A theorem on ROD-hypersmooth equivalence relations in the Solovay model

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It is known that every Borel hypersmooth but non-smooth equivalence relation is Borel bi-reducible to \( E_1 \). We prove a ROD version of this result in the Solovay model.

1 Introduction

It is known since [5] that classical theorems on Borel and analytic sets tend to generalize to all projective, generally, all real-ordinal definable (ROD) sets in the Solovay model. In particular, as one of the authors demonstrated in [2], the fundamental theorem of Glimm-Effros classification for Borel equivalence relations admits such a generalization (although not straightforward). In this note we prove the following theorem:

Theorem 1 (Main Theorem) In the Solovay model, if \( E \) is a ROD-hypersmooth equivalence relation, then either \( E \leq_{\text{ROD}} E_0 \) or \( E \sim_{\text{ROD}} E_1 \). The two cases are incompatible.

This is a partial generalization of a fundamental result on the Borel reducibility, saying that any Borel hypersmooth equivalence relation \( E \) satisfies either \( E \leq_B E_0 \) or \( E \sim_B E_1 \) (Theorem 2.1 in [4], also known as “the third dichotomy theorem”). The generalization is not complete: due to a simple counterexample, we cannot claim that \( E \) is ROD-hyperfinite in the “or” case.

2 Notation

ROD means: real-ordinal-definable. OD\((p)\) means: ordinal-definable in a real \( p \), i.e., definable with \( p \) and any ordinals as parameters.

We consider ROD equivalence relations on (also ROD) sets. If \( E, F \) are ROD equivalence relations on sets \( X, Y \), respectively, then, by analogy with the Borel reducibility, \( E \leq_{\text{ROD}} F \) means that there exists a ROD map \( \vartheta : X \rightarrow Y \) such that \( xE_x' \text{ iff } \vartheta(x)F\vartheta(x') \). (In principle, it is not assumed here that \( X, Y \) carry any topological or other structure.) As usual, \( E \sim_{\text{ROD}} F \text{ iff } E \leq_{\text{ROD}} F \text{ and } F \leq_{\text{ROD}} E \) (ROD bi-reducibility), while \( E <_{\text{ROD}} F \text{ iff } E \leq_{\text{ROD}} F \text{ but } F \nleq_{\text{ROD}} E \) (strict ROD-reducibility).

An equivalence relation \( E \) on \( X \) is ROD-finite iff it is ROD and every \( E \)-class \( [x]_E = \{ y : xEy \} \), \( x \in X \), is finite. A ROD-hyperfinite equivalence relation is any one of the form \( \bigcup_n E_n \), where \( \{ E_n \}_{n \in \mathbb{N}} \) is an increasing chain of ROD-finite equivalence relations.

An equivalence relation \( E \) on a set \( X \) is ROD-smooth iff \( E \leq_{\text{ROD}} D(2^{2^X}) \), i.e., there is a ROD map \( \vartheta : X \rightarrow 2^{2^X} \) such that \( xEy \text{ iff } \vartheta(x) = \vartheta(y) \). A ROD-hypersmooth equivalence relation is an increasing union of ROD-smooth equivalence relations. Obviously all ROD-hyperfinite and all ROD-hypersmooth equivalence relations are ROD.

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Recall that $E_0$ is an equivalence relation on $2^\omega$ defined as follows: $x E_0 y$ iff $x_n = y_n$ for almost all $n$: here we assume that $x = \{x_n\}_{n \in \mathbb{N}}$ and $y = \{y_n\}_{n \in \mathbb{N}}$ belong to $2^\omega$. This is a ROD-hyperfinite, moreover, Borel-hyperfinite equivalence relation. Further, $E_1$ is an equivalence relation on $P(\mathbb{N})^\omega$ defined similarly, i.e., $x E_1 y$ iff $x_n = y_n$ for almost all $n$. $E_1$ is a typical example of a ROD-hypersmooth equivalence relation, indeed, even Borel-hypersmooth equivalence relation.

**Lemma 2** An equivalence relation $E$ is ROD-hypersmooth iff $E \leq_{\text{ROD}} E_1$.

**Proof.** Similar to the Borel case, see [4, 1.3] for the nontrivial direction. \qed

By the Solovay model we mean a $\mathbb{P}^\sharp$-generic extension of $L$, the constructible universe\(^1\), where $\Omega$ is an inaccessible cardinal in $L$. $\mathbb{P}^\sharp = \prod_{\gamma < \Omega} \mathbb{P}_\gamma$ (the product with finite support), and $\mathbb{P}_\gamma = \mathbb{P}^{\mathbb{P}_\gamma} = \bigcup_n \gamma^n$ for every $\gamma < \Omega$.

Assume that $\gamma < \Omega$. Let $T_\gamma[p]$ be the set of all terms $t = (\gamma, \{t_n\}_{n \in \mathbb{N}}) \in L[p]$ such that $t_n \subseteq \mathbb{P}_\gamma$ for all $n$. If $f \in \gamma^n$ (an infinite sequence), then let $t[f] = \{n : \exists m (f \upharpoonright m \in t_n)\}$.

Let $\mathbb{P}_\gamma[p]$ be the set of all over $L[p] \mathbb{P}_\gamma$-generic functions $f \in \gamma^n$. Put $t[w] = \{t[f] : w \subseteq f \in \mathbb{P}_\gamma[p]\}$ for any $w \in \mathbb{P}_\gamma$ and $t \in T_\gamma$. The following result is established, e.g., in [2, Proposition 5].

**Proposition 3** (In the Solovay model) Let $p$ be a real. Then

(i) If $\emptyset \neq X \subseteq P(\mathbb{N})$ is OD($p$), then there exist $\gamma < \Omega$, $w \in \mathbb{P}_\gamma$, and $t \in T_\gamma[p]$ such that $t[w] \subseteq X$.

(ii) If $\gamma < \Omega$, $w \in \mathbb{P}_\gamma$, and $\emptyset \neq X \subseteq t[w]$ is OD($p$), then there exists $w' \in \mathbb{P}_\gamma$ such that $w \subseteq w'$ and $t[w'] \subseteq X$. \qed

### 3 Incompatibility in the main theorem

It suffices to show that $E_1 \not\leq_{\text{ROD}} E_0$ in the Solovay model. The proof that $E_1 \not\leq_{\text{B}} E_0$, moreover, $E_1 \not\leq_{\text{B}} F$ for any countable Borel equivalence relation $F$ in [4, 1.4 and 1.5] actually gives non-reducibility even via Baire measurable functions, i.e., those continuous on a dense $G_\delta$ set. However, it is known (see [5]) that in the Solovay model any ROD function is Baire measurable.

### 4 The partition into cases

This section begins the essential part of the proof of Theorem 1.

We argue in the Solovay model.

Let $E$ be a ROD equivalence relation on a set $X$. Suppose that $E$ is ROD-hypersmooth. We have $E \leq_{\text{ROD}} E_1$ by Lemma 2. Let this be witnessed by a ROD map $\vartheta : X \rightarrow P(\mathbb{N})^\omega$. We put $P = \text{ran } \vartheta$, the full image of $\vartheta$. This is still a ROD set, hence, there is a real $p$ such that $P$ is OD($p$).

The real $p$ is fixed until the end of the proof.

To define the partition into two cases, we need the following notation. If $x \in P(\mathbb{N})^\omega$, then $x|_{\geq n}$ is the restriction of $x$ (a function defined on $\mathbb{N}$) to the domain $[n, \infty)$. If $X \subseteq P(\mathbb{N})^\omega$, then let $X|_{\geq n} = \{x|_{\geq n} : x \in X\}$. Define $X|_{> n}$ and $X|_{> n}$ similarly. In particular, $P(\mathbb{N})|_{\geq n} = P(\mathbb{N})|_{\leq n} = P(\mathbb{N})^n = P(\mathbb{N})|_{n, \infty}$. For a sequence $x \in P(\mathbb{N})^\omega$, let $\text{dep } x$ (the depth of $x$) be the number (finite or $\infty$) of elements of the set

$$J(x) = \{j \geq n : x(j) \notin \text{OD}(p, x|_{> j})\}.$$  

Recall that, in the Solovay model, $x \in \text{OD}(y)$ iff $x \in L[y]$ for any two reals $x$, $y$.

Case 1. All $x \in P = \text{ran } p$ satisfy $\text{dep } x < \infty$.

Case 2. There exist $x \in P$ with $\text{dep } x = \infty$.

The content of the remainder will be to prove $E \leq_{\text{ROD}} E_0$ in Case 1 and $E_1 \leq_{\text{ROD}} E$ in Case 2.

\(^1\) Theorem 1 is true, with some rather clear adjustments of the proof, for the Solovay extensions not necessarily of the constructible universe.
4.1 Case 1

As obviously $E \leq_{\text{ORD}} E_1 \uparrow P$, it suffices to show that $E_1 \uparrow P \leq_{\text{ORD}} E_0$.

Suppose that $x \in P$. If $\text{dep} x = \emptyset$, then let $f(x) = x$. If $\text{dep} x \neq \emptyset$, then (as $\text{dep} x$ is finite) let $n_x$ be the largest $n$ in $\text{dep} x$. Define $f(x) = y \in \mathcal{P}(N)^{\aleph_0}$ so that $x_{\geq n_x} = y_{\geq n_x}$ while $y(j) = \emptyset$ for all $j \leq n_x$. Easily $f$ is a ROD reduction of $E_1 \uparrow P$ to $E_1 \uparrow Q$, where $Q = \text{ran} f$, thus, it suffices to show that $E_1 \uparrow Q \leq_{\text{ORD}} E_0$. The set $Q$ belongs to $\text{OD}(p)$ together with $P$.

Note that by definition any point $x \in Q$ satisfies $\text{dep} x = \emptyset$, so that $x(n) \in \text{OD}(p, x_{\geq n})$ for any $n \in \mathbb{N}$ and $x \in Q$. It follows that $x(n) \in L[p, x_{\geq n}]$ for any $n \in \mathbb{N}$ and $x \in Q$, by known properties of the Solovay model. In other words, $Q \subseteq T = \{x \in \mathcal{P}(N)^{\aleph_0} : \forall n (x(n) \in L[p, x_{\geq n}])\}$, hence, it suffices to prove that $E_1 \uparrow T \leq_{\text{ORD}} E_0$. Note that $T = \text{OD}(p)$.

Fix $x \in T$. For any $n \in \mathbb{N}$ let $\xi_n(x)$ be the order of $x(n)$ in the sense of the canonical well-ordering of $\mathcal{P}(N)^{\aleph_0}$, then $\xi_n(x) < \omega_1^{L[p, x_{\geq n}]}$. Note that still $\xi(x) = \sup_n \xi_n(x) < \omega_1^{L[p, x]}$, because the map $n \mapsto \xi_n(x)$ is $\text{OD}(p, x)$. Now define $\mu(x) = \inf \{\xi(y) : y \in T \wedge y E_1 x\}$. This is $E_1$-invariant, i.e., $\mu(x) = \mu(y)$ whenever $x, y \in T$ and $x E_1 y$; moreover, $\mu(x) < \omega_1^{L[p, x]}$.

Let $W = \{x \in T : \xi(x) = \mu(x)\}$. This is an $\text{OD}(p)$ subset of $T$, and there is a ROD reduction of $E_1 \uparrow T$ to $E_1 \uparrow W$. (Indeed: Let $x \in T$. By definition there is an $m$ such that $\xi(x) \leq \mu(x)$ for all $j \geq m$; let $m_x$ be the least of such numbers $m$. Define $y = g(x) \in \mathcal{P}(N)^{\aleph_0}$ so that $x_{\geq m_x} = y_{\geq m_x}$ while $y(j) = \emptyset$ for all $j > m_x$. Then $y \in W$, under the natural assumption that $\emptyset$ has order $0$ in any relevant well-ordering, and $y E_1 x$. Thus, $g$ is a ROD reduction of $E_1 \uparrow T$ to $E_1 \uparrow W$.) It suffices to prove that $E_1 \uparrow W \leq_{\text{ORD}} E_0$.

By definition, $\xi_n(x) \leq \mu(x) < \omega_1^{L[p, x]}$ for all $x \in W$ and $n \in \mathbb{N}$, hence, if $a \subseteq W_{\geq n}$, then the set $S_W(a) = \{x(n) : x \in W \wedge a = x_{\geq n}\} \subseteq L[p, x]$ is countable in $L[p, x] = L[p, a]$. Thus there exists an $\text{OD}(p)$ map $F$ with $S_W(a) = F(a, k) : k \in \mathcal{P}(N)$ whenever $a \in A = \bigcup_{m \in \mathbb{N}} W_{\geq m}$. Assuming w.l.o.g. that $\mu(x) \geq \omega$ for any $x$. All sets $S_W(a)$, $a \in A$, are strictly countable, hence, we can assume that for any $a \in A$ the partial map $F_a(k) = F(a, k)$ is a bijection of $\mathcal{P}(N)$ onto $\mathcal{P}(N)$. Then for any $x \in W$ and $n$ there is a unique $k = \kappa_n(x)$ such that $x(n) = F(x_{\geq n}, k)$. Let $\kappa(x) = \{\kappa_n(x) : n \in \mathbb{N}\}$. Note that if $x \neq y \in W$ and $x E_1 y$, then $\kappa(x) \neq \kappa(y)$.

The next step is to uniformly define an ordering of any set of the form $[x]_{E_1} \cap W$, $x \in W$, similar to $Z$. Define $\sigma_n(x) = \max\{n, \max_{j \leq n} \kappa_j(x)\}$ for all $x \in W$ and $n$. Define the infinite sequence

$$\sigma(x) = \{\kappa_0(x), \sigma_0(x), \kappa_1(x), \sigma_1(x), \ldots, \kappa_n(x), \sigma_n(x), \ldots\}$$

of natural numbers. Easily if $x, y \in \mathcal{P}(N)^{\aleph_0}$ satisfy $x \leq_{\text{lex}} y$, i.e., $x_{\geq n} = y_{\geq n}$ for some $n$, then still $\sigma(x) \leq \sigma(y)$, i.e., $\sigma(x)_{\geq k} = \sigma(y)_{\geq k}$ for some $k \geq n$. Define, for $x, y \leq_{\text{lex}} y$, $x \leq_{\text{lex}} y$ if $\sigma(x) < \sigma(y)$ (the antilexicographical ordering), meaning that $\sigma(x) < \sigma(y)$, where $k$ is the least number such that $\sigma(x)_{\geq k} = \sigma(y)_{\geq k}$. Easily $\sigma_{\text{lex}}$ orders any $E_0$-class of an element of $\mathcal{P}(N)^{\aleph_0}$ similarly to $Z$, with the only exception of the $E_0$-class of the constant $0$ which is ordered similarly to $N$. It follows that any $E_1$-class $[x]_{E_1} \cap W$, $x \in W$, is ordered by $\leq_{\text{lex}}$ similarly to either $Z$ or $\mathcal{P}(N)^{\aleph_0}$. As a matter of fact, any class ordered similarly to $\mathcal{P}(N)^{\aleph_0}$ can be rearranged, in some trivial manner, to that its order is now $Z$ instead of $\mathcal{P}(N)^{\aleph_0}$. This way we obtain an $\text{OD}(p)$ binary relation $\leq_{\text{lex}}$ which orders every set of the form $[x]_{E_1} \cap W$, $x \in W$, similarly to $Z$. In other words, we have defined an $\text{OD}(p)$ action of $Z$ on whose orbits are exactly $E_1$-classes $[x]_{E_1} \cap W$, $x \in W$.

The rest of the argument involves a construction given in [1]. For any $x \in W$ define $\zeta(x) \in W^Z$ so that $\zeta(x)(0) = x$ and, for any $c \in Z$, $\zeta(x)(c + 1)$ is the $<_{\text{lex}}$-next element of $[x]_{E_1} \cap W$ after $\zeta(x)(c)$. Thus, $\zeta$ is an $\text{OD}(p)$ map $W \rightarrow Z = (\mathcal{P}(N)^{\aleph_0})^Z$. For $\zeta \in Z$ define $\zeta_{\text{lex}}$ iff there is an integer $z \in Z$ such that $\zeta(c) = \eta(c + z)$ for all $c \in Z$. Thus, $F$ is the equivalence relation $E(Z, \mathcal{P}(N)^{\aleph_0})$ on $Z = (\mathcal{P}(N)^{\aleph_0})^Z$, in the sense of [1].

The map $\zeta$ is obviously a reduction of $E_1 \uparrow W$ to $F$, hence, it suffices to show that $F \leq_{\text{ORD}} E_0$. But $[1, 7.1]$ yields a stronger result: $F \leq_{\text{B}} E_0$. \hfill $\Box$

Case 1
4.2 Case 2

Thus, assume that the OD(p) set \( R = \{ x \in P : \text{dep } x = \infty \} \) is non-empty. Our goal is to define an OD(p) subset \( X \subseteq R \) with \( E_1 \leq_B E_1 \upharpoonright X \).

We continue to argue in the Solovay model.

We begin with a reduction to the case when \( J(x) = \{ n : x(n) \not\in L[p, x_{>n}] \} \) is equal to \( \mathbb{N} \) for any \( x \in R \).

Fix, for any \( k \), a recursive bijection \( b_k : \mathcal{P}(\mathbb{N})^{k+1} \times \mathbb{N}^2 \overset{\text{onto}}{\longrightarrow} \mathcal{P}(\mathbb{N}) \). Now let \( x \in R \). Then \( J(x) \subseteq \mathbb{N} \) is infinite; let \( J(x) = \{ j_0, j_1, j_2, \ldots \} \) in the increasing order. For any \( m \), put

\[ y(m) = b_{j_{m+1}}(j_m - j_{m+1} - 1, x \restriction (j_m, j_{m+1}]) \]

(with \( j_{-1} = -1 \) for \( m = 0 \)). The map \( x \mapsto y \) is OD(p), \( x_{E_1} \upharpoonright y_{E_1} \) if \( y \in \mathcal{P}(\mathbb{N}) \), and also \( J(y) = \mathbb{N} \). This observation justifies to assume w.l.o.g. \( x(n) \not\in \text{OD}(p, x_{>n}) \) for any \( x \in R \) and \( n \).

The following construction uses the basic idea of [4, Theorem 2.1], in the form of a splitting construction developed in [3] for the study of “ill”-founded Sacks iterations. Fix a recursive map \( G : \mathbb{N}^2 \to \mathbb{N} \) as assumed in [3] for the study of “ill”-founded Sacks iterations. Fix a recursive map \( \nu : \mathbb{N} \to \mathbb{N} \) such that \( \nu(k) = \max \{ \nu' : k < n \land \nu(k) \neq \nu' \} \).

Let us demonstrate how such a system of sets accomplish Case 2. According to (iii) and (iv), for any \( a \in 2^{\mathbb{N}} \), the intersection \( \bigcap_n X_a \cap a \) contains a single point, let it be \( F(a) \), so that \( F : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})^N \) is continuous and one-to-one.

Define a parallel system of sets \( Y_u \), \( u \in 2^{\mathbb{N}} \), as follows. Put \( Y_\emptyset = \mathcal{P}(\mathbb{N})^N \). Suppose that \( Y_u \) has been defined, \( u \in 2^{\mathbb{N}} \), and \( \phi(n) = j \). Let \( \ell \) be the number of all indices \( k \) with \( \phi(k) = j \), perhaps \( \ell = 0 \). Put \( Y_{u \upharpoonright i} = \{ x \in Y_u : x(\ell) = i \} \) for \( i = 1 \). Each of \( Y_u \) is a clopen set and \( \mathcal{P}(\mathbb{N}) \), and one easily verifies that conditions (i) – (iv) are satisfied for the sets \( Y_u \) (instead of \( X_u \), in particular, for any \( a \in 2^{\mathbb{N}} \), the intersection \( \bigcap_n Y_{a \cap n} = \{ G(a) \} \) is a singleton, and the map \( G \) is continuous and one-to-one. We can define \( G \) explicitly:

\[ G(a) : \ell \mapsto \nu(\ell) \]

We conclude that the map \( \theta(x) = F(G^{-1}(x)) \) is a continuous bijection, hence, a homeomorphism by the compactness of the spaces considered, of \( \mathcal{P}(\mathbb{N}) \) onto the set \( X = \{ F(a) : a \in 2^{\mathbb{N}} \} = \bigcap_n \bigcup_{u \in 2^{\mathbb{N}}} X_u \).

We further assert that \( \theta \) satisfying the following: for each \( y, y' \in \mathcal{P}(\mathbb{N})^N \) and \( m \),

\[ (*) \quad y \upharpoonright \geq m = y' \upharpoonright \geq m \quad \text{iff} \quad \theta(y) \upharpoonright \geq m = \theta(y') \upharpoonright \geq m. \]

Indeed: Let \( y = G(a) \) and \( x = F(a) = \theta(y) \), and similarly \( y' = G(a') \) and \( x' = F(a') = \theta(y') \), where \( a, a' \in 2^{\mathbb{N}} \). Suppose that \( y \upharpoonright \geq m = y' \upharpoonright \geq m \). According to (i)(b) for \( \nu \) and the sets \( Y_u \) we then have \( m \geq \nu(\ell) \) for any \( a \), hence, \( X_{a \cap n} \upharpoonright \geq m = X_{a' \cap n} \upharpoonright \geq m \) for any \( n \) by (i)(a). Assuming now that Polish metrics on all spaces \( X \in \mathcal{P}(\mathbb{N})^N \) are chosen so that \( \text{diam}(Z) \leq \text{diam}(Z) \upharpoonright j \) for all \( Z \subseteq \mathcal{P}(\mathbb{N}) \) and \( j \), we easily obtain that \( x \upharpoonright \geq m = x' \upharpoonright \geq m \), i.e., the right-hand side of \( (*) \). The inverse implication in \( (*) \) is proved similarly.

Thus we have \( (*) \), but this means that \( \theta \) is a continuous reduction of \( E_1 \) to \( E_1 \upharpoonright X \), thus, \( E_1 \leq_B E_1 \upharpoonright X \), as required.

\[ \Box \] Theorem 1 modulo the construction (i) – (iv)

5 The construction

We continue to argue in the Solovay model.

Recall that \( R \subseteq \mathcal{P}(\mathbb{N})^N \) is a fixed non-empty OD(p) set such that \( J(x) = \mathbb{N} \) for each \( x \in R \). According to Proposition 3(i), there is \( \gamma < \Omega \), \( w_0 \in \mathcal{P}_\gamma \), and \( t \in T_\gamma[p] \) such that \( X_\Lambda = t[w_0] \subseteq R \). Let us fix an enumeration

(not OD (p)) \{D_n\}_{n \in \mathbb{N}} of all dense subsets of \(\mathbb{P}_n\), which belong to \(L[p]\). We define, along with sets \(X_u\), a system \(\{w_u\}_{u \in 2^{\omega}}\) of finite sequences \(w_u \in 2^{<\omega}\) satisfying

(v) \(w_u \in D_{\text{dom} u}\) and, for any \(i, w_u \subset w_{u^{-i}}\) and \(t[w_{u^{-i}}] \subseteq X_u \subseteq t[w_u]\).

Prove that this implies (iv). Let \(a \in 2^{\mathbb{N}}\). Then there is \(f \in \gamma^n\) such that \(w_{a^n} \subseteq f\) for any \(n\). This map \(f\) is generic over \(L[p]\), because for all \(n, w_{a^n} \in D_n\) that, \(f \in \mathbb{F}_n[p]\). It follows that \(t[f] \in \cap_n t[w_{a^n}] = \cap_n X_{a^n}\), as required.

To begin with, let a system \(X_u\) of all dense subsets of \(\mathbb{P}_n\) which belongs to \(D_0\). Put \(X_A = t[w_{u}]\). Now suppose that the sets \(X_u \subseteq X_u\) with \(u \in 2^n\) have been defined and satisfy the applicable part of (i)–(iii) and (v).

Lemma 4 If \(u_0 \in 2^n\) and \(X' \subseteq X_{u_0}\) is a non-empty OD\((p)\) set, then there is a system of OD\((p)\) sets \(\emptyset \neq X'_u \subseteq X_u\) with \(X'_0 = X'\), still satisfying (i).

Proof. For any \(u \in 2^n\), let \(X_u' = \{x \in X_u : x \upharpoonright n(u) \in X' \upharpoonright n(u)\}\), where \(n(u) = \nu_p[u, u_0]\). In particular, this gives \(X'_0 = X'\) because, \(\nu_p[0, 0] = -1\). The sets \(X'_u\) are as required, via a routine verification. Q.E.D. Lemma

Step 1. Put \(j = \varphi(n)\) and \(Y_u = X_u \upharpoonright j\). Take any \(u_1 \in 2^n\). Under our assumptions, any element \(x \in X_u\) satisfies \(j \in J(x)\), so that \(x(j) \notin OD\((p, x \upharpoonright j)\)\). Since \(X_u\) is an OD\((p)\) set, it follows that the set \(X_u \upharpoonright (j) = \{x \upharpoonright j : x \in X_u \land x \upharpoonright j = x(j)\}\) is not a singleton, in fact is uncountable. Then there is a number \(l_{u_1}\) having the property that the set

\[Y'_u = \{y \in Y_{u_1} : (\exists x, x' \in X_u) (x \upharpoonright j = y \land l_{u_1} \in x(j) \land l_{u_1} \notin x'(j))\}\]

is non-empty. We now put \(X' = \{x \in X_{u_1} : x \upharpoonright j \in Y'_u\}\) and define OD\((p)\) sets \(\emptyset \neq X'_u \subseteq X_u\) as in the lemma, in particular, \(X'_0 = X'\), \(X'_u \upharpoonright j = Y'_u\), still (i) is satisfied, and in addition

(1) \((\forall y \in X'_u \upharpoonright j)(\exists x, x' \in X_{u_1})\ (x \upharpoonright j = y \land l_{u_1} \in x(j) \land l_{u_1} \notin x'(j))\)

Now take some other \(u_2 \in 2^n\). Let \(\nu = \nu_p[u_1, u_2]\). If \(j > \nu\), then \(X_u \upharpoonright j = X_u \upharpoonright j\), so that we already have, for \(l_{u_2} = l_{u_1}\).

(2) \((\forall y \in X'_u \upharpoonright j)(\exists x, x' \in X_{u_2})\ (x \upharpoonright j = y \land l_{u_2} \in x(j) \land l_{u_2} \notin x'(j))\),

and can pass to some \(u_3 \in 2^n\). Suppose that \(\nu \geq j\). Now things are somewhat nastier. As above there is a number \(l_{u_3}\) such that

\[Y''_{u_3} = \{y \in Y_{u_3} : (\exists x, x' \in X_{u_1}) (x \upharpoonright j = y \land l_{u_3} \in x(j) \land l_{u_3} \notin x'(j))\}\]

is a non-empty OD\((p)\) set, thus, we can define \(X'' = \{x \in X_{u_1} : x \upharpoonright j \in Y''_{u_2}\}\) and maintain the construction of Lemma 4, getting non-empty OD\((p)\) sets \(X''_u \subseteq X'_u\) still satisfying (i) and \(X''_0 = X''\), therefore, we still have (2) for the set \(X''_u\).

Yet it is most important in this case that (1) is preserved, i.e., it still holds for the set \(X''_u\) instead of \(X'_u\) ! Indeed: According to the construction in the proof of Lemma 4, we have \(X''_u = \{x \in X'_u : x \upharpoonright \nu \in X'' \upharpoonright \nu\}\). Thus, although, in principle, \(X''_u\) is smaller than \(X'_u\), for any \(y \in X'' \upharpoonright j\) we have

\[\{x \in X''_u : x \upharpoonright j = y\} = \{x \in X'_u : x \upharpoonright j = y\},\]

simply because now we assume \(\nu \geq j\). This implies that (1) still holds.

Iterating this construction so that each \(u \in 2^n\) is eventually encountered, we obtain, in the end, a system of non-empty OD\((p)\) sets, let us call them “new” \(X_u\) but they are subsets of the “original” \(X_u\), still satisfying (i), and, for any \(u \in 2^n\) a number \(l_u\) such that \(j > \nu_p[u, v]\) implies \(l_u = l_v\)

\((*) (\forall y \in X_u \upharpoonright j)(\exists x, x' \in X_u)\ (x \upharpoonright j = y \land l_u \in x(j) \land l_u \notin x'(j))\).

Step 2. We define the \((n + 1)\)th-level by \(X_{u^{-1}} = \{x \in X_u : l_u \notin x(j)\}\) and \(X_{u^{-1}} = \{x \in X_u : l_u \notin x(j)\}\) for all \(u \in 2^n\), where still \(j = \varphi(n)\). It follows from (*) that all these OD\((p)\) sets are non-empty.

Lemma 5 The system of sets \(\{X_u\}_{u \in 2^{n+1}}\) just defined satisfies (i).

Proof. Let \(s = u^{-i} \land t = v^{-i'}\) belong to \(2^{n+1}\), so that \(u, v \in 2^n\) and \(i, i' \in \{0, 1\}\). Let \(\nu = \nu_p[u, v]\) and \(\nu' = \nu_p[s, t]\).
Case 1. \( \nu \geq j = \varphi(n) \). Then easily \( \nu = \nu' \), so that (i)(b) immediately follows from (i)(b) at level \( n \) for \( X_u \) and \( X_v \). As for (i)(a), we have \( X_u^{> \nu} = X_u^{> \nu'} \) (because by definition \( X_u^{> j} = X_u^{> j} \)), and similarly \( X_u^{> \nu} = X_u^{> \nu'} \), therefore, \( X_u^{> j} = X_u^{> j} \) since \( X_u^{> \nu} = X_u^{> \nu} \) by (i)(a) at level \( n \).

Case 2. \( j > \nu \) and \( i = i' \). Then still \( \nu = \nu' \), thus we have (i)(b). Further, \( X_u^{> \nu} = X_u^{> \nu} \) by (i)(a) at level \( n \), hence, \( X_u^{> j} = X_u^{> j} \) and \( l_u = l_v \) as above. Assuming that, say, \( i = i' = 1 \) and \( l_u = l_v = l \), we conclude that \( X_u^{> \nu} = \{ y \in X_u^{> \nu} : l \in y(j) \} = \{ y \in X_u^{> \nu} : l \in y(j) \} = X_u^{> \nu} \).

Case 3. \( j > \nu \) and \( i \neq i' \), say, \( i = 0 \) and \( i' = 1 \). Now \( \nu' = j \). Yet by definition \( X_u^{> j} = X_u^{> j} \) and \( X_u^{> j} = X_u^{> j} \), so it remains to apply (i)(a) for level \( n \). As for (i)(b), note that by definition \( \ell \not\in x(j) \) for any \( x \in X_u^{> 0} \) while \( l \in x(j) \) for any \( x \in X_u^{> 1} \), where \( l = l_u = l_v \). □ Lemma

Step 3. In addition to (i), we already have (ii) at level \( n + 1 \). To achieve the remaining properties (iii) and (v), consider, one by one, all elements \( s \in 2^{n+1} \), finding, at each such a substep \( s = u \upharpoonright i \) (\( u \in 2^n \) and \( i = 0, 1 \)), a non-empty OD(\( p \)) subset of \( X_u \), and also an extension \( w_x \in 2^{<\omega} \) of \( w_u \), consistent with (iii) and (v). As for (iii), just take a subset whose diameter is \( \leq 2^{-n} \). As for (iv), choose, using Proposition 3(ii), \( w_x \in \mathcal{P}_\gamma \) such that the following holds: \( w_x \in D_{n+1} \), \( w_u \subseteq w_s \), and the set \( [w_x] \) is a subset of the “current value” of \( X_u \). Finally, define the “new” value of \( X_u \) to be \( [w_x] \). Then reduce all other sets \( X_t, t \in 2^{n+1} \), as in Lemma 4 at level \( n + 1 \). Thus ends the substep \( s \). We have to pass to another \( s' \in 2^{n+1} \) and carry out substep \( s' \). And so on, with the consideration of all \( s \in 2^{n+1} \) one by one.

□ Construction and Theorem 1

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References