The relation (4.1) shows that the analog of the inequality (2.2) for the Hilbert transforms of a function of several variables does not hold (for $y \to +0$). As Theorem 1 shows, the relation (2.2) does not hold in the multidimensional case also when $y \to +\infty$. Thus, the analog of the inequality (2.2) for multidimensional Hilbert transforms does not hold; in other words, the multiple Hilbert operator does not have the weak type (1.1).

It is easily verified that if $f \in L^p(E_n)$, $p \in (1, +\infty)$, then for each $B \subset M$ we have

$$\operatorname{mes}\{x:\ |\tilde{f}_B(x)| > y > 0,\ \ x \in E_n\} \leqslant (C/y^p)\int_{E_n} |f(x)|^p\,dx \quad (y \to +0,\ y \to +\infty).$$

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N. N. LUZIN'S PROBLEMS ON IMBEDDABILITY AND DECOMPOSABILITY OF PROJECTIVE SETS

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1. Formulation of the Problems and Basic Results

Let $\omega = \{0, 1, 2, \ldots\}$ be the natural series and $I = {}^{\omega}\omega = \{x\colon x$ is a function from ω to $\omega\}$ be the Baire space [1, p. 154] (homeomorphic to the set of all irrational points of the real line). The projective subsets of the space I^m, $m \geqslant 1$ are obtained from the open sets in these spaces by repeated application of two operations: complementation and projection. By the complement of the set $X \subseteq I^m$ is meant the difference $I^m - X$. By the projection of the set $P \subseteq I^{m+1}$ is meant the set

$$\operatorname{pr} P = \{\langle x_1, \ldots, x_m\rangle\colon \exists y\,(\langle x_1, \ldots, x_m, y\rangle \in P)\}.$$

Projective sets are organized into the projective hierarchy, formed by the classes Σ^1_n, Π^1_n, Δ^1_n. The definition of these classes goes by induction on $n \in \omega$ (see [2, Chap. 8, Sec. 2]):

Σ^1_0 is the collection of all open subsets of the form I^m;

Π^1_n is the collection of all complements of sets of the class Σ^1_n;

Σ^1_{n+1} is the collection of all projections of sets of the class Π^1_n;

$\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$.

A set is called projective if it belongs to one of the projective classes. In the classical system of notation of Luzin [3, p. 586] the classes Σ^1_n, Π^1_n, Δ^1_n have the notation A_n, CA_n, B_n, respectively.

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A set $P \subseteq I^2$ is called single-valued if each of its vertical sections $P_x = \{y: \langle x, y\rangle \in P\}$ contains no more than one point, and is called countable-valued if each P_x is no more than countable. The theory of single-valued and countable-valued sets of the first level of the projective hierarchy assumed its completed form in Chap. IV of the lectures on analytic sets and their applications of Luzin (in [3, pp. 189-224]). It is proved there, in particular, that each single-valued (or countable-valued) set of the class Σ_1^1 can be imbedded in a single-valued (respectively, countable-valued) Δ_1^1-set. Moreover, each countable-valued set of the class Δ_1^1 (or of the class Σ_1^1) can be represented as a union of a countable number of single-valued sets of the same class Δ_1^1 (respectively, of the class Σ_1^1). For proofs of these results, see also [4, pp. 72-82] and [5, Sec. 1].

Having given the definition of the projective classes, Luzin [3, pp. 230-242] poses a series of questions, whose general sense is the following: Can one carry over certain theorems about projective sets of the first level of the projective hierarchy to higher levels? Among these questions are the questions of the validity for various values of n of the following four assertions:

1.1. Each single-valued Σ_n^1-set can be imbedded in a single-valued set of the class Δ_n^1.

1.2. Each countable-valued Σ_n^1-set can be imbedded in a countable-valued set of class Δ_n^1.

1.3. Each countable-valued Δ_n^1-set is the union of a countable number of single-valued sets of class Δ_n^1.

1.4. Each countable-valued Π_{n-1}^1-set is the union of a countable number of single-valued sets of class Π_{n-1}^1.

(The last assertion is formulated by Luzin for the class CA_n, i.e., Π_n^1; the translation by 1 in our formulation is made for uniformity of the formulations of the basic results.)

For n = 0, assertions 1.1, 1.2, 1.3 are trivially true, since a countable-valued open set can only be the empty set, and assertion 1.4 is meaningless. Besides, the Luzin formulation does not touch on the case n = 0: the hierarchy of classes A_n, CA_n, B_n by definition starts with n = 1.

As was noted above, the proof of the truth of the first three assertions for n = 1 is contained in [3]. About the truth or falsity of assertion 1.4 for n = 1 (i.e., about the possibility of representing each countable-valued closed set as the union of a countable number of single-valued closed sets) nothing is known to the author. Novikov and Keldysh [6] found that assertion 1.1 is false for n = 2. The other questions remained open in [6] even for the case n = 2. The question of the validity of assertion 1.4 for n = 2 is singled out by Lyapunov [4, p. 72]. We shall show that for n = 2 each of assertions 1.2, 1.3, 1.4 is false:

THEOREM 1. There exists a single-valued Σ_2^1-set, which cannot be imbedded in any countable-valued set of class Π_2^1.

THEOREM 2. There exists a countable-valued Π_1^1-set, which is not the union of a countable number of single-valued sets of class Σ_2^1 (and all the more for the classes Δ_2^1 and Π_1^1).

As to values $n \geqslant 3$, there is little hope of deciding the questions of the truth of the assertions considered in the classical sense, i.e., of proving or refuting them by means of the axioms of ZF or ZFC (Zermelo—Frankel set theory without the axiom of choice AC or with AC, respectively [2, 7, 8]). Only the question of the consistency of these assertions and of their negations lends itself to investigation.

The simplest method of proof of consistency is the derivation of the assertion considered from the axiom of constructivity V = L (about this axiom, see [2, Chap. 5], or [7, 8]). This axiom is consistent with the axioms of ZFC, in view of which any of its consequences is also consistent with ZFC.

In [6] it is noted that the axiom V = L implies the negation of assertion 1.1 for all n, "starting with some $N \geqslant 3$."

THEOREM 3. If $n \geqslant 3$ and V = L holds, then there exists a single-valued Σ_n^1-set, which cannot be imbedded in any countable-valued set of class Π_n^1.

THEOREM 4. If $n \geqslant 3$ and V = L holds, then there exists a countable-valued $\mathbf{\Pi}^1_{n-1}$-set, which is not the union of a countable number of single-valued sets of class $\mathbf{\Sigma}^1_n$.

Thus, the axiom of constructivity implies the falseness of assertions 1.1-1.4 for all $n \geqslant 3$. Consequently, the negations of these four assertions do not contradict ZFC for $n \geqslant 3$. In other words, the methods of the system ZFC are insufficient to prove one of these assertions for any $n \geqslant 3$. The following theorem gives the consistency of the negations of 1.3 and 1.4 in an even stronger form than is guaranteed by Theorems 2 and 4.

THEOREM 5. Let us assume that the hypothesis of the existence of a strongly inaccessible cardinal is consistent with the theory ZFC. Then this theory is also consistent with the existence of a countable-valued $\mathbf{\Pi}^1_1$-set, which is not the union of a countable number of single-valued projective sets of any class.

The definition of a strongly inaccessible cardinal can be found in [2, Chap. 3] or in [7, Chap. 9], where the naturality of the hypothesis of its existence is also explained. We prove Theorem 5 with the help of the Levy—Solovay model [9], in which all projective sets are Lebesgue measurable. This model is also used in the proof of the following theorem, giving the consistency of assertion 1.2 for any $n \geqslant 3$.

THEOREM 6. Under the hypotheses of Theorem 5, the theory ZFC is consistent with the assertion that each countable-valued projective set can be imbedded in a countable-valued set of class $\mathbf{\Sigma}^1_2$.

The consistency of assertions 1.1, 1.3, 1.4 themselves (and not their negations) remains an open problem.

If one rejects the full axiom of choice AC, replacing it by the principle of dependent choice DC, postulating the possibility of a countable sequence of choices in the case when the nonempty set from which one makes the n-th choice depends on the preceding choices [2, Chap. 2], [8], then Theorems 5 and 6 can be strengthened:

THEOREM 7. Under the hypotheses of Theorem 5, the assertion of the existence of a countable-valued $\mathbf{\Pi}^1_1$-set, which is not the union of a countable number of arbitrary single-valued sets, is consistent with the theory ZF + DC.

THEOREM 8. Under the hypotheses of Theorem 5, the assertion that each countable-valued set can be imbedded in a countable-valued $\mathbf{\Sigma}^1_2$-set is consistent with the theory ZF + DC.

It is easy to see that the assertions with which we are concerned in Theorems 7 and 8 are inconsistent with the stronger theory ZFC.

The content of the present paper is the proof of Theorems 1-8.

2. Basic Facts Used in the Proofs of Theorems 1-4

We use the notation Σ^1_n, Π^1_n, Δ^1_n and $\Sigma^{1,p}_n$, $\Pi^{1,p}_n$, $\Delta^{1,p}_n$ (where $p \in I$) for the classes of the effective hierarchy, see [2, Chap. 8], or [7, Chap. 7]. In a certain sense Σ^1_n is the collection of all $\mathbf{\Sigma}^1_n$-sets with recursive code, and $\Sigma^{1,p}_n$, is the collection of all $\mathbf{\Sigma}^1_n$-sets whose code is recursive with respect to p. The classes of the effective hierarchy contain subsets of spaces of the form $I^m \times \omega^k$ (where $m, k \in \omega$), and not only subsets of the spaces I^m. We note that if $X \in \mathbf{\Sigma}^1_n$, then $X \in \Sigma^{1,p}_n$ for suitable $p \in I$ and analogously for Π and Δ.

By the uniformization principle for the class Γ is meant the assertion that for each set $P \subseteq I^2$ of the class Γ one can find a single-valued set $Q \subseteq P$ of the same class Γ such that pr Q = pr P. The following uniformization theorem of Novikov—Kondo—Addison is proved, e.g., in [2, Chap. 8] or in [7, Chap. 7].

2.1. THEOREM. Let $p \in I$. Then the uniformization principle is valid for the class $\Pi^{1,p}_1$, and also for the class $\mathbf{\Pi}^1_1$.

2.2. COROLLARY. If $p \in I$, then the uniformization principle is valid for the class $\Sigma^{1,p}_2$. It is also valid for the class $\mathbf{\Sigma}^1_2$.

2.3. COROLLARY. If $p \in I$ and the set $X \subseteq I$ of class $\Sigma^{1,p}_2$ is nonempty, then one can find a $\Delta^{1,p}_2$-point $x \in X$.

To prove 2.3 it is necessary to apply 2.2 to the set $\{a\} \times X$, where $a = \omega \times \{0\} \in I$. The proof of 2.2 is achieved by application of Theorem 2.1 to the set of class $\Pi^{1,p}_1$ (or

Π^1_1), whose projection pr Q is equal to G"P, where $P \subseteq I^2$ is a given set of class $\Sigma_2^{1,p}$ (or Σ^1_2), and G is the canonical homeomorphism of I^2 onto I, defined by the condition

$$z = G(x, y) \leftrightarrow \forall i\, (z(2i) = x(i) \wedge z(2i+1) = y(i)).$$

For more details, see [2, Chap. 8, Sec. 4].

The class L[x] of all sets, constructive with respect to a given point $x \in I$, is the smallest class containing x and all ordinals, and which is a model for ZFC (see [5, Sec. 2] or [8]). With each $x \in I$ one can associate a binary relation $<_x$ on I such that the following three conditions hold (see [5, Sec. 2]).

2.4. $<_x$ is a total ordering of the set $L[x] \cap I$ of type $\leqslant \omega_1$.

2.5. $\{\langle x, y, z\rangle: y <_x z\} \in \Sigma^1_2$.

2.6. If $n \geqslant 2$, $p \in I$ and the relation Q(x, y, z, w, . . .) belongs to the class $\Delta_n^{1,p}$, then the relation

$$\exists y <_x z Q(x, y, z, w, \ldots), \quad \forall y <_x z Q(x, y, z, w, \ldots)$$

belongs to the same class $\Delta_n^{1,p}$.

The following theorem, usually called Addison's uniformization theorem in the literature, is proved with the help of 2.4-2.6 by the choice of the $<_{\omega\times\{0\}}$-smallest point in each vertical section.

2.7. THEOREM. Let $p \in I$, $n \geqslant 2$ and V = L hold. Then the uniformization principle is valid for the classes $\Delta_n^{1,p}$ and Δ^1_n.

2.8. COROLLARY. Under the hypotheses of 2.7 the uniformization principle is valid for the classes $\Sigma_n^{1,p}$ and Σ^1_n.

2.9. COROLLARY. Under the hypotheses of 2.7 each nonempty $\Sigma_n^{1,p}$-set $X \subseteq I$ contains a $\Delta_n^{1,p}$-point $x \in X$.

The derivation of 2.8 and 2.9 from 2.7 is the same as the derivation of 2.2 and 2.3 from Theorem 2.1.

2.10. COROLLARY. If $y <_x z$, then $y \in \Delta_2^{1,x,z}$.

Proof. For each $u \in I$ we set $|u| = \{(u)_m: m \in \omega\} - \{(u)_0\}$ where $(u)_m \in I$ is given by the $(u)_m(k) = u(2^m(2k+1) - 1)$. The following set

$$U = \{u \in I: u = \{w: w <_x z\}\}$$

is nonempty according to 2.4 and belongs to $\Delta_2^{1,x,z}$ according to 2.5 and 2.6. Consequently, by 2.3 one can find a point $u \in U$ of class $\Delta_2^{1,x,z}$. But y = $(u)_m$ for some m.

2.11. THEOREM. Let $x \in I$ and φ be a closed Σ^1_2-formula with parameters from L[x]. Then one has the equivalence

$$(\varphi \text{ true}) \leftrightarrow (\varphi \text{ true in } L[x]).$$

This theorem is a special case of Schoenfield's principle of absoluteness (see [7, p. 457] or [5, Sec. 2]). By a Σ^1_n-formula is meant any formula of the form $Q\psi$, where the quantifier prefix Q consists of the alternating quantifiers $\exists$ and $\forall$, in number n + 1, of which the rightmost one is over ω, and all the others are over I, and the leftmost quantifier of Q is the quantifier $\exists$ (over I); finally, ψ must be a recursive formula (with variables over ω as well as over I). The concept of Π^1_n-formula is introduced analogously, only, the quantifier prefix must start with $\forall$.

The class $\Sigma_n^{1,p}$ is formed of exactly those sets which can be defined with the help of a Σ^1_n-formula with only parameter p.

2.12. COROLLARY (from 2.11). If $x \in I$ and the set $u \subseteq \omega$ belongs to the class $\Sigma_2^{1,x}$, then $u \in L[x]$ and in L[x] it is true that $u \in \Sigma_2^{1,x}$.

3. Proof of Theorems 1 and 3

In accordance with 2.2 and 2.8, it is sufficient to prove that if $n \geqslant 2$ and the uniformization principle holds for the class Σ^1_n, then there exists a single-valued Σ^1_n-set, which cannot be imbedded in any countable-valued set of class Π^1_n. We begin the proof of this assertion with a Σ^1_n-set $U \subseteq I^3$, universal in the sense that

(*) for each Σ_n^1-set $X \subseteq I^2$ one can find a point $x \in I$ such that $X = \{\langle y, z\rangle : \langle x, y, z\rangle \in U\}$.

On the existence of universal sets, see [2, Chap. 8].

The set $Z = \{\langle x, z\rangle : \langle x, x, z\rangle \in U\}$ also belongs to the class Σ_n^1. Hence, according to our assumption, one can find a single-valued Σ_n^1-set $P \subseteq Z$ such that pr P = pr Z. We shall show that this set P is also the one sought, i.e., that it is impossible to imbed it in a countable-valued Π_n^1-set.

Let us assume the contrary: $P \subseteq Q$, where the set $Q \subseteq I^2$ of class Π_n^1 is countable-valued. The difference $X = I^2 - Q$ belongs to Σ_n^1. Hence, by (*) one can find a point $x \in I$ such that $X_x = Z_x$. But from the inclusions $P \subseteq Q$ and $P \subseteq Z$ it follows that $P_x \subseteq Q_x \cap Z_x$, i.e., $P_x \subseteq Q_x \cap X_x$ by the choice of x. Hence P_x is empty, since $X = I^2 - Q$. Thus $X_x = Z_x$ is also the empty set in view of the fact that pr P = pr Z. But this cannot be, since X is the complement of a countable-valued set. The contradiction obtained finishes the proof of Theorems 1 and 3.

4. Proof of Theorems 2 and 4

In accord with the hypotheses of these theorems, we fix a natural number $n \geqslant 2$ and we assume that the following condition holds:

4.1. Either n = 2 or $n \geqslant 3 \wedge V = L$.

Under this assumption we shall first construct a countable-valued set of class Σ_n^1, which cannot be decomposed into a countable number of single-valued Σ_n^1-sets. In the course of Sec. 4 the letters x, y, z, w denote points of the space I, and the letters k, l, m denote natural numbers.

Using a suitable version of the theorem on the universal set [2, Chap. 8, Sec. 4], one can choose a Σ_n^1-formula $\theta(k, x, l)$, universal in the sense of this assertion:

4.2. For any $x \in I$ and any $\Sigma_n^{1,x}$-set $u \subseteq \omega$ one can find a $k \in \omega$, such that $u = \{l\colon \theta(k, x, l)\}$. By f_{kx} $(\in {}^{\omega}2)$ we denote the characteristic function of the set $\{l\colon \theta(k, x, l)\}$, and we consider the set

$$U = \{\langle x, f_{kx}\rangle\colon x \in I \wedge k \in \omega\}.$$

The set U is obviously countable-valued. The following two lemmas show that it is exactly the example sought.

4.3. LEMMA. U is not the union of a countable number of single-valued sets of class Σ_n^1.

Proof. Suppose, on the contrary, $U = \bigcup_{m\in\omega} U_m$, where each set U_m is single-valued and belongs to the class Σ_n^1. One can find $x \in I$ such that all U_m are $\Sigma_n^{1,x}$-sets. We consider an arbitrary set $u \subseteq \omega$ such that $u \in \Sigma_n^{1,x} - \Delta_n^{1,x}$. The characteristic function f of the set u belongs to the section U_x by definition of U and by virtue of 4.2. Consequently, f belongs to one of the sets $U_{mx} = \{y : \langle x, y\rangle \in U_m\}$. But U_{mx} is a $\Sigma_n^{1,x}$-set with no more than one point. Consequently, $f \in \Delta_n^{1,x}$ by virtue of 2.3 or 2.9 (for n = 2 and for $n \geqslant 3 \wedge V = L$, respectively). Hence the set $u = \{l : f(l) = 1\}$ also belongs to the class $\Delta_n^{1,x}$, which contradicts the choice of u.

4.4. LEMMA. U is a Σ_n^1-set.

Proof. Being a Σ_n^1-formula, the formula θ has the form $\exists y \psi(y, k, x, l)$, where ψ is a Π_{n-1}^1-formula. By $\psi^*(z, y, k, x, l)$ we denote the formula obtained from ψ by relativization to the set $\mathrm{pred}_x(z) = \{w : w <_x z\}$, i.e., by the change of all quantifiers $\exists w, \forall w$ (over I) to $\exists w <_x z$, $\forall w <_x z$, respectively. By f_{kx}^z we denote the characteristic function of the set $\{l : \theta^*(z, k, x, l)\}$, where θ^* is the formula $\exists y <_x z \psi^*(z, y, k, x, l)$.

With each point $x \in I$ we associate the sets

$$E_x = \{z \in L[x] : \forall k, l \forall y <_x z$$
$$(\psi^*(z, y, k, x, l) \leftrightarrow \psi(y, k, x, l))\},$$

$S_x = \{y \in I\colon y$ can be defined by a formula which is the conjunction of a Σ_n^1-formula with parameter x and a Π_n^1-formula with parameter x$\}$,

$$D_x = \{y \in I\colon y \in \Delta_n^{1,x}\} \ (\subseteq S_x).$$

We prove the following four assertions:

(1) If $k \in \omega$, $x \in I$, $z \in E_x$ and $D_x \subseteq \mathrm{pred}_x(z)$, then $f_{kx} = f^z_{kx}$.

(2) If $\langle x, f\rangle \in U$, then $f \in S_x$.

(3) If $x \in I$, then one can find a $z \in E_x$, such that $S_x \subseteq \mathrm{pred}_x(z)$.

(4) The set $\{\langle x, z\rangle: x \in I \wedge z \in E_x\}$ belongs to Σ^1_n.

Proof of (1). By definition of E_x, it suffices to prove the equivalence

$$\exists y <_x z \psi(y, k, x, l) \leftrightarrow \exists y \psi(y, k, x, l).$$

The implication from left to right is obvious. We shall prove the opposite implication. Let $\exists y \psi(y, k, x, l)$. Then the $\Pi^{1,x}_{n-1}$-set $Y = \{y : \psi(y, k, x, l)\}$ is nonempty. Consequently, Y contains a $\Delta^{1,x}_n$-point $y \in Y$ by virtue of 2.3 or 2.9. Then $y \in D_x$, and hence $y <_x z$ according to the formulation of (1).

The assertion (2) is obvious from the definition of U and the fact that θ is a Σ^1_n-formula.

Proof of (3). According to Proposition 4.1, two cases are possible: $n \geqslant 3 \wedge V = L$, or n = 2.

Case 1: $n \geqslant 3$ and V = L. We consider the $<_x$-smallest point $z \in I$ such that $z \notin \Delta^{1,x}_{n+1}$. Due to 2.10 the set $\mathrm{pred}_x(z)$ is exactly the collection of all $\Delta^{1,x}_{n+1}$-points $w \in I$ whence we quickly get $S_x \subseteq \mathrm{pred}_x(z)$. Moreover, standard model-theoretic arguments allow us to deduce from 2.9 that if the closed Σ^1_{n+1}-formula φ has parameters only from the set $\mathrm{pred}_x(z)$ and φ^* is obtained by relativization of φ to the set $\mathrm{pred}_x(z)$, then one has the equivalence $\varphi \leftrightarrow \varphi^*$ whence it follows that $z \in E_x$.

Case 2: n = 2. Arguing in L[x] analogously to Case 1, we find $z \in L[x] \cap I$ such that $S^{L[x]}_x \subseteq \mathrm{pred}_x(z)^{L[x]}$ and $z \in E^{L[x]}_x$, where $S^{L[x]}_x$ is "the set S_x, defined in L[x]," and so on. But from 2.11 it follows that $S^{L[x]}_x = S_x$, and from 2.5 and 2.11 we get $\mathrm{pred}_x(z)^{L[x]}_x = \mathrm{pred}_x(z)$. The last equation and Theorem 2.11 (for the formula ψ) lead to the relation $E^{L[x]}_x = E_x$. Thus, $z \in E_x$ and $S_x \subseteq \mathrm{pred}_x(z)$.

Finally, the proof of assertion (4) is obtained by direct analysis of the definitions with the help of 2.6. Instead of $z \in L[x]$ it is necessary to write $z <_x x \vee x <_x z \vee x = z$ and to use assertion 2.5. Actually, if $n \geqslant 3$, then the set (4) even belongs to the class Δ^1_n, but we do not need this.

Continuing the proof of Lemma 4.4, we give the proof of this equation:

(5) $U = \{\langle x, f\rangle: \exists k \exists z (z \in E_x \wedge f <_x z \wedge f = f^z_{kx})\}$.

Let $\langle x, f\rangle \in U$, i.e., $f = f_{kx}$ for some k. According to (3), there exists a point $z \in E_x$ such that $S_x \subseteq \mathrm{pred}_x(z)$. Then $f = f^z_{kx}$ by (1), and $f <_x z$ by (2).

Conversely, let $z \in E_x$ and $f = f^z_{kx} <_x z$. If here $D_x \subseteq \mathrm{pred}_x(z)$, then from (1) follows $f = f^z_{kx}$, i.e., $\langle x, f\rangle \in U$ by definition. Now let $D_x \not\subseteq \mathrm{pred}_x(z)$. We take $y \in D_x$ such that $\neg y <_x z$. We note that $z \in L[x]$, since $z \in E_x$. Moreover, $y \in L[x]$ (in the case V = L this is obvious, and in the case n = 2 it follows from 2.12, since $y \in D_x$ means that $y \in \Delta^{1,x}_2$). Hence $z \leqslant_x y$, whence $f <_x y$, and finally, $f \in \Delta^{1,x,y}_2$ in view of 2.12. But $y \in \Delta^{1,x}_n$. Consequently, $f \in \Delta^{1,x}_n$, i.e., f is the characteristic function of the $\Delta^{1,x}_n$-set $\{l: f(l) = 1\}$. This also gives $\langle x, f\rangle \in U$.

Equation (5) is proved. Now Lemma 4.4 is deduced easily from (5) and (4) with the help of 2.5 and 2.6.

Thus, under the assumption 4.1 we have a countable-valued $\mathbf{\Sigma}^1_n$-set U, which is not the union of a countable number of single-valued $\mathbf{\Sigma}^1_n$-sets. We shall show how from this set one can get a $\mathbf{\Pi}^1_{n-1}$-set with the same properties.

4.5. Proposition. Under the hypotheses 4.1, each $\mathbf{\Sigma}^1_n$-set is the projection of a single-valued set of class $\mathbf{\Pi}^1_{n-1}$.

In the case n = 2 this proposition follows in an elementary way from 2.1. In the case $n \geqslant 3 \wedge V = L$ it is proved in [5, Sec. 2].

We apply 4.5 to the Σ_n^1-set G"U (for the definition of the homeomorphism G: I^2 onto I, see Sec. 2). Let the single-valued Π_{n-1}^1-set $Q \subseteq I^2$ be such that G"U = pr Q. The set

$$P = \{\langle x, G(y, z)\rangle : \langle G(x, y), z\rangle \in Q\}$$

belongs to the class Π_{n-1}^1, being the continuous preimage of Q (really G is continuous in both directions). From the countable-valuedness of U and the single-valuedness of Q there follows trivially the countable-valuedness of P. Finally, if P were the union $\bigcup_{m\in\omega} P_m$ of single-valued Σ_n^1-sets P_m, then defining

$$U_m = \{\langle x, y\rangle : \exists z (\langle x, G(y, z)\rangle \in P_m)\},$$

we would get $U = \bigcup_{m\in\omega} U_m$, and each U_m is single-valued and belongs to the class Σ_n^1, which contradicts Lemma 4.3.

The proof of Theorems 2 and 4 is completed.

4.6. Remark. If $n \geqslant 2$ and the axiom of constructivity V = L holds, then each countable-valued Σ_n^1-set $P \subseteq I^2$ is the union of a countable number of single-valued Δ_{n+1}^1-sets. In fact, the set $W = \{\langle x, y\rangle : P_x = |y|\}$ belongs to the class Δ_{n+1}^1 (more precisely, is the intersection of a Σ_n^1-set with a Π_n^1-set). According to 2.7, one can find a single-valued Δ_{n+1}^1-set $Q \subseteq W$ such that pr Q = pr W. For each $m \in \omega$ we set

$$P_m = P \cap \{\langle x, (y)_m\rangle : \langle x, y\rangle \in Q \wedge m \in \omega\}.$$

All the sets P_m are single-valued, belong to the class Δ_{n+1}^1, and their union coincides with the given set P.

5. Proof of Theorems 5 and 6

Here are some words about the Levy–Solovay model [9] used in the proof of these theorems. One fixes a countable transitive $\in$-model M of the theory ZF + V = L with a strictly inaccessible in M cardinal $\Omega \in M$. Each countable transitive model M' of the theory ZFC such that $M \subseteq M'$ and Ω remains a strictly inaccessible cardinal in M' is called an Ω-model. In particular, M itself is an Ω-model.

The set $\mathcal{P}$ (in [9] it is denoted by $\mathcal{P}_\Omega$) consists of all finite sets $p \subseteq \Omega \times \omega \times \Omega$ such that

$$\forall \alpha < \Omega \text{ (the set } \{\langle k, \beta\rangle : \langle \alpha, k, \beta\rangle \in p\} \text{ is a function).}$$

The set $\mathcal{P}$ is ordered inversely by inclusion: $p \leqslant q \leftrightarrow q \subseteq p$. It belongs to each Ω-model.

A set $D \subseteq \mathcal{P}$ is called dense in $\mathcal{P}$ if for each $p \in \mathcal{P}$ one can find a $q \in D$, such that $q \leqslant p$.

Let M' be an Ω-model. The set $G \subseteq \mathcal{P}$ is called $\mathcal{P}$-generic over M' if the following three conditions hold:

1) $p \in \mathcal{P} \wedge q \in G \wedge p \geqslant q \rightarrow p \in G$,

2) $\forall p, q \in G \;\; \exists r \in G \;\; (r \leqslant p \wedge r \leqslant q)$,

3) if the set $D \in M'$, $D \subseteq \mathcal{P}$ is dense in $\mathcal{P}$, then the intersection $G \cap D$ is nonempty.

In this case there exists a smallest countable transitive $\in$-model of the theory ZFC, containing all sets from M' and the set G, denoted by M'[G]. This model is also the generic extension of Levy–Solovay of the model M'.

5.1. Proposition [9, Corollary 3.5, p. 17]. Let $G' \subseteq \mathcal{P}$ be a $\mathcal{P}$-generic set over the Ω-model M', and the point $f \in M'[G'] \cap I$ be M'-defined in M'[G'] (i.e., f is definable in M'[G'] by some $\in$-formula with parameters only from M'). Then $f \in M'$.

5.2. Proposition [9, Arguments 1.7-1.11, pp. 42-46]. If M' and G' are as in 5.1, and the set $X \in M'[G']$, $X \subseteq I$ is M'-defined in M'[G'], and $X \not\subseteq M'$, then X is uncountable in M'[G'].

From this point we fix a $\mathcal{P}$-generic set $G \subseteq \mathcal{P}$ over M. We shall show that the model N = M[G] is the one sought for Theorems 5 and 6 in the sense of the following theorem.

5.3. THEOREM. In N the following two statements are true:

(A) There exists a countable-valued Π_1^1-set which is not the union of a countable number of single-valued projective sets.

(B) Each countable-valued projective set can be imbedded in a countable-valued Σ_2^1-set.

The proof of Theorem 5.3 in a certain sense [10] can be transformed into the proof of Theorems 5 and 6. We begin the proof of Theorem 5.3 with the formulation of another proposition, collecting together several properties of the Levy—Solovay model proved in [9]. The proofs of Propositions 5.1, 5.2, 5.4 are also contained in [5, Sec. 4].

5.4. Proposition. (a) [9, Point 1.4, p. 5]. N is a countable transitive $\in$-model of the theory ZFC with the same ordinal series as M.

(b) [9, Corollary 2, p. 16]. If $x \in N \cap I$, then Ω remains an inaccessible cardinal in M[x], i.e., M[x] will be an Ω-model. Moreover, $M[x] \cap I$ is countable in N.

(c) [9, Point 4.1, p. 18]. If $x \in N \cap I$, then there exists a $\mathscr{P}$-generic over M[x] set $G' \subseteq \mathscr{P}$, such that N = M[x][G'].

The meaning of assertions (b) and (c) is that the model N is constructed identically with respect to all of its submodels of the form $M[x]$, $x \in I$.

5.5. COROLLARY. If $x, f \in N \cap I$ and the point f is M[x]-defined in N, then $f \in M[x]$.

5.6. COROLLARY. If $x \in N \cap I$ and the set $X \in N$, $X \subseteq I$ is countable and M[x]-definable in N, then $X \subseteq M[x]$.

The proof follows from 5.1, 5.2, 5.4.

5.7. LEMMA. Let $x, y \in N \cap I$. Then

$$y \in M[x] \leftrightarrow \text{in } N,\ y \in L[x] \text{ is true.}$$

The proof is trivial: once the axiom of constructivity V = L is true in M by hypothesis, M is the collection of all sets constructive in N.

Proof of assertion (A). We consider the set

$$U = \{\langle x, y\rangle : x, y \in N \cap I \wedge y \in M[x]\}.$$

5.8. LEMMA. U is a countable-valued Σ_2^1-set in N.

Proof. The countable-valuedness of U follows from 5.4 (b). Further, by 5.7, in N, $U = \{\langle x, y\rangle : y \in L[x]\}$, is true, in view of which $U \in \Sigma_2^1$ in N holds according to 2.5 (the notation $y \in L[x]$ should be replaced by $y <_x x \vee x <_x y \vee x = y$).

It remains to verify that U is not in N the union of a countable number of single-valued projective sets. We shall prove a stronger assertion, for whose formulation it is necessary to introduce a concept. The set $X \in N$ is called M — I-definable in N if X is definable in N by some $\in$-formula with parameters only from $M \cup (N \cap I)$. In the model N each projective set and any countable sequence of projective sets are M — I-definable.

5.9. LEMMA. Let $\langle P_m : m \in \omega\rangle \in N$ be an M — I-definable in N sequence of single-valued sets $P_m \subseteq I$. Then $U \neq \bigcup_{m\in\omega} P_m$.

Proof. Suppose, on the contrary, U coincides with the union of the sets P_m. All parameters $p \in I$, figuring in the definition of the sequence of sets P_m can be replaced by some one parameter $x \in N \cap I$. Let $u = \{m \in \omega : \exists y (\langle x, y\rangle \in P_m)\}$. For each $m \in \omega$ there also exists a unique point y such that $\langle x, y\rangle \in P_m$. We denote it by y_m. By the assumption of the contrary, we have the equation

$$U_x = \{y_m : m \in \omega\}.$$

According to the choice of x, the sequence $\langle P_m : m \in \omega\rangle$ is M[x]-definable in N. Hence, too, the sequence $\langle y_m : m \in \omega\rangle$ will be M[x]-definable in N. Consequently, the characteristic function $f \in {}^{\omega}2$ of the set

$$\{2^m \cdot 3^k \cdot 5^i : m \in u \wedge k \in \omega \wedge i = y_m(k)\}$$

is also M[x]-definable in N. Thus, $f \in M[x]$ by 5.5. Whence it is easy to deduce that $\langle y_m : m \in \omega\rangle \in M[x]$ and that the set $U_x = \{y_m : m \in \omega\}$ belongs to M[x] and is countable in M[x]. But $U_x = M[x] \cap I$ by definition of U. We have obtained a contradiction with Cantor's theorem in M[x].

5.10. COROLLARY. U is not in N the union of a countable number of single-valued projective sets.

Now the proof of assertion (A) is obtained from 5.8 and 5.10 with the help of Theorem 2.1, completely analogously to the way in Sec. 4 we got the proof of Theorem 2 from Lemmas 4.3 and 4.4.

Proof of assertion (B). As above we prove a stronger assertion.

5.11. LEMMA. Let the set $P \in N, P \subseteq I^2$ be countable-valued and M − I-definable in N. Then P can be imbedded in N in a countable-valued Σ_2^1-set.

Proof. Let $p \in N \cap I$ be such that the set P is $M \cup \{p\}$ -definable in N. Repeating the proof of Lemma 5.8, it is easy to show that the set

$$Q = \{\langle x, y\rangle: x, y \in N \cap I \wedge y \in M[x, p]\}$$

is countable-valued and belongs to $\Sigma_2^{1,p}$ in N. It remains to verify that $P \subseteq Q$. We fix $x \in I$ and we shall show that $P_x \subseteq Q_x$. According to the choice of x, the set $P_x = \{y: \langle x, y\rangle \in P\}$ is countable and M[x, p]-definable in N. Hence, $P_x \subseteq M[x, p]$ by 5.6. But $Q_x = M[x, p] \cap I$ by definition of Q. The lemma is proved.

This completes the proof of Theorem 5.3 and Theorems 5 and 6 of Sec. 1.

6. Proof of Theorems 7 and 8

To prove these theorems we consider a special submodel N* of the model N from Sec. 5. We begin with several definitions. The set $X \in N$ is said to be $^{\omega}$Ord-definable in N if X is definable in N by some $\in$-formula with parameters only from the collection of all functions $f \in N$ from ω to the ordinal series. The set $X \in N$ is said to be hereditarily $^{\omega}$Ord-definable in N if X itself, all elements of X, all elements of elements of X, etc., are $^{\omega}$Ord-definable in N.

By N* we denote the collection of all hereditarily $^{\omega}$Ord-definable in N sets $X \in N$ (second Levy–Solovay model). The following assertions are proved in [9, pp. 51-52] and also in [5, Point 25].

6.1. N* is a countable transitive model of the theory ZF + DC.

6.2. $N^* \cap I = N \cap I$.

6.3. Each set $X \in N^*$ is M − I-definable in the model N.

6.4. THEOREM. In N* the following two statements are true:

(A) There exists a countable-valued Π_1^1-set which is not the union of a countable number of (arbitrary) single-valued sets.

(B) Each countable-valued set can be imbedded in a countable-valued set of class Σ_2^1.

Proof. (A) According to 6.2, the projective hierarchies of the models N and N* coincide. Hence the set U from the proof of Theorem 5.3 (A) is countable-valued and belongs to the class Σ_2^1 in N*. Moreover, if the sequence $\langle P_m: m \in \omega\rangle$ of single-valued sets $P_m \subseteq I^2$ belongs to N*, then it is M − I-definable in N according to 6.3, and hence $U \neq \bigcup_m P_m$ according to 5.9. Thus, in N* it is true that the set $U \subseteq I^2$ belongs to the class Σ_2^1, is countable-valued, and is not the union of a countable number of single-valued sets. As in Secs. 4 and 5, (A) follows from this.

The derivation of (B) from 5.11 goes analogously.

The results 6.1 and 6.4 show that the model N* is in fact a model for Theorems 7 and 8.

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SPACE OF ANALYTIC FUNCTIONS WITH PRESCRIBED GROWTH NEAR THE BOUNDARY

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Introduction. Let D be a bounded convex domain in the plane, containing the origin. By $d(\lambda)$, $\lambda \in D$, we denote the distance from the boundary of D, i.e.,

$$d(\lambda) = \operatorname{Inf}_{z \in \partial D} |z - \lambda|.$$

Let the function $f(\lambda)$ be analytic inside D. The quantity

$$\rho = \overline{\lim_{d(\lambda)\to 0}} \; (\ln|f(\lambda)|)/(-\ln d(\lambda))$$

will be called the order of $f(\lambda)$. By H(p), p > 0, we denote the space of functions, analytic in D and of order not exceeding p:

$$H(p) = \{f(\lambda) \in H(D)/\forall \varepsilon > 0 \quad \exists C\colon \; |f(\lambda)| \leqslant C \exp(1/d(\lambda))^{p+\varepsilon}\}.$$

In H(p) we introduce the topology defined by the seminorms

$$\|f\|_\varepsilon = \sup_{\lambda \in D} [|f(\lambda)| \exp(-(1/d(\lambda))^{p+\varepsilon})], \quad \varepsilon > 0.$$

We choose a sequence $\varepsilon_n \searrow 0$, $n = 1, 2, \ldots$, and introduce the spaces

$$B_n = \{f(\lambda) \in H(D) \mid \|f\|_{\varepsilon_n} < \infty\}, \quad n = 1, 2, \ldots.$$

The space H(p) may be considered as the projective limit of the Banach spaces B_n:

$$H(p) = \bigcap_{n=1}^{\infty} B_n.$$

By H'(P) we denote the dual space of H(p), equipped with the strong topology. As is known [1], H'(p) is isomorphic to the inductive limit of the dual spaces B_n' of the spaces B_n:

$$H'(p) = \bigcup_{n=1}^{\infty} B_n'. \tag{1}$$

Moreover, let $h(-\theta)$ be the support function of D and let $q \in (0, 1)$. By E_n, n = 1, 2, . . ., we denote the space of entire functions satisfying the condition

$$\sup_{z \in \mathbf{C}} [|f(z)|/\exp(h(\arg z)|z| - |z|^{q+\varepsilon_n})] < \infty,$$

where ε_n, n = 1, 2, . . ., is the sequence chosen above. The expression on the left-hand side of the last inequality defines a norm in E_n. This norm will be denoted by $\|F\|_n$. By P(q) we shall denote the inductive limit of the spaces E_n, n = 1, 2, ...

In this paper we describe the spaces H'(p) in terms of Laplace transforms.

1. The Dual of H(p)

THEOREM 1. Let p > 0 and q = p/(p + 1). Then H'(p) is isomorphic to P(q).

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