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PROBLEM OF THE EXISTENCE OF NONBOREL $AF_{\|}$-SETS

V. G. Kanovei

Absolute G_δ-sets are sets belonging to the class G_δ in some (and then also in any) complete metric (CM) space where they lie. The concepts of absolute A-sets and absolute CA-sets are introduced similarly, and have the analogous property. Each absolute G_δ is metrizable with a complete metric. Therefore, such a set X has the property $F_{\|}$, which says that there are no closed sets $Y \subseteq X$ in X of the first category in Y. The following is proved in [1]:

THEOREM (Gurewicz). If X lies in a separable metric (not necessarily complete) space P, is a CA set in P and has the property $F_{\|}$, then X is a G_δ-set in P.

Hence it follows, in particular, that any separable absolute CA-space with the property $F_{\|}$ is absolutely G_δ, i.e., the implication $G_\delta \to F_{\|}$ is invertible on separable absolute CA-sets. Gurewicz [1] posed the following problem: Does there exist a separable absolute A-set with the property $F_{\|}$ which is not absolute G_δ? We prove that this problem is undecidable.

THEOREM [2]. The following three statements are equivalent and undecidable in the theory ZFC of Zermelo and Frankel with the axiom of choice:

(a) There exists a separable absolute A-set with the property $F_{\|}$, which is not absolute G_δ.

(b) There exists an uncountable CA-set lying in the Baire space $\mathcal{J}$ which does not contain any discontinua (i.e., sets homeomorphic to the Cantor discontinuum).

(c) There exists a non-Borel CA-set $E \subseteq \mathcal{J}$ such that any Borel set $B \subseteq E$ can be embedded in the F_δ-set $U \subseteq E$.

Therefore, Gurewicz's problem is undecidable in ZFC. In the 1960s and 70s this problem was studied by Michael and Stone, San-Raimon, and Ostrovskii (see [2]).

The implication (b) → (a) was proved by Gurewicz himself. Let the CA-set $E \subseteq \mathcal{J}$ be uncountable and contain no discontinua. By the Aleksandrov–Hausdorff theorem, E is non-Borel, i.e., the A-set $X = \mathcal{J} - E$ is also non-Borel. We verify $F_{\|}$ for X. It is sufficient [1] to verify that X does not contain any countable sets which are perfect in X. On the contrary, suppose that $Q \subseteq X$ is such a set. Its closure K in $\mathcal{J}$ is perfect in $\mathcal{J}$, and, therefore, is uncountable. Thus, E contains an uncountable Borel set K − Q. We have a contradiction with the choice of E, by the Aleksandrov–Hausdorff theorem.

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The implication (a) → (c) was proved by Ostrovskii (see [2]). The idea of the proof is as follows. We embed the set X' given by (a) in a separable CM space P'. The latter is the continuous one-one image of the closed $P \subseteq \mathcal{J}: P' = F(P)$. The continuous inverse image $E = F^{-1}(E')$ of the set $E' = P' - X'$ is a CA-set in P and in $\mathcal{J}$. The set E is non-Borel (for otherwise its continuous single-valued image E' would be Borel in P', which would contradict Gurewicz's theorem, stating that X' and E' are non-Borel).

Consider an arbitrary Borel set $B \subseteq E$. Its image $B' = F(B)$ is, as above, Borel in P'. Denote by K the closure of B' in P'. The intersection $M = X' \cap K$ has the property $F_{\|}$, as it is closed in X'. Moreover, since B' is Borel, the set $M = (M \cup B') - B'$ is Borel, and $A = M \cup B'$. Therefore, by Gurewicz's theorem, M is G_δ in A, i.e, $M = A - Z$ for a suitable set $Z \subseteq P'$ in the class F_σ in P'. We have $B' = A - M \subseteq Z \cap K \subseteq E'$, i.e., the set $U' = Z \cap K$ in the class F_σ in P satisfies $B' \subseteq U' \subseteq E'$. The inverse image $U = F^{-1}(U')$ belongs to the class F_σ in P, and in $\mathcal{J}$ it satisfies the relations $B \subseteq U \subseteq E$, which is what we required.

The undecidability in ZFC of the statement (b) was established by Novikov and Soloveev (see [2]). Therefore, to prove the theorem it remains to prove the implication (c) → (b). This is the purpose of this article.

The Baire space $\mathcal{J}$ consists of all the functions defined on the set of natural numbers $\omega = \{0, 1, 2, \ldots\}$, with values in ω. The topology in $\mathcal{J}$ is generated by the system of Baire integrals $\mathcal{J}_u = \{x: u \subset x\}$, where u belongs to the set Seq of all finite strings of elements of ω.

The proof of the implication (c) → (b) uses the representation of CA-sets in $\mathcal{J}$ by means of sieves. For reasons which will become clear later, we shall not work with sieves themselves, but with their codes. A code of a sieve is any set $R \subseteq Q \times \mathrm{Seq}$, where Q is the set of rational numbers. Each code of the sieve R defines a classical Luzin sieve $\langle R_a: a \in Q \rangle$, consisting of sets $R_a = \bigcup_{\langle a, u \rangle \in R} \mathcal{J}_u$ which are open in $\mathcal{J}$. Conversely, any sieve composed of open R_a can be obtained in this manner from its code $R = \{\langle a, u \rangle: \mathcal{J}_u \subseteq R_a\}$.

Let R be a code of a sieve. If $x \in \mathcal{J}$, then the section $R^x = \{a \in Q: x \in R_a\}$ may or may not be totally ordered in the sense of the natural ordering in Q. In the first case we write $x \in [R]$. With each $\nu < \omega_1$ we associate the ν-th constituent $[R]_\nu = \{x \in [R]:$ the order type of R^x is equal to $\nu\}$. The constituents $[R]_\nu$ are all Borel, pairwise nonintersecting, and give [R] in union. [R] itself is a CA-set. Conversely, each CA-set is of the form [R] for a suitable code of the sieve R.

Suppose that (b) does not hold, i.e., any uncountable CA-set contains a discontinuum. It is known (see, for example, Sec. 3 of the supplement to [3]), that in this case, for any $\gamma < \omega_1$, the γ-th uncountable cardinal $\omega^{L[R]}_\gamma$ in the class L[R] of all sets which are constructive with respect to R, is countable in the universe of all sets, i.e., $\omega^{L[R]}_\gamma < \omega_1$ for any $R \subseteq Q \times \mathrm{Seq}$ (we are using the "effective countability" of the set Q × Seq).

Fix a code of the sieve R such that each constituent $[R]_\nu$ can be embedded in a F_σ-set $\subseteq [R]$. We shall prove $\neg$ (c), and thus the theorem, if we can establish that the number of nonempty constituents $[R]_\nu$ is no more than countable (in this case [R] is Borel).

It is known [4] that if the Borel set U is contained in [R], then U is contained in the union of a countable number of substituents $[R]_\mu$. Therefore, the assumption of the uncountability of the number of nonempty constituents reduces to the existence of an uncountable set $S \subseteq \omega_1$ such that $[R]_\nu$ is nonempty for any $\nu \in S$, and for any pair $\mu, \nu \in S$, if $\mu < \nu$ then the "approximation" $[R]_{\leqslant\mu} = \bigcup_{\xi \leqslant \mu} [R]_\xi$ is F_σ-separable from $[R]_\nu$ (i.e., is contained in some F_σ-set which has no common points with $[R]_\nu$). We shall obtain the required contradiction if we find a countable family W of sets $\subseteq \mathcal{J}$ such that for each pair $\mu < \nu < \omega_1$, if the approximation $[R]_{\leq\mu}$ is F_σ-separable from $[R]_\nu$, then the separating set can be chosen in the family W.

To construct W we introduce a coding of F_σ-sets. With each $d \subseteq \omega_1 \times \mathrm{Seq}$ we associate the set

$$X[d] = \bigcup_{\xi < \omega_1} X[d, \xi], \text{ where } X[d, \xi] = \mathcal{J} - \bigcup_{\langle \xi, u \rangle \in d} \mathcal{J}_u.$$

If $\lambda < \omega_1$ and $d \subseteq \lambda \times \mathrm{Seq}$, then the set X[d] belongs to the class F_σ. Conversely, each F_σ-set $\subseteq \mathcal{J}$ is of the form X[d] for a suitable $d \subseteq \omega \times \mathrm{Seq}$.

As W we take the family of all sets of the form X[d], where $d \in L[R]$, $d \subseteq \omega_1 \times \mathrm{Seq}$. It is not at first clear why W is countable. However, we note that the set X[d] is totally defined by the set w_d of all sets $\{u: \langle \xi, u\rangle \in d\}$, $\xi < \omega_1$. Each w_d belongs to L[R] and consists of subsets of Seq. Therefore, the set of all sets w_d has cardinality $\leq \omega^{L[R]}_2$ in the class L[R]. But $\omega^{L[R]}_2 < \omega_1$ (see above). Therefore, the set of all w_d, and by the same token also the family W, is countable in the universe of all sets.

To prove the implication (c) → (b) and the theorem, it remains to prove the following lemmas.

LEMMA 1. Let $\mu < \nu < \omega_1$ and let $[R]_{\leq\mu}$ be F_σ-separable from $[R]_\nu$. Then the separating set can be chosen in W.

Proof. Denote by P the set of all finite strings of ordinals $<\nu$. Order P by inverse inclusion; $p \leq q$ if p is an extension of q. The letters p, q, and r will only be used to denote elements of the set P.

Let $p \in P$ and $t \subseteq P \times \omega \times \mathrm{Seq}$. Define

$$Z_p[t] = \bigcup_{q\leqslant p,\, k\in\omega} Z_{qk}[t], \text{ where } Z_{qk}[t] = \mathcal{J} - \bigcup_{u\in e(t,q,k)} \mathcal{J}_u$$

and $e(t, q, k) = \{u: \exists r, r' \in P\, (r \leqslant r' \wedge r \leqslant q \wedge \langle r', k, u\rangle \in t)\}$. The following lemma is crucial:

LEMMA 2. We can select $p \in P$ and $t \in L[R]$, $t \subseteq P \times \omega \times \mathrm{Seq}$ so that $Z_p[t]$ separates $[R]_{\leq\mu}$ from $[R]_\nu$.

We show that Lemma 1 follows from Lemma 2. The set $P \times \omega$ has cardinality less than the "actual" ω_1 in L[R], since $\nu < \omega_1$. Fix an enumeration $\{\langle q_\xi, k_\xi\rangle: \xi < \lambda\}$ of this set in L[R] $(\lambda < \omega_1)$. Denote by d the set

$$\{\langle \xi, u\rangle: q_\xi \leqslant p \wedge \exists r, r' \in P\, (r \leqslant r' \wedge r \leqslant q_\xi \wedge \langle r', k_\xi, u\rangle \in t)\}.$$

Clearly, $d \subseteq \omega_1 \times \mathrm{Seq}$ and $d \in L[R]$, since $t \in L[R]$. On the other hand, the equation $X[d] = Z_p[t]$ is easily verified.

Thus, to prove Lemma 1 (and the theorem), it is sufficient to prove Lemma 2. We shall first prove it in a countable transitive model M of the theory ZFC. In other words, let $R \in M$ be a code of a sieve, $\mu < \nu < \omega^M_1$, let $L^M[R] = \{y \in M: y \in L[R]$ is true in M$\}$, and let the approximation $[R]_{\leqslant\mu}$ be F_σ-separable from $[R]_\nu$ in M. We are required to select $p \in P$ and $t \in L^M[R]$, $t \subseteq P \times \omega \times \mathrm{Seq}$ such that the set $Z_p[t]$ separates $[R]_{\leqslant\mu}$ from $[R]_\nu$ in M. Then we shall show how to carry the proof over to the case when it does not apply to a model M, but to the universe of all sets.

The construction of the required p and t uses the method of forcing (see Chap. 4 of [3]). As the initial (extendable) models we consider M and $M_0 = L^M[R]$. As a set of forcing conditions we take P; $p \leq q$ means that formulas forced by q are also forced by the condition p. The sets $t \subseteq P \times M$ are called P-terms. If $G \subseteq P$ and $t \subseteq P \times M$, then we define the "inflation" $i_G(t) = \{x: \exists p \in G\, (\langle p, x\rangle \in t)\}$. If the set $G \subseteq P$ is M_0-generic and $y \in M_0$, $d \in M_0[G]$, $d \subseteq y$, then there exists a P-term $d^* \in M$, $d^* \subseteq P \times y$ (the name for d), such that $d = i_G(d^*)$. Moreover, if d already belongs to the model M_0, then we may take $d^* = P \times d$ (the canonical name for $d \in M_0$). All the above also refers to the case when the initial model is M and not M_0.

We continue with the proof of Lemma 2 in M. Fix an M-generic $G \subseteq P$. By the condition of Lemma 1, in M we have:

(1) the approximation $[R]_{<\mu}$ is F_σ-separable from $[R]_\nu$. We intend to prove that the statement (1) is true both in M[G] and in $M_0[G]$. To do this we show how to write (1) in the form of a Σ^1_2-formula with parameters in M (such formulas are absolute).

Denote by $\varphi(R, \nu, x)$ the following formula:

$$\exists f: Q \to \nu \forall a \forall b\, (a, b \in R^x \wedge a < b \to f(a) < f(b)),$$

which expresses the existence of an ordered embedding of R^x in ν, and denote by $\psi(R, \nu, x)$ the conjunction of $\varphi(R, \nu, x)$ with the formula

$$\exists f\colon \nu \to Q \forall \xi \forall \eta\, (\xi < \eta < \nu \to f(\xi), f(\eta) \in R^x \wedge f(\xi) < f(\eta)),$$

which expresses the existence of an inverse embedding. In all the models we are considering, we have the equivalences

(2) $x \in [R]_{\leqslant\mu} \leftrightarrow \varphi(R, \mu, x)$ and $x \in [R]_\nu \leftrightarrow \psi(R, \nu, x)$. Moreover, denote by $\chi(d, x)$ the formula $\exists k \forall u\, (\langle k, u\rangle \in d \to x \notin \mathcal{I}_u)$, which means that

(3) $x \in X[d] \leftrightarrow \chi(d, x)$, for any $x \in I$ and $d \subseteq \omega \times \mathrm{Seq}$.

By (2) and (3), statement (1) can be written thus:

(4) $\exists d \subseteq \omega \times \mathrm{Seq}\, \forall x\, [(\varphi(R, \mu, x) \to \chi(d, x)) \wedge (\psi(R, \nu, x) \to \neg \chi(d, x))]$.

Moreover, the equivalence (1) $\leftrightarrow$ (4) holds in all models of ZFC. The formulas φ and ψ stand on the left of the implication in (4), i.e., after we have gone from $\to$ to $\vee$, their external quantifiers $\exists f$ become $\forall f$. If we now use some suitable mapping ω on the sets $\mu = \{\xi\colon \xi < \mu\}$, ν, Q, R, Seq, $Q \times \mathrm{Seq}$, and replace the sets $\subseteq\omega$ by their characteristic functions, we can rewrite formula (4) in the form $\exists h \in \mathcal{I} \forall z \in \mathcal{I} \Phi$, where the formula Φ contains the parameters from only $\mathcal{I} \cap M$ and quantifiers only from ω. (For example, if we fix a bijection $g \in M$: from ω to μ, then in Φ we have the parameter $y \in \mathcal{I} \cap M$, which changes μ in the sense that $g(i) < g(j) \leftrightarrow y(2^i \cdot 3^j) = 1$.)

We call formulas of this form Σ_2^1-formulas. They are absolute, by Shenfield's theorem (see [3, Supplement, Sec. 2]), i.e., if $M_1 \subseteq M_2$ are two models of ZFC with the same class of ordinals, and a Σ_2^1-formula has parameters only in $M_1 \cap \mathcal{I}$, then this formula is either simultaneously true or simultaneously false in both models.

The statement (1), which is equivalent to (4), is also absolute. Thus, once it is true for M, then it remains true in M[G]. Completely analogous arguments give the truth of (1) in the model $M_0[G] \subseteq M[G]$, if we only show that ν is countable in this model. But the set P is a ν-convoluting set of forcing conditions, in the terminology of Chap. 4 of [3]. This means that $g = \bigcup G$ maps ω onto ν. Moreover, $g \in M_0[G]$. Therefore, ν is countable in $M_0[G]$, which is what we required.

Thus the statement (1) is true in $M_0[G]$, i.e., there exists a set $d \in M_0[G]$, $d \subseteq \omega \times \mathrm{Seq}$ such that, in this model, we have:

(5) the set X[d] separates $[R]_{\leqslant\mu}$ from $[R]_\nu$.

Considering $M_0[G]$ as a generic extension of the model $M_0 = L^M[R]$, in view of the above we obtain a P-term $d^* \in M_0$, $d^* \subseteq P \times (\omega \times \mathrm{Seq})$ such that $d = i_G(d^*)$.

The statement (5) is absolute, in the sense that statement (1) is; this fact can be established roughly by the same arguments used to analyze (1). Therefore, (5) is true in M[G]. By the theorem on the connection between truth and restraint in generic extensions, there exists $p \in G$ such that

(6) $p \Vdash (X[d^*]$ separates $[R^*]_{\leqslant\mu}$ from $[R^*]_{\nu^*})$,

where $\Vdash$ is the forcing corresponding to the initial model M and the set of forcing conditions P.

The following lemma shows that the p and d* we have found (instead of t) are the required ones in the sense of Lemma 2 (in the model M), and this completes the proof of Lemma 2 in M.

<u>LEMMA 3.</u> The following is true in M: $[R]_{\leqslant\mu}$ is contained in $Z_p[d^*]$, but $[R]_\nu$ has no common points with $Z_p[d^*]$.

<u>Proof.</u> Let $x \in M \cap \mathcal{I}$ and let $x \in [R]_{\leqslant\mu}$ be true in M. The section R^x is defined from R and x in an obviously absolute manner. Therefore, $x \in [R]_{\leqslant\mu}$ is also true in M[G]. Therefore, $x \in X[d]$ in M[G], since (5) is true in M[G]. Thus there exist $k \in \omega$ and a condition $q \in G$, $q \leqslant p$ such that $q \Vdash x^* \in X[d^*, k^*]$. We verify that $x \in Z_{qk}[d^*]$ in M; this completes the proof of the first statement of Lemma 3. Suppose not; there exist $u \in \mathrm{Seq}$ and conditions $r, r' \in P$ such that

$$r \leqslant r',\ r \leqslant q,\ \langle r', k, u\rangle \in d^* \text{ and } x \in \mathcal{I}_u.$$

If r $\leq$ r', then $\Vdash \langle k^*, u^*\rangle \in d^*$. Moreover, $r \Vdash x^* \in \mathcal{I}_u$. Therefore, r forces $x^* \notin X[d^*, k^*]$, which contradicts the choice of q and the relation r $\leq$ q.

To prove the second statement of Lemma 3, let $x \in [R]_v$ in M. It is sufficient to verify, for an arbitrary pair $q \in P, q \leqslant p$ and $k \in \omega$ that $x \notin Z_{qk}[d^*]$ in M. Consider an arbitrary M-generic set $H \subseteq P$, containing q. As above, in the extension M[H] we have $x \in [R]_v$. Moreover, once q $\leq$ p, then $p \in H$, and, therefore, corresponding to (6) we obtain $x \notin X[i_H(d^*), k]$ in M[H]. Therefore, there exists a strong u $\in$ Seq such that $x \in \mathcal{I}_u$ and $\langle k, u\rangle \in i_H(d^*)$. By the definition of $i_H(d^*)$, the latter relation reduces to the condition $r' \in H$ so that $\langle r', \langle k, u\rangle\rangle = \langle r', k, u\rangle \in d^*$. Finally, for any pair $r', q \in H$ there exists $r \in H$, such that r $\leq$ r' and r $\leq$ q. Collecting these together,

$$r \leqslant r', r \leqslant q, \langle r', k, u\rangle \in d^* \text{and } x \in \mathcal{I}_u.$$

This enables us to conclude that $x \notin Z_{qk}[d^*]$ in M, which is what we required. Lemma 3 is proved.

The proof of Lemma 2 in the model M is complete.

We now show how to prove Lemma 2 in the universe of all sets, i.e., in its indirect formulation. Essentially, the countability of the model M is only used in order that we may consider generic extensions M[G] and M[H]. If the model M is countable, then for any condition p in the given set of forcing conditions $P \in M$ there exists an M-generic set $G \subseteq P$, containing p (see Chap. 4 of [3]).

Therefore, if we take, instead of M, the universe V of all sets, then we can never construct any generic extensions V[G], since V contains generally all sets, and G (excluding certain trivial cases) cannot belong to an extended model. However, there is a method which allows us to overcome this difficulty. This method consists of considering a definite "fictive" generic extension $V^{(P)}$ of the universe V. The class $V^{(P)}$ basically consists of the names for the elements of the actual extension V[G], as if the latter existed. These names must ensure the inflation of all the elements of the extension, and not only of those contained in the initial model. In connection with this, the construction of P-terms and inflations is slightly changed (see [5]). However, as before, there exists an embedding d $\mapsto$ d* of the universe V in $V^{(P)}$. The reader will find more detailed information about this interpretation of forcing (the Boolean-valued version) in [6].

Having slightly reconstructed the proof of Lemma 2 in the above direction, we obtain the proof of this lemma in the universe of all sets. Thus we have proved Lemmas 2 and 1, and the theorem is proved.

In conclusion, some remarks. A. V. Ostrovskii brought Gurewicz's problem to the attention of the author, and told him of the results connected with it, and in particular the above-mentioned proofs of the applications (b) $\rightarrow$ (a) and (a) $\rightarrow$ (c). He also first made the hypothesis of the pairwise equivalence of (a), (b), and (c), and their undecidability.

Basically, the crucial moment in the proof of the implication (c) $\rightarrow$ (b) is the statement that, if (b) does not hold, then for any code of the sieve R we can select a countable family W of subsets of a Baire space such that for any ordinals $\mu < \nu < \omega_1$, if the approximation $[R]_{<\mu}$ is F_σ-separable from $[R]_\nu$, then the separating set can be chosen in the family W. This theorem also remains true in the case when, instead of the class F_σ, we consider any other Borel class F_α or G_α of Hausdorff (F_1 is F_σ, but G_1 is G_δ). In a slightly weaker form (for $\nu = \mu + 1$; the general case is proved analogously), this theorem can be found in Sec. 3 of the supplement of [3], with an outline of the proof. Some other theorems on constituents are also proved there. These include a theorem which states that if (b) does not hold, then for any code of the sieve R, if all the constituents $[R]_\nu$ belong to the same class F_α or G_α (for example, if they all belong to the class F_σ), then the number of nonempty constituents is no more than countable.

The method used in the proof of our theorem can also be applied to the study of more complex sets with the property $F_\parallel$, which are not absolutely G_δ.

The following question remains open: Is the theorem in this article true for nonseparable sets?

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EXISTENCE OF A FINAL DISTRIBUTION FOR AN IRREDUCIBLE FELLER PROCESS WITH INVARIANT MEASURE

G. P. Klimov

Let X be a metric space, $\mathscr{B}$ the class of its Borel subsets, $\mathfrak{M}(X)$ the set of bounded (multivalued) measures on $(X, \mathscr{B})$, and B(X) the space of bounded real Borel functions on X. Each Markov operator $\mathsf{P}: X \times \mathscr{B} \to [0, 1]$, which is a Borel function in the first argument and a probability measure in the second, generates two operators (which we shall denote by the same symbol P). One operator acts in $\mathfrak{M}(X)$ by the formula

$$(\alpha\mathsf{P})(B) = \int_X \mathsf{P}(x, B)\,\alpha(\mathrm{d}x), \qquad B \in \mathscr{B},$$

and the other acts in B(X) by the formula

$$(\mathsf{P}f)(x) = \int_X f(y)\,\mathsf{P}(x, \mathrm{d}y), \qquad x \in X.$$

We note that any of these three operators defines the two others, since $(\delta_x\mathsf{P})(B) = \mathsf{P}(x, B)$, where δ_x is the Dirac measure, concentrated at the point x, and $(\mathsf{P}1_B)(x) = \mathsf{P}(x, B)$, where 1_B is the indicator of the Borel set B. We also note that if

$$(\alpha, f) = \int_X f(x)\,\alpha(\mathrm{d}x) \quad \text{for} \quad \alpha \in \mathfrak{M}(X), \qquad f \in B(X),$$

then $(\alpha\mathsf{P}, f) = (\alpha, \mathsf{P}f)$.

Consider now the family $\mathscr{P} = \{\mathsf{P}^t: t \in T\}$ of these operators, satisfying the semigroup property $\mathsf{P}^{s+t} = \mathsf{P}^s\mathsf{P}^t$. Here T = {1, 2, ...} or T = (0, ∞). In the case T = {1, 2, ...}, the semigroup $\mathscr{P}$ is defined by one generating operator $P = P^1$.

We shall assume that the operators P^t are Feller, i.e., $P^tC(X) \subset C(X)$, where C(X) is the space of bounded real continuous functions on X.

We also introduce a weak convergence in C(X) and in $\mathfrak{M}(X)$. The sequence $\{f_n\} \subset C(X)$ converges weakly to $f \in C(X)$, if $(\alpha, f_n) \to (\alpha, f)\ \forall \alpha \in \mathfrak{M}(X)$. This is equivalent to pointwise convergence, if the sequence $\{f_n\}$ is uniformly bounded. For $C \subset C(X)$ we say that the sequence $\{\alpha_n\} \subset \mathfrak{M}(X)$ converges C-weakly, or simply weakly, to $\alpha \in \mathfrak{M}(X)$, if $(\alpha_n, f) \underset{C}{\to} (\alpha, f)\ \forall f \in C$. This will be written in the form $\alpha_n \xrightarrow{C} \alpha$ or simply $\alpha_n \to \alpha$.

We shall make the following assumptions.

1. The semigroup $\mathscr{P}$ is irreducible, i.e., for any $x \in X$ and any open set $U \subset X$, there exists $t \in T$, such that $P^t(x, U) > 0$.

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