Monatshefte für Mathematik (2021) 194:105–117
https://doi.org/10.1007/s00605-020-01482-9

Factoring Solovay-random extensions, with application to the reduction property

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Received: 30 November 2018 / Accepted: 3 November 2020 / Published online: 12 November 2020

Abstract
If a real a is random over a model M and $x \in M[a]$ is another real then either (1) $x \in M$, or (2) $M[x] = M[a]$, or (3) $M[x]$ is a random extension of M and $M[a]$ is a random extension of $M[x]$. This result may belong to the old set theoretic folklore. It appeared as Exapmle 1.17 in Jech's book "Multiple forcing" without the claim that $M[x]$ is a random extension of M in (3), but, likely, it has never been published with a detailed proof. A corollary: $\mathbf{\Sigma}^1_n$-Reduction holds for all $n \geq 3$, in models extending the constructible universe $\mathbf{L}$ by κ-many random reals, κ being any uncountable cardinal in $\mathbf{L}$.

Keywords Forcing · Solovay-random extensions · Factoring · Reduction property

Mathematics Subject Classification 03E35 · 03E15

1 Introduction

It is known from Solovay [20], and especially Grigorieff [3] in most general form, that any subextension $\mathbf{V}[x]$ of a generic extension $\mathbf{V}[G]$, generated by a set $x \in \mathbf{V}[G]$, is itself a generic extension $\mathbf{V}[x] = \mathbf{V}[G_0]$ of the same ground universe $\mathbf{V}$, and the whole extension $\mathbf{V}[G]$ is equal to a generic extension $\mathbf{V}[G_0][G_1]$ of the intermediate model $\mathbf{V}[x] = \mathbf{V}[G_0]$. See a more recent treatment of this question in [5,9,13,21].

Communicated by S.-D. Friedman.

This research was funded by Russian Foundation for Basic Research RFBR Grant Number 18-29-13037.

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In particular, it is demonstrated in [9] that if $\mathbb{P} = \langle \mathbb{P} ; \le \rangle \in \mathbf{V}$ is a forcing notion, a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{V}$, $t \in \mathbf{V}$ is a $\mathbb{P}$-name, $x = t[G] \in \mathbf{V}[G]$ is the G-valuation of t, and $x \subseteq \mathbf{V}$, then

(1) there is a set $\Sigma \subseteq \mathbb{P}$ such that $\mathbf{V}[\Sigma] = \mathbf{V}[x]$ and G is Σ-generic over $\mathbf{V}[x]$;
(2) there exists an order $\le_t$ on $\mathbb{P}$ in the ground universe $\mathbf{V}$, such that $p \le q$ implies $p \le_t q$, and Σ itself is $\langle \mathbb{P} ; \le_t \rangle$-generic over $\mathbf{V}$.

However the nature and forcing properties of the derived forcing notions, that is, $\mathbb{P}_0 = \langle \mathbb{P} ; \le_t \rangle \in \mathbf{V}$ and $\mathbb{P}_1(x) = \langle \Sigma ; \le \rangle \in \mathbf{V}[x]$, is not immediately clear.

At the trivial side, we have the Cohen forcing $\mathbb{P} = \mathbb{C} = 2^{<\omega}$. In this case, $\mathbb{P}_0$ and $\mathbb{P}_1(x)$ are countable forcing notions, hence the corresponding extensions, $\mathbf{V} \to \mathbf{V}[x]$ and $\mathbf{V}[x] \to \mathbf{V}[G]$ in the above scheme, are Cohen generic or trivial. As observed in [9], this leads to the following result of set theoretic folklore, perhaps never explicitly appeared in publications, except for Sami [19, Lemma 1.9]. (It can also be derived from some results in [3], especially 4.7.1 and 2.14.1.)

Theorem 1.1 (folklore, Ramez Sami) *Let $a \in 2^\omega$ be Cohen-generic over a ground set universe $\mathbf{V}$. Let x be a real in $\mathbf{V}[a]$. Then precisely one of the following holds*:

(C1) $x \in \mathbf{V}$;
(C2) $\mathbf{V}[x] = \mathbf{V}[a]$;
(C3) (a) $\mathbf{V}[x]$ *is a Cohen-generic extension of* $\mathbf{V}$, *and*
(b) $\mathbf{V}[a]$ *is a Cohen-generic extension of* $\mathbf{V}[x]$.[1] □

A much more complex case is the Levy-Solovay collapse extension of $\mathbf{L}$, the constructible universe. As established in [20], such an extension is equal to a Levy-Solovay extension of $\mathbf{L}[x]$ for any real x it contains.

The following theorem, proved below, is a result of the same type.

Theorem 1.2 *Let $a \in 2^\omega$ be Solovay-random over a ground set universe $\mathbf{V}$. Let x be a real in $\mathbf{V}[a]$. Then we have exactly one of the following*:

(R1) $x \in \mathbf{V}$;
(R2) $\mathbf{V}[x] = \mathbf{V}[a]$;
(R3) (a) $\mathbf{V}[x]$ *is a Solovay-random extension of* $\mathbf{V}$, *and*
(b) $\mathbf{V}[a]$ *is a Solovay-random extension of* $\mathbf{V}[x]$.[2]

This theorem may belong to the old set theoretic folklore. It appeared without further reference as Example 1.17 in Jech's book "Multiple forcing" [4], yet without claim (R3)(a) and with a rather scarse sketch of a proof in terms of Boolean-valued approach to forcing. As far as we know, the full result has never been published with a detailed proof.

Note that Theorem 1.2 contains two separate dichotomies: (R1) vs. (R3)(a) and (R2) vs. (R3)(b). In spite of obvious semblance of Theorem 1.1, Theorem 1.2 takes more effort. Its proof (it begins in Sect. 4) involves some results related rather to real analysis and measure theory.

[1] Theorem 1.1 may fail for intermediate models not generated by reals, in particular those in which the axiom of choice does not hold. For instance, a model M of $\mathbf{ZF}$ is constructed in [14], which lies between $\mathbf{L}$ and $\mathbf{L}[c]$ for a Cohen real c and does not have the form $\mathbf{L}(x)$ for any set x.

[2] It is ***not*** asserted though that the real x itself is Solovay-random over $\mathbf{V}$ in (R3)(a) (in (R2), resp.), and/or the real a itself is Solovay-random over $\mathbf{V}[x]$ in (R3)(b).

2 A corollary: reduction in extensions by random reals

The reduction property for a pointclass K, or simply *K-Reduction*, is the assertion that for any two sets X, Y in K (in the same Polish space) there exist *disjoint* sets $X' \subseteq X$, $Y' \subseteq Y$ in the same class K such that $X' \cup Y' = X \cup Y$.

It is known classically from studies of Kuratowski [16] that Reduction holds for $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$, but fails for $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_2$. As for the higher projective classes, Addison [1] proved that the axiom of constructibility $\mathbf{V} = \mathbf{L}$ implies that Reduction holds for $\mathbf{\Sigma}^1_n$, $n \geq 3$, but fails for $\mathbf{\Pi}^1_n$, $n \geq 3$. On the other hand, by Martin [17], the axiom of projective determinacy **PD** implies that, similarly to projective level 1, $\mathbf{\Pi}^1_n$-Reduction holds for all odd numbers $n \geq 3$, and, similarly to projective level 2, $\mathbf{\Sigma}^1_n$-Reduction holds for all even numbers $n \geq 4$.

Apparently not much is known on Reduction for higher projective classes in generic models. One can expect that rather homogeneous, well-behaved forcing notions produce generic extensions of $\mathbf{L}$ in which Reduction keeps to be true for projective classes $\mathbf{\Sigma}^1_n$ and accordingly fails for $\mathbf{\Pi}^1_n$, $n \geq 3$, while this pattern can be violated in specially designed non-homogeneous extensions. This idea is supported by a few known results. Ramez Sami [19] proved

Theorem 2.1 (Sami) *It is true in any extension of* $\mathbf{L}$ *by* $\aleph_1$ *Cohen reals that if* $n \geq 3$ *then* Σ^1_n*-Reduction holds, and hence* $\mathbf{\Sigma}^1_n$*-Reduction holds, too.*[3] □

On the other hand, it is proved in [8] that Reduction fails for $\mathbf{\Sigma}^1_3$ (and in fact Separation fails for both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$) in a rather complicated model related to an $\aleph_1$-product of forcings similar to Jensen's minimal forcing [6]. See also [10,12] on similar models in which the Uniformization principle fails for $\mathbf{\Pi}^1_2$ (or $\mathbf{\Pi}^1_n$ for a given $n \geq 3$) sets with countable sections, and [11] on some related (and very complex) models of Harrington. The following theorem is the second main result of this paper.

Theorem 2.2 *It is true in any extension of* $\mathbf{L}$ *by* $\aleph_1$ *Solovay-random reals that if* $n \geq 3$ *then* Σ^1_n*-Reduction holds, and hence* $\mathbf{\Sigma}^1_n$*-Reduction holds, too.*

The theorem also holds in models obtained by adding any uncountable (not necessarily $\aleph_1$) number κ of random reals, as such models are elementarily equivalent, with respect to analytical formulas, to the extension by $\aleph_1$ random reals.

Proof (Theorem 2.2, sketch) The idea, due to Sami [19], is to closely emulate Addison's proof of Σ^1_n-Reduction in $\mathbf{L}$. The next "localization lemma" (based on Theorem 1.2) is another key ingredient. Similar results were obtained by Solovay [20], and by Sami [19] (with respect to extensions by $\aleph_1$ Cohen reals).

Now, arguing in an $\aleph_1$-random extension N of $\mathbf{L}$, we suppose that $n \geq 3$, and $X = \{x : \varphi(x)\}$ and $Y = \{x : \psi(x)\}$ are sets of reals, φ and ψ being Σ^1_n formulas. We are going to make use of the following lemma.

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[3] To prove that Σ^1_n-Reduction implies the boldface $\mathbf{\Sigma}^1_n$-Reduction, it suffices to employ a double-universal pair of Σ^1_n sets, as those used in a typical proof that $\mathbf{\Sigma}^1_n$-Reduction and $\mathbf{\Sigma}^1_n$-Separation contradict each other. This argument does not work for Separation though. Recall that the separation property for a pointclass K, or simply *K-Separation*, is the assertion that any two disjoint sets X, Y in K (in the same Polish space) can be separated by a set in $K \cap K^{\complement}$, where $K^{\complement}$ is the pointclass of complements of sets in K.

Lemma 2.3 (Proof see Sect. 8) *If $n \geq 2$ and $\varphi(x)$ is a parameter-free Σ^1_n formula then there is a parameter-free Σ^1_n formula $\varphi^*(x)$ such that if x is a real in an $\aleph_1$-random extension N of $\mathbf{L}$ then $\varphi(x)$ holds in N iff $\mathbf{L}[x] \models \varphi^*(x)$.*

By Lemma 2.3, we have $X = \{x : \mathbf{L}[x] \models \varphi^*(x)\}$ and $Y = \{x : \mathbf{L}[x] \models \psi^*(x)\}$, where φ^* and ψ^* are still Σ^1_n-formulas. Thus $\varphi^*(x)$ is $\exists\, y\, \Phi(x, y)$ and $\psi^*(x)$ is $\exists\, y\, \Psi(x, y)$, Φ and Ψ being Π^1_{n-1}.

Still arguing in N, if $x \in 2^\omega$ then let $<_{\mathbf{L}[x]}$ be the canonical Gödel wellordering of the reals in $\mathbf{L}[x]$, of order type ω_1. The crucial property of this system of order relations says that the *bounded quantifiers* $\forall\, y' <_{\mathbf{L}[x]} y$ and $\forall\, y' \leq_{\mathbf{L}[x]} y$, applied to a Σ^1_n formula, yield a Σ^1_n formula. It follows that the sets

$$X' = \{x : \mathbf{L}[x] \models \exists\, y \left(\Phi(x, y) \wedge \forall\, y' <_{\mathbf{L}[x]} y \neg \Psi(x, y')\right)\}$$
$$Y' = \{x : \mathbf{L}[x] \models \exists\, y \left(\Psi(x, y) \wedge \forall\, y' \leq_{\mathbf{L}[x]} y \neg \Phi(x, y')\right)\}$$

are Σ^1_n, because the relativization to $\mathbf{L}[x]$ does not violate being Σ^1_n ($n \geq 2$). It is easy to check that X' and Y' are as desired.

□ (Theorem 2.2, modulo Lemma 2.3)

3 Two lemmas on random forcing

The proof of Theorem 1.2 makes use of only some basic forcing ideas and some classical results related to real analysis and measure theory. In this section, we present two lemmas on random reals involved in the proof.

Random (or *Solovay-random*) reals, over a set universe $\mathbf{V}$, are usually defined as those reals in 2^ω (or true reals in the unit interval $[0, 1] = \mathbb{I}$) which avoid Borel sets which are coded in $\mathbf{V}$ and null with respect to the usual product probability measure μ_0 on 2^ω (or the true Lebesgue measure λ on $\mathbb{I}$, resp.).

The μ_0-random reals in 2^ω and λ-random reals in $\mathbb{I}$ produce the same generic extensions and thereby both notions can be identified, which is witnessed by the Borel map $f : 2^\omega \xrightarrow{\text{onto}} \mathbb{I}$ with $f(a) = \sum_{a(n)=1} 2^{-n-1}$. It satisfies $\lambda(f[X]) = \mu_0(X)$ for any Borel set $X \subseteq 2^\omega$. Therefore, if $a \in 2^\omega$ and $x = f(a) \in \mathbb{I}$ then a is μ_0-random iff x is λ-random, and $\mathbf{V}[a] = \mathbf{V}[x]$ in this case, of course. There is a general version of such a correspondence, provided by the next lemma.

Lemma 3.1 *Assume that ν is a continuous* (that is, all singletons are null sets) *Borel probability measure defined on 2^ω in a set universe $\mathbf{V}$. Then there is a continuous map $g : 2^\omega \xrightarrow{\text{onto}} \mathbb{I}$, coded in $\mathbf{V}$, such that if $a \in 2^\omega$ and $x = g(a) \in \mathbb{I}$ then a is ν-random over $\mathbf{V}$ iff x is λ-random over $\mathbf{V}$, and in this case $\mathbf{V}[a] = \mathbf{V}[x]$.*

Proof Let $<_{\mathtt{lex}}$ be the lexicographical order on 2^ω, and let $(a, b)_{\mathtt{lex}}$ denote $<_{\mathtt{lex}}$-intervals in 2^ω. Put $g(a) = \nu((0_{\mathtt{lex}}, a)_{\mathtt{lex}})$, where $0_{\mathtt{lex}} \in 2^\omega$ is the $<_{\mathtt{lex}}$-least element: $0_{\mathtt{lex}}(k) = 0$ for each k. Easily g is increasing ($a \leq_{\mathtt{lex}} b$ implies $g(a) \leq g(b)$), hence continuous as all singletons in 2^ω are ν-null.

Moreover g is measure-preserving: if $X \subseteq 2^\omega$ is Borel then $\nu(X) = \lambda(g[X])$. (Compare with the proof of Theorem 17.41 in Kechris [15].)

It follows that a is ν-random iff x is λ-random, whenever $a \in 2^\omega$ and $x = g(a)$. To see that $a \in \mathbf{V}[x]$, note that $J = g^{-1}[x]$ is a closed $\leq_{\texttt{lex}}$-interval in 2^ω, the interior U of which (if non-empty) is a countable union of $\leq_{\texttt{lex}}$-intervals U_n in 2^ω with "rational" endpoints.[4] But each U_n is a Borel set coded in $\mathbf{V}$ (while U itself is not necessarily coded in $\mathbf{V}$). We conclude that each U_n is a ν-null set, hence $a \notin U_n$, and therefore a is equal to an endpoint of J, thus easily $a \in \mathbf{V}[x]$. $\square$

The second lemma in this section belongs to forcing folklore, but we have not been able to find a really suitable reference. Therefore we add a proof for the reader's convenience. See also [22] for a broad consideration of the property of continuous reading of names.

Lemma 3.2 (continuous reading of names) *Let $a_0 \in 2^\omega$ be Solovay-random over a ground set universe $\mathbf{V}$, and $y_0 \in \mathbf{V}[a_0] \cap 2^\omega$. Then there is a continuous function $f : 2^\omega \to 2^\omega$, coded in $\mathbf{V}$ and such that $y_0 = f(a_0)$.*

The result also holds for $y_0 \in \mathbf{V}[a_0] \cap (2^\omega)^\omega$, $y_0 \in \mathbf{V}[a_0] \cap \mathbb{I}$, and $y_0 \in \mathbf{V}[a_0] \cap \mathbb{I}^\omega$.

Proof **We argue in $\mathbf{V}$**. Let $\mathbf{R}$ be the set of all closed μ_0-positive sets $X \subseteq 2^\omega$, the Solovay-random forcing. Let $\dot{y}_0$ be a $\mathbf{R}$-name for y_0, and let $\dot{a}_0$ be a canonical $\mathbf{R}$-name for the principal random real a_0. Consider the set $\mathbf{R}'$ of all conditions $Y \in \mathbf{R}$ such that there is a continuous function $f : 2^\omega \to 2^\omega$ (coded in $\mathbf{V}$), such that Y $\mathbf{R}$-forces $\dot{y}_0 = f(\dot{a}_0)$. It suffices to prove that $\mathbf{R}'$ is dense in $\mathbf{R}$.

Let $X \in \mathbf{R}$. If $n < \omega$ then let D_n consist of all conditions $Y \in \mathbf{R}$ such that $Y \subseteq X$ and Y $\mathbf{R}$-forces $\dot{y}_0(n) = 0$ or $\mathbf{R}$-forces $\dot{y}_0(n) = 1$. Clearly D_n is dense in $\{Y \in \mathbf{R} : Y \subseteq X\}$. Therefore by the CCC property of $\mathbf{R}$ for any n there is a finite pairwise disjoint set $A_n \subseteq D_n$ satisfying $\mu_0(\bigcup A_n) \geq \mu_0(X) \cdot (1 - 2^{-n-2})$. Then $Y = \bigcap_n \bigcup A_n$ is a closed subset of X with $\mu_0(Y) \geq \mu_0(X)/2$, hence $Y \in \mathbf{R}$. Define $f_0 : Y \to 2^\omega$ such that if $a \in Y$, $n < \omega$, and $i = 0, 1$ then $f_0(a)(n) = i$ iff there is a condition $Y' \in A_n$ which contains a and $\mathbf{R}$-forces $\dot{y}_0(n) = i$. Then f_0 is continuous, and any continuous extension $f : 2^\omega \to 2^\omega$ of f_0 witnesses $Y \in \mathbf{R}'$.

The result for the spaces $(2^\omega)^\omega$, $\mathbb{I}$, $\mathbb{I}^\omega$ can be derived by means of suitable continuous maps $2^\omega \xrightarrow{\text{onto}} (2^\omega)^\omega$, $2^\omega \xrightarrow{\text{onto}} \mathbb{I}$, and $2^\omega \xrightarrow{\text{onto}} \mathbb{I}^\omega$. $\square$

4 Proof of Theorem 1.2: case split

Proof (Theorem 1.2, completed in Sect. 7) Let $\boldsymbol{a}_0 \in 2^\omega$ be Solovay-random over the background set universe $\mathbf{V}$. We shall assume that $\boldsymbol{x}_0 \in \mathbf{V}[\boldsymbol{a}_0]$ is a real in the unit segment $\mathbb{I} = [0, 1]$. By Lemma 3.2, there is a continuous map $f_0 : 2^\omega \to \mathbb{I}$, coded in $\mathbf{V}$, such that $\boldsymbol{x}_0 = f_0(\boldsymbol{a}_0)$. Let μ_0 be the usual product probability measure on 2^ω, and λ be the Lebesgue measure on $\mathbb{I} = [0, 1]$.

We have to prove the trichotomy (R1) vs. (R2) vs. (R3) of Theorem 1.2.

First split. *Arguing in* $\mathbf{V}$, consider the set $C = \{x \in \mathbb{I} : \mu_0({f_0}^{-1}[x]) > 0\}$. It is at most countable. Consider the complementary sets $D = {f_0}^{-1}[C]$ and $A_1 = 2^\omega \smallsetminus D$. These are $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$ sets, respectively, coded in $\mathbf{V}$. We identify them with "the same" (*i.e.*, coded by the same codes) sets in the extensions $\mathbf{V}[\boldsymbol{a}_0]$, $\mathbf{V}[\boldsymbol{x}_0]$.

[4] We call a point $b \in 2^\omega$ "rational" iff it is eventual 0 or eventual 1.

Case 1: $\boldsymbol{a}_0 \in D$. Then there is a real $x_1 \in C \subseteq \mathbb{I} \cap \mathbf{V}$ such that $\boldsymbol{a}_0 \in f_0{}^{-1}[x_1]$, hence $\boldsymbol{x}_0 = x_1 \in \mathbf{V}$, and (R1) holds.

Case 2: $\boldsymbol{a}_0 \in A_1$ (it will be clear that $\boldsymbol{x}_0 \notin \mathbf{V}$ in this case). Then $\mu_0(A_1) > 0$ by the randomness. The set $Y_1 = f_0[A_1]$ is analytic, and we have by construction:

(I) if $B \subseteq A_1$ is Borel and $\mu_0(B) > 0$ then f_0 is **not** a constant on B.

Second split. *Still arguing in* $\mathbf{V}$, we let $\mathscr{B}$ be the family of all Borel sets $B \subseteq A_1$ such that $\mu_0(B) > 0$ and f_0 is 1-1 on B. The set $\mathscr{B}$ can be empty or not, but anyway there is a Borel set B_0, equal to a union of $\le \aleph_0$ sets in $\mathscr{B}$, such that $\mu_0(B \smallsetminus B_0) = 0$ for any $B \in \mathscr{B}$. (If $\mathscr{B} = \varnothing$ then $B_0 = \varnothing$ as well.) We let $A_2 = A_1 \smallsetminus B_0$ and $Y_2 = f_0[A_2]$. Thus A_2 is Borel, $Y_2 \subseteq Y_1$ analytic, and

(II) if $B \subseteq A_2$ is Borel and $\mu_0(B) > 0$ then f_0 is **not** 1-1 on B.

Subcase 2a of Case 2: $\boldsymbol{a}_0 \in A_1 \smallsetminus A_2 = B_0$. Then there is a Borel set $B \subseteq A_1$ such that $\boldsymbol{a}_0 \in B$ and f_0 is 1-1 on B. It follows that $\boldsymbol{a}_0 = (f_0 \restriction B)^{-1}(\boldsymbol{x}_0)$ is absolutely definable (in fact Σ^1_1-definable) from $\boldsymbol{x}_0$ and some parameters $p, p' \in 2^\omega \cap \mathbf{V}$ (*i.e.*, codes for f_0, B resp.). We conclude that $\boldsymbol{a}_0 \in \mathbf{V}[\boldsymbol{x}_0]$, thus (R2) holds.

Subcase 2b of Case 2: $\boldsymbol{a}_0 \in A_2$, hence $\mu_0(A_2) > 0$ by the randomness. This is the **key subcase**, mostly considered in the three following sections. The goal will be to get (R3), of course, that is, both (R3)(a) and (R3)(b). □

5 The key subcase, preliminaries

We argue under the assumption of Subcase 2b, *i.e.*, $\boldsymbol{a}_0 \in A_2$, and hence $\mu_0(A_2) > 0$. It holds in $\mathbf{V}$ that there is an $\mathbf{F}_\sigma$ set $A'_2 \subseteq A_2$ of the same measure $\mu_0(A'_2) = \mu_0(A_2)$. The Borel set $A_2 \smallsetminus A'_2$, coded in $\mathbf{V}$, is null, and hence $\boldsymbol{a}_0 \in A'_2$. Therefore there is, in $\mathbf{V}$, a perfect set $A_3 \subseteq A'_2$, satisfying $\boldsymbol{a}_0 \in A_3$ and $\mu_0(A_3) > 0$.

The set R of all open rational intervals $J \subseteq \mathbb{I}$ such that $\mu_0(A_3 \cap f_0{}^{-1}[J]) = 0$ is at most countable. Therefore $\mathbb{A}_0 = A_3 \smallsetminus \bigcup_{J \in R} f_0{}^{-1}[J]$ is a closed subset of A_3, of the same measure $\mu_0(\mathbb{A}_0) = \mu_0(A_3) > 0$ — hence $\boldsymbol{a}_0 \in \mathbb{A}_0$ (by the randomness). To simplify things, define **the restricted function** $f = f_0 \restriction \mathbb{A}_0$. Then f maps $\mathbb{A}_0$ onto the closed set $\mathbb{Y}_0 = f_0[\mathbb{A}_0] = f[\mathbb{A}_0]$ (since generally continuous images of compact sets are compact), the real $\boldsymbol{x}_0 = f_0(\boldsymbol{a}_0) = f(\boldsymbol{a}_0)$ belongs to $\mathbb{Y}_0$, and we have

(III) if J is an open interval in $\mathbb{I}$ and $\mathbb{Y}_0 \cap J \neq \varnothing$ then $\mu_0(f^{-1}[\mathbb{Y}_0 \cap J]) > 0$.

We also define **the restricted measure** $\mu(A) = \mu_0(A)/\mu_0(\mathbb{A}_0)$, for any Borel set $A \subseteq \mathbb{A}_0$, so μ is a continuous probability measure on $\mathbb{A}_0$, and $\boldsymbol{a}_0 \in \mathbb{A}_0$ is μ-random over $\mathbf{V}$. The following two claims are easy corollaries of (I), (II) above, since generally (I), (II) are preserved under the restriction of the domain, so that

(I') if $x \in \mathbb{Y}_0$ then $\mu(f^{-1}[x]) = 0$ (f-preimages of singletons are μ-null);
(II') if $B \subseteq \mathbb{A}_0$ is Borel and $\mu(B) > 0$ then f is **not** 1-1 on B.

Lemma 5.1 *If* $x \in \mathbb{I}$ *then let* $g(x) = \mu(f^{-1}[\mathbb{Y}_0 \cap [0, x)])$, *so* $g : \mathbb{I} \to \mathbb{I}$. □

Lemma 5.2 *The map* g *is continuous,* $\mathtt{rang} = \mathbb{I}$, *and* g *is strictly increasing, except that* $g(x) = g(x')$ *in case when* $x < x'$ *belong to* $\mathbb{I}$ *and* $\mathbb{Y}_0 \cap (x, x') = \varnothing$.

Proof Let $x < x'$ belong to $\mathbb{I}$. Then $g(x) \le g(x')$ is clear. To prove the strict inequality, note that $g(x') - g(x) = \mu(f^{-1}[\mathbb{Y}_0 \cap [x, x')])$, which is strictly positive by (III) provided $\mathbb{Y}_0 \cap (x, x') \ne \varnothing$. The map g is continuous by (I'). □

Lemma 5.3 *The superposition map $F(a) = g(f(a)) : \mathbb{A}_0 \xrightarrow{onto} \mathbb{I}$ is continuous and measure-preserving in the sense that if $X \subseteq \mathbb{I}$ is Borel then $\mu(F^{-1}[X]) = \lambda(X)$, while if $A \subseteq \mathbb{A}_0$ is Borel then $\lambda(F[A]) \ge \mu(A)$.*

Proof Consider any interval $X = [0, m)$ in $\mathbb{I}$, $0 \le m \le 1$; thus $\lambda(X) = m$. By definition, we have $g(x) \in X$ iff $\mu(f^{-1}[\mathbb{Y}_0 \cap [0, x)]) < m$. Therefore the g-preimage $g^{-1}[X]$ is equal to $[0, R)$, where R is the smallest real in $\mathbb{I}$ satisfying the inequality $\mu(f^{-1}[\mathbb{Y}_0 \cap [0, R)]) \ge m$. Then clearly $\mu(f^{-1}[\mathbb{Y}_0 \cap [0, R)]) = m$.

But $f^{-1}[\mathbb{Y}_0 \cap [0, R)] = f^{-1}[g^{-1}[X]] = F^{-1}[X]$. We conclude that $\mu(F^{-1}[X]) = m = \lambda(X)$ for any $X = [0, m)$, as above. By induction, this implies $\mu(F^{-1}[X]) = \lambda(X)$ for any Borel set $X \subseteq \mathbb{I}$, the first claim. The second claim follows, since $A \subseteq F^{-1}[F[A]]$, and any analytic set has a Borel superset of the same measure. □

Corollary 5.4 (under Subcase 2b) *The real $\boldsymbol{y}_0 = F(\boldsymbol{a}_0) = g(\boldsymbol{x}_0) \in \mathbb{I}$ is λ-random over $\mathbf{V}$. Thus the model $\mathbf{V}[\boldsymbol{x}_0] = \mathbf{V}[\boldsymbol{y}_0]$ is a Solovay-random extension of $\mathbf{V}$, so that (R3)(a) holds.* □

Proof To prove the second claim, note that g is "almost" 1-1 (except for possibly 2-element pre-images) on $\mathbb{Y}_0$ by Lemma 5.2, and hence $\mathbf{V}[\boldsymbol{x}_0] = \mathbf{V}[\boldsymbol{y}_0]$. □

6 The key subcase, measure construction

Arguing under the assumption of Subcase 2b, we are going to prove that $\mathbf{V}[\boldsymbol{a}_0]$ is a random extension of $\mathbf{V}[\boldsymbol{x}_0]$, that is, (R3)(b). A measure ν on the set $\Omega = F^{-1}[\boldsymbol{y}_0]$ will be defined in $\mathbf{V}[\boldsymbol{x}_0] = \mathbf{V}[\boldsymbol{y}_0]$, with respect to which $\boldsymbol{a}_0$ is random. We'll make use of the following lemma which combines effects of random forcing and Shoenfield's absoluteness.

Lemma 6.1 *Let $\varphi(x)$ be a combination of Σ^1_1-formulas and Π^1_1-formulas, by means of $\wedge$, $\vee$, $\neg$, and quantifiers over ω, and with reals in $\mathbf{V}$ as parameters. If $\varphi(\boldsymbol{y}_0)$ is true then there is a closed set $Y \subseteq \mathbb{I}$ of positive measure $\lambda(Y) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and satisfying $\varphi(y)$ for all $y \in Y$.*

Proof The set $\{y : \varphi(y)\}$ is measurable, hence, it is true in $\mathbf{V}$ that any Borel set $Y_0 \subseteq \mathbb{I}$ of positive measure contains a perfect subset $Y \subseteq Y_0$ still of positive measure $\lambda(Y) > 0$, satisfying either (1) $\forall\, y \in Y \varphi(y)$ or (2) $\forall\, y \in Y \neg\varphi(y)$. These formulas are Π^1_2 (with a parameter $p \in 2^\omega \cap \mathbf{V}$ for the set Y), hence absolute by Shoenfield's absoluteness. It follows by the randomness of $\boldsymbol{y}_0$ that there is a perfect subset $Y \subseteq \mathbb{I}$ of positive measure, containing $\boldsymbol{y}_0$ and satisfying (1) or (2). But (2) is impossible because of $\varphi(\boldsymbol{y}_0)$. □

Recall that $\mathbb{A}_0 \subseteq 2^\omega$, $\mu(\mathbb{A}_0) = 1$, $f : \mathbb{A}_0 \xrightarrow{\text{onto}} \mathbb{Y}_0 \subseteq \mathbb{I}$, and $F = g \circ f : \mathbb{A}_0 \xrightarrow{\text{onto}} \mathbb{I}$. Suppose that $B \subseteq \mathbb{A}_0$ is a Borel set.

If $X \subseteq \mathbb{I}$ then let $B \Lsh X = B \cap F^{-1}[X] = \{a \in B : F(a) \in X\}$, e.g. $B \Lsh \mathbb{I} = B$. In particular, if $x \in \mathbb{I}$ then let $B \Lsh x = B \cap F^{-1}[x] = \{a \in B : F(a) = x\}$.

Lemma 6.2 *If $B \subseteq \mathbb{A}_0$ and $X \subseteq \mathbb{I}$ are Borel sets then $\mu(B) \le \lambda(F[B])$ and $\mu(B \Lsh X) \le \lambda(X)$.*

Proof Apply Lemma 5.3. To prove the second inequality, put $A = B \Lsh X$. Then $\mu(A) \le \lambda(F[A])$ by Lemma 5.3. However $F[A] = X \cap F[B]$, hence we have $\lambda(F[A]) \le \lambda(X)$. □

If $X \subseteq \mathbb{I}$ is Borel then put $\lambda_B(X) = \mu(B \Lsh X)$; λ_B is a σ-additive Borel measure on $\mathbb{I}$, concentrated on $F[B]$ (that is, $\lambda_B(\mathbb{I} \smallsetminus F[B]) = 0$) and satisfying $\lambda_B(X) \le \lambda(X)$ and $\lambda_B(\mathbb{I}) = \lambda_B(F[B]) = \mu(B)$. Therefore the map $U_B(x) = \lambda_B([0,x)) = \mu(B \Lsh [0,x)) : \mathbb{I} \to \mathbb{I}$ is *non-decreasing* and *Lipschitz*, so that if $x < y$, then $U_B(x) \le U_B(y)$ and $U_B(y) - U_B(x) \le y - x$.

Proposition 6.3 (see e.g. [18], sections 2, 5, 13)

(i) *If $B \subseteq \mathbb{A}_0$ is Borel then a derivative $U'_B(x) < \infty$ exists for λ-almost all $x \in \mathbb{I}$;*
(ii) *if $B_0, B_1, \ldots \subseteq \mathbb{A}_0$ are pairwise disjoint Borel sets, and $B = \bigcup_n B_n$, then we have $U_B(x) = \sum_n U_{B_n}(x)$ for all x, and $U'_B(x) = \sum_n U'_{B_n}(x)$ for λ-almost all $x \in \mathbb{I}$;*
(iii) *if $B \subseteq \mathbb{A}_0$ is a Borel set and $U'_B(x) = 0$ for λ-almost all $x \in \mathbb{I}$, then $U_B(x) = 0$ for all $x \in \mathbb{I}$, and hence $\mu(B) = \lambda_B(\mathbb{I}) = U_B(1) = 0$.* □

Lemma 6.4 *If $C \subseteq \mathbb{A}_0$ and $X \subseteq \mathbb{I}$ are Borel sets, and $B = C \Lsh X$, then $U'_C(x) = U'_B(x)$ for λ-almost all $x \in X$.*

Proof Let $A = C \smallsetminus B$, so that X and $Y = \mathbb{I} \smallsetminus X$ are disjoint Borel sets satisfying $F[A] \subseteq Y$ and $F[B] \subseteq X$. We have $U_C(x) = U_B(x) + U_A(x)$ for all $x \in \mathbb{I}$ and $U'_C(x) = U'_B(x) + U'_A(x)$ (in particular all three derivatives are defined) for λ-almost all $x \in \mathbb{I}$ by Proposition 6.3(ii). Recall that

$$U_A(x) = \mu(C \Lsh (Y \cap [0,x))) = \mu(\{a \in C : F(a) \in Y \cap [0,x)\})$$

by construction. We claim that $U'_A(x) = 0$ for all points $x \in X$ of X-density 1. Indeed suppose that $x \in X$ is such. Fix $\varepsilon > 0$. (We consider the right-side derivative for brevity.) There is $\delta = \delta_\varepsilon > 0$ such that we have $\frac{\lambda(X \cap [x, x+\alpha))}{\alpha} \ge 1 - \varepsilon$ whenever $0 < \alpha < \delta$. Now assume that $0 < \alpha < \delta$. Then

$$U_A(x+\alpha) - U_A(x) = \mu(C \Lsh (Y \cap [x, x+\alpha))) \le \lambda(Y \cap [x, x+\alpha)) \le \alpha\varepsilon$$

by Lemma 6.2 and the choice of δ, and finally $\frac{U_A(x+\alpha) - U_A(x)}{\alpha} \le \varepsilon$. As $\varepsilon > 0$ and $\alpha < \delta_\varepsilon$ are arbitrary in this argument, we can conclude that $U'_A(x) = 0$.

Thus we have $U'_A(x) = 0$ for λ-almost all $x \in X$ by the Lebesgue density theorem, and this implies the lemma. □

Lemma 6.5 *We let $\Omega = f^{-1}[\boldsymbol{x}_0] = F^{-1}[\boldsymbol{y}_0] = \mathbb{A}_0 \Lsh \boldsymbol{y}_0$. This is a closed subset of $\mathbb{A}_0$, containing $\boldsymbol{a}_0$ and coded in $\mathbf{V}[\boldsymbol{y}_0]$ (not necessarily in $\mathbf{V}$).*

Note that if $B \subseteq \mathbb{A}_0$ is a Borel set then $B \Lsh \boldsymbol{y}_0 = B \cap \Omega$. □

Lemma 6.6 *Assume that $\langle P_n\rangle_{n<\omega}$ is a sequence, coded in $\mathbf{V}[\boldsymbol{y}_0]$, of Borel sets $P_n \subseteq \Omega$. Then there is a sequence $\langle B_n\rangle_{n<\omega}$, coded in $\mathbf{V}$, of Borel sets $B_n \subseteq \mathbb{A}_0$, such that $P_n = B_n \upharpoonright\upharpoonright \boldsymbol{y}_0 = B_n \cap \Omega$ for all n.*

Proof There is an ordinal $\rho < \omega_1$ such that each P_n is a $\boldsymbol{\Sigma}^0_\rho$ set coded in $\mathbf{V}[\boldsymbol{y}_0]$. As ω_1 is the same for $\mathbf{V}$ and $\mathbf{V}[\boldsymbol{y}_0]$, there exists a $\boldsymbol{\Sigma}^0_\rho$ set $U \subseteq \mathbb{I} \times \mathbb{A}_0$, coded in $\mathbf{V}$, and universal (in all models with the same ω_1) for all $\boldsymbol{\Sigma}^0_\rho$ sets $X \subseteq \mathbb{A}_0$. By universality, for each n there is a real $z_n \in \mathbb{I} \cap \mathbf{V}[\boldsymbol{y}_0]$ such that $P_n = U_{z_n} = \{a : \langle z_n, a\rangle \in U\}$. By Lemma 3.2, there is a continuous map $\zeta : \mathbb{I} \to \mathbb{I}^\omega$ coded in $\mathbf{V}$ and satisfying $z_n = \zeta(\boldsymbol{y}_0)(n)$ for all n. Let $W = \{\langle n, y, a\rangle \in \omega \times \mathbb{I} \times \mathbb{A}_0 : \langle \zeta(y)(n), a\rangle \in U\}$. Then

$$P_n = U_{z_n} = \{a : \langle \zeta(\boldsymbol{y}_0)(n), a\rangle \in U\} = \{a : \langle n, \boldsymbol{y}_0, a\rangle \in W\} = W_{n\boldsymbol{y}_0} \quad \text{for all} n.$$

Then each $B_n = \{a \in \mathbb{A}_0 : a \in W_{nF(a)}\}$ is a Borel set, and the sequence of all sets B_n is coded in $\mathbf{V}$. Moreover,

$$B_n \cap \Omega = \{a \in \Omega : a \in W_{nF(a)}\} = \Omega \cap W_{n\boldsymbol{y}_0} = \Omega \cap P_n = P_n$$

(since $P_n \subseteq \Omega$), thus $P_n = B_n \cap \Omega = B_n \upharpoonright\upharpoonright \boldsymbol{y}_0$, as required. □

Lemma 6.7 (Definition of the measure ν) *If $P \subseteq \Omega$ is a Borel set coded in $\mathbf{V}[\boldsymbol{y}_0]$ then let $\nu(P) = U'_B(\boldsymbol{y}_0)$, for any Borel set B coded in $\mathbf{V}$ and satisfying $P = B \upharpoonright\upharpoonright \boldsymbol{y}_0$. (Such sets B exist by Lemma 6.6.) Here $U'_B(\boldsymbol{y}_0)$ is defined by Proposition 6.3(i), as $\boldsymbol{y}_0$ is random over $\mathbf{V}$ by Corollary 5.4.* □

Lemma 6.8 *$\nu(P)$ is independent of the choice of B in Definition 6.7.*

Proof Let $C \subseteq \mathbb{A}_0$ be another Borel set such that $P = C \upharpoonright\upharpoonright \boldsymbol{y}_0$. By Lemma 6.1, there is a Borel set $X \subseteq \mathbb{I}$ of positive measure $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $C \upharpoonright\upharpoonright y = B \upharpoonright\upharpoonright y$ holds for all $y \in X$. Therefore the sets $B_1 = B \upharpoonright\upharpoonright X = \bigcup_{y\in X}(B \upharpoonright\upharpoonright y)$ and $C_1 = C \upharpoonright\upharpoonright X = \bigcup_{y\in X}(C \upharpoonright\upharpoonright y)$ coincide with each other. However we have $U'_B(y) = U'_{B_1}(y)$ and $U'_C(y) = U'_{C_1}(y)$ for λ-almost all $y \in X$ by Lemma 6.4. We conclude that $U'_B(y) = U'_C(y)$ for λ-almost all $y \in X$. It follows that $U'_B(\boldsymbol{y}_0) = U'_C(\boldsymbol{y}_0)$, since $\boldsymbol{y}_0 \in X$ is random. □

Thus ν is a well-defined function on Borel sets $P \subseteq \Omega$ in $\mathbf{V}[\boldsymbol{y}_0]$.

7 The key subcase, proof of randomness

To finalize the proof of Theorem 1.2 in Case 2b, we are going to show that $\boldsymbol{a}_0$ is ν-random over $\mathbf{V}[\boldsymbol{y}_0]$. Then it suffices to apply Lemma 3.1, to transform $\boldsymbol{a}_0$ to a "standard" λ-random real in $\mathbb{I}$.

Lemma 7.1 *In $\mathbf{V}[\boldsymbol{y}_0]$, ν is a σ-additive continuous probability measure on Ω.*

Proof (A) To prove $\nu(\Omega) = 1$ take $B = \mathbb{A}_0$. Then $\mathbb{A}_0 \Uparrow \boldsymbol{y}_0 = F^{-1}[\boldsymbol{y}_0] = \Omega$. Lemma 5.3 implies

$$U_{\mathbb{A}_0}(x) = \lambda_{\mathbb{A}_0}([0,x)) = \mu(\mathbb{A}_0 \Uparrow [0,x)) = \mu(F^{-1}[[0,x)]) = \lambda([0,x)) = x\,,$$

and hence $U'_{\mathbb{A}_0}(x) = 1$ for all x. In particular, $\nu(\Omega) = U'_{\mathbb{A}_0}(\boldsymbol{y}_0) = 1$.

(B) Prove the σ-additivity of ν. Lemma 6.6 reduces this to the following claim:

if $\langle C_n \rangle_{n<\omega} \in \mathbf{V}$ *is a sequence of Borel sets* $C_n \subseteq \mathbb{A}_0$ satisfying $(C_k \Uparrow \boldsymbol{y}_0) \cap (C_n \Uparrow \boldsymbol{y}_0) = \varnothing$ for all $k \neq n$, and $C = \bigcup_n C_n$, *then* $U'_C(\boldsymbol{y}_0) = \sum_n U'_{C_n}(\boldsymbol{y}_0)$.

By Lemma 6.1, there is a Borel set $X \subseteq \mathbb{I}$ with $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $(C_k \Uparrow y) \cap (C_n \Uparrow y) = \varnothing$ for all $y \in X$, $k \neq n$. The Borel sets $B_n = C_n \Uparrow X \subseteq \mathbb{A}_0$ are pairwise disjoint, and the set $B = C \Uparrow X$ satisfies $B = \bigcup_n B_n$.

Moreover, we have $U_B(x) = \sum_n U_{B_n}(x)$ for all x, and $U'_B(x) = \sum_n U'_{B_n}(x)$ for λ-almost all $x \in \mathbb{I}$ by Proposition 6.3(ii). Finally, Lemma 6.4 implies that $U'_B(x) = U'_C(x)$ and $U'_{B_n}(x) = U'_{C_n}(x)$ for all n and λ-almost all $x \in X$. It follows that $U'_C(x) = \sum_n U'_{C_n}(x)$ for λ-almost all $x \in X$, hence, $U'_C(\boldsymbol{y}_0) = \sum_n U'_{C_n}(\boldsymbol{y}_0)$ by the randomness, as required.

(C) To prove that ν is continuous, suppose to the contrary that $z_0 \in \Omega$ and $\nu(\{z_0\}) > 0$. By definition there is a Borel set $C \subseteq \mathbb{A}_0$, coded in $\mathbf{V}$ and satisfying $C \Uparrow \boldsymbol{y}_0 = \{z_0\}$ and $U'_C(\boldsymbol{y}_0) > 0$. By Lemma 6.1, there is a Borel set $X \subseteq \mathbb{I}$ with $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $C \Uparrow y$ is a singleton and $U'_C(y) > 0$ for all $y \in X$. Let $B = C \Uparrow X$. Then $B \Uparrow \boldsymbol{y}_0 = \{z_0\}$, $B \Uparrow y$ is a singleton for all $y \in X$, and $U'_B(y) > 0$ for λ-almost all $y \in X$, by Lemma 6.4. It follows that $U_B(1) > 0$, hence $\mu(B) = U_B(1) > 0$. Moreover, by the singleton condition, the preimage $F^{-1}[y] \cap B = B \Uparrow y$ is a singleton for all $y \in F[B] \subseteq X$, or in other words, F is 1-1 on B. Then f is 1-1 on B as well, since $F(a) = g(f(a))$. But this contradicts (II') of Sect. 5. □

Lemma 7.2 *The real* $\boldsymbol{a}_0$ *is* ν*-random over* $\mathbf{V}[\boldsymbol{y}_0]$*, so that (R3)(b) holds.*

Proof Assume that $P \subseteq \Omega$ is a Borel set, coded in $\mathbf{V}[\boldsymbol{y}_0]$, and $\nu(P) = 0$; we have to prove that $\boldsymbol{a}_0 \notin P$. By definition there is a Borel set $C \subseteq \mathbb{A}_0$, coded in $\mathbf{V}$ and satisfying $P = C \Uparrow \boldsymbol{y}_0$ and $U'_C(\boldsymbol{y}_0) = 0$. By Lemma 6.1, there is a closed (here, this is more suitable than Borel) set $X \subseteq \mathbb{I}$ of positive measure $\lambda(X) > 0$, coded in $\mathbf{V}$, containing $\boldsymbol{y}_0$, and such that $U'_C(y) = 0$ for all $y \in X$.

Let $B = C \Uparrow X$. Then $P = B \Uparrow \boldsymbol{y}_0$, and $U'_B(y) = 0$ for λ-almost all $y \in X$ by Lemma 6.4. Note that $F[B] \subseteq X$, thus $U_B(x)$ is a constant inside any open interval disjoint from X. Thus $U'_B(y) = 0$ for all $y \in \mathbb{I} \smallsetminus X$, hence overall $U'_B(y) = 0$ for λ-almost all $y \in \mathbb{I}$. This implies $U_B(x) = 0$ for all $x \in \mathbb{I}$ by Proposition 6.3(iii). Therefore $\lambda_B(\mathbb{I}) = \mu(B) = 0$ by construction. We conclude that $\boldsymbol{a}_0 \notin B$, by the μ-randomness of $\boldsymbol{a}_0$. Then $\boldsymbol{a}_0 \notin P = B \Uparrow \boldsymbol{y}_0$, as required. □

□ (Theorem 1.2)

Corollary 7.3 *If* x, y *are reals in an* $\aleph_1$*-random extension* $N = \mathbf{L}[\langle a_\xi \rangle_{\xi<\omega_1}]$ *of* $\mathbf{L}$*, then* y *belongs to a random extension of* $\mathbf{L}[x]$ *inside* N*.*

Proof We have $x \in N_\alpha = \mathbf{L}[\langle a_\xi \rangle_{\xi<\alpha}]$ and $y \in N_\beta$, for some $\alpha < \beta < \omega_1$. The model N_α is equal to a simple extension of $\mathbf{L}$ by one random real. Thus, by Theorem 1.2, either $N_\alpha = \mathbf{L}[x]$ or N_α is a random extension of $\mathbf{L}[x]$. In addition, N_β is a random extension of N_α. This implies the result. □

8 Proof of the localization lemma

Proof (Lemma 2.3) Let $\mathbf{R}$ be the Solovay-random forcing notion, and $\Vdash_{\mathbf{R}}$ be the associated forcing relation. Let $\nVdash$ be the weakest element of $\mathbf{R}$, and $\check{x}$ be the canonical name for a set x in the ground set universe $\mathbf{V}$.

Claim 8.1 *If $n \geq 2$ and $\varphi(\cdot)$ is a parameter-free Σ^1_n-formula (Π^1_n-formula), then the set $F_\varphi = \{x : \nVdash \Vdash_{\mathbf{R}} \varphi(\check{x})\}$ is Σ^1_n (Π^1_n, resp.).*

Proof We make use of a standard Borel coding system for subsets of 2^ω. It consists of Π^1_1 sets $\mathbf{C} \subseteq 2^\omega$ and $W_+, W_- \subseteq 2^\omega \times 2^\omega$, and an assignment $c \mapsto \mathbf{B}_c \subseteq 2^\omega$, such that (1) $\{\mathbf{B}_c : c \in \mathbf{C}\}$ is exactly the family of all Borel sets $X \subseteq 2^\omega$, and (2) if $c \in \mathbf{C}$ and $x \in 2^\omega$ then $x \in \mathbf{B}_c$ iff $W_+(c, x)$ iff $\neg\, W_-(c, x)$.

To define an associated coding system for Borel maps, let $e \mapsto \langle (e)_n \rangle_{n<\omega}$ be a recursive homeomorphism $2^\omega \xrightarrow{\text{onto}} (2^\omega)^\omega$. Let $\mathbf{CF} = \{e \in 2^\omega : \forall n((e)_n \in \mathbf{C})\}$ — codes of Borel maps $f : 2^\omega \to 2^\omega$. If $e \in \mathbf{CF}$ then define a Borel map $\mathbf{F}_e : 2^\omega \to 2^\omega$ so that $\mathbf{F}_e(x)(n) = 1$ iff $x \in \mathbf{B}_{(e)_n}$, for all $x \in 2^\omega$, $n < \omega$.

If $\varphi(v_1, \dots, v_k)$ is any formula, $e_1, \dots, e_k \in \mathbf{CF}$, and $x \in 2^\omega$, then let $\varphi(e_1, \dots, e_k)[x]$ be the formula $\varphi(\mathbf{F}_{e_1}(x), \dots, \mathbf{F}_{e_k}(x))$, and let

$$\mathbf{Forc}_\varphi = \{\langle c, e_1, \dots, e_k \rangle \in \mathbf{C} \times \mathbf{CF}^k : \mu_0(\mathbf{B}_c) > 0 \wedge \mathbf{B}_c \Vdash_{\mathbf{R}} \varphi(e_1, \dots, e_k)[\mathbf{a}]\},$$

where $\mathbf{a}$ is a canonical name for the random real. We assert the following.

(†) If φ is a Π^1_1 formula then $\mathbf{Forc}_\varphi \in \Sigma^1_2$. If φ is a Σ^1_n formula, $n \geq 2$, then $\mathbf{Forc}_\varphi \in \Sigma^1_n$. If φ is a Π^1_n formula, $n \geq 2$, then $\mathbf{Forc}_\varphi \in \Pi^1_n$.

This is proved by induction. If $\varphi(v)$ is Π^1_1 and $\mu_0(\mathbf{B}_c) > 0$ then $\langle c, e \rangle \in \mathbf{Forc}_\varphi$ iff the set $X = \{x \in \mathbf{B}_c : \neg\, \varphi(\mathbf{F}_e(x))\}$ is null, which can be proved to be Σ^1_2 by coverings with $\mathbf{G}_\delta$ sets. To pass $\Pi^1_n \to \Sigma^1_{n+1}$, assume that $\varphi(v_1) := \exists\, v_2\, \psi(v_1, v_2)$, ψ is Π^1_n. Then $\langle c, e_1 \rangle \in \mathbf{Forc}_\varphi$ iff $\exists\, e_2 \in \mathbf{CF}\, (\langle c, e_1, e_2 \rangle \in \mathbf{Forc}_\psi)$. (We make use of the fact that the random forcing admits Borel reading of names.) Thus if $\mathbf{Forc}_\psi$ is Σ^1_{n+1} then so is $\mathbf{Forc}_\varphi$. To pass $\Sigma^1_n \to \Pi^1_n$, let $\varphi(v)$ be Σ^1_n. Then

$$\langle c, e \rangle \in \mathbf{Forc}_{\neg\,\varphi} \iff \forall\, c' \in \mathbf{C}\, (\mathbf{B}_{c'} \subseteq \mathbf{B}_c \wedge \mu_0(\mathbf{B}_{c'}) > 0 \Longrightarrow \langle c', e \rangle \notin \mathbf{Forc}_\varphi).$$

Thus if $\mathbf{Forc}_\varphi$ is Σ^1_n then $\mathbf{Forc}_{\neg\,\varphi}$ is Π^1_n. This ends the proof of (†).

Finally, $x \in F_\varphi$ iff $\langle c_0, e_x \rangle \in \mathbf{Forc}_\varphi$, where $c_0 \in \mathbf{C}$ satisfies $\mathbf{B}_{c_0} = 2^\omega$, while $e_x \in \mathbf{CF}$ is such that $\mathbf{F}_{e_x}$ is the constant map: $\mathbf{F}_{e_x}(a) = x$ for all $a \in 2^\omega$.

□ (Claim)

To complete the proof of Lemma 2.3, define formulas $\varphi^*(x)$ by induction. If φ is Σ^1_2 or Π^1_2 then $\varphi^* := \varphi$ works by the Shoenfield absoluteness. Suppose that $n \geq 2$, and $\varphi(x)$ is $\exists\, y\, \psi(x, y)$ with $\psi(x, y)$ being Π^1_n, and a Π^1_n-formula ψ^* is defined and satisfies $\psi(x, y) \iff \mathbf{L}[x, y] \models \psi^*(x, y)$ in the universe $N = \mathbf{L}[\langle a_\xi \rangle_{\xi < \omega_1}]$ (a given $\aleph_1$-random extension). We let $\varphi^*(x)$ be the formula $\not\Vdash \,\|\!\!-_{\mathbf{R}} \exists\, y\, (\mathbf{L}[\check{x}, y] \models \psi^*(\check{x}, y))$. This is a Σ^1_{n+1}-formula by Claim 8.1, so it remains to show that the equivalence $\varphi(x) \iff \mathbf{L}[x] \models \varphi^*(x)$ holds in N.

Let x be a real in N satisfying $\varphi(x)$. Thus there is a real $y \in N$ satisfying $\psi(x, y)$, or equivalently, $\mathbf{L}[x, y] \models \psi^*(x, y)$. By Corollary 7.3, y belongs to a random extension of $\mathbf{L}[x]$ inside N. Therefore, as the random forcing is homogeneous, it is true in $\mathbf{L}[x]$ that $\not\Vdash \,\|\!\!-_{\mathbf{R}} \exists\, y\, (\mathbf{L}[\check{x}, y] \models \psi^*(\check{x}, y))$. In other words, $\mathbf{L}[x] \models \varphi^*(x)$.

To prove the converse, assume that $\mathbf{L}[x] \models \big(\not\Vdash \,\|\!\!-_{\mathbf{R}} \exists\, y\, (\mathbf{L}[\check{x}, y] \models \psi^*(\check{x}, y))\big)$. Consider any real $z \in N$ random over $\mathbf{L}[x]$. Then $\exists\, y\, (\mathbf{L}[x, y] \models \psi^*(x, y))$ holds in $\mathbf{L}[x, z]$, so there is a real $y \in \mathbf{L}[x, z]$ satisfying $\mathbf{L}[x, y] \models \psi^*(x, y)$. Then $N \models \psi(x, y)$ by the choice of ψ^*, hence finally $N \models \varphi(x)$.

$\square$ (Lemma 2.3 and Theorem 2.2)

9 Problems

It is natural to figure out the structure of intermediate models of other popular generic extensions, both those by a single real, and more complicated ones. As an example, let $\mathbf{S}$ be the Sacks forcing, and $\mathbf{S}^\omega$ be the countable (= full) support product. Let $N = \mathbf{L}[\langle a_n \rangle_{n<\omega}]$ be a $\mathbf{S}^\omega$-generic extension of $\mathbf{L}$. Assume that x is a real in N. Methods of [7] allow to prove that $\mathbf{L}[x] = \mathbf{L}[\langle a_n \rangle_{n \in u(x)}]$, where $u(x) = \{n : a_n \in \mathbf{L}[x]\}$. If $u(x) \in \mathbf{L}$ then the nature of the factor-extensions $\mathbf{L} \to \mathbf{L}[x]$ and $\mathbf{L}[x] \to N$ depends on the cardinalities of $u(x)$ and $\omega \smallsetminus u(x)$ in a pretty clear way. The case $u(x) \notin \mathbf{L}$ is much less clear. One of the particular questions of interest is the following: if $x, y \in N$ and $u(x) \notin \mathbf{L}$, $u(y) \notin \mathbf{L}$ (not necessarily $u(x) = u(y)$), then are the models $\mathbf{L}[x] = \mathbf{L}[\langle a_n \rangle_{n \in u(x)}]$ and $\mathbf{L}[y]$ elementarily equivalent, and are the extensions $\mathbf{L}[x] \to N$ and $\mathbf{L}[y] \to N$ *similar* in any reasonable way?

The other question is this. Let $N = \mathbf{L}[a]$ be a Cohen-generic extension. (The question is meaningful for extensions of many various types.) Suppose that, in N, E is an OD (ordinal-definable) equivalence relation on an OD set $X \neq \varnothing$ of reals, containing $\leq \aleph_0$ equivalence classes. Say exactly two classes, to begin with. Is it true that there is an OD E-equivalence class? See [2] for a surprising affirmative result (originally by Solovay) for Sacks-generic extensions.

Acknowledgements The authors thank the anonymous referee for their careful reading of the paper and substantial comments and suggestions towards major improvement of the original version of this paper.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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