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On Baire Measurable Homomorphisms of Quotients of the Additive Group of the Reals

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Abstract. The quotient $\mathbb{R}/G$ of the additive group of the reals modulo a countable subgroup G does not admit nontrivial Baire measurable automorphisms.

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0 Introduction

Veličković proved in [6], with reference to earlier technical innovations by Shelah, that the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{fin}$ does not have Baire measurable (BM, in brief)[3)] automorphisms other than the trivial ones, i. e., those generated by bijections between two cofinite subsets of $\mathbb{N}$. (See Farah [2, 3] on more advanced results on automorphisms and homomorphisms of quotients $\mathcal{P}(\mathbb{N})/I$ for different ideals $I \subseteq \mathcal{P}(\mathbb{N})$, in particular, analytic P-ideals.) This raises the problem of the nature of BM automorphisms and homomorphisms of quotients of other similar algebraic structures. We consider this question with respect to quotients of $\mathbb{R}$, the additive group of the reals.

Theorem 1. *Let D, $G \subseteq \mathbb{R}$ be arbitrary subgroups*[4)]*, G being at most countable. Then any Baire measurable homomorphism $\boldsymbol{h} : \mathbb{R}/D \longrightarrow \mathbb{R}/G$ has a lifting of the form $x \longmapsto cx$, where $c \in \mathbb{R}$ satisfies $c \cdot D \subseteq G$.*

(The notion of a lifting and other notions involved will be explained in Section 1.)

Thus, if D, $G \subseteq \mathbb{R}$ are as in the theorem, the only Baire measurable homomorphisms $\mathbb{R}/D \longrightarrow \mathbb{R}/G$ are those generated by maps $x \longmapsto cx$, where a real c satisfies $c \cdot D \subseteq G$. Note that the latter requirement is obviously necessary for $x \longmapsto cx$ to be a lifting of a homomorphism $\mathbb{R}/D \longrightarrow \mathbb{R}/G$. In particular, $\mathbb{R}/\mathbb{Q}$ does not have BM

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3) A map is BM if the pre-image of any open set is an open set modulo a meager set.

4) When speaking of a group $G \subseteq \mathbb{R}$ we always mean a subgroup of the additive group of $\mathbb{R}$.

group automorphisms (and even BM homomorphisms into itself) except for those generated by maps $x \longmapsto cx$, where $c \in \mathbb{Q}$. Another interesting case arises from countable groups G such that $\mathbb{R}/G$ does not admit any BM group automorphism other than the identity, for instance, the group $G = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$, for which there is no $c \neq 1$ such that $c \cdot G = G$, so that $\mathbb{R}/G$ does not admit BM automorphisms other than the identity.

Theorem 1 includes the case when one or both of D, G is the group $\{0\}$. Note that if $G = \{0\}$, then $\mathbb{R}/G$ is $\mathbb{R}$, and it is known classically that the additive group of $\mathbb{R}$ does not admit BM homomorphisms into itself other than $x \longmapsto cx$.

It follows from Theorem 1 that if G is countable while D *un*countable, then there is no BM homomorphism $\mathbb{R}/D \longrightarrow \mathbb{R}/G$. Similarly, there is no BM homomorphism $\mathbb{R}/D \longrightarrow \mathbb{R}$ unless $D = \{0\}$. Another corollary deals with groups $A \subseteq \mathbb{R}^2$, their cross-sections $A_x = \{y : \langle x, y\rangle \in A\}$ and projections $\mathrm{pr}_X A = \{x : A_x \neq \emptyset\}$.

Corollary 2. *Suppose that $A \subseteq \mathbb{R}^2$ is a Borel group, $\mathrm{pr}_X A = \mathbb{R}$, and $Y = A_0$ is countable. Then there is a real c such that $A_x = cx + Y$ for any x.*

Proof. As A is a subgroup of $\mathbb{R}^2$, the map $\boldsymbol{h}(x) = A_x$ is an BM homomorphism $\mathbb{R} \longrightarrow \mathbb{R}/Y$. It remains to apply Theorem 1. □

Suppose that $G \subseteq \mathbb{R}$. We say that a map $H : \mathbb{R} \longrightarrow \mathbb{R}$ is a *G-approximate homomorphism* if $H(x+y) - H(x) - H(y) \in G$ for all x, y.[5)] Then $\boldsymbol{h}(x) = H(x) + G$ is an BM homomorphism $\mathbb{R} \longrightarrow \mathbb{R}/G$, so that, by Theorem 1, we have

Corollary 3. *If $G \subseteq \mathbb{R}$ is a countable group and $H : \mathbb{R} \longrightarrow \mathbb{R}$ an BM G-approximate homomorphism, then there is c such that for all x, $H(x) - cx \in G$.* □

Thus G-approximate homomorphisms are approximable by "true" homomorphisms. It will be demonstrated in Section 6 that Theorem 1, generally speaking, fails in the case when G is an *un*countable Borel subgroup. And, of course, the theorem fails for homomorphisms $\boldsymbol{h}$ which are *not* BM.

The main argument in the proof of Theorem 1 is close to that of [6], but, due to essential differences in the algebraic structure of $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$, especially related to the fact that $\mathbb{R}$ is not a product group, technical details are somewhat different.

1 Homomorphisms and liftings

Every group $G \subseteq \mathbb{R}$ defines the quotient $\mathbb{R}/G$, which consists of cosets $x + G = \{x + g : g \in G\}$ and inherits an abelian (additive) group structure from $\mathbb{R}$, as usual.

If $D, G \subseteq \mathbb{R}$ are groups then a map $\boldsymbol{h} : \mathbb{R}/D \longrightarrow \mathbb{R}/G$ is a *homomorphism* iff $\boldsymbol{h}(X) + \boldsymbol{h}(Y) = \boldsymbol{h}(X + Y)$ for all $X, Y \in \mathbb{R}/D$. In this case, a map $H : \mathbb{R} \longrightarrow \mathbb{R}$ is a *lifting of* $\boldsymbol{h}$ iff $H(x) \in \boldsymbol{h}(x + D)$ for any x.

A map $H : \mathbb{R} \longrightarrow \mathbb{R}$ is *Baire measurable* (BM) iff there is a comeager set $U \subseteq \mathbb{R}$ such that $H \restriction U$ is a Borel function. A homomorphism $\boldsymbol{h}$ is BM iff it has an BM lifting H.

5) One may be interested in another, more numerical notion of approximation. We say that a map $H : \mathbb{R} \longrightarrow \mathbb{R}$ is an *ε-approximate homomorphism* if $|H(x+y) - H(x) - H(y)| < \varepsilon$ for all x, y. Then, for any ε-approximate BM homomorphism $H : \mathbb{R} \longrightarrow \mathbb{R}$ there is $c \in \mathbb{R}$ such that $|H(x) - cx| \leq \varepsilon$ for all x, see Hyers [5]. Farah [4] gives more difficult approximation theorems.

Note that a lifting H of a homomorphism is, by definition, not necessarily a homomorphism itself (i. e., it may not satisfy $H(x+y) = H(x) + H(y)$). Theorem 1 says that, in some cases, if a homomorphism of quotients has an BM lifting, then it has an BM lifting which is a homomorphism, too. Note further that D, the left-hand subgroup in Theorem 1, actually plays little role: indeed, to solve the problem for $\boldsymbol{h} : \mathbb{R}/D \longrightarrow \mathbb{R}/G$ it suffices to apply the same lifting which works for the homomorphism $\boldsymbol{h}'(x) = \boldsymbol{h}(x + D) : \mathbb{R} \longrightarrow \mathbb{R}/G$. After this remark, let us consider a countable group $G \subseteq \mathbb{R}$ and an BM homomorphism $\boldsymbol{h} : \mathbb{R} \longrightarrow \mathbb{R}/G$; our goal will be to find a lifting of $\boldsymbol{h}$ of the form $c \longmapsto cx$.

2 Proof of the Theorem: preliminaries

Let $\mathbb{N}^+ = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \ldots\}$ and let $\mathbb{Q}$ be the set of all rationals, as usual. We shall identify the reals in $\mathbb{T} = \{x \in \mathbb{R} : 0 \le x < 1\}$ with their binary expansions, e. g., reals $x \in \mathbb{T}$ will be sometimes viewed as functions $x : \mathbb{N}^+ \longrightarrow 2 = \{0, 1\}$. Put $\mathrm{D}(x) = \{n : x(n) = 1\}$ for $x \in \mathbb{T}$. If $\mathrm{D}(x)$ is finite, then x is a binary rational number.

For any interval $[k, l)$ in $\mathbb{N}$ let $2^{[k,l)}$ be the set of all functions $s : [k, l) \longrightarrow 2$. Numerically, each $s \in 2^{[k,l)}$ represents the binary rational $\sum_{i=k}^{l-1} 2^{-s(i)}$. For any such a string s define $\mathbb{T}_s = \{x \in \mathbb{T} : s \subset x\}$; this is a subinterval of $\mathbb{T}$.

As $\boldsymbol{h}$ is BM, there is a dense $\boldsymbol{G}_d a$ set $U \subseteq \mathbb{T}$ and a Borel map $H : \mathbb{T} \longrightarrow \mathbb{R}$, continuous on U and satisfying $\boldsymbol{h}(x) = H(x) + G$ for all $x \in U$. Then

(1) $\quad H(y) - H(x) - H(y - x) \in G$ whenever x, y belong to U.

By the choice of U we have $U = \bigcap_n U_n$, where each $U_n \subseteq \mathbb{T}$ is dense open. We can assume that U is $\mathbb{Q}$-invariant, so $(q + U) \cap \mathbb{T} \subseteq U$ for any rational q.

For any n let $U \restriction_{\ge n}$ be the set of all $z \in 2^{[n,\infty)}$ such that $s \cup z \in U$ for all $s \in 2^{[1,n)}$. By the assumption of $\mathbb{Q}$-invariance of U this is equivalent to having it not for all but only for some $s \in 2^{[1,n)}$.

Suppose that s, t belong to $2^{[1,n)}$ and $z \in 2^{[n,\infty)}$. Then, by (1),

$$H_{s,t}(z) = H(s \cup z) - H(t \cup z) \in H(s - t) + G,$$

hence $H_{s,t}(z)$ can take only countably many values, because G is countable. It follows that we can assume that, for all s, t as indicated and any $r \in \mathbb{R}$, the set

$$Z_s^t(r) = \{z \in U \restriction_{\ge n} : H(s \cup z) - H(t \cup z) = r\}$$

is clopen in $U \restriction_{\ge n}$: actually all but countably many of the sets $Z_s^t(r)$ are empty.

3 Generic bisection

We assert that there is a sequence of natural numbers $n_0 = 1 < n_1 < n_2 < n_3 < \cdots$ and, for every j, a string $\sigma_j \in 2^{[n_j, n_{j+1})}$ satisfying the following conditions:

(2) $\quad$ For any $s \in 2^{[1,n_j)}$ there is $t \in 2^{[1,n_j)}$ such that $t \subset H(s \cup \sigma_j \cup z)$ whenever $z \in U \restriction_{\ge n_{j+1}}$.

(3) $\quad$ If $s \in 2^{[1,n_j)}$, then $\mathbb{T}_{s \cup \sigma_j} \subseteq \bigcap_{k \le j} U_k$.

(4) $\quad$ If strings s_1, s_2 belong to $2^{[1,n_j)}$, then there is $r = r(s_1, s_2) \in \mathbb{R}$ such that $H(s_2 \cup \sigma_j \cup z) - H(s_1 \cup \sigma_j \cup z) = r$ for all $z \in U \restriction_{\ge n_{j+1}}$.

(Note that such an $r = r(s_1, s_2) \in \mathbb{R}$ is unique if exists.)

To get (2) use the fact that $H \restriction U$ is continuous. To see that (4) can also be provided, note the following: Suppose that some $n > n_j$ and some $\sigma \in 2^{[n_j,n)}$ have been defined. Recall that, for all s_1, s_2 in $2^{[1,n_j)}$ and $r \in \mathbb{R}$, the set $Z^{s_1 \cup \sigma}_{s_2 \cup \sigma}(r)$ is clopen in $U \restriction_{\geq n}$. It follows that there is some $n' > n$, a string $\sigma' \in 2^{[n,n')}$, and $r = r(s_1, s_2) \in \mathbb{R}$ such that $\sigma' \cup z' \in Z^{s_1 \cup \sigma}_{s_2 \cup \sigma}(r)$ for all $z' \in U \restriction_{\geq n'}$. Take the next pair of s'_1, $s'_2 \in 2^{[1,n_j)}$ and find suitable $r(s'_1, s'_2)$, $n'' > n'$, and $\sigma'' \in 2^{[n',n'')}$, and so on, until all pairs in $2^{[1,n_j)}$ are considered. The final result is as required.

L e m m a 4. *We have* $r(s, s') + r(s', s'') = r(s, s'')$, $r(s, s') = -r(s', s)$, *and also* $r(s, s) = 0$, *whenever* s, s', $s'' \in 2^{[1,n_j)}$ *for some* j.

P r o o f. Choose any $z \in U \restriction_{\geq n_{j+1}}$. Then, by definition,

$$r(s, s') = H(s' \cup \sigma_j \cup z) - H(s \cup \sigma_j \cup z),$$

and similarly for the other pairs, which easily yields the result. □

L e m m a 5. *If* s_1, $s_2 \in 2^{[1,n_j)}$ *and* t, $t' \in 2^{[n_{j+1},n_{j+2})}$, *then*

$$r(s_1 \cup \sigma_j \cup t,\ s_1 \cup \sigma_j \cup t') = r(s_2 \cup \sigma_j \cup t,\ s_2 \cup \sigma_j \cup t').$$

P r o o f. We first note that

$$r(s_1 \cup \sigma_j \cup t,\ s_2 \cup \sigma_j \cup t) = r(s_1 \cup \sigma_j \cup t',\ s_2 \cup \sigma_j \cup t') = r(s_1, s_2).$$

(For take any $z \in U \restriction_{\geq n_{j+3}}$. Then clearly $t \cup \sigma_{j+2} \cup z \in U \restriction_{\geq n_{j+1}}$.) It follows that

$$r(s_1 \cup \sigma_j \cup t,\ s_1 \cup \sigma_j \cup t') = r(s_1 \cup \sigma_j \cup t,\ s_2 \cup \sigma_j \cup t') - r(s_1, s_2),$$
$$r(s_2 \cup \sigma_j \cup t,\ s_2 \cup \sigma_j \cup t') = r(s_1 \cup \sigma_j \cup t,\ s_2 \cup \sigma_j \cup t') - r(s_1, s_2)$$

by Lemma 4, as required. □

D e f i n i t i o n. By Lemma 5, we can define $r(t, t') \in \mathbb{R}$ for all t, $t' \in 2^{[n_{j+1},n_{j+2})}$ so that $r(s \cup \sigma_j \cup t,\ s \cup \sigma_j \cup t') = r(t, t')$ for all $s \in 2^{[1,n_j)}$.

The following is an immediate consequence of Lemma 4.

C o r o l l a r y 6. *If* t, t', $t'' \in 2^{[n_{j+1},n_{j+2})}$, *then* $r(t, t') + r(t', t'') = r(t, t'')$ *and* $r(t, t) = 0$. □

Note that the definition of $r(s, s')$ for $s \in 2^{[1,n_j)}$ involves σ_j, in other words, it retains its intended meaning only in the case that s, s' are assumed to be extended by σ_j. Similarly, the definition of $r(t, t')$ for $t \in 2^{[n_{j+1},n_{j+2})}$ involves σ_{j+2}.

C o r o l l a r y 7. *For all* s_1, s_2, t, t' *as in the definition of* $r(t, t')$,

$$r(s_1 \cup \sigma_j \cup t,\ s_2 \cup \sigma_j \cup t') = r(s_1, s_2) + r(t, t').$$

P r o o f. By Lemma 4 and the first displayed equality in the proof of Lemma 5. □

C o r o l l a r y 8. *Let* s, $s' \in 2^{[1,n_j)}$ *with* $s \restriction [n_k, n_{k+1}) = s' \restriction [n_k, n_{k+1}) = \sigma_k$ *for all* $k < j$, $k = j \pmod 2$. *Then*

$$r(s,\ s') = \textstyle\sum_{k<j,\ k=j \pmod 2} r(s \restriction [n_{k-1}, n_k),\ s' \restriction [n_{k-1}, n_k)).$$ □

The next step is to expand this result for reals in $\mathbb{T}$ (i. e., infinite sequences).

Define $\boldsymbol{o}_j = [n_j, n_{j+1}) \times \{0\}$ and $\mathbf{1}_j = [n_j, n_{j+1}) \times \{1\}$ for any j. Let $\varepsilon = 0, 1$, We put ${}^{\varepsilon}N = \bigcup_i [n_{2i+\varepsilon}, n_{2i+\varepsilon+1})$. Define ${}^{\varepsilon}\sigma \in \mathbb{T}$ by

$${}^{\varepsilon}\sigma \restriction [n_j, n_{j+1}) = \begin{cases} \sigma_j & \text{if } j = \varepsilon \pmod 2, \\ \boldsymbol{o}_j & \text{otherwise.} \end{cases}$$

Similarly, for any $x \in \mathbb{T}$, define ${}^{\varepsilon}x \in \mathbb{T}$ by

$$ {}^{\varepsilon}x \restriction [n_j, n_{j+1}) = \begin{cases} x \restriction [n_j, n_{j+1}) & \text{if } j = \varepsilon \pmod 2, \\ \boldsymbol{o}_j & \text{otherwise.} \end{cases} $$

(Thus $\mathrm{D}({}^{\varepsilon}\sigma) \cup \mathrm{D}({}^{\varepsilon}x) \subseteq {}^{\varepsilon}N$.) Define ${}^{\varepsilon}H(x) = H({}^{\varepsilon}x + {}^{1-\varepsilon}\sigma) - H({}^{1-\varepsilon}\sigma)$. Then by (3) every real of the form ${}^{\varepsilon}x + {}^{1-\varepsilon}\sigma$, where $x \in \mathbb{T}$ and $\varepsilon = 0, 1$, belongs to U. In particular ${}^{0}\sigma = 0 + {}^{0}\sigma$ and ${}^{1}\sigma = 0 + {}^{1}\sigma$ belong to U. Moreover, $x = {}^{0}x + {}^{1}x$, so that, by (1),

(5) $\boldsymbol{h}(x) = {}^{0}H(x) + {}^{1}H(x) + G$ for any $x \in \mathbb{T}$.

Lemma 9. *Suppose that* $\varepsilon = 0, 1$ *and* $x, y \in \mathbb{T}$ *with* $\mathrm{D}(x) \cup \mathrm{D}(y) \subseteq {}^{\varepsilon}N$. *Then* ${}^{\varepsilon}H(x) - {}^{\varepsilon}H(y) = \sum_i r(x \restriction [n_{2i+\varepsilon}, n_{2i+\varepsilon+1}),\, y \restriction [n_{2i+\varepsilon}, n_{2i+\varepsilon+1}))$.

Proof. Let $\varepsilon = 0$, for brevity. For any j, define an approximation $y_j \in \mathbb{T}$ of y such that $y_j \restriction [1, n_j) = y \restriction [1, n_j)$ while $y_j \restriction [n_j, \infty) = x \restriction [n_j, \infty)$. All numbers $x' = x + {}^{1}\sigma$, $y' = y + {}^{1}\sigma$, $y'_j = y_j + {}^{1}\sigma$ belong to U by (1). It follows that

$$ \begin{aligned} H(x') - H(y'_{2i+1}) &= r(x' \restriction [1, n_{2i+1}),\, y' \restriction [1, n_{2i+1})) \\ &= \textstyle\sum_{\nu < i} r(x \restriction [n_{2\nu}, n_{2\nu+1}),\, y \restriction [n_{2\nu}, n_{2\nu+1})) \end{aligned} $$

by Corollary 8. We conclude that

$$ {}^{0}H(x) - {}^{0}H(y) = H(x') - H(y') = \textstyle\sum_i r(x \restriction [n_{2i}, n_{2i+1}),\, y \restriction [n_{2i}, n_{2i+1})), $$

because the reals $y'_{2i+1} \in U$ converge to $y' \in U$ while H is continuous on U. □

4 Additivity

This section contains the key fact: the function $r(\cdot\,,\cdot)$ has some group-theoretic properties, some kind of additivity, true in all but finite cases. The key idea of the proof is as follows: if there were infinitely many exceptions, then there would be infinitely many of them which follow one and the same "pattern", leading to contradiction with the additivity of $\boldsymbol{h}$.

Lemma 10. *The following is true for almost all* j. *Suppose that* $s \le_{\mathrm{lex}} t$ *belong to* $2^{[n_j, n_{j+1})}$. *Then* $r(s,t) = r(\boldsymbol{o}_j, t - s)$. [6)]

Proof. Suppose that the lemma is false. Then there is an infinite set $J \subseteq \mathbb{N}$ and for any $j \in J$ a pair of $s_j \le_{\mathrm{lex}} t_j$ in $2^{[n_j, n_{j+1})}$ such that $r(s_j, t_j) \ne r(\boldsymbol{o}_j, d_j)$, where $d_j = t_j - s_j$. Let us assume that

(i) J contains only even numbers, so that $J = \{2i : i \in I\}$, where $I \subseteq \mathbb{N}^+$;

(ii) $r(s_{2i}, t_{2i}) < r(\boldsymbol{o}_{2i}, d_{2i})$ for all $i \in I$;

(The other cases are similar.) We can also assume that

(iii) $w = \sum_{i \in I} [r(\boldsymbol{o}_{2i}, d_{2i}) - r(s_{2i}, t_{2i})] \notin G$.

(Indeed if the sum w in (iii) belongs to G, then we have a convergent series of infinitely many strictly positive terms. Clearly the set of sums of all subseries has the cardinality of continuum, therefore, as G is countable, we can replace I by an appropriate infinite subset $I' \subseteq I$.) Define reals $x, y, z \in \mathbb{T}$ so that

$$ x \restriction [n_{2i}, n_{2i+1}) = s_{2i}, \quad y \restriction [n_{2i}, n_{2i+1}) = t_{2i}, \quad \text{and} \quad z \restriction [n_{2i}, n_{2i+1}) = d_{2i} $$

for all $i \in I$, while $x \restriction [n_j, n_{j+1}) = y \restriction [n_j, n_{j+1}) = z \restriction [n_j, n_{j+1}) = \boldsymbol{o}_j$ for $j \notin J$.

6) $t - s$ is executed here in the sense of the real number subtraction in $\mathbb{T}$, assuming that each $s \in 2^{[n_j, n_{j+1})}$ is identified with $\sum_{k=n_j}^{n_{j+1}} 2^{-s(k)}$. Note that the lexicographical order coincides with the real number order, so that $t - s \in 2^{[n_j, n_{j+1})}$ whenever $s \le_{\mathrm{lex}} t$ belong to $2^{[n_j, n_{j+1})}$.

Then, by Lemma 9 (with $\varepsilon = 0$), we have ${}^0H(y) - {}^0H(x) = \sum_{i\in I} r(s_{2i}, t_{2i})$, while ${}^0H(z) = \sum_{i\in I} r(\boldsymbol{o}_{2i}, d_{2i})$, so that, by (iii), ${}^0H(x) + {}^0H(z) - {}^0H(y) \notin G$. On the other hand, $\boldsymbol{h}(x) = {}^0H(x) + G$ by (5), and the same for y and z, so that $\boldsymbol{h}(x) + \boldsymbol{h}(z) \neq \boldsymbol{h}(y)$, which contradicts the choice of $\boldsymbol{h}$ as $y = x + z$. □

Choose j_0 big enough for Lemma 10 to be true for all numbers $j \geq j_0$.

Define $\boldsymbol{e}_j \in 2^{[n_j, n_{j+1})}$ by

$$\boldsymbol{e}_j(n) = \begin{cases} 1 & \text{if } n = n_{j+1} - 1, \\ 0 & \text{for all other } n \in [n_j, n_{j+1}). \end{cases}$$

Thus $\boldsymbol{e}_j$ follows $\boldsymbol{o}_j$ in the lexicographical order on $2^{[n_j, n_{j+1})}$. Let $\gamma_j = r(\boldsymbol{o}_j, \boldsymbol{e}_j)$. Since $r(s, s') + r(s', s'') = r(s, s'')$ (Lemma 6), we obtain:

Corollary 11. *For any $j \geq j_0$, if strings $s \leq_{\text{lex}} t$ belong to $2^{[n_j, n_{j+1})}$ then we have $r(s, t) = (t - s) \cdot 2^{n_{j+1}-1} \cdot \gamma_j$. In particular $r(\boldsymbol{o}_j, s) = s \cdot 2^{n_{j+1}-1} \cdot \gamma_j$.* □

(Note that $\boldsymbol{e}_j$, as a number, is equal to $2^{-(n_{j+1}-1)}$.)

Now we figure out the interrelations between neighbouring domains. Note that $\boldsymbol{e}_j = 2^{n_{j+2} - n_{j+1}} \cdot \boldsymbol{e}_{j+1}$.

Lemma 12. *For almost all j we have $\gamma_j = 2^{n_{j+2} - n_{j+1}} \cdot \gamma_{j+1}$.*

Proof. Otherwise we have an infinite set $J \subseteq \mathbb{N}$ containing, say, only even numbers $\geq j_0$, i. e. $J = \{2i : i \in I\}$ for an infinite set $I \subseteq \mathbb{N}$, such that, say, for all $i \in I$, $\gamma_{2i} < 2^{n_{2i+2} - n_{2i+1}} \cdot \gamma_{2i+1}$. Recall that $\mathbf{1}_j = [n_j, n_{j+1}) \times \{1\}$, so that, as a real, $\mathbf{1}_j = (2^{n_{j+1} - n_j} - 1) \cdot \boldsymbol{e}_j$. It follows, by Corollary 11, that $r(\boldsymbol{o}_j, \mathbf{1}_j) = (2^{n_{j+1} - n_j} - 1) \cdot \gamma_j$ for $j \geq j_0$. Thus, for any $i \in I$,

$$r(\boldsymbol{o}_{2i}, \boldsymbol{e}_{2i}) < r(\boldsymbol{o}_{2i+1}, \boldsymbol{e}_{2i+1}) + r(\boldsymbol{o}_{2i+1}, \mathbf{1}_{2i+1}).$$

We can assume, as above, that

$$w = \textstyle\sum_{i\in I} r(\boldsymbol{o}_{2i+1}, \boldsymbol{e}_{2i+1}) + r(\boldsymbol{o}_{2i+1}, \mathbf{1}_{2i+1}) - r(\boldsymbol{o}_{2i}, \boldsymbol{e}_{2i}) \notin G.$$

Define $x, y, z \in \mathbb{T}$ so that $x \restriction [n_{2i+1}, n_{2i+2}) = \boldsymbol{e}_{2i+1}$, $y \restriction [n_{2i+1}, n_{2i+2}) = \mathbf{1}_{2i+1}$ and $z \restriction [n_{2i}, n_{2i+1}) = \boldsymbol{e}_{2i}$ for all $i \in I$, and such that 0 outside of those domains. Then, by Lemma 9, ${}^0H(z) = \sum_{i\in I} r(\boldsymbol{o}_{2i}, \boldsymbol{e}_{2i})$, while ${}^1H(x) = \sum_{i\in I} r(\boldsymbol{o}_{2i+1}, \boldsymbol{e}_{2i+1})$ and ${}^1H(y) = \sum_{i\in I} r(\boldsymbol{o}_{2i+1}, \mathbf{1}_{2i+1})$, so that ${}^1H(x) + {}^1H(y) - {}^0H(z) \notin G$, which leads to contradiction like in the proof of Lemma 10, because by definition $z = x + y$. □

We may assume that Lemma 12 holds for all $j \geq j_0$, together with Corollary 11.

Corollary 13. *If $j \geq j_0$ and $s \in 2^{[n_j, n_{j+1})}$, then $r(\boldsymbol{o}_j, s) = s \cdot 2^{n_{j_0+1}-1} \cdot \gamma_{j_0}$.* □

5 Ending the proof

Let for all $j \geq j_0$ and $s \in 2^{[n_j, n_{j+1})}$, $r(\boldsymbol{o}_j, s) = c \cdot s$, where $c = 2^{n_{j_0+1}-1} \cdot \gamma_{j_0}$. Now, it follows from (5) and Lemma 9 that, for every $x \in \mathbb{T}$ which is small enough, i. e. satisfies $x < x_0 = 2^{-n_{j_0}}$ (or, that is the same, $\mathrm{D}(x) \subseteq [n_{j_0}, \infty)$), we have

$$\begin{aligned} \boldsymbol{h}(x) &= {}^0H(x) + {}^1H(x) + G \\ &= \textstyle\sum_{j\geq j_0} r(\boldsymbol{o}_j, x \restriction [n_j, n_{j+1})) + G \\ &= \textstyle\sum_{j\geq j_0} c \cdot (x \restriction [n_j, n_{j+1})) + G \\ &= c \cdot x + G. \end{aligned}$$

It easily follows that then $\boldsymbol{h}(x) = cx + G$ for all $x \in \mathbb{R}$. Indeed if, say, $x \geq x_0$, then take $m \in \mathbb{N}^+$ big enough for $x' = 2^{-m} \cdot x$ to satisfy $x' < x_0$. Then $\boldsymbol{h}(x') = c \cdot x' + G$ by the above. However $\boldsymbol{h}(x) = m\,\boldsymbol{h}(x')$. □ (Theorem 1)

6 Counterexample with an uncountable group

The following example shows that Theorem 1 fails, generally speaking, for uncountable Borel groups $G \subseteq \mathbb{R}$ and, say, $D = \mathbb{Q}$. Let us consider $\mathbb{R}^2$ as the product of two copies of the additive group of the reals. Define, for any set $A \subseteq \mathbb{R}^2$,

$$\mathrm{pr}_X\, A = \{x : \exists y\,(\langle x, y\rangle \in A)\}, \quad \text{and} \quad \mathrm{pr}_Y\, A = \{y : \exists x\,(\langle x, y\rangle \in A)\},$$

and $A_x = \{y : \langle x, y\rangle \in A\}$ for any x (a cross-section).

Proposition 14.[7] *There is a Borel subgroup A of $\mathbb{R}^2$ such that*

(i) $\mathrm{pr}_X\, A = \mathbb{R}$;

(ii) *for any real c, the set A does not completely include the line $y = cx$;*

(iii) *if $x - x' \in \mathbb{Q}$, then $A_x = A_{x'}$.*

Proof. Let $Y \subseteq \mathbb{R}$ be an uncountable closed set such that $q_1 y_1 + \cdots + q_n y_n \neq 0$ whenever $q_1, \ldots, q_n \in \mathbb{Q} \setminus \{0\}$, while $y_1, \ldots, y_n$ are pairwise different elements of Y. (In particular $0 \notin Y$.) Let F be a Borel $1-1$ map of $\mathbb{R}$ onto Y. Define A to be the set of all points of the form

$$\langle q + q_1 x_1 + \cdots + q_n x_n,\ q_1 F(x_1) + \cdots + q_n F(x_n)\rangle \in \mathbb{R}^2,$$

where $q, q_1, \ldots, q_n \in \mathbb{Q}$ and $x_1, \ldots, x_n \in \mathbb{R}$. Clearly A is a Borel group satisfying (i) and (iii). Let us show that (ii) also holds. First of all A does not contain any point of the form $\langle x, 0\rangle$, except for $\langle q, 0\rangle$ for $q \in \mathbb{Q}$. Now let $c \neq 0$. If A entirely includes the line $y = cx$, then $\mathrm{pr}_Y A = \mathbb{R}$. Then Y is a Borel basis of $\mathbb{R}$ as a $\mathbb{Q}$-vectorspace, which is impossible. (Indeed, if Y contains a rational r, then the $\mathbb{Q}$-closure of $Y \setminus \{r\}$ is a Borel selector for the Vitali equivalence relation, which is impossible. If Y does not contain a rational then $1 = q_1 y_1 + \cdots + q_n y_n$ for some $y_i \in Y$ and rationals $q_i \neq 0$. Replace q_1 by 1 in Y, getting the first case.) □

Assume that A is such a group. Then $G = A_0$ is a Borel subgroup of $\mathbb{R}$.

Example 15. *An BM homomorphism $\boldsymbol{h} : \mathbb{R}/\mathbb{Q} \longrightarrow \mathbb{R}/G$ without a "good" lifting.*

By (iii), we can define a homomorphism $\boldsymbol{h} : \mathbb{R}/\mathbb{Q} \longrightarrow \mathbb{R}/G$ by $\boldsymbol{h}(x + \mathbb{Q}) = A_x$ for any $x \in \mathbb{R}$. We observe that $\boldsymbol{h}$ is Baire measurable: indeed, it is clear that $F(x) \in A_x = \boldsymbol{h}(x + \mathbb{Q})$ for any x. Let us fix $c \in \mathbb{R}$. Then $x \longmapsto cx$ does not lift $\boldsymbol{h}$: otherwise $cx \in A_x$ for any x, which is a contradiction with (ii). □

7 Some questions

Question 1. Generalize Corollary 2 on Borel groups $A \subseteq \mathbb{R}$ not necessarily satisfying $\mathrm{pr}_X A = \mathbb{R}$. □

Let $R = \mathrm{pr}_X A$ for such a group. Then R is a Borel (since the cross-sections are countable) subgroup of $\mathbb{R}$. If R is divisible and (unlike $\mathbb{R}$) has a Borel Hamel basis (over $\mathbb{Q}$) $H \subseteq R$, then A is easily Borel isomorphic to $R \times A_0$.

[7] This example, with the exception of requirement (iii), was communicated by G. Hjorth in May 1998 and presented here with his permission.

Question 2. Find uncountable subgroups G of $\mathbb{R}$ which still satisfy Theorem 1. (Farah [2, 3] found a family of uncountable Borel ideals in $\mathcal{P}(\mathbb{N})$, called *nonpathological analytic P-ideals*, which admit a certain analog of our Theorem 1.) $G = \mathbb{R}$ is a trivial example. Are there less trivial examples ?

It would be interesting to get results, similar to Theorem 1, for Polish groups other than $\mathbb{R}$.

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