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A Model in Which Well-Orderings of the Reals First Appear at a Given Projective Level, Part II

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Abstract: We consider the problem of the existence of well-orderings of the reals, definable at a certain level of the projective hierarchy. This research is motivated by the modern development of descriptive set theory. Given $n \geq 3$, a finite support product of forcing notions similar to Jensen's minimal- Δ_3^1 -real forcing is applied to define a model of set theory in which there exists a good Δ_n^1 well-ordering of the reals, but there are no Δ_{n-1}^1 well-orderings of the reals (not necessarily good). We conclude that the existence of a good well-ordering of the reals at a certain level $n \geq 3$ of the projective hierarchy is strictly weaker than the existence of a such well-ordering at the previous level $n - 1$. This is our first main result. We also demonstrate that this independence theorem can be obtained on the basis of the consistency of \mathbf{ZFC}^- (that is, a version of \mathbf{ZFC} without the Power Set axiom) plus 'there exists the power set of ω ', which is a much weaker assumption than the consistency of \mathbf{ZFC} usually assumed in such independence results obtained by the forcing method. This is our second main result. Further reduction to the consistency of second-order Peano arithmetic \mathbf{PA}_2 is discussed. These are new results in such a generality (with $n \geq 3$ arbitrary), and valuable improvements upon earlier results. We expect that these results will lead to further advances in descriptive set theory of projective classes.

Keywords: forcing; projective well-orderings; projective classes; Jensen's forcing**MSC:** 03E15; 03E35

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1. Introduction

This paper is written as a continuation of our earlier paper [1] under the same title, which thereby has to be viewed as Part I of this paper.

Problems related to the well-orderability of the real line \mathbb{R} emerged in the early years of set theory. The axiom of choice \mathbf{AC} implies that every set can be well-ordered, yet \mathbf{AC} does not yield a concrete construction of any particular well-ordering of \mathbb{R} . The famous discussion between Baire, Borel, Hadamard, and Lebesgue in [2] presents related issues widely discussed by mathematicians early in the 20th century.

Then, studies in descriptive set theory demonstrate that no well-ordering of \mathbb{R} belongs (as a set of pairs) to the first-level projective classes Σ_1^1 , Π_1^1 , see e.g., Sierpinski [3]. This was a consequence of Luzin's theorem [4] saying that sets in $\Sigma_1^1 \cup \Pi_1^1$ are Lebesgue measurable. (We refer to Moschovakis' monograph [5] in matters of both modern and early notation systems and early history of descriptive set theory. Yet, we may recall that Σ_1^1 consists of all continuous images of Borel sets in Polish spaces, Π_n^1 consists of all complements of Σ_n^1 sets, Σ_{n+1}^1 consists of continuous images of Π_n^1 sets, and $\Delta_n^1 = \Sigma_n^1 \cap \Pi_n^1$, for all $n \geq 1$).

For the sake of brevity, we let $\mathbf{WO}(\Gamma)$ be the hypothesis saying:

"There is a well-ordering of the real numbers which belongs to Γ as a set of pairs."

Here, Γ is a given class of subsets of Polish spaces. Typical examples include projective classes $\Sigma_n^1, \Pi_n^1, \Delta_n^1$ defined as above, and their *effective* subclasses resp. $\Sigma_n^1, \Pi_n^1, \Delta_n^1$, defined the same way but beginning with *effective Borel* sets, i.e., those that admit a Borel construction from an effectively (that is, computably) defined sequence of rational cubes.

Here, we can limit ourselves to classes Δ_n^1 and Δ_n^1 . Indeed,

$$\mathbf{WO}(\Sigma_n^1) \iff \mathbf{WO}(\Pi_n^1) \iff \mathbf{WO}(\Delta_n^1) \text{ and } \mathbf{WO}(\Sigma_n^1) \iff \mathbf{WO}(\Pi_n^1) \iff \mathbf{WO}(\Delta_n^1),$$

because if a well-ordering \preceq of the reals is say Σ_n^1 then it is Π_n^1 as well since $x \preceq y$ is equivalent to $x = y \vee y \not\preceq x$. Therefore, the result above can be summarized as $\neg \mathbf{WO}(\Delta_n^1)$.

At the next projective level, Gödel [6] proved that $\mathbf{WO}(\Delta_2^1)$ is consistent with the axioms of the Zermelo–Fraenkel set theory **ZFC**. This was established by a concrete definition of a Δ_2^1 well-ordering \leq_L of the reals in the constructible universe **L**. Then, Addison [7] distinguished a crucial property of \leq_L now known as *goodness*. Namely, a Δ_n^1 -good well-ordering is defined to be any Δ_n^1 well-ordering \preceq such that the class Δ_n^1 is closed under \preceq -bounded quantification. In other words, it is required that if $P(y, x)$ is a binary Δ_n^1 relation on the reals, then the following relations

$$Q(z, x) := \exists y \preceq x P(z, y) \text{ and } R(z, x) := \forall y \preceq x P(z, y)$$

belong to Δ_n^1 as well. The result by Gödel–Addison then claims that, in **L**, \leq_L is a Δ_2^1 -good well-ordering of the reals. It follows that the existence of such a well-ordering is a consequence of the axiom of constructibility $\mathbf{V} = \mathbf{L}$, and hence it is consistent with **ZFC**. The Δ_2^1 -goodness of \leq_L is behind many key results on sets of the second projective level, see for instance ([5], Section 5A).

As for the opposite direction, studies in the early years of modern set theory (see, e.g., Levy [8] and Solovay [9]) demonstrated that the non-existence statement, saying that there is no well-ordering of \mathbb{R} definable by any set-theoretic formula with ordinal and real parameters (this includes $\Sigma_\infty^1 = \bigcup_n \Sigma_n^1$ as a small part), is consistent as well.

Modern research in connection with projective well-orderings touches on such issues as connections with forcing axioms [10,11], connections with large cardinals [12,13], connections with cardinal characteristics of the continuum [14,15], connections with the structure and properties of projective sets [16–19], and others. The following theorem contributes to this research field. The theorem is the first principal result of this paper.

Theorem 1. *Let $n \geq 3$. Then, there exists a generic extension of **L**, in which:*

- (i) $\mathbf{WO}(\Delta_n^1)$ is true, and moreover, there is a ω_1 -long Δ_n^1 -good well-ordering of the reals;
- (ii) $\mathbf{WO}(\Delta_{n-1}^1)$ is false, that is, there are no Δ_{n-1}^1 well-orderings of the reals, of any kind, i.e., not necessarily good.

Therefore, it is consistent that “ $\mathbf{WO}(\Delta_n^1)$ holds, even by means of a Δ_n^1 -good well-ordering, and in the same time the stronger statement $\mathbf{WO}(\Delta_{n-1}^1)$ fails”.

As an immediate corollary of Theorem 1, we conclude that, for any $n \geq 3$, the hypothesis $\mathbf{WO}(\Delta_{n-1}^1)$ is strictly stronger than $\mathbf{WO}(\Delta_n^1)$ because there exists a model in which the latter holds whereas the former fails. Thus, the strict ascending condition $\Delta_{n-1}^1 \subsetneq \Delta_n^1$ of the classes Δ_n^1 is adequately reflected in the property of the existence of a well-ordering of the reals in a given class.

Theorem 1 significantly strengthens a theorem in our previous paper [1], where we defined a generic extension of **L** in which there is a Δ_n^1 -good well-ordering but there do not exist Δ_{n-1}^1 -good well-orderings of the reals. Thus, Theorem 1 improves the result in [1] by eliminating the goodness property in part (ii). This improvement required some crucial modifications in the proof of the theorem in part (ii) in this paper. Indeed in [1] we were able to use some well-known consequences of the Δ_{n-1}^1 -goodness, in particular, the basis theorem saying that all non-empty Σ_{n-1}^1 sets of reals contain Δ_{n-1}^1 elements.

This consequence is not available in the context of claim (ii) of the theorem since the Δ_{n-1}^1 -goodness is not assumed. To circumvent this difficulty, this paper introduces an entirely new technique of working with the auxiliary forcing relation forc , developed in Sections 23–30 of this paper.

The other direction of the paper belongs to the context of the second-order Peano arithmetic PA_2 and related set and class theories. Theory PA_2 governs the interrelations between the natural numbers and sets of natural numbers, and is widely assumed to lay down working foundations for essential parts of modern mathematics including whatever is (or can be) developed by means of the theory of projective sets, see e.g., Simpson [20].

In particular, claims (i) and (ii) of Theorem 1 can be adequately presented by certain formulas of the language of PA_2 based on suitable universal formulas for classes Σ_n^1 and Σ_{n-1}^1 . Therefore, for any given $n \geq 3$, the statement (i) + (ii) of Theorem 1 is essentially a formula, say Φ_n , of the language of PA_2 , whose consistency is established by the theorem. Thus, it becomes a natural problem to prove the consistency of Φ_n as in Theorem 1 on the base of tools close to PA_2 , rather than (much stronger) ZFC tools. The next theorem, our second main result, solves this problem on the basis of ZFC^- (minus stands for the absence of the Power Set axiom), which is a theory equiconsistent with PA_2 and thereby a substantial approximation towards PA_2 .

Theorem 2 (in $\text{ZFC}^- + \text{‘}\mathcal{P}(\omega)\text{ exists’}$). *Let $n \geq 3$. Then, the conjunction of (i) and (ii) of Theorem 1 is consistent with PA_2 .*

Further reduction to pure PA_2 will be the topic of our subsequent planned paper.

2. Outline of the Proof

Given $n \geq 3$ as in Theorem 1, a generic extension of \mathbf{L} , the constructible universe, was defined in [1], in which there exist Δ_n^1 -good well-orderings of the reals, but no Δ_{n-1}^1 -good well-orderings. Here, to prove our main results, Theorems 1 and 2, we make use of a modified model. This model involves a product forcing construction in \mathbf{L} , earlier applied in [18,21] for models with various effects related to the property of separation in the projective hierarchy, and also in [17] for a model in which the full basis theorem holds in the effective projective hierarchy (all non-empty Σ_∞^1 sets of reals contain Σ_∞^1 elements), in the absence of a Σ_∞^1 well-ordering of the reals, for generic models with counterexamples to the countable axiom of choice AC_ω and dependent choices DC in [22], to name a few examples.

Following the earlier papers [1,17,18], we make use of a sequence of forcings $\mathbb{P}(\xi)$, $\xi < \omega_1$, defined in \mathbf{L} such that the product forcing $\mathbb{P} = \prod_{\xi} \mathbb{P}(\xi)$ adds a sequence of generic reals to \mathbf{L} , uniformly Π_{n-1}^1 -definable in two arguments. Each forcing notion $\mathbb{P}(\xi)$ in this construction is a set of perfect trees $T \subseteq 2^{<\omega}$, similarly to the Jensen minimal forcing defined in [23]. See more in ([24], 28A) on Jensen’s forcing. Infinite finite-support products of Jensen’s forcing were first considered by Enayat [25], as demonstrated in [1], following this modification of Jensen–Enayat construction results in the existence of Δ_n^1 -good well-orderings in \mathbb{P} -generic extensions, thus witnessing (i) of Theorem 1.

Yet, a substantial modification of the Jensen–Enayat forcing construction is maintained in this paper, in order to get rid of using countable models of ZFC^- (i.e., ZFC without the Power Set axiom). Different tools based on such models were used in earlier papers, e.g., in [1,18], in particular, for evaluating the complexity of various sets, like e.g., the forcing notion itself. However, as one of our goals is to reduce the whole complexity of the construction of the models required, we have to remove models of ZFC^- from our instrumentarium. Getting rid of models of ZFC^- is thereby a **principal technical achievement** of this paper.

We begin in Sections 3–7 with a rather routine material related to arboreal forcings (those with perfect trees in $2^{<\omega}$ as forcing conditions) and their countable finite-support products called *multiforcings*, as well as finite tuples of trees called *multitrees*. The principal *refinement* relation $\pi \sqsubset \varrho$ between multiforcings π, ϱ is introduced in Sec-

tion 7. Roughly speaking, its meaning consists in the requirement that every multitree in $\mathbf{MT}(\varrho)$ (all multitrees related to ϱ) has to be meager in every multitree in $\mathbf{MT}(\pi)$.

The second part of the paper (Sections 8–15) develops the background for the above-mentioned technical achievement. It is based on the notion of *sealing refinement* $\pi \sqsubset_D \varrho$ (D being a dense subset of $\mathbf{MT}(\pi)$), which means that, besides $\pi \sqsubset \varrho$, every multitree $p \in \mathbf{MT}(\varrho)$ is covered by a finite collection of D -extendable multitrees in $\mathbf{MT}(\pi)$ (Definition 11). The following *transitivity* property takes place: if multiforcings π, ϱ, δ satisfy $\pi \sqsubset_D \varrho \sqsubset \delta$ then $(\pi \cup^{\text{cw}} \varrho) \sqsubset_D \delta$, where \cup^{cw} is the component-wise union of multiforcings. We consider different types of dense sets to be sealed, including those which govern a kind of Cantor-Bendixson derivative procedure in Sections 12 and 13.

Corollary 12 summarizes the transitivity property as above for different versions of \sqsubset_D . Theorem 4 proves the existence of sealing refinements. Theorem 5 provides consequences for generic extensions.

The next part of the paper (Sections 16–22) presents the key constructions involved in the proof of Theorem 1. We fix a natural number $n \geq 3$ as in Theorem 1, and consider the constructible universe \mathbf{L} as the ground model. Theorem 6 in Section 19 introduces a ω_1 -long \sqsubset -increasing sequence $\vec{\pi} \in \mathbf{L}$ of countable multiforcings, whose properties include; first, sealing a sufficient amount of dense sets during the course of the construction; second, a sort of definable genericity in \mathbf{L} ; and third, a definability requirement—as in Definition 23. The subsequent key forcing notion $\mathbb{P} \in \mathbf{L}$ (which depends on $\vec{\pi}$) is defined in Section 20. Its properties include CCC by Theorem 7. Then, we consider \mathbb{P} -generic extensions of \mathbf{L} , called key models. The main results regarding key models are: Theorem 8, which characterizes generic reals; and Theorem 9, which provides a Δ^1_n -good well-ordering in the generic model considered, with (i) of Theorem 1 as a consequence. Along with Theorem 7, these are the main results of this part of the paper.

Claim (ii) of Theorem 1 involves one more important technical tool related to the above-defined key forcing notion \mathbb{P} . It turns out that the \mathbb{P} -forcing relation of Σ^1_{n-1} formulas is equivalent (up to level $n - 1$ of the projective hierarchy of formulas) to a certain auxiliary forcing relation *forc* defined and studied in the following part of the paper (Sections 23–30). Theorem 11 proves the equivalence. This auxiliary forcing is invariant with respect to permutations of indices $\xi < \omega_1$ (Theorem 12), whereas the forcing \mathbb{P} itself is absolutely not invariant in that sense. Such a *hidden invariance* plays a crucial role in the construction. It was applied in [1] in the proof that \mathbb{P} -generic extensions satisfy a weaker version of (ii) only for Δ^1_{n-1} -good well-orderings. Here, we make use of the invariance to prove, using Theorem 13, that the full version of (ii) also holds in the \mathbb{P} -generic extensions. Theorems 11–13 are the main results related to *forc*, and the introduction and the whole treatment of the auxiliary forcing in a form compatible with the system of sealing relations without any reference to countable models of \mathbf{ZFC}^- is our second principal technical achievement.

The final part of the paper contains Section 31 with a short proof of Theorem 2 and a brief discussion of its possible reduction to a theory weaker than $\mathbf{ZFC}^- + \text{‘}\omega_1 \text{ exists’}$. We finish in Section 32 with conclusions and problems.

Part I: Basic Constructions

Here, we present a rather routine material on arboreal forcing notions, i.e., those with perfect trees in $2^{<\omega}$ in the role of forcing conditions. Then, in Section 6, we consider countable finite-support products of arboreal forcing notions, called *multiforcings*, as well as finite tuples of trees called *multitrees*. We introduce and study a principal *refinement* relation between arboreal forcing notions in Section 4 and between multiforcings in Section 7.

3. Trees and Arboreal Forcing Notions

Recall that $2^{<\omega}$ is the set of all tuples (i.e., finite sequences) of 0, 1. If $t \in 2^{<\omega}$ and $i = 0, 1$, then $t \hat{\ } i$ is the extension of t by i taking the rightmost position. If $s, t \in 2^{<\omega}$, then

- $s \sqsubseteq t$ if and only if t extends s ;
- $s \subset t$ if and only if $s \sqsubseteq t$ but $s \neq t$.

Generally, \subset denotes a **strict** inclusion (the equality “=” not allowed) in all cases in this paper, i.e., the same as \subsetneq . The non-strict inclusion is \subseteq . The length of t is denoted by $\text{lh}(t)$, and we put $2^n = \{t \in 2^{<\omega} : \text{lh}(t) = n\}$, the set all tuples of length n .

Trees in $2^{<\omega}$ are considered. Thus, $T \subseteq 2^{<\omega}$ is a *tree* if $t \in T \implies s \in T$ holds for all tuples $s \subset t$ in $2^{<\omega}$. Then, the *body*

$$[T] = \{a \in 2^\omega : \forall n (a \upharpoonright n \in T)\} \subseteq 2^\omega$$

is a closed set in 2^ω . A tree $T \subseteq 2^{<\omega}$ is:

- *Pruned*, if T contains no \subseteq -maximal tuples;
- *Perfect*, if it is pruned and has no *isolated branches*;
- We let **PT** contain all *perfect trees* $\emptyset \neq T \subseteq 2^{<\omega}$;
- If $s \in T \in \mathbf{PT}$ then we put $T \upharpoonright_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$; clearly $T \upharpoonright_s \in \mathbf{PT}$ as well.

If $T \in \mathbf{PT}$ then $[T]$ is a perfect set in 2^ω .

Definition 1. If $S, T \in \mathbf{PT}$, then define $S \perp T$ (S, T are incompatible) if $[S] \cap [T] = \emptyset$; this is equivalent to $S \cap T$ being finite. Then, $S \not\perp T$ means the negation of $S \perp T$.

A set $A \subseteq \mathbf{PT}$ is an *antichain* if $S \perp T$ holds for all $S \neq T$ in A .

Definition 2 (arboreal forcing notions). A set $P \subseteq \mathbf{PT}$ is an *arboreal forcing* if $u \in T \in P$ implies $T \upharpoonright_u \in P$. We define **AF** to be the set of all arboreal forcings P . Any $P \in \mathbf{AF}$ is:

- *Regular*, if, for all trees $S, T \in P$, the intersection $[S] \cap [T]$ is clopen in $[S]$ or in $[T]$;
- *Special*, if $P = \{T \upharpoonright_s : s \in T \in A\}$ for some finite or countable antichain $A \subseteq P$ —note that in this case the antichain A is unique and the forcing P has to be countable.

Note that every special arboreal forcing is regular.

Example 1. For any $s \in 2^{<\omega}$, define $T[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subseteq s\}$. Then, $T[s] \in \mathbf{PT}$ and $T[s] = (2^{<\omega}) \upharpoonright_s, \forall s$. Then, $\mathbb{P}_{\text{coh}} = \{T[s] : s \in 2^{<\omega}\}$ is the Cohen forcing, a regular and special arboreal forcing. The set **PT** itself is a non-regular arboreal forcing.

Definition 3 (perfect kernels). The *perfect kernel* of a tree $T \subseteq \omega^{<\omega}$ is the set

$$\mathbf{ker}(T) = \{s \in T : \text{there exists a perfect tree } S \text{ with } s \in S \subseteq T \upharpoonright_s\}.$$

This is the largest perfect tree $K \subseteq T$.

Definition 4 (meet of perfect trees). If $S, T \in \mathbf{PT}$ then let $S \wedge T = \mathbf{ker}(S \cap T)$.

The intersection $S \cap T$ may not even be pruned, but $S \wedge T$ is a perfect (or empty) tree, $[\mathbf{ker}(S \cap T)] \subseteq [S] \cap [T]$, and the difference $([S] \cap [T]) \setminus [\mathbf{ker}(S \cap T)]$ is at most countable.

Lemma 1. Let P be a regular arboreal forcing. Then,

- (i) If $T_1, \dots, T_n \in P$ and $X = [T_1] \cap \dots \cap [T_n] \neq \emptyset$ then X is a finite union of sets of the form $[S], S \in P$, and then $T = T_1 \wedge \dots \wedge T_n = \mathbf{ker}(T_1 \cap \dots \cap T_n)$ is a perfect tree equal to a finite union of trees in P , and $[T_1] \cap \dots \cap [T_n] = [T]$.
- (ii) Any trees $S, T \in P$ are P -compatible (i.e., some tree $R \in P$ satisfies $R \subseteq S \cap T$) if and only if $[S] \cap [T] \neq \emptyset$, equivalently, $S \not\perp T$.

Proof. (i) Assume that $n = 2$. By the regularity assumption, let $X = [T_1] \cap [T_2]$ be clopen in say $[T_1]$. Then, there are tuples $s_1, \dots, s_k \in T_1$ such that $X = [T]$, where $T = \mathbf{ker}(T_1 \cap T_2) = T_1 \upharpoonright_{s_1} \cup \dots \cup T_1 \upharpoonright_{s_k}$. However, $T_1 \upharpoonright_{s_i} \in P$ as $P \in \mathbf{AF}$. As for $n > 2$, proceed by induction.

(ii) is an easy corollary of (i). \square

Lemma 2. *If $T \in P \in \mathbf{AF}$ and $S \in \mathbf{PT}$, $T \not\subseteq S$, then there exists a tree $T' \in P$ satisfying $T' \subseteq T$ and $[T'] \cap [S] = \emptyset$.*

Proof. Let $T' = T \upharpoonright_s$, where $s \in T \setminus S$. \square

4. Refinements of Arboreal Forcings

In this section, we introduce the key notion of refinement of arboreal forcings. We remind that if $\mathbb{P} = \langle \mathbb{P}; \leq \rangle$ is any poset then a set $D \subseteq \mathbb{P}$ is:

- Dense in \mathbb{P} in case $\forall p \in \mathbb{P} \exists q \in D (q \leq p)$;
- Open dense in \mathbb{P} if in addition $\forall p \in \mathbb{P} \forall q \in D (p \leq q \implies p \in D)$;
- Pre-dense in \mathbb{P} if the set $D' = \{p \in \mathbb{P} : \exists q \in D (p \leq q)\}$ is dense in \mathbb{P} .

An arboreal forcing Q is a *refinement* of an arboreal forcing P , in symbol $P \sqsubset Q$, if:

- (1) Q is dense in $P \cup Q$, so that for any $T \in P$ there is $Q \in Q$ with $Q \subseteq T$;
- (2) For any $T \in Q$ we have $T \subseteq^{\text{fin}} \cup P$, meaning that there exists a finite $D \subseteq P$ satisfying $T \subseteq \cup D$, or equivalently $[T] \subseteq \cup_{S \in D} [S]$;
- (3) If $T \in Q$ and $S \in P$ then the intersection $[S] \cap [T]$ is clopen in $[S]$, and $S \not\subseteq T$ —it follows that $\mathbb{P} \cap \mathbb{Q} = \emptyset$ and the set $[S] \cap [T]$ is meager in $[S]$.

Thus, trees in the refinement Q define closed sets that are essentially smaller in the sense of category than the trees of the original arboreal forcing P do.

Lemma 3. *Assume that $P \sqsubset Q$ are arbitrary regular arboreal forcings. Then:*

- (i) *The union $P \cup Q$ is regular, too, and Q is open dense in $P \cup Q$;*
- (ii) *If $S \in P$, $T \in Q$, and $S \not\subseteq T$, then $S \wedge T$ is a finite union of trees $T \upharpoonright_s \in Q$, $s \in T$;*
- (iii) *If $S, S' \in P$, $T \in Q$, and $T \subseteq S_1 \cap S_2$, then there are trees $R \in P$ and $T' \in Q$ satisfying $T' \subseteq T$ and $T' \subseteq R \subseteq S_1 \cap S_2$.*

Proof. To prove the regularity of $P \cup Q$ in (i), make use of (3). To prove (ii) apply (3) once again. Finally prove (iii). By Lemma 1(i), there are trees $R_1, \dots, R_n \in P$ such that $[S_1] \cap [S_2] = [R_1] \cup \dots \cup [R_n]$. It follows by (ii) that there is a tuple $s \in T$ such that $T' = T \upharpoonright_s \subseteq R_i$. We observe that $T' \in Q$ as Q is an arboreal forcing. Put $R = R_i$. \square

Lemma 4. *If $P \sqsubset Q \sqsubset R$ are arboreal forcings then $P \sqsubset R$, $P \sqsubset (Q \cup R)$, $(P \cup Q) \sqsubset R$.*

Proof. Prove $P \sqsubset R$. Properties (1), (2) are rather obvious. To check (3), let $T \in R$ and $S \in P$. By (2), there is a finite $D \subseteq Q$ with $T \subseteq \cup D$. If $U \in D$, then $[T] \cap [U]$ is clopen in $[U]$ and $[U] \cap [S]$ is clopen in $[S]$. Thus, $[T] \cap [S]$ is clopen in $[S]$. To see that $S \not\subseteq T$, assume otherwise. Then, $S \subseteq \cup D$, and hence there is a tree $U \in D \subseteq Q$ such that $[U] \cap [S]$ is not meager in $[S]$. On the other hand, $[U] \cap [S]$ is clopen in $[U]$ by (3). It follows that there are tuples $u, s \in 2^{<\omega}$ satisfying $U \upharpoonright_u = S \upharpoonright_s$. However, $U \upharpoonright_u \in Q$ and $S \upharpoonright_s \in P$. This contradicts (3).

The relations $P \sqsubset (Q \cup R)$ and $(P \cup Q) \sqsubset R$ are easy consequences. \square

Lemma 5 (see Lemma 5.2 in [18]). *Let $\langle P_\alpha \rangle_{\alpha < \gamma}$ be any \sqsubset -increasing sequence of special arboreal forcings. Then, $P = \bigcup_{\alpha < \gamma} P_\alpha$ is a regular arboreal forcing.*

In addition, if $0 < \mu < \gamma$ then $\bigcup_{\alpha < \mu} P_\alpha = P_{<\mu} \sqsubset P_{\geq\mu} = \bigcup_{\mu \leq \alpha < \gamma} P_\alpha$.

Moreover, if $0 < \mu < \gamma$ then P_μ is pre-dense in P and $P_{\geq\mu}$ is dense in P .

Proof. If $S \in P_\alpha$ and $T \in P_\beta$, $\alpha < \beta$, then $P_\alpha \sqsubset P_\beta$; hence, $[S] \cap [T]$ is clopen in $[T]$ by (3) above. This implies the regularity of P . The additional claims are elementary as well. \square

5. Sealing Refinements: Arboreal Forcings

Assume that $P \sqsubset Q$. Then, a dense set $D \subseteq P$ is not dense in $P \cup Q$ any more. Generally speaking, it may not even be *pre-dense* in $P \cup Q$. Yet, it happens that there is a special type of dense sets called *sealed dense* that preserves pre-density under refinements, and a special type of refinements that turns dense sets into sealed dense sets.

The case of arboreal forcings considered here is a simplified introduction into the more important case of multiforcings in the next section.

Definition 5. Let $P \subseteq \mathbf{PT}$ be an arboreal forcing. A set $D \subseteq P$ is sealed dense, if 1) it is open, i.e., $\forall S \in P \forall T \in D (S \subseteq T \implies S \in D)$, and 2) if $S \in P$ then $S \subseteq^{\text{fin}} \bigcup D$.

Lemma 6. Assume that D is a sealed dense set in $P \in \mathbf{AF}$. Then, D is open dense.

Proof. To prove the density, assume that $S \in P$. Then, $S \subseteq T_1 \cup \dots \cup T_n$, where $T_i \in D$ for all i . At least one of the intersections $[S] \cap [T_i]$ is not meager in $[S]$. Then, there is a tuple $t \in S$ such that $T = S \upharpoonright_t \subseteq T_i$. Then, $T \in P$; hence, $T \in D$ by the openness. \square

Lemma 7. Assume that $P \sqsubset Q$ are arboreal forcings, and D is a sealed dense set in P . Then, $D \uparrow Q = \{U \in Q : \exists S \in D (U \subseteq S)\}$ is a sealed dense set in Q .

Proof. Let $U \in Q$. By (2), in Section 4, the tree U is covered by a finite set of trees in P , hence, by a finite set T_1, \dots, T_n of trees $T_i \in D$ because D is sealed dense in P . Then, any intersection $[U] \cap [T_i]$ is clopen in $[U]$ by (3) in Section 4; hence, $[U] \cap [T_i]$ is equal to a finite union of sets $[V]$, $V \in Q$. Thus, overall, $[U]$ itself is equal to a finite union of sets $[V]$, where $V \in Q$ is such that $V \subseteq T_i$ for some $T_i \in D$. It remains to note that each such V belongs to $D \uparrow Q$. \square

Thus, the sealed denseness is preserved by the refinement operation. The next lemma shows that dense sets give rise to a sealed dense set by a certain kind of refinement.

Definition 6. If $P \sqsubset Q$ are arboreal forcings and $D \subseteq P$, then $P \sqsubset_D Q$ means that every tree $T \in Q$ is covered by a finite union of trees in D .

Lemma 8 (see Lemma 5.4 in [18]). Assume that P, Q, R are arboreal forcings, $D \subseteq P$, and $P \sqsubset_D Q \sqsubset R$. Then, $P \sqsubset_D (Q \cup R)$ and $P \cup Q \sqsubset_D R$.

6. Multiforcings and Multitrees

By a **multiforcing**, we understand any map $\pi : |\pi| \rightarrow \mathbf{AF}$ such that $|\pi| = \text{dom } \pi \subseteq \omega_1$. The set of all multiforcings is denoted by \mathbf{MF} . We can represent an arbitrary $\pi \in \mathbf{MF}$ in the form of an indexed set $\pi = \langle P_\xi \rangle_{\xi \in |\pi|}$, with $P_\xi \in \mathbf{AF}$ for each $\xi \in |\pi|$, where all components $P_\xi = P_\xi^\pi = \pi(\xi)$, $\xi \in |\pi|$, are arboreal forcings. Say that π is:

- *Small*, in case both $|\pi|$ and each set $\pi(\xi) = P_\xi^\pi$, $\xi \in |\pi|$, are countable;
- *special*, in case each $\pi(\xi) = P_\xi^\pi$ is special (then countable) as in Definition 2;
- *Regular*, in case all $\pi(\xi) = P_\xi^\pi$ are regular as in Definition 2.

Similarly, a *multitree* is any function $p : |p| \rightarrow \mathbf{PT}$ with a finite domain $|p| = \text{dom } p \subseteq \omega_1$. Let \mathbf{MT} be the set of all multitrees. Any multitree $p \in \mathbf{MT}$ can be represented in the form $p = \langle T_\xi^p \rangle_{\xi \in |p|}$, where $T_\xi^p = p(\xi) \in \mathbf{PT}$ for all $\xi \in |p|$. The set \mathbf{MT} is **ordered component-wise**: $q \leq p$ (q is stronger than p) if $|p| \subseteq |q|$ and $T_\xi^q \subseteq T_\xi^p$ for all $\xi \in |p|$.

Let $\pi = \langle P_\xi \rangle_{\xi \in |\pi|}$ be a multiforcing. Any multitree $p \in \mathbf{MT}$ is called a π -*multitree*, if $|p| \subseteq |\pi|$ and for each $\xi \in |p|$ the tree $p(\xi) = T_\xi^p$ belongs to $P_\xi = \pi(\xi)$. Clearly, the collection $\mathbf{MT}(\pi)$ of all π -multitrees can be identified with the *finite support* product $\prod_{\xi \in |\pi|} P_\xi$ of the arboreal forcings $P_\xi = \pi(\xi)$ involved.

Definition 7. If $\mathbf{p} \in \mathbf{MT}$ then define $[\mathbf{p}] = \prod_{\xi \in |\mathbf{p}|} [\mathbf{p}(\xi)]$, the finite Cartesian product of the perfect sets $[\mathbf{p}(\xi)]$, $\xi \in |\mathbf{p}|$. If $|\mathbf{p}| \subseteq X \subseteq \omega_1$, then let $[\mathbf{p}] \uparrow X = \{x \in (2^\omega)^X : x \upharpoonright |\mathbf{p}| \in [\mathbf{p}]\}$, this is a cylinder in $(2^\omega)^X$ based on $[\mathbf{p}]$.

Definition 8 (extension). If $D \subseteq \mathbf{MT}$ and φ is an arbitrary multiforcing then we define $D \uparrow \varphi = \{\mathbf{q} \in \mathbf{MT}(\varphi) : \exists \mathbf{p} \in D (\mathbf{q} \leq \mathbf{p})\}$.

Corollary 1 (of Lemma 1(i)). Let π be a regular multiforcing, $\mathbf{p}, \mathbf{q} \in \mathbf{MT}(\pi)$, $X = |\mathbf{p}| \cup |\mathbf{q}|$. Then, the intersection $([\mathbf{p}] \uparrow X) \cap ([\mathbf{q}] \uparrow X)$ is equal to a finite (perhaps empty) union of sets $[\mathbf{w}]$, where $\mathbf{w} \in \mathbf{MT}(\pi)$ and $|\mathbf{w}| = X$.

Definition 9. Multitrees \mathbf{p}, \mathbf{q} are incompatible, in symbol $\mathbf{p} \perp \mathbf{q}$, if $\mathbf{p}(\xi) \perp \mathbf{q}(\xi)$, or equivalently, $[\mathbf{p}(\xi)] \cap [\mathbf{q}(\xi)] = \emptyset$, holds for some index $\xi \in |\mathbf{p}| \cap |\mathbf{q}|$, and compatible otherwise. As usual, a set $A \subseteq \mathbf{MT}$ of pairwise incompatible multitrees is called an antichain.

Given a multiforcing π , multitrees \mathbf{p}, \mathbf{q} are π -compatible, if there exists a multitree $\mathbf{r} \in \mathbf{MT}(\pi)$ such that $\mathbf{r} \leq \mathbf{p}$ and $\mathbf{r} \leq \mathbf{q}$, and otherwise are π -incompatible, in symbol $\mathbf{p} \perp^\pi \mathbf{q}$. Sets $A \subseteq \mathbf{MT}$ of pairwise π -incompatible multitrees are π -antichains.

If multitrees are incompatible, then they are π -incompatible for any π . The next corollary shows that the inverse is true for regular multiforcings.

Corollary 2 (of Lemma 1(ii)). Let π be a regular multiforcing and $\mathbf{p}, \mathbf{q} \in \mathbf{MT}(\pi)$. Then, \mathbf{p}, \mathbf{q} are π -compatible if \mathbf{p}, \mathbf{q} are compatible as in Definition 9.

It follows that being an antichain is equivalent to being a π -antichain.

Corollary 3 (of Lemma 2). Let π be a regular multiforcing and $\mathbf{p} \in \mathbf{MT}(\pi)$, $\mathbf{r} \in \mathbf{MT}$. If $\mathbf{p}(\xi) \not\leq \mathbf{r}(\xi)$ for at least one $\xi \in |\mathbf{r}| \cap |\pi|$, then there exists a multitree $\mathbf{q} \in \mathbf{MT}(\pi)$, $\mathbf{q} \leq \mathbf{p}$, satisfying $\mathbf{q} \perp \mathbf{r}$.

Let π, φ be multiforcings. Define a multiforcing $\sigma = \pi \cup^{\text{cw}} \varphi$ (the component-wise union), so that $|\sigma| = |\pi| \cup |\varphi|$ and

$$\sigma(\xi) = \begin{cases} \pi(\xi) & \text{iff } \xi \in |\pi| \setminus |\varphi|, \\ \varphi(\xi) & \text{iff } \xi \in |\varphi| \setminus |\pi|, \\ \pi(\xi) \cup \varphi(\xi) & \text{iff } \xi \in |\varphi| \cap |\pi|. \end{cases}$$

If $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \lambda}$ is a sequence of multiforcings, then the component-wise union $\pi = \cup^{\text{cw}} \vec{\pi} = \cup_{\alpha < \lambda} \pi_\alpha \in \mathbf{MF}$ is accordingly defined so that $|\pi| = \cup_{\alpha < \lambda} |\pi_\alpha|$ and $\pi(\xi) = \cup_{\alpha < \lambda, \xi \in |\pi_\alpha|} \pi_\alpha(\xi)$ for all $\xi \in |\pi|$. We observe that \cup^{cw} does not preserve regularity.

Definition 10 (component-wise meet of multitrees). Let π be a regular multiforcing. Say that a finite set of multitrees $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbf{MT}(\pi)$ is compatible as a whole if for any index $\xi \in \cup_i |\mathbf{p}_i|$, we have $\cap_i [\mathbf{p}_i(\xi)] \neq \emptyset$. (Here, and below, it is understood that $\mathbf{p}_i(\xi) = 2^{<\omega}$ whenever $\xi \notin |\mathbf{p}_i|$). In such a case, let us define a multitree

$$\mathbf{p} = \wedge_i^{\text{cw}} \mathbf{p}_i = \mathbf{p}_1 \wedge^{\text{cw}} \dots \wedge^{\text{cw}} \mathbf{p}_n$$

so that $|\mathbf{p}| = \cup_i |\mathbf{p}_i|$ and $\mathbf{p}(\xi) = \wedge_i \mathbf{p}_i(\xi) = \mathbf{ker}(\cap_i \mathbf{p}_i(\xi))$ for all $\xi \in |\mathbf{p}|$.

Corollary 4 (of Lemma 1(i)). Suppose that π is a regular multiforcing, and a finite set of multitrees $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbf{MT}(\pi)$ is compatible as a whole. Then, $\mathbf{p} = \wedge_i^{\text{cw}} \mathbf{p}_i$ is a multitree, and $[\mathbf{p}]$ is a finite union of sets of the form $[\mathbf{q}]$, where $\mathbf{q} \in \mathbf{MT}(\pi)$, $|\mathbf{q}| = |\mathbf{p}|$.

Remark 1 (forcing). Let $P \in \mathbf{AF}$ be an arboreal forcing. We may treat P as a forcing notion, so that if $T \subseteq T'$ then T is a stronger condition. Clearly, P adjoins a real in 2^ω .

If $\pi = \langle P_\xi \rangle_{\xi \in |\pi|} \in \mathbf{MF}$ is a multiforcing then the set $\mathbf{MT}(\pi)$, ordered as above, is accordingly viewed as a forcing notion which adjoins a generic sequence $\langle x_\xi \rangle_{\xi \in |\pi|}$, where every $x_\xi = x_\xi[G] \in 2^\omega$ is a P_ξ -generic real. Reals of the form $x_\xi[G]$ will be called principal generic reals in the extension by a $\mathbf{MT}(\pi)$ -generic set G .

7. Refinements of Multiforcings

Here, we extend the notion of refinement to multiforcings in component-wise way.

Let π, φ be arbitrary multiforcings. Then, φ is said to be a refinement of π , symbolically $\pi \sqsubset \varphi$, if $|\pi| \subseteq |\varphi|$ and we have $\pi(\xi) \sqsubset \varphi(\xi)$ in \mathbf{AF} for all $\xi \in |\pi|$.

Corollary 5 (of Lemma 4). *If $\pi \sqsubset \varphi \sqsubset \sigma$ are multiforcings then $\pi \sqsubset \sigma$, $\pi \sqsubset (\varphi \cup^{cw} \sigma)$, and $(\pi \cup^{cw} \varphi) \sqsubset \sigma$.*

Corollary 6 (of Lemma 3). *Let $\pi \sqsubset \varphi$ be regular multiforcings. Then, so is $\pi \cup^{cw} \varphi$, and $\mathbf{MT}(\varphi)$ is an open dense set in $\mathbf{MT}(\pi \cup^{cw} \varphi)$. Moreover, if $p, p' \in \mathbf{MT}(\pi)$, $q \in \mathbf{MT}(\varphi)$, and $q \leq p, p'$, then there are multitrees $r \in \mathbf{MT}(\pi)$, $q' \in \mathbf{MT}(\varphi)$ satisfying $q' \leq q$ and $q' \leq r \leq p, p'$.*

Corollary 7 (of Lemma 1(i)). *Let $\pi \sqsubset \varphi$ be regular multiforcings, $p \in \mathbf{MT}(\pi)$, $q \in \mathbf{MT}(\varphi)$, $X = |p| \cup |q|$. Then, the intersection $([p] \uparrow X) \cap ([q] \uparrow X)$ is a finite (perhaps empty) union of sets of the form $[w]$, where $w \in \mathbf{MT}(\varphi)$ and $|w| = X$.*

Remark 2. *It follows from the above that the relations \sqsubset, \sqsubset are strict partial orders on sets resp. \mathbf{AF}, \mathbf{MF} . In addition, if π, φ are multiforcings and $|\pi| \subseteq |\varphi|$, then the relations $\pi \sqsubset \varphi$ and $\pi \sqsubset \varphi'$ are equivalent, where $\varphi' = \varphi \upharpoonright |\pi|$.*

Part II: Sealing Refinements

The first goal of this Part is to introduce a notion of sealing refinements for multiforcings, similar to the sealing refinements for arboreal forcings as in Section 5. This is a considerably more difficult case because obtaining adequate, working definitions both of the sealed density and the sealing refinements are somewhat less obvious. In particular, the notion of sealing refinement $\pi \sqsubset_D \varphi$ (D being a dense subset of $\mathbf{MT}(\pi)$), stipulates, that, besides $\pi \sqsubset \varphi$, every multitree $p \in \mathbf{MT}(\varphi)$ is covered by a finite collection of D -extendable multitrees in $\mathbf{MT}(\pi)$ (Definition 11). We consider different types of dense sets to be sealed, including those that govern a kind of Cantor-Bendixon derivative procedure in Sections 12 and 13.

Corollary 12 summarizes the transitivity property as above for different versions of \sqsubset_D . Theorem 4 proves the existence of sealing refinements. Theorem 5 provides consequences for generic extensions. These are main results of Part II.

8. Sealing Refinements

Suppose that u is a multitree and D a set of multitrees. Define $u \subseteq^{fin} \vee D$, if there exists a finite subset $D' \subseteq D$ such that 1) $|v| = |u|$ for all $v \in D'$, and 2) $[u] \subseteq \bigcup_{v \in D'} [v]$. (Regarding $[u]$ we refer to Definition 7).

Definition 11. *Let π be a multiforcing and $D \subseteq \mathbf{MT}$.*

- *A multitree p is D -extendable if there exists a multitree $q \in D$ satisfying $p = q \upharpoonright |p|$.*
 - *If $X \subseteq \omega_1$ is finite then let $D_{ext}^X(\pi) = \{p \in \mathbf{MT}(\pi) : |p| = X \wedge p \text{ is } D\text{-extendable}\}$.*
- (1) *D is sealed dense in $\mathbf{MT}(\pi)$ if D is open in $\mathbf{MT}(\pi)$ and $u \subseteq^{fin} \vee D_{ext}^{|u|}(\pi)$ holds for every $u \in \mathbf{MT}(\pi)$.*
 - (2) *A multiforcing φ seals D over π , symbolically $\pi \sqsubset_D \varphi$, if $\pi \sqsubset \varphi$ and, for every $u \in \mathbf{MT}(\varphi)$ with $|u| \subseteq |\pi|$, the relation $u \subseteq^{fin} \vee D_{ext}^{|u|}(\pi)$ holds.*

(3) A multiforcing φ seals D over π in the old sense, symbolically $\pi \sqsubset_D^{\text{old}} \varphi$, if $\pi \sqsubset \varphi$ and the next condition is true:

- if $p \in \text{MT}(\pi)$, $u \in \text{MT}(\varphi)$, $|u| \subseteq |\pi|$, $|u| \cap |p| = \emptyset$, then there exists $q \in \text{MT}(\pi)$ satisfying $q \leq p$, also $|q| \cap |u| = \emptyset$, and finally $u \subseteq^{\text{fin}} \bigvee D_q^{|u|}$, where

$$D_q^{|u|} = \{w \in \text{MT}(\pi) : |w| = |u| \wedge w \cup q \in D\}.$$

The old definition (3) of sealing refinements was given in [18] on the basis of earlier studies [22,26]. We use here a more flexible definition by (2).

Lemma 9. If π, φ are arbitrary multiforcings, $D \subseteq \text{MT}(\pi)$, and $\pi \sqsubset_D^{\text{old}} \varphi$ then $\pi \sqsubset_D \varphi$.

Proof. Apply (3) with $p = \Lambda$ (the empty multitree). \square

We will use the notation $D \uparrow \varphi$ as in Definition 8 in the following lemmas.

Lemma 10. Suppose that π, φ are regular multiforcings, and $D \subseteq \text{MT}(\pi)$ any set. Then:

- (i) If D is sealed dense in $\text{MT}(\pi)$ then D is open dense, and moreover, if $u \in \text{MT}(\pi)$, $X = |u|$, then there is a D -extendable multitree $v \in \text{MT}(\pi)$ with $v \leq u$ and $v = X$;
- (ii) If D is sealed dense in $\text{MT}(\pi)$, then $\pi \sqsubset \varphi$ implies $\pi \sqsubset_D \varphi$;
- (iii) If $\pi \sqsubset_D \varphi$ then $D \uparrow \varphi$ is sealed dense in $\text{MT}(\varphi)$ whereas D itself is pre-dense in $\text{MT}(\pi)$ and in $\text{MT}(\pi \cup^{\text{cw}} \varphi)$;
- (iv) If $\pi \sqsubset_D \varphi$, and $u \in \text{MT}(\varphi)$, $X = |u|$, then there exists a $(D \uparrow \varphi)$ -extendable multitree $v \in \text{MT}(\varphi)$ with $v \leq u$ and $v = X$;
- (v) If D is sealed dense in $\text{MT}(\pi)$, and $\pi \sqsubset \varphi$, then the set $D \uparrow \varphi$ is sealed dense in $\text{MT}(\varphi)$ and open dense in both $\text{MT}(\varphi)$ and $\text{MT}(\pi \cup^{\text{cw}} \varphi)$.

Proof. (i) As the openness of D is given, prove the ‘moreover’ claim. Let $u \in \text{MT}(\pi)$, $X = |u|$. Then, $[u] \subseteq [p_1] \cup \dots \cup [p_n]$, where the multitrees $p_i \in \text{MT}(\pi)$ satisfy $|p_i| = X$ and are D -extendable. Then, u is compatible with at least one p_i , and hence π -compatible by Corollary 2, so that there is a multitree $v \in \text{MT}(\pi)$ with $v \leq p_i$, $v \leq u$, and still $|v| = X$. It remains to be shown that v is D -extendable.

By the choice of p_i , there exists a multitree $q \in D$ with $X \subseteq Y = |q|$ and $q \upharpoonright X = p_i$. Define a multitree $w \in \text{MT}(\pi)$ so that $|w| = Y$, $w \upharpoonright X = v$, and $w \upharpoonright (Y \setminus X) = q \upharpoonright (Y \setminus X)$. Then, clearly, $w \leq q$, and hence $w \in D$ by the openness of D .

(ii) Let $u \in \text{MT}(\varphi)$, $|u| = X \subseteq |\pi|$. By $\pi \sqsubset \varphi$, there exists a finite collection $F \subseteq \text{MT}(\pi)$ of sets $p \in \text{MT}(\pi)$ such that $|p| = X$ and $[u] \subseteq \bigcup_{p \in F} [p]$. However, each $[p]$, $p \in F$, is covered by a finite union of sets $[q]$, where q is D -extendable.

(iii) The openness of $D \uparrow \varphi$ in $\text{MT}(\varphi)$ is obvious. To prove the sealed density, let $u \in \text{MT}(\varphi)$. Since $\text{MT}(\pi \cup^{\text{cw}} \varphi)$ is a product, we can assume that $X = |u| \subseteq |\pi|$. As $\pi \sqsubset_D \varphi$, $[u] \subseteq [p_1] \cup \dots \cup [p_n]$ holds, where the multitrees $p_i \in \text{MT}(\pi)$ satisfy $|p_i| = X$ and are D -extendable; hence, there are $p'_i \in D$ such that $X \subseteq |p'_i|$ and $p_i = p'_i \upharpoonright X$.

For each p_i , it follows by Corollary 7 that there exists a finite set $V(i)$ of multitrees $v \in \text{MT}(\varphi)$ satisfying still $|v| = X$ and $[u] \cap [p] = \bigcup_{v \in V(i)} [v]$. Let $V = \bigcup_{i \leq n} V(i)$. Then, $[u] = \bigcup_{v \in V} [v]$, so it remains to show that each $v \in V$ is $(D \uparrow \varphi)$ -extendable in $\text{MT}(\varphi)$.

Let $v \in V(i)$ and $q = p'_i$, so that $q \in D$ and $p_i = q \upharpoonright X$. As $u \leq p$, there is a multitree $w \in \text{MT}(\varphi)$ with $w \leq q$ and $w \upharpoonright X = v$. Then, $w \in D \uparrow \varphi$; therefore, w witnesses that v is $(D \uparrow \varphi)$ -extendable. This completes the proof that $D \uparrow \varphi$ is sealed dense in $\text{MT}(\varphi)$.

To prove the pre-density of D in $\text{MT}(\pi)$, let $p \in \text{MT}(\pi)$. As $\pi \sqsubset \varphi$, there is $u \in \text{MT}(\varphi)$, $u \leq p$. There exists $v \in D \uparrow \varphi$, $v \leq u$, by the above. Then, $v \leq$ some $q \in D$. Thus, v witnesses that p, q are compatible; therefore, p, q are π -compatible by Corollary 2.

To prove (iv) make use of (iii) and apply (i) for $D \uparrow \varphi$.

Finally, (v) easily follows from (i) (to infer the open density in $\text{MT}(\varphi)$), (ii), (iii), and Corollary 6 (to infer the open density in $\text{MT}(\pi \cup^{\text{cw}} \varphi)$). \square

Thus, in the case of multiforcings, the sealed density is preserved by the refinement operation, and just a dense set D converts to a sealed dense set by the refinement \sqsubset_D .

Lemma 11. *Let π, φ be regular multiforcings and $D_1 \cup D_2 \subseteq \mathbf{MT}(\pi)$. Then:*

- (i) *If D_1, D_2 are sealed dense sets in $\mathbf{MT}(\pi)$ then $D = D_1 \cap D_2$ is sealed dense as well;*
- (ii) *If D_1, D_2 are open dense sets in $\mathbf{MT}(\pi)$, $\pi \sqsubset_{D_1} \varphi$, and $\pi \sqsubset_{D_2} \varphi$, then we have $\pi \sqsubset_D \varphi$.*

Proof. (i) Both D_1, D_2 are open dense in $\mathbf{MT}(\pi)$, and hence, so is D . Now, let $u \in \mathbf{MT}(\pi)$, $X = |u|$. As D_1 is sealed dense, we have $[u] \subseteq [u_1] \cup \dots \cup [u_n]$, where the multitrees $u_i \in \mathbf{MT}(\pi)$ satisfy $|u_i| = X$ and are D_1 -extendable. In other words, for any u_i there is a multitree $p_i \in D_1$ such that $X \subseteq |p_i|$ and $u_i = p_i \upharpoonright X$. Let $X_i = |p_i|$. As D_2 is sealed dense, we have $[p_i] \subseteq [p_i^1] \cup \dots \cup [p_i^{n(i)}]$, where the multitrees $p_i^k \in \mathbf{MT}(\pi)$ satisfy $|p_i^k| = X_i$ and are D_2 -extendable. Thus, for any p_i^k there is a multitree $w_k^i \in D_2$ such that $X_i \subseteq |w_k^i|$ and $p_i^k = w_k^i \upharpoonright X_i$. Finally, each set $u_k^i = w_k^i \upharpoonright X_i$ is D -extendable (to w_k^i), and $[u] \subseteq \bigcup_{ik} u_k^i$.

(ii) Let $u \in \mathbf{MT}(\varphi)$, $X = |u| \subseteq |\pi|$. The sets $D_1 \upharpoonright \varphi$ and $D_2 \upharpoonright \varphi$ are sealed dense in $\mathbf{MT}(\varphi)$ by Lemma 10(iii), hence so is $D' = (D_1 \upharpoonright \varphi) \cap (D_2 \upharpoonright \varphi)$ by (i). It follows that we can w.l.o.g. assume that u is already D' -extendable, so that there is multitrees $v \in \mathbf{MT}(\varphi)$, $p_1 \in D_1$, and $p_2 \in D_2$ such that $X \subseteq Y = |v|$ and $v \leq p_1, p_2, u = v \upharpoonright X$.

Then, $[v] \subseteq ([p_1] \upharpoonright Y) \cap ([p_2] \upharpoonright Y)$, and on the other hand we have $([p_1] \upharpoonright Y) \cap ([p_2] \upharpoonright Y) = [r_1] \cup [r_2] \cup \dots \cup [r_m]$ by Corollary 4, where $r_i \in \mathbf{MT}(\pi)$, $|r_i| = Y$. Furthermore, $r_i \in D = D_1 \cap D_2$ by the open density assumption in (ii).

For each i , if $[v] \cap [r_i] \neq \emptyset$, then, by $\pi \sqsubset \varphi$, there are multitrees $w_1^i, w_2^i, \dots, w_{n(i)}^i \in \mathbf{MT}(\varphi)$ such that $|w_j^i| = Y$ for all i, j , and $[v] \cap [r_i] = \bigcup_{j \leq n(i)} [w_j^i]$. We observe that $w_j^i \leq r_i$ for all i, j by construction; hence, $w_j^i \in (D_1 \cap D_2) \upharpoonright \varphi$.

Now, let $u_j^i = w_j^i \upharpoonright X$. Then, $u_j^i \in \mathbf{MT}(\varphi)$, $|u_j^i| = X$, $[u] = \bigcup_{i,j} [u_j^i]$, and each u_j^i is $((D_1 \cap D_2) \upharpoonright \varphi)$ -extendable (to w_j^i), as required. \square

Lemma 12. *Let $\pi \sqsubset_D \varphi \sqsubset \delta$ be regular multiforcings, $D \subseteq \mathbf{MT}(\pi)$. Then:*

- (i) $\pi \sqsubset_D (\varphi \cup^{cw} \delta)$;
- (ii) $\pi \sqsubset_D \delta$;
- (iii) $\varphi \sqsubset_E \delta$, where $E = D \upharpoonright \varphi = \{q \in \mathbf{MT}(\varphi) : \exists p \in D (q \leq p)\}$;
- (iv) $(\pi \cup^{cw} \varphi) \sqsubset_E \delta$ and $(\pi \cup^{cw} \varphi) \sqsubset_D \delta$.

Proof. (i) Corollary 5 implies $\pi \sqsubset (\varphi \cup^{cw} \delta)$. Let $u \in \mathbf{MT}(\varphi \cup^{cw} \delta)$, $|u| = X \subseteq |\pi|$. As $\varphi \sqsubset \delta$, there is a finite collection $U \subseteq \mathbf{MT}(\varphi)$ satisfying $|v| = X$ for all $v \in U$, and $[u] \subseteq \bigcup_{v \in U} [v]$. As $\pi \sqsubset_D \varphi$, we obtain $v \subseteq^{fin} \bigvee D_{ext}^X(\pi)$ for any $v \in U$, and hence $u \subseteq^{fin} \bigvee D_{ext}^X(\pi)$.

(ii) The relation $\pi \sqsubset_D \sigma$ is an easy corollary.

(iii) Let $u \in \mathbf{MT}(\delta)$, $|u| = X \subseteq |\pi|$. As $\varphi \sqsubset \delta$, there exists a finite $U \subseteq \mathbf{MT}(\varphi)$ satisfying $|v| = X$ for each $v \in U$, and $[u] \subseteq \bigcup_{v \in U} [v]$. Then, we have $v \subseteq^{fin} \bigvee D_{ext}^X(\pi)$ for every $v \in U$ since $\pi \sqsubset_D \varphi$. In other words, there exists a finite $W \subseteq D_{ext}^X(\pi)$ with $[u] \subseteq \bigcup_{v \in U} [v] \subseteq \bigcup_{w \in W} [w]$. The multitrees in U can be refined using Corollary 7, so that we obtain a finite collection $U' \subseteq \mathbf{MT}(\varphi)$ satisfying

- (1) Still $|v'| = X$ for any $v' \in U'$;
- (2) $\bigcup_{v' \in U'} [v'] = \bigcup_{v \in U} [v]$;
- (3) If $v' \in U'$ then $v' \leq w$ for some $w \in W$ — and hence easily $v' \in E_{ext}^X(\varphi)$.

It follows that $[u] \subseteq^{fin} \bigvee E_{ext}^X(\varphi)$, as required.

(iv) The relation $(\pi \cup^{cw} \varphi) \sqsubset_E \delta$ is an easy corollary of (3).

Finally, $(\pi \cup^{cw} \varphi) \sqsubset_D \delta$ is established by the set W in the proof of (iii). \square

9. Two Examples

Here, we consider two important types of dense sets that can be made sealed dense.

Example 2. If π is a multiforcing and $p_0 \in \mathbf{MT}$ (not necessarily $p_0 \in \mathbf{MT}(\pi)$), then

(*) the set $D_{p_0}(\pi) = \{q \in \mathbf{MT}(\pi) : |p_0| \subseteq |q| \wedge (q \leq p_0 \vee p_0 \perp q)\}$ is open dense in $\mathbf{MT}(\pi)$ by Corollary 3 in case $|p_0| \subseteq |\pi|$, whereas if $|p_0| \not\subseteq |\pi|$ then $D_{p_0}(\pi) = \emptyset$.

If φ is another multiforcing then we write $\pi \sqsubset_{p_0} \varphi$ instead of $\pi \sqsubset_{D_{p_0}(\pi)} \varphi$, and say that φ seals p_0 over π . Note that $|p_0| \subseteq |\pi|$ in this case.

In addition, if $D_{p_0}(\pi)$ is sealed dense in $\mathbf{MT}(\pi)$ then we say that p_0 is sealed by π . Still $|p_0| \subseteq |\pi|$ in this case.

Corollary 8. Assume that $\pi \sqsubset \varphi \sqsubset \delta$ are regular multiforcings, and $p_0 \in \mathbf{MT}$. Then:

- (i) If p_0 is sealed by π then $\pi \sqsubset_{p_0} \varphi$;
- (ii) If $\pi \sqsubset_{p_0} \varphi$ then p_0 is sealed by φ , while $D_{p_0}(\pi)$ is open dense in $\mathbf{MT}(\pi)$.

Proof. We first recall (*) in Example 2 and observe that $(D_{p_0}(\pi) \uparrow \varphi) \subseteq D_{p_0}(\varphi)$. Then, to prove (i), (ii) apply Lemma 10. \square

If π is a multiforcing and p, q are π -incompatible multitrees in \mathbf{MT} (not necessarily in $\mathbf{MT}(\pi)$), then it is well possible that p, q become φ -compatible for another multiforcing φ , even with $\pi \sqsubset \varphi$. To inhibit such a case, the following condition is introduced.

Example 3. Let π be a multiforcing. If $p, q \in \mathbf{MT}$ (not necessarily $\in \mathbf{MT}(\pi)$), then let

$$N_{pq}(\pi) = \{r \in \mathbf{MT}(\pi) : r \perp p \vee r \perp q\}.$$

The set $N_{pq}(\pi)$ is open dense in $\mathbf{MT}(\pi)$ by Corollary 3, provided $|p| \cup |q| \subseteq |\pi|$ and p, q are π -incompatible, but if p, q are π -compatible, then $N_{pq}(\pi)$ is not dense in $\mathbf{MT}(\pi)$.

We define $\pi \sqsubset_{pq} \varphi$ to mean that $|p| \cup |q| \subseteq |\pi|$, p and q are π -incompatible, and $\pi \sqsubset_{N_{pq}(\pi)} \varphi$. In this case, we say that φ seals $p \perp^\pi q$ over π .

If $|p| \cup |q| \subseteq |\pi|$, p and q are π -incompatible, and $N_{pq}(\pi)$ is sealed dense in $\mathbf{MT}(\pi)$, then we say that $p \perp^\pi q$ is sealed by π .

The following corollary reinterprets some key results above in terms of \sqsubset_{pq} .

Corollary 9. Let $\pi \sqsubset \varphi$ and δ be regular multiforcings and $p, q \in \mathbf{MT}$. Then:

- (i) If $p \perp^\pi q$ is sealed by π then $\pi \sqsubset_{pq} \varphi$;
- (ii) If $\pi \sqsubset_{pq} \varphi$ then $p \perp^\varphi q$ is sealed by φ , while $N_{pq}(\pi)$ is open dense in $\mathbf{MT}(\pi)$;
- (iii) If $p \perp^\pi q$ is sealed by π , $p', q' \in \mathbf{MT}$, and $p' \leq p, q' \leq q, |p'| \cup |q'| \subseteq |\pi|$, then $p' \perp^\pi q'$ is sealed by π as well.

Proof. The proof is similar to Corollary 8. We make use of Lemma 10 and Lemma 12(i),(iv), in view of the fact that $(N_{pq}(\pi) \uparrow \varphi) \subseteq N_{pq}(\varphi)$. As for the extra item (iii), we obviously have $N_{pq}(\pi) \subseteq N_{p'q'}(\pi)$ provided $p' \leq p$ and $q' \leq q$. \square

10. Real Names and Direct Forcing

In this section, a notational system for names of reals in 2^ω is introduced. It is appropriate for dealing with forcing notions $\mathbf{MT}(\pi)$.

Definition 12. We let a real name be any $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times 2)$ such that the sets $K_{ni}^{\mathbf{c}} = \{p \in \mathbf{MT} : \langle p, n, i \rangle \in \mathbf{c}\}$ satisfy the following condition: given $n < \omega$, any $p \in K_{n0}^{\mathbf{c}}, q \in K_{n1}^{\mathbf{c}}$ are incompatible, i.e., $p \perp q$ (Definition 9). Let $K_n^{\mathbf{c}} = K_{n0}^{\mathbf{c}} \cup K_{n1}^{\mathbf{c}}; K_n^{\mathbf{c}} \subseteq \mathbf{MT}(\pi)$.

A real name \mathbf{c} is small if every set $\mathbf{K}_n^{\mathbf{c}}$ is finite or countable — then both the set $|\mathbf{c}| = \bigcup_n \bigcup_{p \in \mathbf{K}_n^{\mathbf{c}}} |p|$, and \mathbf{c} itself, are countable as well.

Given a multforcing π , a real name \mathbf{c} is:

- π -complete, whenever every collection $\mathbf{K}_n^{\mathbf{c}} \uparrow \pi = \{p \in \mathbf{MT}(\pi) : \exists q \in \mathbf{K}_n^{\mathbf{c}} (p \leq q)\}$ (the π -cone of $\mathbf{K}_n^{\mathbf{c}}$) is pre-dense (and then clearly open dense) in $\mathbf{MT}(\pi)$.
- sealed π -complete, whenever each set $\mathbf{K}_n^{\mathbf{c}} \uparrow \pi$ is sealed dense in $\mathbf{MT}(\pi)$.

It is not assumed here that $\mathbf{c} \subseteq \mathbf{MT}(\pi) \times (\omega \times 2)$, or equivalently, $\mathbf{K}_n^{\mathbf{c}} \subseteq \mathbf{MT}(\pi), \forall n$.

Suppose that \mathbf{c} is a real name. Say that a multitree p :

- Directly forces $\mathbf{c}(n) = i$, where $n < \omega$ and $i = 0, 1$ —in case there is a multitree $q \in \mathbf{K}_{ni}^{\mathbf{c}}$ such that $p \leq q$;
- Directly forces $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$,—in case p directly forces $\mathbf{c}(n) = i$ for all $n < \text{lh}(s)$, where $i = s(n)$;
- Directly forces $\mathbf{c} \notin [T]$, where $T \in \mathbf{PT}$ —in case there is a tuple $s \in 2^{<\omega} \setminus T$ such that p directly forces $s \subset \mathbf{c}$.

Lemma 13. Let π be a multforcing, $p \in \mathbf{MT}(\pi)$, $n < \omega$, \mathbf{c} a π -complete real name, $T \in \mathbf{PT}$. There exists $i = 0, 1$ and a multitree $q \in \mathbf{MT}(\pi)$, $q \leq p$, which directly forces $\mathbf{c}(n) = i$. There exists $s \in T$ and a multitree $q \in \mathbf{MT}(\pi)$, $q \leq p$, which directly forces $\mathbf{c} \notin [T \upharpoonright_s]$.

Proof. See Lemma 9.2 in [18]. \square

The definition of direct forcing is associated with the following notion of genericity.

Definition 13. Suppose that π is a multforcing. A set $G \subseteq \mathbf{MT}(\pi)$ is π -generic if:

- (1) For any $p \leq q$ in $\mathbf{MT}(\pi)$, $p \in G$ implies $q \in G$.
 - (2) If $p, q \in G$ then there is $r \in G$ with $r \leq p, r \leq q$.
- Say that G is π -generic over a given π -complete real name \mathbf{c} , if in addition
- (3) G intersects every set of the form $\mathbf{K}_n^{\mathbf{c}} \uparrow \pi, n < \omega$.

In this case, we define a real $\mathbf{c}[G] \in 2^\omega$ as follows: $\mathbf{c}[G](n) = i$ if $G \cap \mathbf{K}_{ni}^{\mathbf{c}} \neq \emptyset$.

Lemma 14 (obvious). Suppose that π is a multforcing and \mathbf{c} is a π -complete real name. Let $G \subseteq \mathbf{MT}(\pi)$ be π -generic over \mathbf{c} . If some $p \in G$ directly forces $\mathbf{c}(n) = i$, or $s \subset \mathbf{c}$, or $\mathbf{c} \notin [T]$, then resp. $\mathbf{c}[G](n) = i, s \subset \mathbf{c}[G], \mathbf{c}[G] \notin [T]$.

Example 4. If $\xi < \omega_1$, then let \dot{x}_ξ be a real name such that each set $\mathbf{K}_{ni}^{\dot{x}_\xi}$ consists of a single multitree $\mathbf{P}_{ni}^{\dot{x}_\xi}$, satisfying $|\mathbf{P}_{ni}^{\dot{x}_\xi}| = \{\xi\}$ (a singleton), and $\mathbf{P}_{ni}^{\dot{x}_\xi}(\xi) = T_{ni}$, where $T_{ni} = \{s \in 2^{<\omega} : \text{lh}(s) \leq n \vee s(n) = i\}$. Then, \dot{x}_ξ is a small real name, π -complete for any multforcing π . If a set $G \subseteq \mathbf{MT}(\pi)$ is π -generic over \dot{x}_ξ , then the real $\dot{x}_\xi[G]$ is identical to the real $x_\xi[G]$ (see Remark 1). In other words, \dot{x}_ξ is a canonical name for $x_\xi[G]$.

11. Sealing Real Names and Avoiding Refinements

Here, we develop the idea of Definition 11 in the context of dense sets generated by real names.

Definition 14. Let $\pi \sqsubset \varphi$ be multforcings and \mathbf{c} be a real name. We define that φ seals \mathbf{c} over π , in symbol $\pi \sqsubset_{\mathbf{c}} \varphi$, in case φ seals each set

$$\mathbf{K}_n^{\mathbf{c}} \uparrow \pi = \{p \in \mathbf{MT}(\pi) : \exists q \in \mathbf{K}_n^{\mathbf{c}} (p \leq q)\},$$

over π , i.e., $\pi \sqsubset_{\mathbf{K}_n^{\mathbf{c}} \uparrow \pi} \varphi$, in the sense of Definition 11.

Corollary 10. *Suppose that π, φ, σ are regular multiforcings and c is a real name. Then:*

- (i) *If $\pi \sqsubset_c \varphi$ then the name c is π -complete, $(\pi \cup^{c^\omega} \varphi)$ -complete, and sealed φ -complete;*
- (ii) *If $\pi \sqsubset \varphi$ and c is sealed π -complete, then c is sealed φ -complete.*

Proof. To prove (i), (ii) apply Lemma 10 and observe that $(K_n^c \uparrow \pi) \uparrow \varphi \subseteq K_n^c \uparrow \varphi$. \square

If π is a multiforcing then the forcing notion $\mathbf{MT}(\pi)$ adjoins a family of *principal generic reals* $x_\xi = x_\xi[G] \in 2^\omega$, $\xi \in |\pi|$, where every x_ξ is $\pi(\xi)$ -generic over the ground set universe. Obviously many more reals are added. The next definition provides a sufficient condition for a π -complete real name c to generate **not** a real of the form x_ξ .

Definition 15. *Suppose that π is a multiforcing and $\xi \in |\pi|$. A real name c is called non-principal at ξ over π , if the next set $D_\xi^c(\pi)$ is open dense in $\mathbf{MT}(\pi)$:*

$$D_\xi^c(\pi) = \{p \in \mathbf{MT}(\pi) : \xi \in |p| \text{ and } p \text{ directly forces } c \notin [p(\xi)]\}.$$

It will be demonstrated by Theorem 5(i) below that the non-principality at ξ implies that c is **not** a name of the real $x_\xi[G]$. Moreover, the avoidance condition in the following definition will be demonstrated to imply that c is a name of a non-generic real.

Definition 16. *Let π be a multiforcing and $Y \subseteq \mathbf{PT}$ be a set of trees (e.g., $Y = \pi(\xi)$ for some $\xi \in |\pi|$). A real name c is said to avoid Y over π , if for each tree $Q \in Y$, the set*

$$D_Q^c(\pi) = \{r \in \mathbf{MT}(\pi) : r \text{ directly forces } c \notin [Q]\}$$

is sealed dense (then open dense) in $\mathbf{MT}(\pi)$ in the sense of Definition 11.

Let π, φ be multiforcings, $\pi \sqsubset \varphi$, $Y \subseteq \mathbf{PT}$ be a set of trees. We write $\pi \sqsubset_Y^c \varphi$, if for each tree $Q \in Y$, φ seals the set $D_Q^c(\pi)$ over π — that is formally $\pi \sqsubset_{D_Q^c(\pi)}^c \varphi$.

The relation $\pi \sqsubset_Y^c \varphi$ will be applied mainly in case $Y = \varphi(\xi)$ for some $\xi \in |\pi|$.

Theorem 11.1 in [18] demonstrates that if π is a small regular multiforcing, $\xi \in |\pi|$, and a real name c is non-principal at ξ over π (in the sense of Definition 15) then there is a special multiforcing φ with $\pi \sqsubset_{\varphi(\xi)}^c \varphi$ (as in Definition 16). This fact will be used in the proof of Theorem 4 below.

Lemma 15. *Let $\pi \sqsubset \varphi$ be regular multiforcings, $Y \subseteq \mathbf{PT}$ be a set of trees, c be a real name.*

- (i) *If $\pi \sqsubset_Y^c \varphi$ then c avoids Y over φ ;*
- (ii) *If c avoids Y over π then c avoids Y over φ as well.*

Proof. (i) Let $Q \in Y$. The set $D = D_Q^c(\pi) \uparrow \varphi$ is sealed dense in $\mathbf{MT}(\varphi)$ by Lemma 10(iii). However, clearly, $D \subseteq D_Q^c(\varphi)$; thus, the set $D_Q^c(\varphi)$ is sealed dense in $\mathbf{MT}(\varphi)$ as well.

(ii) Let $Q \in Y$. The set $D = D_Q^c(\pi)$ is sealed dense in $\mathbf{MT}(\pi)$ by the avoidance assumption. Thus, $D \uparrow \varphi$ is a sealed dense set in $\mathbf{MT}(\varphi)$ by Lemma 10(iii). However, clearly, $(D \uparrow \varphi) \subseteq D_Q^c(\varphi)$. It follows that the set $D_Q^c(\varphi)$ is sealed dense in $\mathbf{MT}(\varphi)$ as well. \square

12. Inductive Analysis of Well-Foundedness

Here, we accomplish some work related to the combinatorial description of forcing of well-founded trees. This will be applied in Part IV as a tool to define an auxiliary forcing relation for formulas in Σ_1^1 and Π_1^1 via the well-foundedness of certain trees.

A set $\tau \subseteq \mathbf{MT} \times \omega^{<\omega}$ is called a *tree-name*, if whenever $s \subset t$ belong to $\omega^{<\omega}$ and $p \in \mathbf{MT}$ then $\langle p, t \rangle \in \tau \implies \langle p, s \rangle \in \tau$. Following Section 10, say that a multitree $q \in \mathbf{MT}$ directly forces $s \in \tau$ if $\langle q_0, s \rangle \in \tau$ for some $q_0 \in \mathbf{MT}(\pi)$ such that $q_0 \geq q$.

Definition 17. *Assume that π is a multiforcing and $\tau \subseteq \mathbf{MT} \times \omega^{<\omega}$ is a tree-name.*

If $p \in \mathbf{MT}$ and $s \in \omega^{<\omega}$ then define

$$\begin{aligned} \mathbf{MT}(\pi)_{\leq p} &= \{q \in \mathbf{MT}(\pi) : q \leq p\}; \\ W_{\leq p}^s(\tau) &= \{q \in \mathbf{MT} : q \leq p \wedge \exists j (q \text{ directly forces } s \hat{\ } j \in \tau)\}; \\ W_{\leq p}^s(\tau, \pi) &= W_{\leq p}^s(\tau) \cap \mathbf{MT}(\pi)_{\leq p}; \\ {}^+W_{\leq p}^s(\tau, \pi) &= \{q \in \mathbf{MT}(\pi) : q \in W_{\leq p}^s(\tau, \pi) \text{ or } q \text{ is incompatible with } p\}. \end{aligned}$$

Let the derivative $\tau'_\pi \subseteq \tau$ contain all pairs $\langle p, s \rangle \in \tau$ such that $W_{\leq p}^s(\tau, \pi)$ is dense in $\mathbf{MT}(\pi)_{\leq p}$ (then clearly open dense too), so that $\forall r \in \mathbf{MT}(\pi)_{\leq p} \exists q \in W_{\leq p}^s(\tau, \pi) (q \leq r)$. This is equivalent to saying that $W_{\leq p}^s(\tau)$ is dense in $\mathbf{MT}(\pi)$ below p .

Note that τ'_π is a tree-name. Define a descending sequence of tree-names $\tau^\nu_\pi, \nu < \omega_1$, by transfinite induction, so that $\tau^0_\pi = \tau, \tau^{\nu+1}_\pi = (\tau^\nu_\pi)'_\pi$ for $\nu < \omega_1$, and $\tau^\lambda_\pi = \bigcap_{\nu < \lambda} \tau^\nu_\pi$ for limit λ . Then, eventually $\tau^{\nu+1}_\pi = \tau^\nu_\pi$ for some $\nu = \nu_\pi(\tau) < \omega_1$, and we let $\tau^\infty_\pi = \tau^\nu_\pi$ for this index ν . Thus, $\tau^\infty_\pi \subseteq \tau$ is a tree-name as well, and $(\tau^\infty_\pi)'_\pi = \tau^\infty_\pi$.

Lemma 16. Let $\tau \subseteq \mathbf{MT} \times \omega^{<\omega}$ be a tree-names, $\pi \sqsubset \varphi$ be regular multiforcings, and $\sigma = \pi \cup^{c\omega} \varphi$. Then, $\tau^\nu_\sigma = \tau^\nu_\varphi$ for all ν , and accordingly $\tau^\infty_\sigma = \tau^\infty_\varphi$.

Proof. It suffices to prove that just $\tau'_\sigma = \tau'_\varphi$; all further inductive steps are similar. Recall that $\mathbf{MT}(\varphi)$ is open dense in $\mathbf{MT}(\sigma)$ by Corollary 6. It follows that one and the same set $W_{\leq p}^s(\tau)$ is dense in $\mathbf{MT}(\varphi)_{\leq p}$ if it is dense in $\mathbf{MT}(\sigma)_{\leq p}$. \square

Definition 18. A set $G \subseteq \mathbf{MT}(\pi)$ is π -generic over τ if G is π -generic as in Definition 13, and G intersects every set of the form ${}^+W_{\leq p}^s(\tau'_\pi, \pi)$, dense in $\mathbf{MT}(\pi)$, where $\langle p, s \rangle \in \tau$ and $\nu \leq \nu_\pi(\tau)$. Put $\tau[G] = \{s \in \omega^{<\omega} : \exists p \in G (p \text{ directly forces } s \in \tau)\}$.

Thus, $\tau[G]$ is a tree is $\omega^{<\omega}$ because τ is a tree-name.

For any tree $T \subseteq \omega^{<\omega}$, let T' be the pruned derivative, that consists of all $s \in T$ that are not terminal nodes in T , and let T^∞ be the pruned kernel, the largest subtree $S \subseteq T$ with no terminal nodes, that consists of all $s \in T$ that belong to infinite branches $B \subseteq T$.

Lemma 17. Assume that π is a multiforcing, $\tau \subseteq \mathbf{MT} \times \omega^{<\omega}$ is a tree-name, and a set $G \subseteq \mathbf{MT}(\pi)$ is π -generic over τ . Then:

- (i) $\tau'_\pi[G]$ is the pruned derivative of the tree $\tau[G] \subseteq \omega^{<\omega}$;
- (ii) $\tau^\infty_\pi[G]$ is the pruned kernel of $\tau[G] \subseteq \omega^{<\omega}$;
- (iii) G remains π -generic over τ'_π and over τ^∞_π .

Proof. (i) The contrary assumption results in the two following cases.

Case 1: some $s \in \tau'_\pi[G]$ is maximal in $\tau[G]$. In particular, we have multitrees $p \in G$ and p_0 such that $\langle p_0, s \rangle \in \tau'_\pi$ and $p \leq p_0$. By definition, the set $W_{\leq p_0}^s(\tau)$ has to be dense in $\mathbf{MT}(\pi)_{\leq p_0}$. Therefore, as G is generic over τ , and $p \in G$, some $q \in W_{\leq p_0}^s(\tau)$ belongs to G as well. By definition, q directly forces $s \hat{\ } j \in \tau$ for some $j < \omega$. Then, there is a multitree q_0 satisfying $q \leq q_0$ and $\langle q_0, s \hat{\ } j \rangle \in \tau$. However, $s \hat{\ } j \in \tau[G]$, contrary to the choice of s .

Case 2: a tuple $s \hat{\ } j$ belongs to $\tau[G]$ but s does not belong to $\tau'_\pi[G]$. Then, we have $\langle p_0, s \hat{\ } j \rangle \in \tau, p \in G, p \leq p_0$. It follows that $\langle p_0, s \rangle \in \tau'_\pi$, and hence $s \in \tau'_\pi[G]$, contrary to the choice of s .

Claim (ii) is a corollary of (i). To check (iii) note that $\tau^{1+\nu}_\pi = (\tau'_\pi)^\nu_\pi$ for all ν . \square

Corollary 11. Under the assumptions of the lemma, let $p \in G$.

- (i) If p directly forces $s \in \tau^\infty_\pi$ then $\tau[G]$ has an infinite chain containing s .

(ii) If no condition $q \in \mathbf{MT}(\pi)$, $q \leq p$ directly forces $s \in \tau_\pi^\infty$, then $\tau[G]$ is well-founded over s .

Proof. By Lemma 17, $\tau_\pi^\infty[G]$ is the the pruned kernel of $\tau[G]$. Thus, $\tau[G]$ includes an infinite chain containing some $s \in \tau[G]$ if $s \in \tau_\pi^\infty[G]$. This easily implies both items. \square

13. Absoluteness of the Derivative

The key result of this section will be to show that, under certain restrictions, the pruned derivative operation introduced in Section 12 is absolute with respect to refinements of the multiforcings involved. We need, however, to introduce another property of the form of *sealing* of dense sets, as in Definition 11.

Definition 19. Let π be a multiforcing, and $\tau \subseteq \mathbf{MT} \times \omega^{<\omega}$ is a tree-name, as in Section 12. Say that τ is sealed in π , if the following conditions hold:

- (a) If $p \in \text{dom } \tau = \{p : \exists s (\langle p, s \rangle \in \tau)\}$ then $|p| \subseteq |\pi|$,
- (b) If $v \leq v_\pi(\tau)$, $\langle p, s \rangle \in \tau_\pi^v$, and $D = {}^+W_{\leq p}^s(\tau_\pi^v, \pi)$ (see Section 12) is a set dense (then open dense as well) in $\mathbf{MT}(\pi)$, then D is sealed dense in $\mathbf{MT}(\pi)$.

Let φ be another multiforcing with $\pi \sqsubset \varphi$, so that φ is a refinement of π . Say that φ seals τ over π , symbolically $\pi \sqsubset_\tau \varphi$, if the next two conditions hold:

- (c) Just as (a) above;
- (d) If $v \leq v_\pi(\tau)$ and $\langle p, s \rangle \in \tau_\pi^v$, and the set $D = {}^+W_{\leq p}^s(\tau_\pi^v, \pi)$ is dense (then open dense) in $\mathbf{MT}(\pi)$, then $\pi \sqsubset_D \varphi$.

The following claims show the effect of \sqsubset_τ in terms of Lemma 16.

Lemma 18 (obvious). Let $\tau \subseteq \mathbf{MT} \times \omega^{<\omega}$ be a tree-name, $\pi \sqsubset_\tau \varphi$ be regular multiforcings. Then $\pi \sqsubset_{\tau_\pi^v} \varphi$ for all v , and accordingly $\pi \sqsubset_{\tau_\pi^\infty} \varphi$.

Theorem 3. Let $\tau \subseteq \mathbf{MT} \times \omega^{<\omega}$ be a tree-name, $\pi \sqsubset \varphi$ be regular multiforcings, $\pi \sqsubset_r \varphi$ holds for all $r \in \text{dom } \tau$, and $\sigma = \pi \cup^{\text{cw}} \varphi$. Then, the following holds:

- (i) If $\langle r, s \rangle \in \tau$, then $W_{\leq r}^s(\tau, \pi) \uparrow \varphi \subseteq W_{\leq r}^s(\tau, \varphi)$ and $D_r(\pi) \uparrow \varphi \subseteq D_r(\varphi)$;
- (ii) If $\langle p, s \rangle \in \tau$, then the set $W_{\leq p}^s(\tau, \pi) \uparrow \varphi$ is dense in $W_{\leq p}^s(\tau, \varphi)$;
- (iii) If τ is sealed in π , then $\pi \sqsubset_\tau \varphi$;
- (iv) If $\pi \sqsubset_\tau \varphi$, then $\tau_\pi^v = \tau_\varphi^v = \tau_\sigma^v$ for all v , and accordingly $\tau_\pi^\infty = \tau_\varphi^\infty = \tau_\sigma^\infty$;
- (v) If $\pi \sqsubset_\tau \varphi$, then τ is sealed in φ ;
- (vi) If $\pi \sqsubset_\tau \varphi$, $s \in \omega^{<\omega}$, $p \in \mathbf{MT}$, and $\pi \sqsubset_{p q_0} \varphi$ (see Example 3) holds for all $q_0 \in \text{dom } \tau$ such that p, q_0 are π -incompatible— then the following are equivalent:
 - (1) No multitree $r \in \mathbf{MT}(\pi)_{\leq p}$ directly forces $s \in \tau_\pi^\infty$;
 - (2) No multitree $r \in \mathbf{MT}(\sigma)_{\leq p}$ directly forces $s \in \tau_\sigma^\infty$;
 - (3) No multitree $r \in \mathbf{MT}(\varphi)_{\leq p}$ directly forces $s \in \tau_\varphi^\infty$.

Proof. (i) Let $u \in \mathbf{MT}(\varphi)$ belong to $W_{\leq r}^s(\tau, \pi) \uparrow \varphi$. There is a multitree $q \in W_{\leq r}^s(\tau, \pi)$ with $u \leq q$. Then, q directly forces $s \wedge j \in \tau$, some j , and hence, so does u ; thus, $u \in W_{\leq r}^s(\tau, \varphi)$.

(ii) Let $u \in W_{\leq p}^s(\tau, \varphi)$, meaning that $u \in \mathbf{MT}(\varphi)$, $u \leq p$, and also $u \leq r$, where $\langle r, s \wedge j \rangle \in \tau$ for some $j < \omega$. Let $X = |u|$. Note that both r, p belong to $\text{dom } \tau$; therefore, we have $\pi \sqsubset_r \varphi$ and $\pi \sqsubset_p \varphi$, and then $\pi \sqsubset_{D_r(\pi) \cap D_p(\pi)} \varphi$ by Lemma 11(ii). It follows by Lemma 10(iv) that there is a $((D_r(\pi) \cap D_p(\pi)) \uparrow \varphi)$ -extendable multitree $v \in \mathbf{MT}(\varphi)$, $v \leq u$, with $|v| = X$. In other words, there are multitrees $w \in \mathbf{MT}(\varphi)$ with $X \subseteq Y = |w|$ and $w \upharpoonright X = v$, and $q \in D_r(\pi) \cap D_p(\pi)$ with $w \leq q$. As $w \leq v \leq u \leq p, r$, the multitree q cannot be incompatible with p and with r . Therefore, $q \leq r, p$. This implies $q \in W_{\leq p}^s(\tau, \pi)$, and hence, $w \in q \in W_{\leq p}^s(\tau, \pi) \uparrow \varphi$. This ends the proof.

(iii) Suppose that $\nu \leq \nu_\pi(\tau)$, $\langle p, s \rangle \in \tau_\pi^\nu$, and the set $D = {}^+W_{\leq p}^s(\tau_\pi^\nu, \pi)$ is dense in $\mathbf{MT}(\pi)$. Then, $\pi \sqsubset_D \varphi$ holds by Lemma 10(ii), as required.

(iv) In view of Lemmas 16 and 18, it suffices to prove $\tau'_\varphi \subseteq \tau'_\pi \subseteq \tau'_\sigma$.

Step 1: $\tau'_\pi \subseteq \tau'_\sigma$. Let $\langle p, s \rangle \in \tau'_\pi$. Then, $\langle p, s \rangle \in \tau$, and by definition, the set $W_{\leq p}^s(\tau, \pi)$ is dense in $\mathbf{MT}(\pi)_{\leq p}$. We conclude that $D = {}^+W_{\leq p}^s(\tau, \pi)$ (see Section 12) is open dense in the whole $\mathbf{MT}(\pi)$, and hence, $\pi \sqsubset_D \varphi$ holds. Then, $D \uparrow \varphi$ is open dense in $\mathbf{MT}(\sigma)$ by Lemma 10(iii),(v). Accordingly, $W_{\leq p}^s(\tau, \pi) \uparrow \varphi$ is open dense in $\mathbf{MT}(\sigma)_{\leq p}$. Then, $W_{\leq p}^s(\tau, \varphi)$, a bigger set by (i), is open dense in $\mathbf{MT}(\sigma)_{\leq p}$, too. Thus, $\langle p, s \rangle \in \tau'_\sigma$.

Step 2: $\tau'_\varphi \subseteq \tau'_\pi$. Assume that $\langle p, s \rangle \in \tau'_\varphi$, so that $\langle p, s \rangle \in \tau$ and the set $W_{\leq p}^s(\tau, \varphi)$ is open dense in $\mathbf{MT}(\varphi)_{\leq p}$. Then, the set $W_{\leq p}^s(\tau, \pi) \uparrow \varphi$ is dense in $\mathbf{MT}(\varphi)_{\leq p}$ as well by (ii). We claim that the set $W_{\leq p}^s(\tau, \pi)$ is dense in $\mathbf{MT}(\pi)_{\leq p}$.

Indeed, let $p_1 \in \mathbf{MT}(\pi)_{\leq p}$. Then, there is $q_1 \in \mathbf{MT}(\varphi)$, $q_1 \leq p_1$. By (ii), there exists a pair $q_2 \leq p_2$ of multitrees $p_2 \in W_{\leq p}^s(\tau, \pi)$ and $q_2 \in \mathbf{MT}(\varphi)$ such that $q_2 \leq q_1$. Therefore, q_2 witnesses that the multitrees p_1, p_2 in $\mathbf{MT}(\pi)$ are compatible, and hence, compatible right in $\mathbf{MT}(\pi)$ by Corollary 2. Thus, we have established that the set $W_{\leq p}^s(\tau, \pi)$ is at least pre-dense, and then obviously dense in $\mathbf{MT}(\pi)_{\leq p}$, as required.

(v) Prove (b) of Definition 19 for φ . In view of (iv) and Lemma 18, it suffices to only consider the case $\nu = 0$, i.e., given $\langle p, s \rangle \in \tau$, and assuming that the set $D = {}^+W_{\leq p}^s(\tau, \varphi)$ is dense in $\mathbf{MT}(\varphi)$, we have to prove that D is sealed dense in $\mathbf{MT}(\varphi)$.

By definition, the set $W_{\leq p}^s(\tau, \varphi)$ is dense (then open dense) in $\mathbf{MT}(\varphi)_{\leq p}$. It follows by (ii) that the set $W_{\leq p}^s(\tau, \pi) \uparrow \varphi$ is also dense in $\mathbf{MT}(\varphi)_{\leq p}$. We conclude (see Step 2 above) that the set $W_{\leq p}^s(\tau, \pi)$ itself is dense in $\mathbf{MT}(\pi)_{\leq p}$. Then, the set $E = {}^+W_{\leq p}^s(\tau, \pi)$ itself is dense in $\mathbf{MT}(\pi)p$. Therefore, we have $\pi \sqsubset_E \varphi$ because $\pi \sqsubset_\tau \varphi$ is assumed.

It follows by Lemma 10(iii) that $E \uparrow \varphi$ is sealed dense in $\mathbf{MT}(\varphi)$. However, easily $E \uparrow \varphi \subseteq D$ by (ii). This ends the proof that D is sealed dense in $\mathbf{MT}(\varphi)$.

(vi) Recall that $\mathbf{MT}(\varphi)$ is dense in $\mathbf{MT}(\sigma)$ by Corollary 6. Therefore, (iv) implies that (2) \iff (3). Moreover, (iv) implies as well that (2) \implies (1) simply because $\mathbf{MT}(\pi) \subseteq \mathbf{MT}(\sigma)$. It remains to be proven that conversely (1) \implies (3).

Let (3) fail, that is, we assume that $u \in \mathbf{MT}(\varphi)_{\leq p_0}$, $X = |u|$, and u directly forces $s \in \tau_\sigma^\infty$. Then, by (iv) u directly forces $s \in \tau_\pi^\infty$; hence, $u \leq q_0$ holds for some q_0 with $\langle q_0, s \rangle \in \tau_\pi^\infty$. Thus, p, q_0 are φ -compatible (by u). It follows that p, q_0 are π -compatible as well. Indeed, otherwise we have $\pi \sqsubset_{pq_0} \varphi$ by the assumptions of (vi). This implies that $p \perp^\varphi q$ is sealed by φ by Corollary 9(ii). However, this contradicts the φ -compatibility of p, q_0 . Finally, the π -compatibility of p, q_0 means $\neg(1)$. \square

14. Combining Refinement Types

The properties of generic refinements considered above in Sections 5, 8, 9, and 11 are summarized by the next definition.

Definition 20. Let $\pi \sqsubset \varphi$ be multiforcings and M be any set. We define $\pi \sqsubset_M \varphi$ to mean that the following conditions hold:

- (1) If $\xi \in |\pi|$, $D \subseteq \pi(\xi)$, $D \in M$, D is pre-dense in $\pi(\xi)$, then we have $\pi(\xi) \sqsubset_D \varphi(\xi)$;
- (2) If $D \subseteq \mathbf{MT}(\pi)$, $D \in M$, D is open dense in $\mathbf{MT}(\pi)$, then we have $\pi \sqsubset_D \varphi$;
- (3) If $p \in M \cap \mathbf{MT}$ and $|p| \subseteq |\pi|$, then $\pi \sqsubset_p \varphi$;
- (4) If $p, q \in M \cap \mathbf{MT}$, $|p| \cup |q| \subseteq |\pi|$, and p, q are π -incompatible, then $\pi \sqsubset_{pq} \varphi$;
- (5) If $c \in M$ and c is a π -complete real name then we have $\pi \sqsubset_c \varphi$;
- (6) If $c \in M$ is a π -complete real name, non-principal at $\xi \in |\pi|$ over π , then $\pi \sqsubset_{\varphi(\xi)}^c \varphi$;
- (7) If $\tau \in M$, $\tau \subseteq \mathbf{MT} \times \omega^{<\omega}$ is a tree-name, then $\pi \sqsubset_\tau \varphi$.

Corollary 12. Let M be a countable set, π, φ, δ be regular multiforcings, and $\pi \sqsubset_M \varphi \sqsubset \delta$. Then, $\pi \sqsubset_M (\varphi \cup^{\text{cw}} \delta)$ and $(\pi \cup^{\text{cw}} \varphi) \sqsubset_M \delta$.

Proof. Our basic reference is Lemma 12(i)(iv), which has to be applied for those sets D involved in the definition of $\pi \sqsubseteq_M \varphi$ above (Definition 20). \square

This follows a refinement existence result.

Theorem 4. *If π is a small regular multiforcing and M a countable set, then there exists a special multiforcing φ satisfying $|\pi| = |\varphi|$ and $\pi \sqsubseteq_M \varphi$.*

Proof (sketch). The proof is based on some rather difficult results in [18] which we make use of here without proofs.

First of all, we can assume that $M \in \text{HC} =$ all hereditarily countable sets, since all elements in $M \setminus \text{HC}$ are irrelevant. Let $\mathfrak{M} \subseteq \text{HC}$ be the (countable) set containing π, M , every $\xi \in |\pi|$, and every element of M . Let \mathfrak{M}^+ contain all sets $X \subseteq \text{HC}$, \in -definable over HC , with sets in \mathfrak{M} allowed as parameters.

Definition 7.1 in [18] introduces the notion of \mathfrak{M} -generic refinements. Lemma 7.2 and Theorem 7.3 in [18] prove the existence of a special multiforcing φ , which satisfies $|\varphi| = |\pi|$ and is an \mathfrak{M} -generic refinement of a given small regular multiforcing π . If φ is such, then Theorem 8.1 in [18] proves the relation $\pi \sqsubseteq_D^{\text{old}} \varphi$, and hence, $\pi \sqsubseteq_D \varphi$, for all open dense sets $D \in \mathfrak{M}^+$, $D \subseteq \text{MT}(\pi)$. This implies (1)–(5) and (7) of Definition 20 because all dense sets involved there belong to \mathfrak{M}^+ by construction.

Finally, (6) of Definition 20 is separately established by Theorem 11.1 in [18]. We may note that (6) of Definition 20 differs from other items of this definition in that the list of the dense sets involved depends on the new multitree φ (the one claimed to exist). Therefore, it needs a special theorem in [18], namely Theorem 11.1. \square

15. Consequences for Generic Extensions

Lemma 19 shows that real names provide a suitable representation of reals in $\text{MT}(\pi)$ -generic extensions. Then, corollaries for non-principal names will be the subject of Theorem 5.

Lemma 19. *Assume that π is a regular multiforcing in the ground set universe \mathbf{V} , and $G \subseteq \text{MT}(\pi)$ is a $\text{MT}(\pi)$ -generic set over \mathbf{V} .*

- (i) *If $x \in 2^\omega$ is a real in $\mathbf{V}[G]$ then there exists a π -complete real name $\mathbf{c} \in \mathbf{V}$, $\mathbf{c} \subseteq \text{MT}(\pi) \times \omega \times 2$, satisfying $x = \mathbf{c}[G]$.*
- (ii) *Let $\text{MT}(\pi)$ be a CCC forcing in \mathbf{V} , and $\mathbf{c} \in \mathbf{V}$, $\mathbf{c} \subseteq \text{MT}(\pi) \times \omega \times 2$ be a π -complete real name. Then, there exists a **small** π -complete real name $\mathbf{d} \in \mathbf{V}$, $\mathbf{d} \subseteq \text{MT}(\pi) \times \omega \times 2$, such that every condition in $\text{MT}(\pi)$ forces the equality $\mathbf{c}[G] = \mathbf{d}[G]$ over \mathbf{V} .*

As usual, the CCC property means here that every π -antichain (i.e., antichain in $\text{MT}(\pi)$, see Definition 9 and Corollary 2) $A \subseteq \text{MT}(\pi)$ is at most countable.

Proof. Claim (i) is a partial case of a general forcing theorem. To prove claim (ii), consider open dense set $K_n^c \uparrow \pi = \{p \in \text{MT}(\pi) : \exists q \in K_n^c (p \leq q)\}$, choose maximal antichains $A_n \subseteq K_n^c \uparrow \pi$ in those sets, note that each A_n is countable by CCC, and finally, define $\mathbf{d} = \{\langle p, n, i \rangle : p \in A_{ni}\}$, where $A_{ni} = \{p \in A_n : \exists q \in K_{ni}^c (p \leq q)\}$. \square

Theorem 5. *Suppose that π is a regular multiforcing, and $\xi \in |\pi|$. Then, the following holds.*

- (i) *If $\text{MT}(\pi)$ is a CCC forcing, $G \subseteq \text{MT}(\pi)$ is a set generic over the ground set universe \mathbf{V} , $x \in 2^\omega$ is a real in $\mathbf{V}[G]$, and $x \neq x_\xi[G]$, then there exists a small π -complete real name $\mathbf{c} \subseteq \text{MT}(\pi) \times (\omega \times 2)$, non-principal at ξ over π , satisfying $x = \mathbf{c}[G]$.*
- (ii) *If $\mathbf{c} \subseteq \text{MT}(\pi) \times (\omega \times 2)$ is a π -complete real name that avoids $\pi(\xi)$ over π , φ is a regular multiforcing, $\pi \sqsubseteq \varphi$, and $G \subseteq \text{MT}(\pi \cup^{\text{cw}} \varphi)$ is generic over \mathbf{V} , then $\mathbf{c}[G] \notin \bigcup_{Q \in \pi(\xi)} [Q]$.*

Proof. (i) By a general forcing theorem, there exists a π -complete real name \mathbf{c} such that $x = \mathbf{c}[G]$ and $\mathbf{MT}(\pi)$ forces that $\mathbf{c} \neq x_{\zeta}[G]$. We can assume that \mathbf{c} is small, by Lemma 19 (as $\mathbf{MT}(\pi)$ is CCC). Let us prove that \mathbf{c} is non-principal at ζ over π , meaning that the set

$$D_{\zeta}^{\mathbf{c}}(\pi) = \{p \in \mathbf{MT}(\pi) : p \text{ directly forces } \mathbf{c} \notin [p(\zeta)]\}.$$

is open dense in $\mathbf{MT}(\pi)$. As the openness is clear, it remains to prove the density.

Let $q \in \mathbf{MT}(\pi)$. Then, by the choice of \mathbf{c} , q $\mathbf{MT}(\pi)$ -forces $\mathbf{c} \neq x_{\zeta}[G]$. Thus, we may assume that, for some n , the inequality $\mathbf{c}(n) \neq x_{\zeta}[G](n)$ is $\mathbf{MT}(\pi)$ -forced by q . By Lemma 13, there is a tuple $s \in \omega^{n+1}$ and a multitree $p \in \mathbf{MT}(\pi)$, $p \leq q$, such that p directly forces the sentence $s \subseteq \mathbf{c}$. It remains to be checked that $s \notin p(\zeta)$. Indeed, assume otherwise: $s \in p(\zeta)$. Then, the tree $T = p(\zeta) \upharpoonright_s$ belongs to $\mathbf{MT}(\pi)$. Define a multitree r by $r(\zeta) = T$ and $r(\zeta') = p(\zeta')$ for all $\zeta' \neq \zeta$. Then, $r \in \mathbf{MT}(\pi)$ and we have $r \leq p \leq q$. However, r directly forces both $\mathbf{c}(n)$ and $x_{\zeta}[G](n)$ to be equal to the same number $\ell = s(n)$, which contradicts the choice of n .

(ii) Suppose towards the contrary that $Q \in \pi(\zeta)$ and $\mathbf{c}[G] \in [Q]$. Lemma 15 (ii) implies that \mathbf{c} avoids $\pi(\zeta)$ over φ as well. Lemma 10 implies that the set $D_Q^{\mathbf{c}}(\pi) \upharpoonright \varphi$ is open dense in $\mathbf{MT}(\pi \cup^{\text{cw}} \varphi)$. Therefore, the set $D_Q^{\mathbf{c}}(\pi)$ itself is pre-dense in $\mathbf{MT}(\pi \cup^{\text{cw}} \varphi)$. We conclude that $G \cap D_Q^{\mathbf{c}}(\pi) \neq \emptyset$ by the genericity, so that some multitree $r \in G$ directly forces $\mathbf{c} \notin [Q]$. It follows that $\mathbf{c}[G] \notin [Q]$, which is a contradiction. \square

Part III: The Forcing and the Model

Here, we present the key forcing constructions of the proof of Theorem 1.

We consider the constructible universe \mathbf{L} as the ground model.

Fix a natural number $n \geq 3$ as in Theorem 1.

Theorem 6 in Section 19 introduces a ω_1 -long \sqsubset -increasing sequence $\vec{\pi} \in \mathbf{L}$ of special multiforcings, whose properties include: first, sealing many dense sets during the course of the construction; second, a sort of definable genericity in \mathbf{L} ; and third, a definability requirement—as in Definition 23. The subsequent key forcing notion $\mathbb{P} \in \mathbf{L}$ (which depends on $\vec{\pi}$) is defined in Section 20. Its properties include CCC by Theorem 7. Then, we consider \mathbb{P} -generic extensions of \mathbf{L} , called key models. The main results about key models are Theorem 8, which characterizes generic reals, and Theorem 9, which provides a Δ_n^1 -good well-ordering, with (i) of Theorem 1 as a consequence. Along with Theorem 7, they are the main results of this Part.

We begin with routine stuff on \sqsubset -increasing sequences of special multiforcings.

16. Increasing Sequences of Multiforcings

Based on Remark 2, we consider \sqsubset -increasing sequences of multiforcings. Let

$$\mathbf{MFsp} = \{\pi \in \mathbf{MF} : \pi \text{ is a special, hence small multiforcing}\}.$$

Thus, a multiforcing $\pi \in \mathbf{MF}$ (the set of all multiforcings) belongs to \mathbf{MFsp} if $|\pi| \subseteq \omega_1$ is finite or countable and each $\pi(\zeta)$, $\zeta \in |\pi|$, is a special forcing. (See Sections 3 and 6).

- If $\kappa \leq \omega_1$, then let $\vec{\mathbf{MF}}_{\kappa}$ be the set of all \sqsubset -increasing sequences $\vec{\pi} = \langle \pi_{\alpha} \rangle_{\alpha < \kappa}$ of multiforcings $\pi_{\alpha} \in \mathbf{MFsp}$, of length $\text{dom}(\vec{\pi}) = \kappa$, which are domain-continuous, so that if $\lambda < \kappa$ is a limit ordinal then $|\pi_{\lambda}| = \bigcup_{\alpha < \lambda} |\pi_{\alpha}|$.
- Let $\vec{\mathbf{MF}} = \bigcup_{\kappa < \omega_1} \vec{\mathbf{MF}}_{\kappa}$ (\sqsubset -increasing sequences of countable length).
- The set $\vec{\mathbf{MF}} \cup \vec{\mathbf{MF}}_{\omega_1}$ is ordered by the relations \subseteq, \subset of extension of sequences. Thus, $\vec{\pi} \subset \vec{\rho}$ means that a sequence $\vec{\rho}$ properly extends $\vec{\pi}$.
- If $\vec{\pi} \in \vec{\mathbf{MF}}_{\kappa}$, then let $\bigcup^{\text{cw}} \vec{\pi} = \bigcup_{\alpha < \kappa}^{\text{cw}} \pi_{\alpha}$ (the component-wise union), $\mathbf{MT}(\vec{\pi}) = \mathbf{MT}(\pi)$, $|\vec{\pi}| = |\bigcup^{\text{cw}} \vec{\pi}| := \bigcup_{\alpha < \kappa} |\vec{\pi}(\alpha)|$ (a subset of ω_1).

Lemma 20. Assume that $\vec{\pi} \in \overline{\mathbf{MF}}_\kappa$ and $0 < \gamma < \lambda < \kappa$. Let $\pi_\alpha = \vec{\pi}(\alpha)$ for all α . Then:

- (i) The multiforcings $\pi = \bigcup_{\alpha < \kappa}^{\text{cw}} \pi_\alpha$, $\pi_{<\mu} = \bigcup_{\alpha < \mu}^{\text{cw}} \pi_\alpha$, $\pi_{\geq\mu} = \bigcup_{\mu \leq \alpha < \kappa}^{\text{cw}} \pi_\alpha$, $\pi_{>\mu} = \bigcup_{\mu < \alpha < \kappa}^{\text{cw}} \pi_\alpha$ are regular, and we have: $\pi_{<\mu} \sqsubset \pi_\mu \sqsubset \pi_{>\mu}$, $\pi_{<\mu} \sqsubset \pi_{\geq\mu}$, and $\pi_\mu \sqsubset \pi_\lambda$;
- (ii) The set $\mathbf{MT}(\pi_\mu)$ is pre-dense in $\mathbf{MT}(\pi)$ and $\mathbf{MT}(\pi_{\geq\mu})$ is dense in $\mathbf{MT}(\pi)$.

Proof. To prove (i),(ii) apply Lemma 5. \square

The following is a related form of \sqsubset -type definitions.

Definition 21. Let $\vec{\pi} \in \overline{\mathbf{MF}}_\kappa$, $\vec{\varphi} \in \overline{\mathbf{MF}}_\lambda$, $\kappa < \lambda$, and M be any set. Define $\vec{\pi} \sqsubset_M \vec{\varphi}$, if $\vec{\pi} \sqsubset \vec{\varphi}$ (i.e., φ extends π) and $\bigcup^{\text{cw}} \vec{\pi} \sqsubset_M \vec{\varphi}(\kappa)$, where $\bigcup^{\text{cw}} \vec{\pi} = \bigcup_{\alpha < \kappa}^{\text{cw}} \vec{\pi}(\alpha)$ is the component-wise union and $\vec{\varphi}(\kappa)$ is the first term in $\vec{\varphi}$ absent in $\vec{\pi}$.

Lemma 21. Assume that M is a countable set. Then:

- (i) If $\kappa < \lambda < \omega_1$ and $\vec{\pi} \in \overline{\mathbf{MF}}_\kappa$, then there exists a sequence $\vec{\varphi} \in \overline{\mathbf{MF}}_\lambda$ such that $\vec{\pi} \sqsubset_M \vec{\varphi}$;
- (ii) If $\kappa < \lambda \leq \omega_1$, $\vec{\pi} \in \overline{\mathbf{MF}}_\kappa$, $\vec{\varphi} \in \overline{\mathbf{MF}}_\lambda$, $\vec{\pi} \sqsubset_M \vec{\varphi}$, and a set $D \in M$ is open dense in $\mathbf{MT}(\vec{\pi})$, then $\pi = \bigcup^{\text{cw}} \vec{\pi} \sqsubset_D \varphi_{\geq\kappa} = \bigcup_{\kappa \leq \alpha < \lambda}^{\text{cw}} \vec{\varphi}(\alpha)$, so that D is pre-dense in $\mathbf{MT}(\vec{\varphi})$.

Proof. (i) Let $\pi = \bigcup^{\text{cw}} \vec{\pi}$. By Theorem 4, there is a special multiforcing σ satisfying $|\pi| = |\sigma|$ and $\pi \sqsubset_{M'} \sigma$, where $M' = M \cup \{D_\alpha : \alpha < \kappa\}$ and

$$D_\alpha = \vec{\pi}(\alpha) \uparrow \pi = \{p \in \mathbf{MT}(\pi) : \exists q \in \mathbf{MT}(\vec{\pi}(\alpha))(p \leq q)\}.$$

Each D_α is open dense in $\mathbf{MT}(\pi)$ because $\mathbf{MT}(\vec{\pi}(\alpha))$ is pre-dense by Lemma 20(ii). Then, using Theorem 4 for $M = \{D_\alpha : \alpha < \gamma\}$ in iteration, we define by transfinite induction special multiforcings σ_γ , $\kappa \leq \gamma < \lambda$, such that $\sigma_0 = \sigma$ and the sequence $\vec{\sigma} = \langle \sigma_\gamma \rangle_{\kappa \leq \gamma < \lambda}$ is just \sqsubset -increasing. Now, let $\vec{\varphi} = \vec{\pi} \cup \vec{\sigma}$, that is, $\vec{\varphi}(\alpha) = \vec{\pi}(\alpha)$ for $\alpha < \kappa$ but $\vec{\varphi}(\gamma) = \sigma_\gamma$ for $\kappa \leq \gamma < \lambda$. Then, $\vec{\varphi} \in \overline{\mathbf{MF}}_\lambda$ and $\vec{\pi} \sqsubset_M \vec{\varphi}$ by construction.

(ii) We have $\pi \sqsubset_M \varphi_{\geq\kappa}$ by Corollary 12, hence in particular $\pi \sqsubset_D \varphi_{\geq\kappa}$. It follows by Lemma 10(iii) that D is a pre-dense set in $\mathbf{MT}(\pi \cup^{\text{cw}} \varphi_{\geq\kappa}) = \mathbf{MT}(\vec{\varphi})$. \square

17. Definability Lemma

Recall that HC is the set of all hereditarily countable sets. Thus, $X \in \text{HC}$ if the transitive closure $\text{TC}(X)$ is at most countable. Note that $\text{HC} = \mathbf{L}_{\omega_1}$ under $\mathbf{V} = \mathbf{L}$.

We use the standard notation $\Sigma_n^{\text{HC}}, \Pi_n^{\text{HC}}, \Delta_n^{\text{HC}}$ (slanted lightface Σ, Π, Δ) for classes of parameter-free definability in HC (no parameters allowed), and $\Sigma_n(\text{HC}), \Pi_n(\text{HC}), \Delta_n(\text{HC})$ for full definability in HC (parameters from HC allowed). We will make use of the following known result, see e.g., Lemma 25.25 in Jech [24]: if $X \subseteq 2^\omega$ and $n \geq 1$ then

$$X \in \Sigma_n^{\text{HC}} \iff X \in \Sigma_{n+1}^1, \quad \text{and} \quad X \in \Sigma_n(\text{HC}) \iff X \in \Sigma_{n+1}^1, \tag{1}$$

and similar equivalences for the classes Π, Π, Δ, Δ instead of Σ, Σ .

Lemma 22 (in \mathbf{L}). The following ternary relation belongs to the class $\Delta_1^{\text{HC}} = \Delta_1^{\mathbf{L}_{\omega_1}}$:

$$\vec{\pi}, \vec{\varphi} \in \overline{\mathbf{MF}} \wedge M \in \text{HC} \wedge \vec{\pi} \sqsubset_M \vec{\varphi}.$$

Proof. Note first of all that $\overline{\mathbf{MF}} \subseteq \text{HC}$, so that the claim makes sense. The proof goes on by routine verification that all sets and relations involved are definable by Δ_0 formulas, i.e., those with only bounded quantifiers over suitable countable sets such as ω or $2^{<\omega}$, despite the fact that their prima facie definitions may include quantifiers over uncountable sets such as 2^ω . Consider for instance the relation

$$C(S, T) := S, T \in \mathbf{PT} \wedge [S] \cap [T] \text{ is clopen in } [S],$$

that participates in several definitions, e.g., in the definition of *regular* arboreal forcing (Definition 2), in the definition of refinements in Section 4, etc. We observe that, because of the compactness of 2^ω , if $S, T \in \mathbf{PT}$ then for $[S] \cap [T]$ to be clopen in $[S]$ it is necessary and sufficient that there exists a finite set $U \subseteq S$ such that

$$\bigcup_{t \in U} S \upharpoonright_t = \{r \in S \cap T : (S \cap T) \upharpoonright_r \text{ is infinite}\},$$

and this condition is obviously Δ_0 . Thus, this implies that the refinement relations \sqsubset and \sqsubset_D between arboreal forcings (Sections 4 and 5) are definable by Δ_0 formulas.

To check that $\pi \sqsubset_D \varrho$ as a ternary relation (Definition 11) is definable by Δ_0 formula, it suffices to prove the Δ_0 definability of the relation $[u] \subseteq \bigcup_{v \in D'} [v]$ (see the beginning of Section 8), where it is assumed that $u \in \mathbf{MT}$, $D' \subseteq \mathbf{MT}$ is finite, and $|v| = |u|$ for all $v \in D'$. Then, the relation $[u] \subseteq \bigcup_{v \in D'} [v]$ is equivalent to the following:

if $s_\xi \in u(\xi)$ for all $\xi \in |u|$ then there is $v \in D'$ such that $s_\xi \in v(\xi)$ for all $\xi \in |u|$.

However, this condition is Δ_0 as required. \square

18. Auxiliary Diamond Sequence

We argue in \mathbf{L} . Let us recall the technique of diamond sequences in \mathbf{L} .

Lemma 23 (\diamond_{ω_1} in \mathbf{L}). *There is a Δ_1^{HC} sequence $\langle S_\alpha \rangle_{\alpha < \omega_1}$ of sets $S_\alpha \subseteq \alpha$, such that*

(*) *if $X \subseteq \omega_1$ then the set $\{\alpha < \omega_1 : S_\alpha = X \cap \alpha\}$ is stationary in ω_1 , so that it has a non-empty intersection with each club (i.e., a closed unbounded set) $C \subseteq \omega_1$.*

Proof. The existence of a sequence satisfying (*) is the *diamond principle* \diamond_{ω_1} , see ([24], Theorem 13.21). The Δ_1^{HC} -definability (see is achieved by taking the $\leq_{\mathbf{L}}$ -least possible S_α at each step α , where $\leq_{\mathbf{L}}$ is the Gödel's well-ordering of \mathbf{L} , see ([24], p. 558). \square

Definition 22 (in \mathbf{L}). *We fix a sequence $\langle S_\alpha \rangle_{\alpha < \omega_1}$ given by Lemma 23.*

We let $c_\alpha = \alpha$ th element of $\text{HC} = \mathbf{L}_{\omega_1}$ in the sense of $\leq_{\mathbf{L}}$; thus $\text{HC} = \{c_\alpha : \alpha < \omega_1\}$.

If $Z \subseteq \text{HC}$ and $\alpha < \omega_1$ then let $(Z)_{<\alpha} = \{c_\xi \in Z : \xi < \alpha\}$.

If $\alpha < \omega_1$ then let $A_\alpha = \{c_\xi : \xi \in S_\alpha\}$. Then, $\langle A_\alpha \rangle_{\alpha < \omega_1}$ is still a Δ_1^{HC} sequence.

Let $A_\alpha^n = \{a : \langle n, a \rangle \in A_\alpha\}$.

Let $\mathfrak{M}(\alpha) = \{A_\alpha^n : n < \omega\}$. Then, $\langle \mathfrak{M}(\alpha) \rangle_{\alpha < \omega_1}$ is still a Δ_1^{HC} sequence.

An ordinal $\gamma < \kappa$ is a crucial ordinal for a sequence $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \kappa} \in \overline{\mathbf{MF}}_\kappa$ if the relation $(\bigcup_{\alpha < \gamma} \pi_\alpha) \sqsubset_{\mathfrak{M}(\gamma)} \pi_\gamma$ holds. This is equivalent to $\vec{\pi} \upharpoonright \gamma \subset_{\mathfrak{M}(\gamma)} \vec{\pi}$.

We obtain the following lemma as an easy corollary.

Lemma 24 (in \mathbf{L}). (i) *If $Z \subseteq \text{HC}$ then the set $W' = \{\alpha < \omega_1 : A_\alpha = (Z)_{<\alpha}\}$ is stationary.*

(ii) *If $Z_n \subseteq \text{HC}$ for all n then the set $W'' = \{\alpha < \omega_1 : \forall n (A_\alpha^n = (Z_n)_{<\alpha})\}$ is stationary.*

Proof. To prove (i), let $X = \{\alpha < \omega_1 : c_\alpha \in Z\}$. The set $W = \{\alpha < \omega_1 : S_\alpha = X \cap \alpha\}$ is then stationary. However, easily $W = W'$. To prove (ii) put $Z = \{\langle n, x \rangle : n < \omega \wedge x \in Z_n\}$ and apply (i). \square

19. The Key Sequence

The next theorem (Theorem 6) is a crucial step towards the construction of the forcing notion that will prove Theorem 1. The construction employs some ideas related to *definable generic* transfinite constructions, and it will go on by a transfinite inductive definition of a sequence $\vec{\pi} \in \overline{\mathbf{MF}}_{\omega_1}$ in \mathbf{L} from countable subsequences. The result can be viewed as a maximal branch in $\overline{\mathbf{MF}}$, *generic* with respect to all sets of a given complexity.

Definition 23 (in \mathbf{L}). *From now on a number $\mathfrak{n} \geq 3$ as in Theorem 1 is fixed.*

A sequence $\vec{\pi} \in \overline{\mathbf{MF}}$ blocks a set W if either $\vec{\pi}$ belongs to W (a positive block) or no sequence $\vec{\varphi} \in W \cap \overline{\mathbf{MF}}$ extends $\vec{\pi}$ (a negative block).

Any sequence $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \omega_1} \in \overline{\mathbf{MF}}_{\omega_1} \cap \mathbf{L}$, satisfying (in \mathbf{L}) the following four conditions (A)–(D) for this n , will be called a key sequence:

- (A) The set $|\vec{\pi}| = \bigcup_{\alpha < \kappa} |\vec{\pi}(\alpha)|$ is equal to ω_1 .
- (B) Every $\gamma < \omega_1$ is a crucial ordinal for $\vec{\pi}$ in the sense of Definition 22.
- (C) If in fact $n \geq 4$ and $W \subseteq \overline{\mathbf{MF}}$ is a boldface $\Sigma_{n-3}(\text{HC})$ set (a definition with parameters), then there exists an ordinal $\gamma < \omega_1$ such that the subsequence $\vec{\pi} \upharpoonright \gamma$ blocks W — so that either $\vec{\pi} \upharpoonright \gamma \in W$, or there is no sequence $\varphi \in W$ extending $\vec{\pi} \upharpoonright \gamma$.
- (D) The sequence $\vec{\pi}$ belongs to the definability class Δ_{n-2}^{HC} .

Theorem 6 (in \mathbf{L}). *There exists a key sequence $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \omega_1} \in \overline{\mathbf{MF}}_{\omega_1}$.*

Proof. We argue under $\mathbf{V} = \mathbf{L}$, with $n \geq 3$ fixed. In case $n \geq 4$, let $\mathbf{un}_n(p, x)$ be a universal Σ_{n-3} formula. In other words, the collection of all boldface $\Sigma_{n-3}(\text{HC})$ sets $X \subseteq \text{HC}$ is equal to the family of all sets of the form $Y_n(a) = \{x \in \text{HC} : \text{HC} \models \mathbf{un}_n(a, x)\}$, $a \in \text{HC}$.

Claim 1. *If $n \geq 4$ then $\{\langle \vec{\pi}, a \rangle : \vec{\pi} \in \overline{\mathbf{MF}} \wedge a \in \text{HC} \wedge \vec{\pi} \text{ blocks } Y_n(a)\}$ is a Δ_{n-2}^{HC} set.*

Proof (Claim). We skip a routine verification that $\overline{\mathbf{MF}}$ is Δ_1^{HC} . Further, if $\vec{\pi} \in \overline{\mathbf{MF}}$ and $a \in \text{HC}$ then for $\vec{\pi}$ to block $Y_n(a)$ it is necessary and sufficient that

$$\underbrace{\vec{\pi} \in Y_n(a)}_{\Sigma_{n-3}^{\text{HC}}} \vee \underbrace{\neg \exists \vec{\varphi} \left(\underbrace{\vec{\varphi} \in \overline{\mathbf{MF}} \wedge \vec{\varphi} \text{ extends } \vec{\pi}}_{\Delta_1^{\text{HC}}} \wedge \underbrace{\vec{\varphi} \in Y_n(a)}_{\Sigma_{n-3}^{\text{HC}}} \right)}_{\Pi_{n-3}^{\text{HC}}}.$$

This is a disjunction of Σ_{n-3}^{HC} and Π_{n-3}^{HC} , hence, Δ_{n-2}^{HC} , and we are finished. \square

Coming back to the proof of the theorem, a sequence $\vec{\pi}[\alpha] \in \overline{\mathbf{MF}}$ is defined by induction on $\alpha < \omega_1$. To begin with, we put $\vec{\pi}[0] = \emptyset$ (the empty sequence).

Step $\alpha \rightarrow \alpha + 1$. Assume that $\vec{\pi}[\alpha] \in \overline{\mathbf{MF}}$ is already defined. Put $\kappa = \text{dom } \vec{\pi}[\alpha]$, $\mathfrak{M} = \mathfrak{M}(\alpha)$, and let p_α be the α -th element of $\text{HC} = \mathbf{L}_{\omega_1}$ in the sense of the Gödel well-ordering $\leq_{\mathbf{L}}$ of \mathbf{L} . By Lemma 21(i), there is a sequence $\vec{\tau} \in \overline{\mathbf{MF}}_{\kappa+1}$ with $\vec{\pi}[\alpha] \subset_{\mathfrak{M}} \vec{\tau}$, and then a sequence $\vec{\varphi} \in \overline{\mathbf{MF}}_{\kappa+2}$ with $\vec{\tau} \subset \vec{\varphi}$. If $\alpha \notin |\vec{\varphi}(\kappa + 1)|$ then we trivially extend the last term $\vec{\varphi}(\kappa + 1)$ of the construction by $\vec{\varphi}(\kappa + 1)(\alpha) = \mathbb{P}_{\text{coh}}$ (see Example 1).

Finally if $n \geq 4$ then there is a sequence $\vec{\pi} \in \overline{\mathbf{MF}}$ satisfying $\vec{\varphi} \subset \vec{\pi}$ and blocking the set $Y_n(p_\alpha)$, while if $n = 3$ then simply put $\vec{\pi} = \vec{\varphi}$.

Thus, overall we have:

- (*) $\vec{\pi}[\alpha] \subset_{\mathfrak{M}} \vec{\pi}$, $\kappa + 1 < \text{dom } \vec{\pi}$, $\alpha \in |\vec{\varphi}(\kappa + 1)|$, and $\vec{\pi}$ blocks $Y_n(p_\alpha)$ in case $n \geq 4$.

Finally we let $\vec{\pi}[\alpha + 1]$ be the $\leq_{\mathbf{L}}$ -least one of all sequences $\vec{\pi} \in \overline{\mathbf{MF}}$ satisfying (*).

Note the role of the blanket assumption $\mathbf{V} = \mathbf{L}$ in this construction (step $\alpha \rightarrow \alpha + 1$); otherwise, the $\leq_{\mathbf{L}}$ -least choice of $\vec{\pi}[\alpha + 1]$ could not be executed.

Limit step. If $\lambda < \omega_1$ is a limit ordinal then we obviously define $\vec{\pi}[\lambda] = \bigcup_{\alpha < \lambda} \vec{\pi}[\alpha]$.

We have $\alpha < \beta \implies \vec{\pi}[\alpha] \subset \vec{\pi}[\beta]$ by construction; hence, $\vec{\pi} = \bigcup_{\alpha} \vec{\pi}[\alpha] \in \overline{\mathbf{MF}}_{\omega_1}$.

Let us check (D) of Definition 23. Note first of all that the relation $R(\vec{\pi}, \vec{\varphi}, \mathfrak{M}) := \vec{\pi} \subset_{\mathfrak{M}} \vec{\varphi}$ is Δ_1^{HC} by Lemma 22. Easily “to block $Y_n(p)$ ” is a Δ_{n-2}^{HC} relation by Claim 1 above. On the other hand, it is known that choosing the $\leq_{\mathbf{L}}$ -least element in each non-empty section of a Δ_k^{HC} set under $\mathbf{V} = \mathbf{L}$ results in a set (transversal) of the same class Δ_k^{HC} . Therefore, the assignment $\alpha \mapsto \mathfrak{M}(\alpha)$ is Δ_1^{HC} as well. With these estimations, a routine calculation shows that the relation (*) still is a Δ_{n-2}^{HC} relation (in \mathbf{L}). This helps to easily accomplish the verification of (D), which we leave to the reader.

To check (A) of Definition 23, note that $\alpha \in |\bigcup^{\text{cw}} \vec{\pi}[\alpha + 1]|$ by construction.

To check (C) of Definition 23 ($n \geq 4$), note that any boldface $\Sigma_{n-3}(\text{HC})$ set $W \subseteq \overrightarrow{\mathbf{MF}}$ is equal to $Y_n(p_\alpha)$ for some $\alpha < \omega_1$, so $\gamma = \text{dom } \vec{\pi}[\alpha + 1]$ is as required.

Finally, (B) holds by construction. \square

Definition 24 (in \mathbf{L}). From now on we fix a key sequence $\vec{\pi} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$, given by Theorem 6 for the number $n \geq 3$ fixed by Definition 23. It satisfies (A)–(D) of Definition 23. We call this fixed $\vec{\pi} \in \mathbf{L}$ the key sequence.

Lemma 25. Assume that $n \geq 4$. Let $W \subseteq \overrightarrow{\mathbf{MF}}$ be a $\Sigma_{n-3}(\text{HC})$ set dense in $\overrightarrow{\mathbf{MF}}$. Then, there exists an ordinal $\gamma < \omega_1$ satisfying $\vec{\pi} \upharpoonright \gamma \in W$.

Proof. By (C) of Definition 23, $\vec{\pi} \upharpoonright \gamma$ blocks W for some ordinal $\gamma < \omega_1$. The negative block is rejected because W is dense. Therefore, $\vec{\pi} \upharpoonright \gamma \in W$. \square

20. The Key Forcing Notion

Based on Definition 24, we introduce some derived notions.

Definition 25 (in \mathbf{L}). Using the key sequence $\vec{\pi} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1}$, we define the regular multforcing $\mathbb{P} = \bigcup_{\alpha < \omega_1}^{\text{cw}} \mathbb{P}_\alpha \in \mathbf{MF}$, and the forcing notion $\mathbb{P} = \mathbf{MT}(\mathbb{P}) = \mathbf{MT}(\vec{\pi})$.

We put $\mathbb{P}_\alpha = \mathbf{MT}(\mathbb{P}_\alpha)$, $\mathbb{P}_{<\gamma} = \bigcup_{\alpha < \gamma}^{\text{cw}} \mathbb{P}_\alpha$, $\mathbb{P}_{<\gamma} = \mathbf{MT}(\mathbb{P}_{<\gamma}) = \mathbf{MT}(\vec{\pi} \upharpoonright \gamma) = \bigcup_{\gamma < \alpha} \mathbb{P}_\alpha$.

We also put $\mathbb{P}_{\geq\gamma} = \bigcup_{\gamma \leq \alpha < \omega_1}^{\text{cw}} \mathbb{P}_\alpha$.

If $\xi < \omega_1$, then, following (A) of Definition 23, we let $\alpha(\xi) < \omega_1$ be the smallest ordinal α with $\xi \in |\mathbb{P}_\alpha|$. Thus, an arboreal forcing notion $\mathbb{P}_\alpha(\xi) \in \mathbf{AF}$ is defined whenever the inequality $\alpha(\xi) \leq \alpha < \omega_1$ holds. Moreover, $\langle \mathbb{P}_\alpha(\xi) \rangle_{\alpha(\xi) \leq \alpha < \omega_1}$ is a \square -increasing sequence of special forcings $\mathbb{P}_\alpha(\xi) \in \mathbf{AF}$, thus $\mathbb{P}(\xi) = \bigcup_{\alpha(\xi) \leq \alpha < \omega_1} \mathbb{P}_\alpha(\xi) \in \mathbf{AF}$.

We will call \mathbb{P} the key multforcing below, and accordingly the set $\mathbb{P} = \mathbf{MT}(\mathbb{P})$ will be our key forcing notion. The following lemmas present principal properties of \mathbb{P} in the ground universe \mathbf{L} , and of according \mathbb{P} -generic models in the next section.

Lemma 26 (in \mathbf{L}). The sequences $\langle \alpha(\xi) \rangle_{\xi < \omega_1}$ (of ordinals) and $\langle \mathbb{P}_\alpha(\xi) \rangle_{\xi < \omega_1, \alpha(\xi) \leq \alpha < \omega_1}$ (of arboreal forcings) belong to the definability class Δ_{n-2}^{HC} .

Proof. The following double equivalence

$$\begin{aligned} \alpha < \alpha(\xi) &\iff \exists \pi (\pi = \mathbb{P}_\alpha \wedge \xi \in \text{dom } \pi) &&\iff \\ &\iff \forall \pi (\pi = \mathbb{P}_\alpha \implies \xi \in \text{dom } \pi) &&. \end{aligned}$$

holds by construction. Yet, “ $\pi = \mathbb{P}_\alpha$ ” is a Δ_{n-2}^{HC} formula by (D) of Definition 23. Therefore, the sequence $\langle \alpha(\xi) \rangle_{\xi < \omega_1}$ is Δ_{n-2}^{HC} as well. The other sequence is treated similarly. \square

Lemma 27 (in \mathbf{L}). (i) $\mathbb{P}_\alpha(\xi)$ is pre-dense in $\mathbb{P}(\xi)$ whenever $\xi < \omega_1$ and $\alpha(\xi) \leq \alpha < \omega_1$.

(ii) \mathbb{P} is a regular multforcing and $|\mathbb{P}| = \omega_1$, thus $\mathbb{P} = \prod_{\xi < \omega_1} \mathbb{P}(\xi)$ (finite support).

(iii) $\mathbb{C}' = \{ \gamma < \omega_1 : |\mathbb{P}_{<\gamma}| = \gamma \}$ is a club (closed unbounded) in ω_1 , where $\mathbb{P}_{<\gamma} = \bigcup_{\alpha < \gamma}^{\text{cw}} \mathbb{P}_\alpha$.

Proof. (i), (ii) Use Lemma 5. To check that $|\mathbb{P}| = \omega_1$ recall (A) of Definition 23.

(iii) is clear. \square

The next lemma claims that \mathbb{P} satisfies CCC.

Theorem 7 (in \mathbf{L}). The forcing notion \mathbb{P} satisfies CCC. Therefore \mathbb{P} -generic extensions of \mathbf{L} preserve cardinals.

Proof. Let $Z \subseteq \mathbb{P}$ be a maximal antichain. Then, $D = \{p \in \mathbb{P} : \exists q \in Z (p \leq q)\}$ is open dense in \mathbb{P} . In terms of Definition 22, the set \mathbb{C} of all *limit* ordinals $\gamma < \omega_1$, such that

(*) $(Z)_{<\gamma} = Z \cap \mathbb{P}_{<\gamma}$, $(Z)_{<\gamma}$ is a maximal antichain in $\mathbb{P}_{<\gamma} = \mathbf{MT}(\vec{\pi} \upharpoonright \gamma)$, $(D)_{<\gamma}$ is open dense in $\mathbb{P}_{<\gamma}$, and the equality $(D)_{<\gamma} = \{p \in \mathbb{P}_{<\gamma} : \exists q \in (Z)_{<\gamma} (p \leq q)\}$ holds, is a club. Therefore, by Lemma 24(ii), there is an ordinal $\gamma \in \mathbb{C}$ such that $(Z)_{<\gamma} = A_\gamma^0$ and $(D)_{<\gamma} = A_\gamma^1$, and hence $(D)_{<\gamma} \in \mathfrak{M}(\gamma)$.

Note that $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}(\gamma)} \mathbb{P}_\gamma$, or equivalently $\vec{\pi} \upharpoonright \gamma \subset_{\mathfrak{M}(\gamma)} \vec{\pi}$, by (B) of Definition 23.

However, $D' = (D)_{<\gamma} \in \mathfrak{M}(\gamma)$ is open dense in $\mathbb{P}_{<\gamma}$ by (*). Therefore, Lemma 21(ii) implies that D' remains pre-dense in the whole set $\mathbb{P} = \mathbf{MT}(\vec{\pi})$, and hence, $A' = (A)_{<\gamma}$ itself by (*) remains a maximal antichain in \mathbb{P} . We conclude that $A = A'$ is countable. \square

Corollary 13 (in L). *Let a set $D \subseteq \mathbb{P}$ be pre-dense in \mathbb{P} . There is an ordinal $\gamma < \omega_1$ such that $D \cap \mathbb{P}_{<\gamma}$ is already pre-dense in \mathbb{P} .*

Proof. We can w.l.o.g. assume that D is even dense. Let $A \subseteq D$ be a maximal antichain in D . Then, A is a maximal antichain in \mathbb{P} since D is dense. Then, $A \subseteq \mathbb{P}_{<\gamma}$ for some ordinal $\gamma < \omega_1$ by Theorem 7. However, A is pre-dense in \mathbb{P} . \square

21. The Key Model

We aim to prove Theorem 1 using \mathbb{P} -generic extensions of \mathbf{L} , which we call **key models**. We will mostly argue in \mathbf{L} and in $\omega_1^{\mathbf{L}}$ -preserving generic extensions, in particular, in \mathbb{P} -generic extensions of \mathbf{L} (cardinal-preserving by Theorem 7). Therefore, we will always have $\omega_1^{\mathbf{L}} = \omega_1$. This allows us to view things so that $|\mathbb{P}| = \omega_1$ (rather than $\omega_1^{\mathbf{L}}$).

Definition 26. *Let a set $G \subseteq \mathbb{P}$ be generic over the constructible set universe \mathbf{L} . If $\xi < \omega_1$, then, following the remark in the end of Section 6,*

- We put $G(\xi) = \{p(\xi) : p \in G \text{ and } \xi \in |p|\} \subseteq \mathbb{P}(\xi)$;
- We define $x_\xi = x_\xi[G] \in 2^\omega$ as the unique real which belongs to $\bigcap_{T \in G(\xi)} [T]$;
- We finally let $\mathbf{X} = \mathbf{X}[G] = \langle x_\xi[G] \rangle_{\xi < \omega_1} = \{ \langle \xi, x_\xi[G] \rangle : \xi < \omega_1 \}$.

To conclude, the forcing notion \mathbb{P} adjoins an array $\mathbf{X}[G]$ of reals $x_\xi[G]$ to \mathbf{L} , where every real $x_\xi = x_\xi[G] \in 2^\omega \cap \mathbf{L}[G]$ is $\mathbb{P}(\xi)$ -generic over \mathbf{L} , and we have $\mathbf{L}[G] = \mathbf{L}[\mathbf{X}[G]]$.

Theorem 8. *Assume that a set $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over \mathbf{L} , $\xi < \omega_1$, and $x \in \mathbf{L}[G] \cap 2^\omega$. Then, the following statements are equivalent:*

- (1) $x = x_\xi[G]$;
- (2) the real x is $\mathbb{P}(\xi)$ -generic over \mathbf{L} ;
- (3) we have $x \in \bigcap_{\alpha(\xi) \leq \alpha < \omega_1} \bigcup_{T \in \mathbb{P}_\alpha(\xi)} [T]$.

Proof. The implication (1) \implies (2) is routine (see Remark 1). To check (2) \implies (3) note that, by Lemma 27(i), all sets $\mathbb{P}_\alpha(\xi)$ are pre-dense in $\mathbb{P}(\xi)$. Finally, prove (3) \implies (1).

Suppose that $x \in \mathbf{L}[G] \cap 2^\omega$ but $x \neq x_\xi[G]$, i.e., (1) fails. As $\mathbb{P} = \mathbf{MT}(\mathbb{P})$ is CCC by Theorem 7, Theorem 5(i) implies the existence of a small \mathbb{P} -complete real name $\mathbf{c} \in \mathbf{L}$, such that $x = \mathbf{c}[G]$, $\mathbf{c} \subseteq \mathbb{P} \times \omega \times 2$, and \mathbf{c} is non-principal at ξ over \mathbb{P} , meaning that

$$D = D_\xi^{\mathbf{c}}(\mathbb{P}) = \{p \in \mathbb{P} = \mathbf{MT}(\mathbb{P}) : p \text{ directly forces } \mathbf{c} \notin [p(\xi)]\}$$

is a set open dense in $\mathbb{P} = \mathbf{MT}(\mathbb{P})$. By the smallness, \mathbf{c} is a $\mathbb{P}_{<\gamma_0}$ -complete real name for some ordinal $\gamma_0 < \omega_1$.

In terms of Definition 22, the set \mathbb{C} of all *limit* ordinals $\gamma < \omega_1$, such that

(*) $\gamma \geq \gamma_0$ and the set $(D)_{<\gamma}$ satisfies $(D)_{<\gamma} = D \cap \mathbb{P}_{<\gamma}$ and is open dense in $\mathbb{P}_{<\gamma}$,

is a club. Therefore, by Lemma 24(ii), there is an ordinal $\gamma \in \mathbb{C}$ such that $(D)_{<\gamma} = A_\gamma^0 \in \mathfrak{M}(\gamma)$ and $\mathbf{c} = A_\gamma^1 \in \mathfrak{M}(\gamma)$. Then, \mathbf{c} is non-principal at ξ over $\mathbb{P}_{<\gamma}$. On the other hand, $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}(\gamma)} \mathbb{P}_\gamma$, by (B) of Definition 23. It follows that $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}(\gamma)} \mathbb{P}_{\geq\gamma}$ by Lemma 21(ii). Then, we have $\mathbb{P}_{<\gamma} \sqsubset_\xi \mathbb{P}_{\geq\gamma}$ as well by Definition 20(6), since $\mathbf{c} \in \mathfrak{M}(\gamma)$ and because of the non-principality of \mathbf{c} .

Now, Theorem 5(ii) with $\pi = \mathbb{P}_{<\gamma}$ and $\varphi = \mathbb{P}_{\geq\gamma}$ implies $x = \mathbf{c}[G] \notin \bigcup_{Q \in \mathbb{P}_{\geq\gamma}(\xi)} [Q]$. (We observe that $\pi \cup^{c.w} \varphi = \mathbb{P}$).

In particular, x does not belong to $\bigcup_{Q \in \mathbb{P}_\gamma(\xi)} [Q]$. Thus, (3) fails, as required. \square

Corollary 14. *Let a set $G \subseteq \mathbb{P}$ be \mathbb{P} -generic over \mathbf{L} . Then, it holds in $\mathbf{L}[G]$ that $\mathbf{X}[G]$ belongs to the definability class Π_{n-2}^{HC} , and hence, to class Π_{n-1}^1 by (1) of Section 20.*

Proof. By Theorem 8, it is true in $\mathbf{L}[G]$ that $\langle \xi, x \rangle \in \mathbf{X}[G]$ if and only if

$$\forall \alpha < \omega_1 \exists T \in \mathbb{P}_\alpha(\xi) (\alpha(\xi) \leq \alpha \implies x \in [T]),$$

which can be re-written as

$$\forall \mu < \omega_1 \forall \alpha < \omega_1 \forall Y \exists T \in Y (Y = \mathbb{P}_\alpha(\xi) \wedge \mu = \alpha(\xi) \wedge \mu \leq \alpha \implies x \in [T]).$$

Note that the equalities $\mu = \alpha(\xi)$ and $Y = \mathbb{P}_\alpha(\xi)$ belong to the class Δ_{n-2}^{HC} by Corollary 26. This implies that the whole relation is Π_{n-2}^{HC} , since the quantifier $\exists T \in Y$ is bounded. \square

22. Well-Orderings in the Key Model

According to the following theorem, the key model satisfies (i) of Theorem 1. The reals are treated as points of 2^ω , the Cantor space. The proof see Theorem 2 in [1].

Theorem 9. *Assume that a set $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over \mathbf{L} . Then, in $\mathbf{L}[G]$, there is a Δ_n^1 -good well-ordering of 2^ω of length ω_1 , and hence (i) of Theorem 1 holds.*

Our final step is to prove the result complementary to Theorem 9, that is, the key model also fulfills (ii) of Theorem 1. This will need some more effort. We will argue under the following assumption.

Assumption 1. *We assume that $n \geq 4$ from now on.*

This leaves aside the case $n = 3$ in (ii) of Theorem 1. Therefore, this case requires a separate consideration to justify the assumption. Assume that $n = 3$. We assert that (ii) of Theorem 1 holds in $\mathbf{L}[G]$ (which is the key model), where G is an arbitrary set \mathbb{P} -generic over \mathbf{L} . Suppose towards the contrary that (ii) of Theorem 1 fails, so that there is a Δ_2^1 well-ordering of the reals. (We even do not assume that the well-ordering is good). Then, Theorem 25.39 in [24] implies that $2^\omega \subseteq \mathbf{L}[x]$ in $\mathbf{L}[G]$ for some real $x \in 2^\omega$ in $\mathbf{L}[G]$. Yet this cannot be the case for the key models $\mathbf{L}[G]$ we consider.

Indeed, arguing in $\mathbf{L}[G]$, suppose to the contrary that $x \in 2^\omega \cap \mathbf{L}[G] = \mathbf{L}[\langle x_\xi[G] \rangle_{\xi < \omega_1}]$ and $2^\omega \cap \mathbf{L}[G] \subseteq \mathbf{L}[x]$. Theorem 7 then implies that there exists an ordinal $\lambda < \omega_1 = \omega_1^{\mathbf{L}}$ satisfying $x \in \mathbf{L}[\langle x_\xi[G] \rangle_{\xi < \lambda}]$. However, the real $y = x_\lambda[G]$ does not belong to $\mathbf{L}[\langle x_\xi[G] \rangle_{\xi < \lambda}]$ by the product forcing theory. We conclude that $y \notin \mathbf{L}[x]$, which contradicts the choice of x .

Part IV: Non-existence of Simpler Well-orderings

Claim (ii) of Theorem 1 involves one more important technical tool related to the above-defined key forcing notion \mathbb{P} . It turns out that the \mathbb{P} -forcing relation of Σ_{n-1}^1 formulas is equivalent (up to level $n - 1$ of the projective hierarchy of formulas) to a certain auxiliary forcing relation forc defined and studied in Sections 23–30 below. Theorem 11 proves the equivalence. This auxiliary forcing is invariant with respect to permutations of indices

$\xi < \omega_1$ (Theorem 12), whereas the forcing \mathbb{P} itself is, generally speaking, not invariant in that sense. Such a *hidden invariance* plays a crucial role in the construction. Here, we make use of the invariance to prove, using Theorem 13, that the full version of (ii) holds in \mathbb{P} -generic extensions of \mathbf{L} . Theorems 11–13 are the main results of this Part.

23. Auxiliary Forcing Relation

We argue in \mathbf{L} . We make use of the second-order arithmetic language. It involves variables k, l, m, n, \dots (type 0) assumed to run over ω , and variables a, b, x, y, \dots (type 1) over 2^ω . The atomic formulas are only those of the form $x(k) = n$. Consider the extension \mathcal{L} of this language, which allows us to substitute natural numbers for variables of type 0, and **small real names** (Definition 12) $\mathbf{c} \in \mathbf{L}$ for variables of type 1.

- We define natural classes $\mathcal{L}\Sigma_n^1, \mathcal{L}\Pi_n^1$ of \mathcal{L} -formulas, as usual.
- Given a formula φ in $\mathcal{L}\Sigma_n^1$ (resp., $\mathcal{L}\Pi_n^1$), let φ^- be the result of canonical transformation of $\neg \varphi$ to the $\mathcal{L}\Pi_n^1$ (resp., $\mathcal{L}\Sigma_n^1$) form.

Now, we introduce a relation $\mathbf{p} \text{ forc}_\pi \varphi$ between multitrees \mathbf{p} , small multiforcings $\pi \in \overrightarrow{\mathbf{MF}}$, and closed \mathcal{L} -formulas φ in $\mathcal{L}\Sigma_n^1 \cup \mathcal{L}\Pi_n^1, n \geq 1$, which will approximate the true \mathbb{P} -forcing relation. The definition proceeds by induction on the \mathcal{L} -structure of φ .

1°. Let π be a small regular multiforcing, $\mathbf{p} \in \mathbf{MT}$ (not necessarily $\mathbf{p} \in \mathbf{MT}(\pi)$), and φ be a closed $\mathcal{L}\Sigma_1^1$ formula. We assume that φ has the following canonical Σ_1^1 form:

- (f1) $\varphi := \exists x \forall m R(x \upharpoonright m, \mathbf{c}_1 \upharpoonright m, \dots, \mathbf{c}_k \upharpoonright m)$, where $R \subseteq (\omega^{<\omega})^{k+1}$ is a recursive relation and every $\mathbf{c}_i \subseteq \mathbf{MT} \times (\omega \times 2)$ is a small real name.

Consider a tree-name $\tau = \tau(R)$ which consists of all pairs $\langle q, s \rangle \in \mathbf{MT} \times \omega^{<\omega}$ such that there exist tuples $t_1, \dots, t_k \in \omega^m$, where $m = \text{lh}(s)$, and multitrees $\mathbf{r}_j^i \in \mathbf{K}_{j:t_i(j)}^{\mathbf{c}_i}, 1 \leq i \leq k, j < m$ (see Definition 12), satisfying:

- (I) $R(s \upharpoonright n, t_1 \upharpoonright n, \dots, t_k \upharpoonright n)$ for all $n \leq m$;
- (II) q is equal to the multitree $\bigwedge_{1 \leq i \leq k, j < m} \mathbf{r}_j^i$ —see Section 6 on \bigwedge^{cw} , therefore q satisfies $q \leq \mathbf{r}_j^i$ for all i, j , and hence q directly forces $t_i \subseteq \mathbf{c}_i$ for all $1 \leq i \leq k$.

We define $\mathbf{p} \text{ forc}_\pi \varphi$ if the following conditions (a)–(d) and (e1) hold:

- (a) Every $q \in \text{dom } \tau$ is sealed by π (see Example 2);
- (b) If $q \in \text{dom } \tau$ and $\mathbf{p} \perp^\pi q$ then $\mathbf{p} \perp^\pi q$ is sealed by π (see Example 3);
- (c) Every name \mathbf{c}_i in φ is sealed π -complete (see Definition 12);
- (d) $\tau = \tau(R)$ is sealed in π (see Definition 19);
- (e1) \mathbf{p} directly forces $\Lambda \in \tau_\pi^\infty$, i.e., there is a multitree $\mathbf{p}_0 \geq \mathbf{p}$ with $\langle \mathbf{p}_0, \Lambda \rangle \in \tau_\pi^\infty$.

2°. Let π be a small regular multiforcing, $\mathbf{p} \in \mathbf{MT}$ (not necessarily $\mathbf{p} \in \mathbf{MT}(\pi)$), and ψ be a closed $\mathcal{L}\Pi_1^1$ formula. We assume that ψ has the following canonical Π_1^1 form

- (f2) $\psi := \forall x \exists m \neg R(x \upharpoonright m, \mathbf{c}_1 \upharpoonright m, \dots, \mathbf{c}_k \upharpoonright m)$, where $R \subseteq (\omega^{<\omega})^{k+1}$ is a recursive relation and every \mathbf{c}_i is a small real name.

We define a tree-name $\tau = \tau(R)$ as above. Then, define $\mathbf{p} \text{ forc}_\pi \psi$ if conditions (a)–(d), as above, hold and the following condition (e2) holds too instead of (e1) above:

- (e2) if $\mathbf{p}_1 \in \mathbf{MT}(\pi), \mathbf{p}_1 \leq \mathbf{p}$, then \mathbf{p}_1 does not directly force $\Lambda \in \tau_\pi^\infty$.

3°. If $\varphi(x)$ is a $\mathcal{L}\Pi_n^1$ formula, $n \geq 1$, then we define $\mathbf{p} \text{ forc}_\pi \exists x \varphi(x)$ if and only if there exists a small real name \mathbf{c} such that $\mathbf{p} \text{ forc}_\pi \varphi(\mathbf{c})$.

4°. If φ is a closed $\mathcal{L}\Pi_n^1$ formula, $n \geq 2$, then define $\mathbf{p} \text{ forc}_\pi \varphi$ if and only if there is no special multiforcing ϱ and $\mathbf{p}' \in \mathbf{MT}(\varrho)$ such that $\pi \sqsubset \varrho, \mathbf{p}' \leq \mathbf{p}$, and $\mathbf{p}' \text{ forc}_\varrho \varphi^-$.

Remark 3. If $\mathbf{p} \text{ forc}_\pi \varphi$ holds then it is not necessary that $\mathbf{p} \in \mathbf{MT}(\pi)$, and it is not necessary that every name \mathbf{c} in φ satisfies $\mathbf{c} \subseteq \mathbf{MT}(\pi) \times (\omega \times 2)$.

Definition 27. Given a class K of the form $\mathcal{L}\Sigma_n^1, \mathcal{L}\Pi_n^1$ ($n \geq 1$), we let $\mathbf{FORC}[K]$ contain all triples $\langle \pi, p, \varphi \rangle$ satisfying $p \text{ forc}_\pi \varphi$.

Then, $\mathbf{FORC}[K]$ is a subset of HC. Recall that if $\mathbf{V} = \mathbf{L}$ then $\text{HC} = \mathbf{L}_{\omega_1}$.

Lemma 28 (in \mathbf{L}). $\mathbf{FORC}[\mathcal{L}\Sigma_1^1]$ and $\mathbf{FORC}[\mathcal{L}\Pi_1^1]$ belong to Δ_1^{HC} .
 Given any $n \geq 2$, $\mathbf{FORC}[\mathcal{L}\Sigma_n^1]$ belongs to Σ_{n-1}^{HC} and $\mathbf{FORC}[\mathcal{L}\Pi_n^1]$ belongs to Π_{n-1}^{HC} .

Proof. Relations such as “being a small regular multiforcing”, “being a formula in $\mathcal{L}\Sigma_n^1, \mathcal{L}\Pi_n^1$ ”, $p \in \mathbf{MT}(\pi)$, etc., are definable in $\text{HC} = \mathbf{L}_{\omega_1}$ by bounded formulas, hence Δ_1^{HC} . Such also are the operations $R \mapsto \tau(R)$, and $\tau \mapsto \tau'_\pi, \tau \mapsto \tau_\pi^\infty$ (provided π is small, as in $1^\circ, 2^\circ$). This wraps up the Δ_1^{HC} estimation for the cases of $\mathcal{L}\Sigma_1^1$ and $\mathcal{L}\Pi_1^1$.

The inductive step by 3° is quite simple.

Now, for the step by 4° , assume that $n \geq 2$, and $\mathbf{FORC}[\mathcal{L}\Sigma_{n-1}^1] \in \Sigma_{n-1}^{\text{HC}}$ is already established. Then, $\langle \pi, p, \varphi \rangle \in \mathbf{FORC}[\mathcal{L}\Pi_n^1]$ if π is a small regular multiforcing, $p \in \mathbf{MT}$, φ is a closed $\mathcal{L}\Pi_n^1$ formula, and, by 4° , there is no triple $\langle \varrho, p', \psi \rangle \in \mathbf{FORC}[\mathcal{L}\Sigma_n^1]$ such that ϱ is a special multiforcing, $\pi \sqsubset \varrho, p' \in \mathbf{MT}(\varrho), p' \leq p$, and ψ is φ^- . This clearly implies the estimation Π_{n-1}^{HC} of $\mathbf{FORC}[\mathcal{L}\Pi_n^1]$ as required. \square

24. Elementary Properties of the Auxiliary Forcing

We still argue in \mathbf{L} .

Lemma 29. Assume that $\pi \sqsubset \varrho$ are small regular multiforcings, $p, p' \in \mathbf{MT}, p' \leq p, \varphi$ is an \mathcal{L} -formula. Then $p \text{ forc}_\pi \varphi$ implies $p' \text{ forc}_\varrho \varphi$.

Proof. If φ is a formula in $\mathcal{L}\Sigma_1^1$ as in 1° of Section 23, and $p \text{ forc}_\pi \varphi$, which is witnessed by (a)–(d) and (e1), then $q \text{ forc}_\varrho \varphi$ also holds.

Indeed, condition (a) transfers from π to ϱ by Corollary 8(i), (ii).

Condition (b) transfers to ϱ by Corollary 9(i)(ii) (with same p) and (iii) (to $p' \leq p$).

Condition (c) transfers to ϱ by Corollary 10 (ii).

Condition (d) transfers to ϱ by Theorem 3(iii)(v).

Finally, (e1) transfers to ϱ because $\tau_\pi^\infty = \tau_\varrho^\infty$ by Theorem 3 (iv).

The $\mathcal{L}\Pi_1^1$ case is rather similar, yet the transfer of (e2) from π to ϱ deserves attention. Note that all premises of Theorem 3(vi) hold for π, ϱ, p , and $s = \Lambda$. That is, $\pi \sqsubset_\tau \varrho$ holds by Theorem 3(iii) and (d), whereas $\pi \sqsubset_{pq_0} \varrho$ holds for all $q_0 \in \text{dom } \tau, \pi$ -incompatible with p , by Corollary 9(i) and (b).

Now, condition (e2) for π and p means that (1) of Theorem 3(vi) fails for π, p , and $s = \Lambda$. Therefore, (3) fails as well, that is, no $p'_1 \in \mathbf{MT}(\varrho), p'_1 \leq p$, directly forces $\Lambda \in \tau_\varrho^\infty$. However, $p' \leq p$, and therefore, we have (e2) for ϱ and p' (instead of π, p), as required.

The induction step \exists , as in 3° , is pretty elementary.

Now, for the induction step \forall , as in 4° , assume that $n \geq 2$ and φ is a closed formula in $\mathcal{L}\Pi_n^1$ satisfying $p \text{ forc}_\pi \varphi$. Suppose that $q \text{ forc}_\varrho \varphi$ fails. Then, by 4° there exist: a special multiforcing ϱ' and a multitree $q' \in \mathbf{MT}(\varrho')$ such that $\varrho \sqsubset \varrho', q' \leq q$, and $q' \text{ forc}_{\varrho'} \varphi^-$. However, then we have $\pi \sqsubset \varrho'$ and $q' \leq p$. We conclude that $p \text{ forc}_\pi \varphi$ fails by 4° . \square

Lemma 30 (in \mathbf{L}). Let π be a small regular multiforcing, φ a formula in $\mathcal{L}\Sigma_n^1, n \geq 1, p \in \mathbf{MT}(\pi)$, and if $n = 1$ then $\exists q \in \mathbf{MT}(\pi) (q \leq p)$. Then $p \text{ forc}_\pi \varphi$ implies $\neg(p \text{ forc}_\pi \varphi^-)$.

Recall that φ^- is the canonical transformation of $\neg\varphi$ to the prenex form.

Proof. If $n \geq 2$ then the result follows from definition 4° . Therefore, let $n = 1$, so that, by the contrary assumption and Lemma 29, there exists $q \in \mathbf{MT}(\pi)$ such that both $q \text{ forc}_\pi \varphi$ and $q \text{ forc}_\pi \varphi^-$. However, then (e1) immediately contradicts (e2) with $p_1 = q$. \square

Lemma 31 (in **L**). Let π be a small regular multiforcing, $p \in \mathbf{MT}(\pi)$, φ a formula in $\mathcal{L}\Sigma_n^1$, $n \geq 1$, and each name c in φ is π -complete. Then, there is a special multiforcing ϱ and a multitree $q \in \mathbf{MT}(\varrho)$ such that $\pi \sqsubset \varrho$, $q \leq p$, and either $q \text{ forc}_\varrho \varphi$ or $q \text{ forc}_\varrho \varphi^-$.

Proof. Suppose that $n = 1$, so that φ has the canonical form (f1) with a recursive R . Then, $\tau = \tau(R) \subseteq \mathbf{MT} \times \omega^{<\omega}$ is a tree-name. As each name in φ is π -complete, Theorem 4 gives a special multiforcing ϱ satisfying $\pi \sqsubset_r \varrho$ for each multitree r in $\text{dom } \tau$, $\pi \sqsubset_c \varrho$ for each name c in φ , and $\pi \sqsubset_\tau \varrho$.

Case 1: some $q \in \mathbf{MT}(\varrho)$, $q \leq p$ directly forces $\Lambda \in \tau$. Then, $q \text{ forc}_\varrho \varphi$ by 1°.

Case 2: not case 1. Then, we have $p \text{ forc}_\varrho \varphi^-$ by 2° of Section 23.

If $n \geq 2$, then the result follows from definition 4° of Section 23. \square

25. Forcing the First Level Formulas

The following theorem shows that the auxiliary forcing relation is properly connected with the ordinary forcing at least for formulas in $\mathcal{L}\Sigma_1^1 \cup \mathcal{L}\Pi_1^1$.

Theorem 10 (in **L**). Let $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \omega_1} \in \overline{\mathbf{MF}}_{\omega_1}$ and $Q = \mathbf{MT}(\vec{\pi})$. Assume that $\alpha < \omega_1$ and $p \in \mathbf{MT}(\pi_\alpha)$, φ is a formula in $\mathcal{L}\Sigma_1^1 \cup \mathcal{L}\Pi_1^1$, and $p \text{ forc}_{\pi_\alpha} \varphi$. Then, p Q -forces the sentence $\varphi[G]$ over the universe in the ordinary sense.

Proof. Case 1: φ is a formula in $\mathcal{L}\Sigma_1^1$, of the canonical form (f1), that is,

$$\varphi := \exists x \forall m R(x \upharpoonright m, c_1 \upharpoonright m, \dots, c_k \upharpoonright m), \tag{2}$$

where $R \subseteq (\omega^{<\omega})^{k+1}$ is a recursive relation and every $c_i \subseteq \mathbf{MT} \times (\omega \times 2)$ is a small real name, and $p \text{ forc}_{\pi_\alpha} \varphi$, so that properties (a)–(d) and (e1) of Section 23 hold for $\pi = \pi_\alpha$, $\tau = \tau(R)$, and p . In particular, (*) p directly forces $\Lambda \in \tau_{\pi_\alpha}^\infty$ by (e1).

Now, consider any set $G \subseteq Q$, generic over the given universe \mathbf{V} and containing p ; the goal is to prove that $\varphi[G]$ holds in $\mathbf{V}[G]$. The following lemma simplifies the task.

\square

Lemma 32. The set $G_\alpha = G \cap \mathbf{MT}(\pi_\alpha)$ is π_α -generic over τ in the sense of Definition 18, and is π_α -generic over each name c_i occurring in φ , in the sense of Definition 13.

In addition, $c_i[G] = c_i[G_\alpha]$ for any such name c_i , as well as $\tau[G] = \tau[G_\alpha]$.

Proof (Lemma). First of all, we have to check (2) of Definition 13 for G_α . Thus, let $u, v \in \mathbf{MT}(\pi_\alpha)$ belong to G_α . However, u, v are sealed by π_α by (a) of Section 23, thus the sets

$$D_u(\pi_\alpha) = \{q \in \mathbf{MT}(\pi) : |u| \subseteq |q| \wedge (q \leq u \vee q \perp u)\}$$

and $D_v(\pi_\alpha)$ are sealed dense in $\mathbf{MT}(\pi_\alpha)$. Then, $D_u(\pi_\alpha) \cap D_v(\pi_\alpha)$ is sealed dense in $\mathbf{MT}(\pi_\alpha)$ as well by Lemma 11(i). Therefore, if $\alpha < \beta < \kappa = \text{dom}(\vec{\pi})$ then

$$(D_u(\pi_\alpha) \cap D_v(\pi_\alpha)) \upharpoonright \pi_\beta$$

is a sealed dense, and therefore open dense, set in $\mathbf{MT}(\pi_\beta)$ by Lemma 10(iii). We conclude, by the genericity of G , that there is a multitree $w \in G$ that belongs to $(D_u(\pi_\alpha) \cap D_v(\pi_\alpha)) \upharpoonright \pi_\beta$ for some $\beta > \alpha$. Then, there is a multitree $q \in D_u(\pi_\alpha) \cap D_v(\pi_\alpha)$ satisfying $w \leq q$. We have $q \in G$ since $w \in G$. On the other hand, $u, v \in G$ as well; therefore, q cannot be incompatible with u, v . It follows that $q \leq u$ and $q \leq v$.

This ends the proof of (2) of Definition 13 for G_α .

Now, check the special condition of Definition 18. Let $\langle p, s \rangle \in \tau$, $v \leq v_{\pi_\alpha}(\tau)$, and let the set ${}^+W_{\leq p}^s(\tau_{\pi_\alpha}^v, \pi_\alpha)$ be dense in $\mathbf{MT}(\pi_\alpha)$; prove that ${}^+W_{\leq p}^s(\tau_{\pi_\alpha}^v, \pi_\alpha) \cap G \neq \emptyset$. Note that τ is sealed in $\mathbf{MT}(\pi_\alpha)$ by (d) of Section 23. It follows that ${}^+W_{\leq p}^s(\tau_{\pi_\alpha}^v, \pi_\alpha)$ is sealed

dense in $\mathbf{MT}(\pi_\alpha)$ (see Definition 19(b)). Therefore, by Lemma 10(iii), if $\beta \geq \alpha$, then the set ${}^+W_{\leq p}^s(\tau_{\pi_\alpha}^\nu, \pi_\alpha) \uparrow \pi_\beta$ is dense in $\mathbf{MT}(\pi_\beta)$. It follows, by the genericity of G , that there is a multiree $w \in G \cap ({}^+W_{\leq p}^s(\tau_{\pi_\alpha}^\nu, \pi_\alpha) \uparrow \pi_\beta)$, for some $\beta > \alpha$. Then, we have a multiree $q \in {}^+W_{\leq p}^s(\tau_{\pi_\alpha}^\nu, \pi_\alpha)$ satisfying $w \leq q$. Then, $q \in G$ since $w \in G$, and we are finished.

The genericity over the occurring names is verified similarly, starting from (c).

Finally, prove the additional part of the lemma. To check $\tau[G] = \tau[G_\alpha]$, assume that $s \in \tau[G]$, meaning that there is $q \in G$ with $\langle q, s \rangle \in \tau$, and thus, $q \in \text{dom } \tau$. It follows by (a) of Section 23 that q is sealed by π_α , in other words, the set

$$D_q(\pi_\alpha) = \{r \in \mathbf{MT}(\pi) : |q| \subseteq |r| \wedge (r \leq q \vee r \perp q)\}$$

is sealed dense in $\mathbf{MT}(\pi)$. Then, arguing as above we prove that $G \cap (D_q(\pi_\alpha) \uparrow \pi_\beta) \neq \emptyset$ for some $\beta > \alpha$, and hence there exists $r \in G_\alpha \cap D_q(\pi_\alpha)$. However, $r \perp q$ is impossible since $q \in G$ either, so we have $r \leq q$. Thus, r witnesses that $s \in \tau[G_\alpha]$.

The proof that $c_i[G] = c_i[G_\alpha]$ for any name c_i in φ is similar. \square

We return to the proof of Theorem 10.

We know that p directly forces $\Lambda \in \tau_{\pi_\alpha}^\infty$. The set G_α is π_α -generic over τ by Lemma 32. Therefore, by Corollary 11, $\tau[G_\alpha]$ is ill-founded, i.e., has an infinite chain, and hence, $\tau[G]$ is ill-founded because $\tau[G_\alpha] = \tau[G]$ still by Lemma 32.

Our goal is to prove that the sentence $\varphi[G] := \exists x \forall m R(x \upharpoonright m, y_1 \upharpoonright m, \dots, y_k \upharpoonright m)$ (see Formula (2) above) holds in $\mathbf{V}[G]$, where $y_i = c_i[G] = c_i[G_\alpha]$, $i = 1, \dots, k$. We put

$$T_{y_1, \dots, y_k} = \{s \in \omega^{<\omega} : \forall n \leq \text{lh}(s) R(s \upharpoonright n, y_1 \upharpoonright n, \dots, y_k \upharpoonright n)\}.$$

Thus, $T_{y_1, \dots, y_k} \subseteq \omega^{<\omega}$ is a tree, and $\varphi[G]$ is true if T_{y_1, \dots, y_k} is ill-founded. However, $\tau[G]$ is ill-founded, see just above. Thus, it remains to be shown that $T_{y_1, \dots, y_k} = \tau[G]$.

Let $s \in \tau[G]$, $m = \text{lh}(s)$. We have $\langle q, s \rangle \in \tau$ for some $q \in G$. By definition, there are tuples $t_1, \dots, t_k \in \omega^m$ satisfying (I) and (II) of Section 23; in particular, q directly forces $t_i \subseteq c_i$ by (III) for all $i = 1, \dots, k$, and hence, $t_i = y_i \upharpoonright m$ for all i . Thus, $s \in T_{y_1, \dots, y_k}$ by (I).

Conversely let $s \in T_{y_1, \dots, y_k}$, that is, $R(s \upharpoonright n, t_1 \upharpoonright n, \dots, t_k \upharpoonright n)$ holds for all $n \leq m = \text{lh}(s)$, where $t_i = y_i \upharpoonright m$ for all i . As $y_i = c_i[G]$, there is a family of conditions $r_j^i \in K_{j, t_i(j)}^{c_i} \cap G$, $1 \leq i \leq k$, $j < m$. Then, the multiree $q = \bigwedge_{1 \leq i \leq k, j < m} r_j^i$ belongs to G as well as G is generic, and by definition we have $\langle q, s \rangle \in \tau$. It follows that $s \in \tau[G]$.

Thus, $T_{y_1, \dots, y_k} = \tau[G]$, as required.

Case 2: φ is a formula in \mathcal{LPI}_1^1 . We write ψ instead of φ . Thus, let ψ be a \mathcal{LPI}_1^1 formula of the form (f2), i.e., essentially the negation of φ above, and let $p \text{ forc}_{\pi_\alpha} \psi$, so that properties (a)–(d) and (e2) of Section 23 hold for $\pi = \pi_\alpha$, $\tau = \tau(R)$, and p . In particular, if $p_1 \in \mathbf{MT}(\pi)$, $p_1 \leq p$, then p_1 does not directly force $\Lambda \in \tau_{\pi_\alpha}^\infty$, by (e2). Given any $G \subseteq Q$, generic over L and containing p , the goal is to prove that $\psi[G]$ holds in $\mathbf{V}[G]$.

As $p \in G_\alpha$ and G_α is π_α -generic over τ by Lemma 32, Corollary 11(ii) implies that $\tau[G_\alpha] = \tau[G]$ is well-founded. Then, the tree T_{y_1, \dots, y_k} , defined as above, is well-founded either, since it is equal to $\tau[G_\alpha]$. Therefore, $\psi[G]$ holds, as required.

26. Forcing inside the Key Sequence

It is implied by Theorem 11 below that the forcing relation forc_π , considered with the terms $\pi = \mathbb{P}_\alpha$ of the key sequence $\vec{\pi}$, really approximates the true \mathbb{P} -forcing relation at level $\mathfrak{n} - 1$ and below. Recall that $\mathfrak{n} \geq 4$ is assumed (see Assumption 1).

We argue in L. Recall that the key sequence $\vec{\pi} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1} \in \overline{\mathbf{MF}}_{\omega_1}$, satisfying (A), (B), (C), (D) of Definition 23 was defined by Theorem 6, $\mathbb{P} = \bigcup_{\alpha < \omega_1} \mathbb{P}_\alpha$ is the key multforcing, and $\mathbb{P} = \mathbf{MT}(\vec{\pi}) = \mathbf{MT}(\mathbb{P})$ is our forcing notion.

Definition 28. We write $p \text{ forc}_\alpha \varphi$ instead of $p \text{ forc}_{\mathbb{P}_\alpha} \varphi$, for the sake of brevity. Let $p \text{ forc } \varphi$ mean: $p \text{ forc}_\alpha \varphi$ for some $\alpha < \omega_1$.

Lemma 33 (in **L**). Assume that $p \in \mathbb{P}$, $\alpha < \omega_1$, and $p \text{ forc}_\alpha \varphi$. Then, the following holds:

- (i) If $\alpha \leq \beta < \omega_1$, $q \in \mathbb{P}_{<\beta} = \mathbf{MT}(\vec{\pi} \upharpoonright \beta)$, and $q \leq p$, then $q \text{ forc}_\beta \varphi$;
- (ii) If $q \in \mathbb{P}$, $q \leq p$, then there exists $\beta, \alpha \leq \beta < \omega_1$, such that $q \text{ forc}_\beta \varphi$;
- (iii) If $q \in \mathbb{P}$ and $q \text{ forc } \varphi^-$ then $p \perp q$;
- (iv) It follows that, first, if $p, q \in \mathbb{P}$, $q \leq p$, and $p \text{ forc } \varphi$ then $q \text{ forc } \varphi$, and second, $p \text{ forc } \varphi$ and $p \text{ forc } \varphi^-$ are incompatible.

Proof. Lemma 29 implies (i). To prove (ii), choose an ordinal β with $\alpha < \beta < \omega_1$, satisfying $q \in \mathbf{MT}(\vec{\pi} \upharpoonright \beta)$, and apply (i). To check (iii), we observe that p, q are incompatible in \mathbb{P} , as otherwise (i) leads to contradiction. On the other hand, multitrees incompatible in \mathbb{P} are \perp by Corollary 2. \square

Theorem 11. Assume that $1 \leq n \leq \aleph - 2$ and ψ is a closed formula in $\mathcal{L}\Pi_n^1 \cup \mathcal{L}\Sigma_{n+1}^1$, with all names small and \mathbb{P} -complete, and $p \in \mathbb{P}$. Then $p \text{ P-forces } \psi[G]$ over **L** in the usual sense, if and only if $p \text{ forc } \psi$.

Proof. Let \Vdash be the ordinary \mathbb{P} -forcing relation over **L**.

Part 1: ψ is a $\mathcal{L}\Pi_n^1$ formula, of the canonical form (f2) in Section 23, with a recursive R . Define the tree-name $\tau = \tau(R)$ by (I), (II) in Section 23.

If $p \text{ forc } \varphi$ then $p \text{ forc}_{c_{\mathbb{P}_\gamma}} \varphi$ for some $\gamma < \omega_1$, and then $p \Vdash \varphi[G]$ by Theorem 10.

To prove the converse suppose that $p \Vdash \varphi[G]$. By the choice of φ , there is an ordinal $\lambda_0 < \omega_1$ such that all names in φ are $\mathbb{P}_{<\lambda_0}$ -complete, where $\mathbb{P}_{<\lambda_0} = \bigcup_{\alpha < \lambda_0} \mathbb{P}_\alpha$. Thanks to Lemma 24(ii), the set C of all ordinals $\lambda \geq \lambda_0$ such that

(*) $\mathfrak{M}(\lambda)$ contains all names in ψ , all multitrees in $\text{dom } \tau$, and p, τ as well,

is stationary. Therefore, there is an ordinal $\lambda \in C$, $\lambda > \lambda_0$.

Property (B) in Definition 23 implies $\mathbb{P}_{<\lambda} \sqsubset_{\mathfrak{M}(\lambda)} \mathbb{P}_\lambda$. In particular, it follows by (*), that $\mathbb{P}_{<\lambda} \sqsubset_q \mathbb{P}_\lambda$ for all $q \in \text{dom } \tau$, $\mathbb{P}_{<\lambda} \sqsubset_{pq} \mathbb{P}_\lambda$ for all $q \in \text{dom } \tau$, $\mathbb{P}_{<\lambda}$ -incompatible with p , $\mathbb{P}_{<\lambda} \sqsubset_c \mathbb{P}_\lambda$ for all names c in ψ , and $\mathbb{P}_{<\lambda} \sqsubset_\tau \mathbb{P}_\lambda$.

We conclude that properties (a)–(d) of Section 23 hold for \mathbb{P}_λ in the role of π (and for the given p, ψ). Indeed, to check (a), recall that $\mathbb{P}_{<\lambda} \sqsubset_q \mathbb{P}_\lambda$ holds for all $q \in \text{dom } \tau$ by the above; hence, every $q \in \text{dom } \tau$ is sealed by \mathbb{P}_λ by Corollary 8(ii). To check (b), (c), (d) for \mathbb{P}_λ argue similarly but refer to resp. Corollary 9(ii), Corollary 10(i), Theorem 3(v).

We claim that (e2) of Section 23 also holds, i.e., if $p_1 \in \mathbf{MT}(\mathbb{P}_\lambda)$, $p_1 \leq p$, then p_1 does not directly force $\Lambda \in \tau_{\mathbb{P}_\lambda}^\infty$. Indeed, suppose to the contrary that $p_1 \in \mathbf{MT}(\mathbb{P}_\lambda)$, $p_1 \leq p$ is a counterexample, so that p_1 directly forces $\Lambda \in \tau_{\mathbb{P}_\lambda}^\infty$. Then, $p_1 \text{ forc}_{c_{\mathbb{P}_\gamma}} \varphi$, where φ is ψ^- , by the definition in 1° of Section 23. We conclude that $p_1 \Vdash \varphi[G]$ by Theorem 10, that is, $p_1 \Vdash \neg\psi[G]$. However, this contradicts the assumption that $p \Vdash \varphi[G]$.

Part 2: the step $\mathcal{L}\Pi_n^1 \rightarrow \mathcal{L}\Sigma_{n+1}^1$ ($n \geq 1$). Let $\varphi(x)$ be a $\mathcal{L}\Pi_n^1$ formula. Suppose that $p \text{ forc } \exists x \varphi(x)$. Then, by definition, we have $p \text{ forc } \varphi(c)$ for a small real name c . The inductive hypothesis implies $p \Vdash \varphi(c)[G]$; hence, $p \Vdash \exists x \varphi(x)[G]$. To prove the converse, let $p \Vdash \exists x \varphi(x)[G]$. As \mathbb{P} is CCC, there exists a small real name c (in **L**) satisfying $p \Vdash \varphi(c)[G]$. Then, $p \text{ forc } \varphi(c)$ by the inductive hypothesis; therefore, $p \text{ forc } \exists x \varphi(x)$.

Part 3: the step $\mathcal{L}\Sigma_n^1 \rightarrow \mathcal{L}\Pi_n^1$ ($2 \leq n \leq \aleph - 2$). Let φ be a closed $\mathcal{L}\Sigma_n^1$ formula, and $p \text{ forc } \varphi^-$. Lemma 33(iv) implies that no multitree $q \in \mathbb{P}$, $q \leq p$, satisfies $q \text{ forc } \varphi$. We conclude that $p \Vdash \varphi^- [G]$ by the inductive hypothesis.

Conversely, let $p \Vdash \varphi^- [G]$. There is an ordinal $\lambda < \omega_1$ such that every name in φ is $\mathbb{P}_{<\lambda}$ -complete, where $\mathbb{P}_{<\lambda} = \bigcup_{\alpha < \lambda} \mathbb{P}_\alpha$. Consider the set U of all sequences $\vec{\pi} \in \overline{\mathbf{MF}}$, of successor length $\text{dom}(\vec{\pi}) = \kappa + 1$, such that $\kappa > \lambda$ and there is a multitree $q \in \mathbf{MT}(\vec{\pi})$, $q \leq p$, such that $q \text{ forc}_{c_{\vec{\pi}(\kappa)}} \varphi$. Then, U is a Σ_{n-1} (HC) set (defined with φ, p_0 as parameters) by Lemma 28; hence, U belongs to $\Sigma_{\aleph-3}$ (HC). (Recall that $\aleph \geq 4$ by Assumption 1).

Therefore, using Theorem 6 (and (C) of Definition 23) there is an ordinal $\beta < \omega_1$ such that the restricted sequence $\vec{\pi} \upharpoonright \beta$ blocks U .

Case 1: $\vec{\pi} \upharpoonright \beta \in U$, so that $\beta = \kappa + 1$, $\kappa > \lambda$, and there is a multitree $q \in \mathbf{MT}(\vec{\pi} \upharpoonright \beta)$ satisfying $q \leq p$ and $q \text{ forc}_{\mathbb{P}_\kappa} \varphi$. Then, $q \Vdash \varphi[\underline{G}]$ holds by the inductive hypothesis, which contradicts the choice of p . We conclude that Case 1 is impossible.

Case 2: no sequence in U extends $\vec{\pi} \upharpoonright \beta$. We can assume that $\beta > \lambda$ and β is a successor, $\beta = \kappa + 1$. (If not, replace β by $\max\{\beta + 1, \lambda + 1\}$). We assert that $p \text{ forc}_\kappa \varphi^-$. Indeed, otherwise, 4° implies that there exists a special multiforcing ϱ and a multitree $q \in \mathbf{MT}(\varrho)$, satisfying $\mathbb{P}_\kappa \sqsubset \varrho$, $q \leq p$, and $q \text{ forc}_\varrho \varphi$. However, then the extended sequence $\vec{\varrho} = (\vec{\pi} \upharpoonright \beta) \hat{\ } \varrho$ belongs to U . However, $\vec{\pi} \upharpoonright \beta \subseteq \vec{\pi}$, which contradicts the Case 2 assumption. Thus, $p \text{ forc } \varphi^-$, as required. \square

27. Permutation Invariance

The theory of forcing admits various invariance theorems. Theorem 12 is related to the invariance of the auxiliary forcing under permutations.

We argue in L. Let PERM be the set of all bijections $h : \omega_1 \xrightarrow{\text{onto}} \omega_1$, satisfying $h = h^{-1}$ and such that the *non-identity domain* $\mathbf{NI}(h) = \{\xi : h(\xi) \neq \xi\}$ is at most countable. Bijections in PERM will be called *permutations*.

We extend the action of any $h \in \text{PERM}$ as follows:

- If p is a multitree then hp is a multitree, $|hp| = h''|p| = \{h(\xi) : \xi \in |p|\}$, and $(hp)(h(\xi)) = p(\xi)$ for each $\xi \in |p|$;
- If $\pi \in \mathbf{MT}$ is a multiforcing then accordingly $h \cdot \pi = \pi \circ (h^{-1})$ is a multiforcing, $|h \cdot \pi| = h''|\pi|$ and $(h \cdot \pi)(h(\xi)) = \pi(\xi)$ for each $\xi \in |\pi|$;
- If $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times \omega)$ is a real name, then we define $hc = \{\langle hp, n, i \rangle : \langle p, n, i \rangle \in \mathbf{c}\}$, so that hc is a real name as well;
- If $\vec{\pi} = \langle \pi_\alpha \rangle_{\alpha < \kappa} \in \overline{\mathbf{MF}}$, then $h\vec{\pi} = \langle h \cdot \pi_\alpha \rangle_{\alpha < \kappa}$, this is still a sequence in $\overline{\mathbf{MF}}$;
- If $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ is a \mathcal{L} -formula (all names indicated), then $h\varphi$ is $\varphi(h\mathbf{c}_1, \dots, h\mathbf{c}_n)$.

Many notions and relations defined above are rather obviously PERM-invariant. Thus, $p \in \mathbf{MT}(\pi)$ if $hp \in \mathbf{MT}(h \cdot \pi)$, $\pi \sqsubset^{\text{old}} \varphi$ if $h \cdot \pi \sqsubset^{\text{old}} h \cdot \varphi$, et cetera. As the next lemma shows, the invariance also holds with respect to the relation *forc*. An obvious proof by induction on n is left to the reader. (See Theorem 24.1 in [18]).

Theorem 12. Assume that $h \in \text{PERM}$, π is a small regular multiforcing, $p \in \mathbf{MT}$, $n \geq 1$, and a closed formula φ belongs to $\mathcal{L}\Pi_n^1 \cup \mathcal{L}\Sigma_{n+1}^1$. Then, $p \text{ forc}_\kappa \varphi$ if $(hp) \text{ forc}_{h\pi} (h\varphi)$.

28. Embedding Multiforcings in the Key Sequence

The following lemma proves that any special multiforcing admits a suitable embedding into the key sequence $\vec{\pi} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1}$, due to the generic properties of the latter.

In L, if $\vartheta < \omega_1$, then we define shift permutations $h_1[\vartheta], h_2[\vartheta] \in \text{PERM}$ so that

$$\mathbf{NI}(h_1[\vartheta]) = [0, \vartheta \cdot 2) \text{ and } h_1[\vartheta](\xi) = h_1[\vartheta]^{-1}(\xi) = \vartheta + \xi \text{ for all } \xi < \vartheta,$$

$$\mathbf{NI}(h_2[\vartheta]) = [0, \vartheta) \cup [\vartheta \cdot 2, \vartheta \cdot 3) \text{ and } h_2[\vartheta](\xi) = h_2[\vartheta]^{-1}(\xi) = \vartheta \cdot 2 + \xi \text{ for } \xi < \vartheta,$$

where as usual, $\vartheta \cdot 2 = \vartheta + \vartheta$ and $\vartheta \cdot 3 = \vartheta + \vartheta + \vartheta$. In other words, $h_1[\vartheta], h_2[\vartheta]$ are order-preserving shifts between $[0, \vartheta)$ and resp. $[\vartheta, \vartheta \cdot 2), [\vartheta \cdot 2, \vartheta \cdot 3)$.

Lemma 34 (in L). Let σ be a special multiforcing, $\alpha < \omega$. There exist a special multiforcing ϱ , and ordinals $\nu, \theta < \omega_1$, such that $\sigma \sqsubset \varrho$, $\alpha < \min\{\nu, \theta\}$, $|\varrho| \subseteq \theta$, and the multiforcings $\varrho_1 = h_1[\theta] \cdot \varrho$, $\varrho_2 = h_2[\theta] \cdot \varrho$ satisfy $\varrho_1, \varrho_2 \subseteq \mathbb{P}_\nu$, i.e., $\varrho_1 = \mathbb{P}_\nu \upharpoonright |\varrho_1|$ and $\varrho_2 = \mathbb{P}_\nu \upharpoonright |\varrho_2|$.

Under the conditions of the lemma, it follows by the definition of $h_1[\theta]$ and $h_2[\theta]$ above that $|\varrho_1| \subseteq [\theta, \theta \cdot 2)$ and $|\varrho_2| \subseteq [\theta \cdot 2, \theta \cdot 3)$. This lemma will have two applications

below. One of them (the proof of Lemma 35) will really need the effect of both $h_1[\theta]$ and $h_2[\theta]$. The other one (Section 30) will involve only $h_1[\theta]$.

Proof. We argue in L. First of all, fix any special multiforcing ϱ satisfying $\sigma \sqsubset \varrho$ and $|\varrho| = |\sigma|$. Let U be the set of all sequences $\vec{\pi} \in \overline{\mathbf{MF}}$ such that:

- (+) there are ordinals $\nu < \text{dom}(\vec{\pi}), \theta < \omega_1$ $|\varrho| \subseteq [0, \theta)$, and the shifted multiforcings $\varrho_1 = h_1[\theta] \cdot \varrho, \varrho_2 = h_2[\theta] \cdot \varrho$ satisfy $\varrho_1, \varrho_2 \subseteq \mathbb{P}_\nu$.

Easily, U is a $\Sigma_1(\text{HC})$ set (with ϱ as the only parameter of the Σ_1 definition in HC); therefore, a $\Sigma_{n-3}(\text{HC})$ set, because $n \geq 4$ by Assumption 1. It follows by Definition 23(C) that there exists an ordinal $\nu < \omega_1$ such that $\vec{\pi} \upharpoonright \nu$ blocks U . We can w.l.o.g. assume that $\nu = \gamma + 1$ is a successor; otherwise, just substitute $\nu + 1$ for ν . We put $\pi = \mathbb{P}_\gamma$.

Case 1: no sequence in U extends $\vec{\pi} \upharpoonright \nu$. To demonstrate that this is inconsistent, let $\theta < \omega_1$ be the least ordinal satisfying $|\pi| \cup |\varrho| \subseteq [0, \theta)$. Let π' be any special multiforcing satisfying $\pi \sqsubset \pi'$ and still $|\pi'| = |\pi| \subseteq [0, \theta)$.

We define $\varrho_1 = h_1[\theta] \cdot \varrho, \varrho_2 = h_2[\theta] \cdot \varrho$. Thus, $\pi, \varrho_1, \varrho_2$ are special multiforcings with disjoint domains $|\pi'| \subseteq [0, \theta), |\varrho_1| \subseteq [\theta, \theta \cdot 2), |\varrho_2| \subseteq [\theta \cdot 2, \theta \cdot 3)$. It follows that the simple union $\sigma = \pi' \cup \varrho_1 \cup \varrho_2$ is still a multiforcing, and by the way $\pi \sqsubset \sigma$ since $\pi \sqsubset \pi'$. It follows that the extended sequence $\vec{\pi} = (\vec{\pi} \upharpoonright \nu) \hat{\ } \sigma$ belongs to $\overline{\mathbf{MF}}$ and $(\vec{\pi} \upharpoonright \nu) \subset \vec{\pi}, \text{dom}(\vec{\pi}) = \nu + 1$, and $\vec{\pi}(\nu) = \sigma = \pi' \cup \varrho_1 \cup \varrho_2$, so that $\varrho_1, \varrho_2 \subseteq \vec{\pi}(\nu)$.

We conclude that $\vec{\pi} \in U$, but this contradicts the Case 1 assumption.

Case 2: $\vec{\pi} = \vec{\pi} \upharpoonright \lambda \in U$. Let this be witnessed by an ordinal $\nu < \lambda$ and θ, ϱ as in (+). Then, $\varrho_1 = h_1[\theta] \cdot \varrho, \varrho_2 = h_2[\theta] \cdot \varrho$ satisfy $\varrho_1, \varrho_2 \subseteq \vec{\pi}(\nu) = \mathbb{P}_\nu$, and hence, this accomplishes the proof of the lemma. \square

29. The Non-Well-Orderability Claim, Part I

Here, we begin the proof of Theorem 1 in part (ii). It will be completed in the end of Section 30. We are going to establish the following even somewhat stronger result.

Theorem 13. Assume that a set $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over \mathbf{L} . Then, in $\mathbf{L}[G]$, there is no Σ^1_{n-1} well-orderings of the reals, and moreover, no Σ^1_{n-1} relation well-orders the set $\{x_\xi[G] : \xi < \omega^1_\lambda\}$.

Our plan is to infer a contradiction from the next assumption contrary to Theorem 13.

Assumption 2. Assume that $\Phi(x, y)$ is a Σ^1_{n-1} parameter-free formula, $\alpha < \omega_1, \mathbb{p} \in \mathbb{P}_\alpha = \mathbf{MT}(\mathbb{P}_\alpha)$, and \mathbb{p} \mathbb{P} -forces, over \mathbf{L} , that “the relation $<_\Phi$, defined by $x <_\Phi y$ if $\Phi(x, y)$, strictly well-orders the set $\{x_\xi[G] : \xi < \omega^1_\lambda\}$ in $\mathbf{L}[G]$ ”.

We begin with the next lemma. See Example 4 regarding real names of the form \dot{x}_ξ .

Lemma 35 (in L). Under Assumption 2, suppose that σ is a small regular multiforcing, $\xi \in |\sigma|$, and $s \in \mathbf{MT}(\sigma)$. Then, there exists a special multiforcing π , a multitree $p \in \mathbf{MT}(\pi)$, and an ordinal $\eta \in |\pi|$, such that $\sigma \sqsubset \pi, p \leq s$, and $p \text{ forc}_\pi \Phi(\dot{x}_\eta, \dot{x}_\xi)$.

Proof. We argue in \mathbf{L} . We recall that $\mathbb{p} \in \mathbb{P}_\alpha = \mathbf{MT}(\mathbb{P}_\alpha)$. By Lemma 34, there exist a special multiforcing ϱ , and ordinals $\nu, \theta < \omega_1$, such that $\sigma \sqsubset \varrho, \alpha < \nu, |\varrho| \cup |\mathbb{p}| \subseteq \theta$, and the shifted multiforcings $\varrho_1 = h_1[\theta] \cdot \varrho, \varrho_2 = h_2[\theta] \cdot \varrho$ satisfy $\varrho_1, \varrho_2 \subseteq \mathbb{P}_\nu$.

Let $\xi_1 = h_1[\theta](\xi), \xi_2 = h_2[\theta](\xi)$. Pick a multitree $q \in \mathbf{MT}(\varrho)$ with $q \leq s$. Then

$$q_1 = h_1[\theta]q \in \mathbf{MT}(\varrho_1) \quad \text{and} \quad q_2 = h_2[\theta]q \in \mathbf{MT}(\varrho_2),$$

hence, $q_1, q_2 \in \mathbb{P}_\nu$. We observe that $q_1 \leq s_1 = h_1[\theta]s$ and $q_2 \leq s_2 = h_2[\theta]s$.

Note that the sets $|\mathbb{p}|, |q_1|, |q_2|$ are subsets of the pairwise disjoint intervals resp. $[0, \theta), [\theta, \theta \cdot 2), [\theta \cdot 2, \theta \cdot 3)$. Then, the simple union $p' = \mathbb{p} \cup q_1 \cup q_2$ is still a multitree in $\mathbb{P}_{\leq \nu}$ stronger than each of \mathbb{p}, q_1, q_2 .

Then, by the choice of \mathbb{p} , there exists a condition $\mathbf{p}'' \in \mathbb{P}$, $\mathbf{p}'' \leq \mathbf{p}'$, which either \mathbb{P} -forces $\Phi(x_{\xi_1}[\mathbb{G}], x_{\xi_2}[\mathbb{G}])$ or \mathbb{P} -forces $\Phi(x_{\xi_2}[\mathbb{G}], x_{\xi_1}[\mathbb{G}])$. Let \mathbf{p}'' \mathbb{P} -force say $\Phi(x_{\xi_2}[\mathbb{G}], x_{\xi_1}[\mathbb{G}])$ over \mathbf{L} . We can assume that $\mathbf{p}'' \in \mathbf{MT}(\mathbb{P}_\mu)$ for some $\mu < \omega_1$, satisfying $\mu \geq \nu + 2$ and $[0, \theta \cdot 3] \subseteq |\mathbb{P}_\mu|$. Using Theorem 11, we have

$$(1) \quad \mathbf{p}'' \text{ forc}_{\mathbb{P}_\lambda} \Phi(\dot{x}_{\xi_2}, \dot{x}_{\xi_1}), \text{ for some } \lambda < \omega_1.$$

Here, λ can be chosen large enough for $\lambda > \mu$ by Lemma 29. Then, there exists a multitree $\mathbf{r} \in \mathbf{MT}(\mathbb{P}_\lambda) = \mathbb{P}_\lambda$ satisfying $\mathbf{r} \leq \mathbf{p}''$ and $|\mathbf{r}| = |\mathbf{p}''|$. Lemma 29 implies:

$$(2) \quad \mathbf{r} \text{ forc}_{\mathbb{P}_\lambda} \Phi(\dot{x}_{\xi_2}, \dot{x}_{\xi_1}),$$

and then acting by $\mathbf{h}_1[\theta] = (\mathbf{h}_1[\theta])^{-1}$ on (2), we obtain by Theorem 12:

$$(3) \quad \mathbf{p} \text{ forc}_\pi \Phi(\dot{x}_{\xi_2}, \dot{x}_{\xi_1}) \text{—by Theorem 12,}$$

where $\pi = \mathbf{h}_1[\theta] \mathbb{P}_\lambda$ and $\mathbf{p} = \mathbf{h}_1[\theta] \mathbf{r} \in \mathbf{MT}(\pi)$ because $\mathbf{h}_1[\theta](\xi_1) = (\mathbf{h}_1[\theta])^{-1}(\xi_1) = \xi$.

It remains to be observed that $\sigma \sqsubset \varrho$; hence, $\sigma_1 \sqsubset \varrho_1$, and further $\sigma_1 \sqsubset \mathbb{P}_\nu$ because $\varrho_1 \subseteq \mathbb{P}_\nu$ by construction. Therefore, $\sigma_1 \sqsubset \mathbb{P}_\lambda$ since $\nu < \lambda$. Acting by $\mathbf{h}_1[\theta] = \mathbf{h}_1[\theta]^{-1}$, we obtain $\sigma = \mathbf{h}_1[\theta] \sigma_1 \sqsubset \pi$. We similarly obtain $\mathbf{p} \leq \mathbf{s} = \mathbf{h}_1[\theta] \mathbf{s}_1$ because $\mathbf{r} \leq \mathbf{p}'' \leq \mathbf{p}' \leq \mathbf{q}_1 \leq \mathbf{s}_1$ by construction. This ends the proof of the lemma, with $\eta = \xi_2$. \square

The goal of the next lemma is to strengthen Lemma 35 to the effect that a whole dense set of conditions with the same property will be obtained.

Lemma 36 (in \mathbf{L}). *Under Assumption 2, there is a sequence $\vec{\pi} = \langle \pi_k \rangle_{k < \omega} \in \overrightarrow{\mathbf{MF}}$ satisfying the following for all $\xi \in |\vec{\pi}|$:*

- (i) *If $\mathbf{q} \in \mathbf{MT}(\pi)$, where $\pi = \bigcup^{\text{cw}} \vec{\pi} = \bigcup_{k < \omega} \pi_k$, then there is a condition $\mathbf{p} \in \mathbf{MT}(\pi)$, $\mathbf{p} \leq \mathbf{q}$, and an ordinal $\eta \in |\vec{\pi}|$, such that $\mathbf{p} \text{ forc}_{\vec{\pi}} \Phi(\dot{x}_\eta, \dot{x}_\xi)$;*
- (ii) *The set $D_\xi[\pi] = \{ \mathbf{p} \in \mathbf{MT}(\pi) : \exists \eta \in |\vec{\pi}| \exists k < \omega (\mathbf{p} \text{ forc}_{\pi_k} \Phi(\dot{x}_\eta, \dot{x}_\xi)) \}$ is dense (then in fact open dense by Lemma 29) in $\mathbf{MT}(\pi)$.*

Proof. (i) Using Lemma 34, we define π_k by induction so that for each k there is a certain pair of $\xi_k \in |\pi_k|$ and $\mathbf{q}_k \in \mathbf{MT}(\pi_k)$, satisfying:

$$\exists \mathbf{p}_k \in \mathbf{MT}(\pi_{k+1}) \exists \eta_k \in |\pi_{k+1}| (\mathbf{p}_k \leq \mathbf{q}_k \wedge \mathbf{p}_k \text{ forc}_{\pi_{k+1}} \Phi(\dot{x}_{\eta_k}, \dot{x}_{\xi_k})).$$

Moreover, the enumeration by ξ_k and \mathbf{q}_k can be arranged so that for each ordinal $\xi \in |\vec{\pi}|$ and condition $\mathbf{q} \in \mathbf{MT}(\pi)$ there exists k such that $\xi_k = \xi$ and $\mathbf{q}_k = \mathbf{q}$. However, $\vec{\pi}$ is as required. Claim (ii) is just a reformulation of (i). \square

Corollary 15 (in \mathbf{L}). *Under Assumption 2, let $\vec{\pi} \in \overrightarrow{\mathbf{MF}}_\omega$ satisfy (ii) of Lemma 36: if $\xi \in |\pi|$ then $D_\xi[\pi]$ is open dense in $\mathbf{MT}(\pi)$, where $\pi = \bigcup^{\text{cw}} \vec{\pi}$. There is a special multiforcing ϱ , such that $\pi_k \sqsubset \varrho$ for each k , and each set $D_\xi[\pi] \upharpoonright \varrho$, $\xi \in |\vec{\pi}|$, is sealed dense in $\mathbf{MT}(\varrho)$.*

Proof. Consider the (countable) collection of all sets $D_\xi[\pi]$, $\xi \in |\pi|$. By Lemma 21(i), there exists an extension $\vec{\varrho} \in \overrightarrow{\mathbf{MF}}_{\omega+1}$ of $\vec{\pi}$, by the rightmost term $\varrho = \vec{\varrho}(\omega)$, satisfying $\pi \sqsubset_{D_\xi[\pi]} \varrho$ for all $\xi \in |\pi|$. Then, each set $D_\xi[\pi] \upharpoonright \varrho$ is sealed dense in ϱ by Lemma 10(iii). \square

30. The Non-Well-Orderability Claim, Part II

Still arguing under the conditions of Assumption 2, we proceed with the following construction.

- (I) Pick $\vec{\pi} = \langle \pi_k \rangle_{k < \omega} \in \overrightarrow{\mathbf{MF}}_\omega$ by Lemma 36, so that

$$D_\xi[\pi] = \{ \mathbf{p} \in \mathbf{MT}(\pi) : \exists \eta \in |\pi| \exists k (\mathbf{p} \text{ forc}_{\pi_k} \Phi(\dot{x}_\eta, \dot{x}_\xi)) \}$$

is an open dense set in $\mathbf{MT}(\pi)$ for any $\xi \in |\pi|$, where $\pi = \bigcup^{\text{cw}} \vec{\pi} = \bigcup_{k < \omega} \pi_k$.

- (II) Then, pick a special multforcing φ by Corollary 15, so that $\pi_k \sqsubset \varphi$ for each k , and if $\zeta \in |\pi|$ then $\pi \sqsubset_{D_\zeta[\pi]} \varphi$, and hence, the set $D_\zeta[\vec{\pi}] \uparrow \varphi$ is sealed dense in $\mathbf{MT}(\varphi)$.
- (III) By Lemma 34, there exist a special multforcing χ , and ordinals $\nu, \theta < \omega_1$, such that $\varphi \sqsubset \chi$, $|\chi| \subseteq \theta$, and the multforcing $\chi_* = h_1[\theta] \cdot \chi$ satisfies $\chi_* \subseteq \mathbb{1}_\nu$. Accordingly, we let $\varphi_* = h_1[\theta] \cdot \varphi$, $\vec{\pi}_* = h_1[\theta] \cdot \vec{\pi}$, $\pi_* = h_1[\theta] \cdot \pi$, $\pi_{*k} = h_1[\theta] \cdot \pi_k$, so that $\pi_{*k} \sqsubset \varphi_* \sqsubset \chi_*$ for all k , and $\pi_* \sqsubset \varphi_* \sqsubset \chi_*$. If $\zeta' \in |\pi_*|$ then put

$$D_{\zeta'}[\pi_*] = \{p' \in \mathbf{MT}(\pi_*) : \exists \eta' \in |\pi_*| \exists k < \omega (p' \text{ forc}_{\pi_{*k}} \Phi(\dot{x}_{\eta'}, \dot{x}_{\zeta'}))\}.$$

Lemma 37. *If $\zeta' \in |\pi_*|$ and τ_* is a special multforcing, $\chi_* \sqsubset \tau$, then the set $D_{\zeta'}[\pi_*] \uparrow \tau$ is sealed dense and open dense in $\mathbf{MT}(\tau)$.*

Proof. Let us make use of the action of $h_1[\theta]$ on (II) above. We assert that

$$(*) \quad D_{\zeta'}[\pi_*] = h_1[\theta] \text{''} D_\zeta[\pi] := \{h_1[\theta] \cdot p : p \in D_\zeta[\pi]\}.$$

Indeed, suppose that $\zeta \in |\pi|$ and $p \in D_\zeta[\pi]$ show that $p' = h_1[\theta] \cdot p \in D_{\zeta'}[\pi_*]$, where accordingly $\zeta' = h_1[\theta](\zeta)$. By definition there is an ordinal $\eta \in |\vec{\pi}|$, such that $p \text{ forc}_{\pi_k} \Phi(\dot{x}_\eta, \dot{x}_\zeta)$. Then, $p' \text{ forc}_{\pi_{*k}} \Phi(\dot{x}_{\eta'}, \dot{x}_{\zeta'})$ by Theorem 12, where $\eta' = h_1[\theta](\eta)$. This completes the proof of (*) from right to left. The inverse implication is similar.

Now, it follows from (*) and (II) that each set $D_{\zeta'}[\pi_*] \uparrow \varphi_*$, where $\zeta' \in |\pi_*|$ is sealed dense in $\mathbf{MT}(\varphi_*)$. However, we have $\varphi_* \sqsubset \chi_* \sqsubset \tau$. It follows that the set $D_{\zeta'}[\pi_*] \uparrow \tau$ is sealed dense and open dense in $\mathbf{MT}(\tau)$ simply because $D_{\zeta'}[\pi_*] \uparrow \varphi_* \uparrow \tau \subseteq D_{\zeta'}[\pi_*] \uparrow \tau$, whereas $D_{\zeta'}[\pi_*] \uparrow \varphi_* \uparrow \tau$ is sealed dense and open dense in $\mathbf{MT}(\tau)$ by Lemma 10(ii),(iii). \square

We now obtain a related the pre-density result in the context of the the key sequence $\vec{\pi}$. Recall that $p \text{ forc } \varphi$ means that $p \text{ forc}_{\mathbb{1}_\alpha} \varphi$ holds for some $\alpha < \omega_1$. (See Definition 28).

Lemma 38. *If $\zeta' \in |\pi_*|$, then the following set is open dense in $\mathbb{P} = \mathbf{MT}(\vec{\pi})$:*

$$D_{\zeta'} = \{s \in \mathbb{P} : \exists \eta' \in |\pi_*| (s \text{ forc } \Phi(\dot{x}_{\eta'}, \dot{x}_{\zeta'}))\}.$$

Proof. The openness holds by Lemma 29. To show the density, pick any $p_0 \in \mathbb{P}$. The goal is to find an ordinal $\eta' \in |\pi_*|$, and a multitree $s \in \mathbf{MT}(\vec{\pi}) = \mathbb{P}$ such that $s \leq p_0$ and $s \text{ forc } \Phi(\dot{x}_{\eta'}, \dot{x}_{\zeta'})$. As $\vec{\pi}$ is \sqsubset -increasing, there is an ordinal $\gamma > \nu + 1$ and a stronger multitree $p_1 \in \mathbf{MT}(\mathbb{1}_\gamma)$, $p_1 \leq p_0$. It follows from (III) that $\chi_* \subseteq \mathbb{1}_\nu \sqsubset \mathbb{1}_{\nu+1}$; hence, $\chi_* \sqsubset \mathbb{1}_{\nu+1}$, which implies $\chi_* \sqsubset \mathbb{1}_{\gamma'}$ for each $\gamma' \geq \nu + 1$; therefore, $\chi_* \sqsubset \mathbb{1}_\gamma$ for the ordinal γ chosen just above. However, the set $D_{\zeta'}[\pi_*] \uparrow \mathbb{1}_\gamma$ is open dense in $\mathbf{MT}(\mathbb{1}_\gamma)$ by Lemma 37 (with $\tau = \mathbb{1}_\gamma$). It follows that there is a multitree $q \in D_{\zeta'}[\pi_*] \uparrow \mathbb{1}_\gamma$, $q \leq p_1 \leq p$.

Then, $p_1 \leq r$, where $r \in D_{\zeta'}[\pi_*]$, and hence, $r \text{ forc}_{\pi_{*k}} \Phi(\dot{x}_{\eta'}, \dot{x}_{\zeta'})$ for some $\eta' \in |\pi_*|$ and k . We conclude that then $q \text{ forc}_{\mathbb{1}_\gamma} \Phi(\dot{x}_{\eta'}, \dot{x}_{\zeta'})$ by Lemma 29; thus, $q \in D_{\zeta'}$, as required. \square

Let \Vdash be the \mathbb{P} -forcing relation over \mathbf{L} . It essentially coincides with forc by Proposition 11. Therefore, the lemma implies the following corollary.

Corollary 16. *If $\zeta' \in |\vec{\pi}_*|$ then the following set is open dense in \mathbb{P} :*

$$D_{\zeta'} = \{s \in \mathbf{MT}(\vec{\pi}) : \exists \eta' \in |\vec{\pi}_*| (s \Vdash \Phi(\dot{x}_{\eta'}[G], \dot{x}_{\zeta'}[G]))\}.$$

Proof of Theorem 13. It follows from Corollary 16 that if G is \mathbb{P} -generic over \mathbf{L} , then the set $\{x_{\zeta'}[G] : \zeta' \in |\pi_*|\}$ contains no $<_\Phi$ -minimal element, which contradicts Assumption 2. The contradiction negates Assumption 2 and thereby proves Theorem 13. \square

Combining Theorems 13 and 9, we complete the proof of Theorem 1.

Part V: Final

This final Part contains Section 31 with a short proof of Theorem 2 and a brief discussion of its possible reduction to a theory weaker than $ZFC^- + \mathcal{P}(\omega)$ exists'. We finish in Section 32 with conclusions and problems.

31. Proof of Theorem 2 and Comments

First of all, we recall that ZFC^- is a subtheory of ZFC obtained as follows:

- (a) We exclude the Power Set axiom **PS**;
- (b) The *well-orderability axiom* **WA**, which claims that every set can be well-ordered, is substituted for the usual set-theoretic Axiom of Choice **AC** of ZFC ;
- (c) The Separation schema is preserved, but the Replacement schema (which happens to be not sufficiently strong in the absence of **PS**) is substituted with the *Collection* schema: $\forall X \exists Y \forall x \in X (\exists y \Phi(x, y) \implies \exists y \in Y \Phi(x, y))$.

A comprehensive account of main features of ZFC^- is given in e.g., [27–29].

Proof of Theorem 2. Arguing in ZFC^- , let us drop to the subuniverse L^- of all constructible sets in the ZFC^- universe of discourse. Then, L^- satisfies ZFC^- too, and if $\mathcal{P}(\omega)$ exists then $\mathcal{P}(\omega) \cap L^- \in L^-$ exists in L^- . Thus, instead of $ZFC^- + \mathcal{P}(\omega)$ exists', **we argue in the theory $ZFC^- + (V = L) + \mathcal{P}(\omega)$ exists'**, whose universe is L^- .

Now, the existence of the power set $\mathcal{P}(\omega) = \{X : X \subseteq \omega\}$ leads to the existence of sets such as ω_1 and $HC = L_{\omega_1}$, and basically, the existence of all sets involved in the construction of the key forcing notion \mathbb{P} (including \mathbb{P} itself). After this remark, all arguments in the proof of Theorem 1 in Parts I, II, III, and IV above naturally go through, giving the proof of Theorem 2 by means of a \mathbb{P} -generic extension of L^- . □

It is really interesting to further reduce the assumptions of Theorem 2 down to PA_2 (see [20,30,31] and elsewhere on second-order Peano arithmetic PA_2) or ZFC^- without the extra assumption of the existence of $\mathcal{P}(\omega)$, or the associated class theory GBc^- , which is formalized in a two-sorted language with separate variables and quantifiers for sets and classes, so that lower-case letters are used for set variables, whereas upper-case letters are used for class variables. The minus $-$ still reflects the absence of the Power Set axiom. The axiomatization of GBc^- (see e.g., [29]) includes axioms for sets (exactly those of ZFC^-) and those for classes. In particular, (1) extensionality for classes; (2) the class replacement axiom asserting that every class function restricted to a set is a set; and (3) a predicative comprehension schema asserting that every collection of sets, definable by a formula with no quantified class variables, is a class.

Theories PA_2 , ZFC^- , and GBc^- have been known to be equiconsistent for a while, see e.g., [20,30,31] for PA_2 vs. ZFC^- , and [32–34] for ZFC^- vs. GBc^- .

Such objects as ω_1 and HC are legitimate classes in GBc^- , and such are all ZFC^- -sets that play any role in the proof of Theorem 1 above, with one notable exception. The exceptional case being the $\Delta_1^{HC} \diamond_{\omega_1}$ -sequence used in Lemma 24. The ZFC construction of such a sequence (as e.g., in [24]) can be maintained as a proper class in $GBc^- + \text{'all sets are constructible'}$ as well as in $ZFC + (V = L)$. However, unfortunately, the proof of the \diamond_{ω_1} -property of the resulting sequence does not go through in GBc^- because the ZFC proof involves ordinals beyond ω_1 , and hence, does not directly translate to the GBc^- level. This will be the subject of our forthcoming paper aimed at solving this technical obstacle by means of recently discovered methods as, e.g., in [35,36].

32. Conclusions and Problems

In this study, the method of finite-support products of Jensen's forcing was applied to the problem of obtaining a model of ZFC in which, for a given $n \geq 3$, there is a Δ_n^1 -good well-ordering of the reals, but no well-orderings of the reals exist in the class Δ_{n-1}^1 at the preceding level of the hierarchy. This is achieved by Theorem 1, our **first main result**. We also demonstrate that this theorem can be obtained on the basis of the consistency of ZFC^-

(i.e., **ZFC** sans the Power Set axiom) plus the claim that $\mathcal{P}(\omega)$ exists, which is a much weaker assumption than the consistency of **ZFC** usually assumed in such independence results obtained by forcing method. This is achieved by Theorem 2, our **second main result**. Two **principal technical achievements**, related to getting rid of countable models of **ZFC**[−] as a technical tool and according treatment of the auxiliary forcing, were mentioned in Section 2. These are new results in such a generality (with $n \geq 3$ arbitrary), and valuable improvements upon our earlier results in [1]. They may lead to further progress in studies of the projective hierarchy.

From our study, it is concluded that the technique of *definable generic* inductive construction of forcing notions in **L** that carry *hidden automorphisms*, developed for Jensen-type product forcing in our earlier papers [17,18,21], succeeds to solve other important descriptive set theoretic problems of the same kind, using Theorems 1 and 2.

These results (Theorems 1 and 2) continue the series of recent research such as a model [37] in which there is Π_n^1 real singleton $\{a\}$ that codes a cofinal map $f : \omega \rightarrow \omega_1^L$, while every Σ_n^1 set $X \subseteq \omega$ is constructible, and hence, cannot code a cofinal map $\omega \rightarrow \omega_1^L$, and another model [38], in which there is a non-ROD-uniformizable Π_n^1 set with countable cross-sections, while all Σ_n^1 sets with countable cross-sections are Δ_{n+1}^1 -uniformizable—in addition to the research already mentioned in Section 2 above.

This study may also be a contribution to the search for solutions of several similar and still open problems related to the projective hierarchy, such as separation of the countable **AC** at different levels of the projective hierarchy, a similar problem for the principle **DC** of dependent choices, and a critically significant problem posed by S. D. Friedman in ([39], p. 209) and ([40], p. 602): assuming the consistency of an inaccessible cardinal, find a model for a given n in which all Σ_n^1 sets of reals are Lebesgue measurable and have the Baire and perfect set properties, while there is a Δ_{n+1}^1 well-ordering of the reals.

From the result of Theorem 1, the following more concrete problems arise.

Problem 1. *Prove that it is true in the key model $\mathbf{L}[G]$ of Section 20 that there is no boldface Δ_{n-1}^1 well-ordering of the reals.*

The *boldface* specification means that the real parameters are allowed in the definitions of pointsets, whereas they are not allowed in the *lightface* case. This is a principal difference.

Problem 2. *Prove a version of Theorem 1 with the additional requirement that the negation $2^{\aleph_0} > \aleph_1$ of the continuum hypothesis holds in the generic extension considered.*

The model for Theorem 1 introduced in Section 20 definitely satisfies the continuum hypothesis $2^{\aleph_0} = \aleph_1$. The problem of obtaining models of **ZFC** in which $2^{\aleph_0} > \aleph_1$ and there is a projective well-ordering of the real line, has been known since the beginning of modern set theory. See, e.g., problem 3214 in an early survey [41] by Mathias. Harrington [42] solved this problem using a generic model in which $2^{\aleph_0} > \aleph_1$ and there is a Δ_3^1 well-ordering of the continuum, using a combination of methods based on such coding forcing notions as the almost-disjoint forcing [43] and a forcing by Jensen and Johnsråten [44]. Such a different forcing notion as the product/iterated Sacks forcing [45,46] may also be of interest here.

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