation F on pairwise disjoint sets. These sets are $X_k = \{k\} \times X$, and the copies F_k are defined so that $\langle k, x\rangle \mathsf{F}_k \langle k, y\rangle$ if and only if $x \mathsf{F} y$.

Accordingly, under the same conditions, $\mathbb{N}\mathsf{F}$ is the equivalence relation defined on the set $\mathbb{N} \times X$ so that $\langle k, x\rangle \mathbb{N}\mathsf{F} \langle j, y\rangle$ if and only if $k = j$ and $x \mathsf{F} y$. This $\mathbb{N}\mathsf{F}$ can be viewed as the union of countably many independent copies F_k of F on pairwise disjoint sets.

Theorem 1. *Assume that* F *is a Borel equivalence relation satisfying* $\mathbb{N}\mathsf{F} \le_{\mathrm{B}} \mathsf{F}$*, all* F*-equivalence classes are* σ*-compact, and* E *is a Borel equivalence relation on a Borel set* $X = \bigcup_k X_k$*, where all* X_k *are Borel sets. Assume that* $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{F}$ *for all* k*. Then* $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$*.*

Proof. It suffices to prove the following somewhat more elementary lemma. It shows that a pair of, perhaps, incompatible reduction maps can be converted to a reduction map on the common domain.

Lemma 2. *Assume that* E *is a Borel equivalence relation defined on the union* $X \cup Y$ *of disjoint Borel sets* X *and* Y*, while* F *is a Borel equivalence relation with* σ*-compact equivalence classes on the union* $P \cup Q$ *of two disjoint Borel sets* P *and* Q*,* F*-independent in the sense that* $p \not\mathsf{F} q$ *for all* $p \in P$*,* $q \in Q$*.*

In this case, if f *and* g *are Borel reductions of* $\mathsf{E} \restriction X$*,* $\mathsf{E} \restriction Y$ *to* $\mathsf{F} \restriction P$ *and* $\mathsf{F} \restriction Q$*, respectively, then there is a Borel reduction* $h : X \cup Y \to P \cup Q$ *of* E *to* F *such that* $h \restriction X = f$*.*

To *deduce the theorem from the lemma,* let $Y = \operatorname{dom} \mathsf{F}$ (the Borel domain of F). Then $\mathbb{N}\mathsf{F}$ is an equivalence relation on $\mathbb{N} \times Y$, and $\langle k, x\rangle \mathbb{N}\mathsf{F} \langle j, y\rangle$ if and only if $k = j$ and $x \mathsf{F} y$. We prove the theorem under the assumption that the sets X_k are pairwise disjoint. (Otherwise consider the sets $X'_k = X_k \setminus \bigcup_{j<k} X_j$, obviously pairwise disjoint. Their union is obviously equal to the union of the given sets X_k.)

*Institute for Information Transmission Problems, Moscow, Russia, e-mail: kanovei@mccme.ru, lyubetsk@ippi.ru.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 358, 2008, pp. 189–198. Original article submitted April 10, 2007.

 1072-3374/09/1585-0708

Put $Y_k = \{k\} \times Y$. Then $\mathbb{N} \times Y$ is the pairwise disjoint union of Borel sets Y_k, and we have a copy F_k of F defined on Y_k, that is, $\langle k, x\rangle \mathsf{F}_k \langle k, y\rangle$ if and only if $x\mathsf{F}y$. Then F_k coincides with the restriction $\mathbb{N}\mathsf{F} \restriction Y_k$ of $\mathbb{N}\mathsf{F}$ to Y_k, and the sets Y_k and Y_j are $\mathbb{N}\mathsf{F}$-independent whenever $k \neq j$. Put $Y'_n = Y_0 \cup \dots \cup Y_n$ and $X'_n = X_0 \cup \dots \cup X_n$.

We define a system of Borel maps $h_n : X'_n \to Y'_n$ satisfying the following two conditions:

(i) h_n is a reduction of $\mathsf{E} \restriction X'_n$ to $\mathbb{N}\mathsf{F} \restriction Y'_n$;

(ii) h_{n+1} extends h_n.

If such a sequence $\{h_n\}$ has been defined, then $h = \bigcup_n h_n$ is obviously a Borel reduction of the relation E (defined on the set $X = \bigcup_n X_n$) to $\mathbb{N}\mathsf{F}$; therefore $\mathsf{E} \leq_{\mathrm{B}} \mathbb{N}\mathsf{F}$ and $\mathsf{E} \leq_{\mathrm{B}} \mathsf{F}$.

The definition of the system of maps h_n goes on by induction. First of all, for every k, since $\mathsf{E} \restriction X_k \leq_{\mathrm{B}} \mathsf{F}$, there is a Borel reduction $\vartheta_k : X_k \to Y_k$ of the relation $\mathsf{E} \restriction X_k$ to F_k, that is, to $\mathbb{N}\mathsf{F} \restriction Y_k$. This allows us to immediately define $h_0 = \vartheta_0$.

Now we carry out the step $n \to n+1$. Assume that a Borel map $h_n : X'_n \to Y'_n$ satisfying (i) has been defined. The sets Y'_n and Y_{n+1} are $\mathbb{N}\mathsf{F}$-independent by the above. In addition, the $\mathbb{N}\mathsf{F}$-equivalence classes are essentially the same as the F-classes, and hence they are σ-compact sets. Therefore Lemma 2 is equally applicable to $\mathbb{N}\mathsf{F}$ and to F. It follows that there is a Borel reduction $\eta : X'_{n+1} \to Y'_{n+1}$ of $\mathsf{E} \restriction X'_{n+1}$ to $\mathbb{N}\mathsf{F} \restriction Y'_{n+1}$ extending h_n. It remains to put $h_{n+1} = \eta$. This completes the step and the derivation of Theorem 1 from Lemma 2.

Proof of Lemma 2. The difficulty consists in the fact that the sets X, Y are not necessarily E-independent, that is, there can exist points $x \in X$ and $y \in Y$ with $x\mathsf{E}y$. In this case, we have to define $h(y)$ not from $g(y)$, but rather as a point in P that is F-equivalent to $f(x)$ in P for some $x \in X$ satisfying $x\mathsf{E}y$.

Thus the key problem is to find an appropriate definition of the values $h(y)$ for elements $y \in Y$ satisfying $g(y) \in \operatorname{ran} U$, where

$$U = \{\langle p, q\rangle \in P \times Q : \exists x \in X\, \exists y \in Y\, (x\mathsf{E}y \wedge f(x)\mathsf{F}p \wedge g(y)\mathsf{F}q)\}$$

and, as usual, $\operatorname{ran} U = \{q : \exists p\, (\langle p, q\rangle \in U)\}$. In general, it would suffice to find a map φ defined on the set

$$Y' = \{y \in Y : \exists x \in X\, (x\mathsf{E}y)\},$$

and with values in X, such that $\varphi(y)\mathsf{E}y$ for all $y \in Y'$, and then put $h(y) = f(\varphi(y))$ for $y \in Y'$. Yet the construction of such a map φ amounts to the problem of uniformization of the set

$$\{\langle y, x\rangle \in Y \times X : x\mathsf{E}y\},$$

which is not always possible in the class of Borel uniformizations. Therefore we apply a more complicated argument.

Note that U is a $\mathbf{\Sigma}^1_1$ set (that is, a Suslin set, see [2, 4] (in Russian) on the modern theory of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ sets). Moreover, since the maps f and g are reductions of the relation E to F, we conclude that U is a subset of the $\mathbf{\Pi}^1_1$ set

$$W = \{\langle p, q\rangle \in P \times Q : \forall \langle p', q'\rangle \in U\, (p\mathsf{F}p' \longleftrightarrow q\mathsf{F}q')\}.$$

Indeed, assume that $\langle p, q\rangle \in U$. There exist $x \in X$ and $y \in Y$ such that $x\mathsf{E}y$ and $f(x)\mathsf{F}p$, $f(y)\mathsf{F}q$. Consider any other pair $\langle p', q'\rangle \in U$, and let $x' \in X$ and $y' \in Y$ satisfy $x'\mathsf{E}y'$, as well as $f(x')\mathsf{F}p'$, $f(y')\mathsf{F}q'$. If, for instance, $p\mathsf{F}p'$, then we have $x\mathsf{E}x'$ because f is a reduction, and hence $y\mathsf{E}y'$ and $q\mathsf{F}q'$.

Therefore, by the Luzin first separation theorem (see [15] or, for instance, [2, 3]), there is an intermediate Borel set V satisfying $U \subseteq V \subseteq W$.

Moreover, it turns out that V can be chosen among invariant sets. Indeed, the sets U and W are F-*invariant*, in the sense that if $p\mathsf{F}p'$ and $q\mathsf{F}q'$, then the pairs $\langle p, q\rangle$ and $\langle p', q'\rangle$ either simultaneously belong to U, or simultaneously do not belong to U, and the same for W. In this case, the *invariant* separation theorem holds (see, e.g., [8, 11]), that is, there exists a Borel set V that satisfies $U \subseteq V \subseteq W$ and is F-invariant in the same sense.[1]

[1] The following is a simple proof of the invariant separation theorem in the case under consideration, given here for the reader's convenience. By the ordinary separation theorem, there is a Borel set V_0 such that $U \subseteq V_0 \subseteq W$. The set

$$U_1 = [V_0]_{\mathsf{F}} = \{\langle p, q\rangle \in P \times Q : \exists \langle p', q'\rangle \in V_0\, (p\mathsf{F}p' \wedge q\mathsf{F}q')\}$$

obviously belongs to $\mathbf{\Sigma}^1_1$ and is F-invariant, and $V_0 \subseteq U_0 \subseteq W$ because W is F-invariant too. Once again, by the separation theorem, there is a Borel set V_1 such that $U_1 \subseteq V_1 \subseteq W$. Consider the $\mathbf{\Sigma}^1_1$ set $U_2 = [V_1]_{\mathsf{F}}$, and so on. This results in an infinite increasing sequence of sets U_n and V_n such that the set $U' = \bigcup_n U_n = \bigcup_n V_n$ is Borel (by the V_n-representation) and F-invariant (by the U_n-representation) and still satisfies $V \subseteq U' \subseteq W$.

The second important property of the set U is that it is a *bijection modulo* F, in the sense that the equivalence

$$p\mathsf{F}p' \longleftrightarrow q\mathsf{F}q'$$

holds for any two pairs $\langle p, q\rangle$ and $\langle p', q'\rangle$ in U. (Indeed, suppose that pairs $\langle p, q\rangle$ and $\langle p, q\rangle$ belong to U, so that there exist points $x, x' \in X$ and $y, y' \in Y$ such that $x\mathsf{E}y$, $x'\mathsf{E}y'$, and also $f(x)\mathsf{F}p$ and $f(y)\mathsf{F}q$, and, accordingly, $f(x')\mathsf{F}p'$ and $f(y')\mathsf{F}q'$. If $p\mathsf{F}p'$, then $f(x)\mathsf{F}f(x')$, therefore $x\mathsf{E}x'$, since f is a reduction. Thus $y\mathsf{E}y'$, and hence $q\mathsf{F}q'$.)

The sets W and V do not necessarily satisfy this requirement, since there can exist pairs $\langle p, q\rangle$ such that p is not F-equivalent to any $p' = f(x)$, $x \in X$. We are going to define such a Borel set V that satisfies the requirement and still is a superset of U. Note that $U \subseteq R$, where the $\mathbf{\Pi}^1_1$ set R is defined as follows:

$$R = \{\langle p', q'\rangle \in V : \forall \langle p, q\rangle \in V\,(p\mathsf{F}p' \longleftrightarrow q\mathsf{F}q')\}.$$

The set R is obviously F-invariant together with V. Therefore, again by the invariant separation theorem, there is an F-invariant Borel set S such that $U \subseteq S \subseteq R$.

Yet the set R, and hence S as well, are bijections modulo F. (Indeed, if pairs $\langle p', q'\rangle$ and $\langle p'', q''\rangle$ belong to V, and say $p'\mathsf{F}p''$, then taking the second pair as $\langle p, q\rangle$ in the definition of R, we obtain $q'\mathsf{F}q''$.) Therefore, by the F-invariance of S (see above), we have: if $q \in Q$, then the cross-section $S_q = \{p : \langle p, q\rangle \in S\}$ is either empty or equal to the F-equivalence class $[p']_{\mathsf{F}} = \{p : p\mathsf{F}p'\}$ of a suitable element $p' \in P$ satisfying $\langle p', q\rangle \in S$. Therefore every cross-section S_q is σ-compact under the conditions of the lemma.

It follows, by the known Arsenin–Kunugui–Shchegolkov theorem for Borel sets with σ-compact cross-sections, the set $Z = \operatorname{ran} S = \{q : \exists p\,(\langle p, q\rangle \in R)\}$ is Borel and, moreover, there exists a uniformizing Borel map $\vartheta : Z \to P$, that is, $\langle \vartheta(q), q\rangle \in S$ for all $q \in Z$ (for more on this theorem, see, e.g., [15, 35.H] and also the papers [7, 17]).

However, by definition we have $\operatorname{ran} U \subseteq Z$ and $p\mathsf{F}\vartheta(q)$ for every pair $\langle p, q\rangle \in U$. In addition, Z is an F-invariant set, that is, $q \in Z \wedge q'\mathsf{F}q$, $q' \in Z$. This allows us to accomplish the proof of the lemma (and the theorem) with the following definition of a Borel reduction of E to F. We naturally put $h(x) = f(x)$ for all $x \in X$. If $y \in Y$ and $g(y) \notin Z$, then put $h(y) = g(y)$. But if $g(y) \in Z$, then we define $h(y) = \vartheta(g(y))$. $\square$

The condition of the σ-compactness of the equivalence classes somewhat reduces the field of applications of Theorem 1, but the latter still remains rather substantial, see below. The role of this condition is clear: it guarantees a suitable choice of an element in an F-equivalence class by means of a Borel function (the function ϑ at the end of the proof of Lemma 2). Here we make use of the uniformization theorem for Borel sets with σ-sections, the strongest known uniformization theorem applicable in this context.[2] It hardly can be expected in the case under consideration that uniformization theorems for sets with "large" cross-sections (for instance, those of positive measure, or nonmeager; for such theorems, see [15]) are applicable in a reasonable way, since it is clear that any equivalence relation has only countably many "large" equivalence classes.

Another opportunity would be to apply the uniformization theorem for sets with σ-compact sections for the equivalence relation E rather than F. Namely, to obtain a Borel map $\varphi : Y' \to X$ satisfying $\varphi(y)\mathsf{E}y$ for all $y \in Y'$ in the proof of Lemma 2. But this would require the σ-compactness of all sets of the form $[x] \cap X$, that is, all sets of the form $[x] \cap X_k$, in the context of Theorem 1. But such a condition provides restrictions to both E and the sets X_k. Yet such a version works and can be useful in the case where all E-equivalence classes are countable. In this case, the countability and hence the σ-compactness as well do not depend on the nature of the sets X_k.

3. Applications

Note that the requirement $\mathbb{N}\mathsf{F} \le_{\mathrm{B}} \mathsf{F}$ in the theorem holds for typical Borel equivalence relations F. (Borel equivalence relations with infinitely many equivalence classes that do not satisfy this condition were originally defined in [14]. A somewhat simplified construction is given in [18]. Yet all counterexamples known so far are rather artificial and quite complicated.) The requirement of the σ-compactness of the F-equivalence classes is more restrictive, of course. Yet it holds for all countable equivalence relations F (those with finite and countable equivalence classes), in particular, for E_0, as well as for (noncountable) equivalence relations E_1 and $\boldsymbol{\ell}^\infty$.

Recall that E_0 is defined on the set $2^{\mathbb{N}}$ of all infinite dyadic sequences so that $\{i_n\}\mathsf{E}_1\{j_n\}$ if and only if $i_n = j_n$ for almost all (except for finitely many) n. The equivalence relation E_1 is defined similarly on the set $\mathbb{R}^{\mathbb{N}}$ of all infinite real sequences: $\{x_n\}\mathsf{E}_1\{y_n\}$ if and only if $x_n = y_n$ for almost all n. The relation $\boldsymbol{\ell}^\infty$ is defined differently

[2] Recall that an arbitrary Borel set is not necessarily uniformizable by a Borel set. The Novikov–Kondo theorem states that it is uniformizable by sets in a wider class $\mathbf{\Pi}^1_1$, but this would lead to non-Borel maps h in the proof of the lemma.

on the same set: $\{x_n\}\boldsymbol{\ell}^\infty\{y_n\}$ if and only if there is a number $C > 0$ such that $|x_n - y_n| < C$ for all n. It is known that $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_1 <_{\mathrm{B}} \boldsymbol{\ell}^\infty$ (see [5] or [10] for a wider spectrum of similar equivalence relations).

It is worthwhile to mention the equality $\Delta_{\mathbb{R}}$ on the real line, which can be regarded as an equivalence relation and obviously satisfies both conditions of Theorem 1 as F.

There are three types of Borel equivalence relations E related to $\Delta_{\mathbb{R}}$, E_0, E_1:

smooth: those satisfying $\mathsf{E} \le_{\mathrm{B}} \Delta_{\mathbb{R}}$, i.e., Borel reducible to $\Delta_{\mathbb{R}}$;

hyperfinite: those satisfying $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ and countable (that is, all equivalence classes are at most countable), see [9];

hypersmooth: those satisfying $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$, see [16].

All smooth equivalence relations are hyperfinite, while all hyperfinite ones are hypersmooth, and neither of the two inclusions is invertible.

Corollary 3. *For each of these three classes of Borel equivalence relations (smooth, hyperfinite, hypersmooth), the following holds.*

Assume that E is a Borel equivalence relation on a Borel set $X = \bigcup_k X_k$, where all X_k are Borel sets, too. If for every k the restricted relation $\mathsf{E} \restriction X_k$ belongs to any of the three mentioned classes, then the relation E itself belongs to the same class.

This case of Theorem 1 has been known for hyperfinite (and, most likely, for smooth) equivalence relations since long ago. Indeed, it is mentioned in [9], although we are not able to locate a reference.

Theorem 1 and Corollary 3 can be useful for upper estimates of the complexity of Borel equivalence relations in cases where, in the context of the problem under consideration, the domain of a given equivalence relation is split into countably (or finitely) many parts on which this equivalence relation can be investigated separately. This happens in the proofs of some complicated dichotomy theorems (see, e.g., [6, 12, 13]), where the first case, i.e., the case of a regular domain, implies a partition into subdomains defined in accordance with the position at which the regularity begins in a certain representation of a given point as a sequence.

The first author acknowledges support from RFBR, grants 06-01-00608 and 07-01-00445. The second author acknowledges support from RFBR 07-01-00445 and MNTC 2766.

Translated by V. G. Kanovei.

REFERENCES

1. J. Barwise (ed.), *Handbook of Mathematical Logic. Part* II. *Set Theory* [Russian translation, with a supplement by V. G. Kanovei], Nauka, Moscow (1982).
2. A. M. Vershik, "Theory of orbits," in: *Dynamical Systems* – 2, Vol. 2, *Itogi Nauki i Tekhniki*, VINITI, Moscow (1985), pp. 89–105.
3. V. Kanovei, "Supplement. Luzin's projective hierarchy: the current state of the theory," in: *Handbook of Mathematical Logic. Part* II. *Set Theory* [Russian translation], J. Barwise (ed.), Nauka, Moscow (1982), pp. 273–364.
4. V. Kanovei, "Topologies generated by effectively Suslin sets and their applications in descriptive set theory," *Uspekhi Mat. Nauk*, **51**, No. 3, 385–417 (1996).
5. V. G. Kanovei and V. A. Lyubetsky, "On some classical problems in descriptive set theory," *Uspekhi Mat. Nauk*, **58**, No. 5, 839–927 (2003).
6. V. G. Kanovei and V. A. Lyubetsky, *Modern Set Theory: Foundations of Descriptive Dynamics* [in Russian], Nauka, Moscow (2007).
7. V. Kanovei and M. Reeken, "Some new results on the Borel irreducibility of equivalence relations," *Izv. Ross. Akad. Nauk Ser. Mat.*, **67**, 59–82 (2003).
8. E. Shchegolkov, "On the uniformization of certain B-sets," *Dokl. Akad. Nauk SSSR*, **59**, 1065–1068 (1948).
9. J. Burgess and D. Miller, "Remarks on invariant descriptive set theory," *Fund. Math.*, **90**, No. 1, 53–75 (1975).
10. R. Dougherty, S. Jackson, and A. S. Kechris, "The structure of hyperfinite Borel equivalence relations," *Trans. Amer. Math. Soc.*, **341**, No. 1, 193–225 (1994).
11. Su Gao, "Equivalence relations and classical Banach spaces," in: *Mathematical Logic in Asia. Proceedings of the 9th Asian Logic Conference*, Novosibirsk, Russia (2005), pp. 70–89.

12. L. A. Harrington, A. S. Kechris, and A. Louveau, "A Glimm–Effros dichotomy for Borel equivalence relation," *J. Amer. Math. Soc.*, **3**, No. 4, 903–928 (1990).
13. G. Hjorth, "Actions by the classical Banach spaces," *J. Symbolic Logic*, **65**, No. 1, 392–420 (2000).
14. G. Hjorth and A. S. Kechris, "Recent developments in the theory of Borel reducibility," *Fund. Math.*, **170**, No. 1–2, 21–52 (2001).
15. G. Hjorth and A. S. Kechris, *Rigidity Theorems for Actions of Product Groups and Countable Borel Equivalence Relations*, Mem. Amer. Math. Soc., **177**, No. 833 (2005).
16. A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York (1995).
17. A. S. Kechris and A. Louveau, "The classification of hypersmooth Borel equivalence relations," *J. Amer. Math. Soc.*, **10**, No. 1, 215–242 (1997).
18. S. M. Srivastava, "Selection and representation theorems for σ-compact valued multifunctions," *Proc. Amer. Math. Soc.*, **83**, No. 4, 775–780 (1981).
19. S. Thomas, "Some applications of superrigidity to Borel equivalence relations," in: *Set Theory* (Piscataway, New Jersey, 1999), *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, **58**, Amer. Math. Soc., Providence, Rhode Island (2002), pp. 129–134.