AN ANSWER TO LUZIN'S QUESTION ABOUT THE SEPARABILITY OF CA-CURVES

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We obtain an affirmative answer to the following question, posed by Luzin [1]: Do there exist two CA-curves, one of which lies under the other, that are nonseparable B by means of a set?

We give some definitions [1]. An everywhere-defined function \( y = f(x) \) on \( OX \) whose graph is a CA-set (as a subset of the plane \( OXY \)) is called a CA-curve in the plane \( OXY \). See, e.g., [2, p. 586] for the definition of the classes \( B \) (of the Borel sets), \( A \) (of the analytic sets), and \( CA \) (of the analytic complements).

A curve \( y = f(x) \) lies under a curve \( y = g(x) \) if \( f(x) < g(x) \) for all points \( x \) of the axis \( OX \).

Let \( E \subseteq OXY \). Each point \( x \) of the axis \( OX \) determines the section \( E_x = \{ y: \langle x, y \rangle \in E \} \) of the set \( E \) by the ordinate with abscissa \( x \). The projection of \( E \) on the axis \( OX \) is the set \( \text{Pr } E = \{ x: E_x \text{ is nonempty} \} \).

Luzin [1] called two CA-curves \( y = f(x) \) and \( y = g(x) \), the first of which lies under the second, separable B by means of a set if there exists a plane Borel set \( E \) such that \( \text{Pr } E = OX \) and \( f(x) < y < g(x) \) whenever \( x \in OX \) and \( \langle x, y \rangle \in E \).

THEOREM 1. There exist two CA-curves \( y = f(x) \) and \( y = g(x) \), the first of which lies under the second, that are not separable B by means of a set.

We start the proof of this theorem by fixing a CA-curve \( y = f(x) \) that is not a B-curve (this means that the graph \( \{ \langle x, f(x) \rangle : x \in OX \} \) is not a Borel set). The existence of such curves has been proved in [1].

For all \( n \in \omega \) and \( x \in OX \) we set \( g_n(x) = f(x) + 2^{-n} \). We have the family of the CA-curves \( y = g_n(x) \) (each of them is, in fact, a CA-curve, since it is obtained from the CA-curve \( y = f(x) \) by vertical parallel translation). In addition, the curve \( y = f(x) \) lies under each of the curves \( y = g_n(x) \).

Now to prove Theorem 1 it is sufficient to prove the following lemma.

LEMMA 1. There exists an \( n \) such that the curves \( y = f(x) \) and \( y = g_n(x) \) are not separable B by means of a set.

Proof. Let us assume the contrary. Then for each \( n \) there exists a plane Borel set \( E_n \) with \( \text{Pr } E_n = OX \) such that \( f(x) < y < f(x) + 2^{-n} \) whenever \( \langle x, y \rangle \in E_n \).

In order to obtain a contradiction, let us consider the sets \( W_n = \{ \langle x, y \rangle: \text{there exists a } y' \in E_n, \text{ such that } |y - y'| < 2^{-n} \} \).

LEMMA 2. Each \( W_n \) is an A-set.

Proof. The proof is simple. \( W_n \) is the projection on the plane \( OXY \) of the space Borel set \( \{ \langle x, y, z \rangle: \langle x, z \rangle \in E_n \land |y - z| < 2^{-n} \} \).

But the projections of Borel sets are A-sets.


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Continuing the proof of Lemma 1, we prove one more auxiliary lemma.

**Lemma 3.** The graph \( F = \{ \langle x, f(x) \rangle : x \in OX \} \) of the function \( y = f(x) \) satisfies the equation \( F = \bigcap_{n \in \mathbb{N}} W_n \).

**Proof.** Let points \( x \) and \( y \) be such that \( y = f(x) \). We prove that \( \langle x, y \rangle \in W_n \). By the choice of the sets \( E_n \), for a given \( n \) there exists a point \( y' \) such that \( \langle x, y' \rangle \in E_n \), and it satisfies the relation \( |y' - f(x)| < 2^{-n} \). This means that \( \langle x, y \rangle \in W_n \).

Conversely, let a pair \( \langle x, y \rangle \) belong to all the sets \( W_n \). We prove that \( y = f(x) \). Let us assume the contrary. Then there exists a natural number \( n \) such that \( |y - f(x)| > 2^{-n+1} \). By the definition of \( W_n \), we can choose a \( y' \) such that \( \langle x, y' \rangle \in E_n \) and \( |y' - f(x)| < 2^{-n} \). But by the choice of the sets \( E_n \), the inequality \( |y' - f(x)| < 2^{-n} \) must be fulfilled. Now we have \( |y - f(x)| < 2^{-n} + 2^{-n} = 2^{-n+1} \), which contradicts the choice of \( n \).

This contradiction completes the proof of the equality \( y = f(x) \) and Lemma 3.

We return to the proof of Lemma 1. By Lemmas 2 and 3, the graph \( F = \bigcap_{n \in \mathbb{N}} W_n \) is an \( A \)-set [3, p. 347]. But this contradicts the choice of the curve \( y = f(x) \), by which \( F \) cannot be an \( A \)-set. (Each set that belongs to both the classes \( A \) and \( CA \) is a Borel set by the Suslin theorem.)

This contradiction completes the proof of Lemma 1 and Theorem 1.

**LITERATURE CITED**


**LOCAL TORELLI THEOREM FOR BUNDLES ON MANIFOLDS WITH \( K = 0 \)**

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**Section 1**

By the local Torelli theorem we shall mean the injectivity of the differential of the period map. The definition of the period map and the calculation of its differential are contained in Griffiths [1].

In Sec. 1 we shall prove the following theorem.

**Theorem 1.** Let \( f: V \rightarrow B \) be a surface with a bundle of elliptic curves with nontrivial functional invariant without multiple fibers and suppose \( |K| \) has no fixed components. Then the local Torelli theorem holds for \( V \).

**Proof.** In [2] the author gives a method for verifying the local Torelli theorem for periods of \( n \)-forms.

We consider the commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & H^0(V, \Omega^1(-K)) \rightarrow H^0(V, \Omega^1) \rightarrow H^0(K, \Omega^1|_K) \\
\downarrow \omega_1 & & \downarrow \omega_1 \\
0 & \rightarrow & H^0(V, \Omega^1) \rightarrow H^0(V, \Omega^1(K)) \rightarrow H^0(K, \Omega^1(K)|_K).
\end{array}
\]