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Nonstandard Set Theory in $\in$-Language

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Abstract—A sufficiently convenient set theory in the standard $\in$-language applicable to nonstandard analysis is proposed.

Key words: *nonstandard set theory, Hrbaček theory, internal set, well-founded set, saturated elementary extensions,* **ZFC** *universe.*

Nonstandard set theories form one of the two known systems of foundations of nonstandard analysis. (The second one consists in using nonstandard extensions of mathematical structures in the "standard" **ZFC** universe.) A typical nonstandard set theory (for instance, Nelson's internal set theory **IST** [1]; see also [2–5]) organizes the universe of sets in such a way that objects of regular mathematics, called *standard*, coexist and interact with *nonstandard* objects (for instance, infinitesimal numbers). In so doing, the class $\mathbb{S}$ of all standard sets is singled out by mean of the undefinable *standardness predicate* $\operatorname{st} x$ (read as "x is standard"). In other words, these nonstandard set theories are formulated in st-$\in$-*language* containing st and $\in$ as atomic predicates.

In this paper we propose a set theory in $\in$-*language* strong enough to formalize nonstandard analysis. It is called the *simplified Hrbaček theory* or **SHST**. In short, **SHST** is the theory of the $\in$-structure of the **HST** universe in which the **HST** axioms (those of the Hrbaček nonstandard set theory, which uses st in its language; see below) are true. The **SHST** theory proves the existence of saturated elementary extensions. Another property of **SHST** is the existence of a (Boolean) **ZFC** interpretation such that the class of all standard sets of the interpretation is isomorphic to the **ZFC** universe. In particular, **SHST** and **ZFC** are equiconsistent and any theorem of **SHST** about standard sets is a theorem of **ZFC** (about all sets).

The main idea underlying the construction of the **SHST** axioms is the fact that the class $\mathbb{S}$ (which does not seem to be $\in$-definable in **HST**) has a $\in$-definable isomorphic copy: the class $\mathbb{V}$ of all well-founded sets (Kawaï's observation [4]). This makes it possible to replace $\mathbb{S}$ as the "standard" universe by $\mathbb{V}$ and use the obvious $\in$-definability of $\mathbb{V}$. The class $\mathbb{I}$ of *internal* sets (the elementary extension of $\mathbb{S}$) also admits a $\in$-definition.

1. HRBAČEK'S THEORY

This theory was introduced by Hrbaček [2]. An improved version was presented in detail in [3], but, for convenience, we list of the axioms of **HST** with brief comments. Recall that **HST** is a theory in the st-$\in$-language; $\operatorname{st} x$ means that x is standard; and $\mathbb{S} = \{x : \operatorname{st} x\}$ denotes the class of all standard sets. Elements of standard sets are called *internal* sets, $\operatorname{int} x$ is the formula $\exists^{\rm st} y\ (x \in y)$ (x is internal), and $\mathbb{I} = \{x : \operatorname{int} x\}$ is the class of all internal sets. The quantifiers $\exists^{\rm st}$ and $\forall^{\rm int}$ below are understood in the obvious way: "there exists standard" and "for any internal."

The *axioms of the universe* are all axioms of **ZFC**, *except* the regularity, degree, and choice axioms. The separation and substitution schemas are formulated in st-$\in$-language.

Transitivity of $\mathbb{I}$. $\forall^{\rm int} x\ \forall y \in x\ (\operatorname{int} y)$.

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Regularity over $\mathbb{I}$. $\forall X \neq \varnothing\ \exists x \in X\ (x \cap X \subseteq \mathbb{I})$.

ZFC$^{\mathrm{st}}$. All formulas of the form Φ^{st} (Φ relativized to $\mathbb{S}$), where Φ is a **ZFC** axiom.

Carry-over. All sentences of the form $\Phi^{\mathrm{st}} \iff \Phi^{\mathrm{int}}$, where Φ is a closed $\in$-formula with standard parameters.

Standardization. $\forall X\ \exists^{\mathrm{st}} Y\ (X \cap \mathbb{S} = Y \cap \mathbb{S})$ (for any X there exists a standard Y that contains the same standard elements).

These axioms suffice to define the class $\mathbb{V}$ of all well-founded sets (i.e., elements of transitive sets X such that $\in \restriction X$ is well-founded) and the $\in$-isomorphism $x \mapsto {}^*x$ of $\mathbb{V}$ onto $\mathbb{S}$ (${}^*x$ is defined as the only standard set u containing all sets of the form ${}^*y$, $y \in x$, and no other standard elements). It follows that $\mathbb{V}$ is a transitive class interpreting **ZFC** and closed under taking subsets. Furthermore, $x \mapsto {}^*x$ is an elementary embedding (in the $\in$-language) of $\mathbb{V}$ in $\mathbb{I}$ by carry-over (see [3, Sec. 1]).

In **HST**, *cardinals, ordinals, natural numbers* are $\mathbb{V}$-notions, so that a natural number is understood as a set $n \in \mathbb{V}$ which is a natural number in $\mathbb{V}$ (briefly, a $\mathbb{V}$-natural number). The set of all natural numbers is denoted by ω.

Sets of the same cardinality as $n = \{0, 1, \dots, n-1\}$, where $n \in \omega$, are called *finite*. Sets of the same cardinality as $x \in \mathbb{V}$ are called *sets of standard size* or, briefly, SS-*sets*.

Let us formulate the last two essential axioms of **HST**.

Saturation of $\mathbb{I}$. If $\mathcal{X} \subseteq \mathbb{I}$ is an SS-set and $\bigcap \mathcal{X}' \neq \varnothing$ for any *finite* $\mathcal{X}' \subseteq \mathcal{X}$ (the finite intersection property), then $\bigcap \mathcal{X}' \neq \varnothing$.

SS-choice. The axiom of choice holds for the case in which the choice function is to be defined on an SS-family of (nonempty) sets.

Saturation allows us to obtain diverse nonstandard sets. The axiom of SS-choice partly compensates for the absence of the complete choice axiom, which contradicts **HST** as well as the degree and regularity axioms.

Theorem 1 (see [3]). *The* **HST** *and* **ZFC** *theories are equiconsistent. Furthermore,* **HST** *has a Boolean interpretation in* **ZFC**, *in which the class $\mathbb{S}$ is provably $\in$-isomorphic to the basic universe* **ZFC**. *Consequently, if* **HST** *proves that a closed $\in$-formula Φ is true in $\mathbb{V}$ (or, equivalently, in $\mathbb{S}$), then Φ is a theorem in* **ZFC**.

Unexpectedly, it turns out that the class $\mathbb{I}$ is directly $\in$-definable in **HST**. We say that a set x is *quasiinternal* if there exists an ω-sequence $\{x_n\}_{n\in\omega}$ such that $x \in x_{n+1} \in x_n$ for all $n \in \omega$.

Proposition 1 (HST). *Classes of internal and quasiinternal sets coincide.*

Proof. [3]. Let $x \in \mathbb{I}$. Reasoning within the universe $\mathbb{I}$, we define $y_k = y_{k-1} \cup \{y_{k-1}\}$ by induction on $k \in {}^*\omega$ starting with $y_0 = x$. Choose an arbitrary $\nu \in {}^*\omega \setminus \omega$, and set $x_n = y_{\nu-n}$ for all $n \in \omega$.

The converse implication readily follows from the regularity over $\mathbb{I}$. □

2. THE SIMPLIFIED HRBAČEK THEORY

The **SHST** theory includes the following groups (i)–(iv) of axioms.

(i) Similarly to **HST**, all the axioms of **ZFC** except the regularity, degree, and choice axioms. (Separation and substitution in the st-$\in$-language.)

This suffices to introduce the class $\mathbb{V}$ of all well-founded sets and to prove its transitivity.

(ii) All formulas of the form Φ^{wf} (Φ relativized to $\mathbb{V} = \{x : \mathrm{wf}\, x\}$), where Φ is an axiom of **ZFC**, and $\mathrm{wf}\, x$ says, "x is well-founded."

Further, suppose that ${}^q\mathbb{I}$ denotes in the $\in$-language the *class of all quasiinternal sets* (see above). We add

(iii) axioms of transitivity of the class ${}^q\mathbb{I}$, regularity over ${}^q\mathbb{I}$, saturation of ${}^q\mathbb{I}$, and SS-choice (as in the **HST** theory, but for ${}^q\mathbb{I}$).

As regards the carry-over, we cannot borrow its formulation directly from **HST**: **SHST** does not ensure any suitable embeddings of $\mathbb{V}$ in ${}^q\mathbb{I}$. However, the following wording is quite acceptable.

(iv) Simplified carry-over. All formulas of the form $\Phi^{\text{wf}} \iff \Phi^{q\text{-int}}$, where Φ is a closed $\in$-formula with parameters from ω. (It is hardly possible to invoke a larger range of parameters: the problem is that $\mathbb{V} \cap \mathbb{I} = H\omega$ (hereditarily finite sets) in **HST**, but parameters from $H\omega$ reduce to ω.)

(Here ${}^{q\text{-int}}$ denotes the relativization to ${}^q\mathbb{I}$.) This axiom needs a comment, because at first, it is not clear that $\omega \subseteq {}^q\mathbb{I}$. The plan is to accept the simplified carry-over in the parameter-free version, which readily implies that ${}^q\mathbb{I}$ is a transitive $\in$-model of **ZFC**, and hence, $\omega \subseteq {}^q\mathbb{I}$. Then we accept the simplified carry-over entirely.

Notice that **SHST** is a subtheory of the $\in$-part of **HST**. (To prove the simplified carry-over in **HST**, we verify by induction on x in **HST** that ${}^*x = x$ for all $x \in \omega$; then $\Phi^{\text{wf}} \iff \Phi^{\text{int}}$, since $x \mapsto {}^*x$ is an elementary embedding of $\mathbb{V}$ into $\mathbb{I}$.) Thus, **SHST** satisfies Theorem 1. The following lemma shows that **SHST** ensures the existence of elementary extensions.

Lemma 1 (SHST). *For any transitive $X \in \mathbb{V}$ there exists a transitive ${}^*X \in {}^q\mathbb{I}$ and an elementary embedding of $\langle X\,;\in\rangle$ into $\langle {}^*X\,;\in\rangle$.*

Proof. The carry-over and saturation of **SHST** yield the transitive set ${}^*X \in {}^q\mathbb{I}$ such that the structures $\langle X\,;\in\rangle$ and $\langle {}^*X\,;\in\rangle$ are elementarily equivalent. Let us construct an elementary embedding of $\langle X\,;\in\rangle$ into $\langle {}^*X\,;\in\rangle$.

By the choice of ${}^*X$ and the saturation of ${}^q\mathbb{I}$ if $n \in \omega$, then for any n-tuple $\langle x_1, \dots, x_n\rangle \in X^n$ there exists an n-tuple $\langle r_1, \dots, r_n\rangle \in {}^*X^n$ such that

(A) for any $\in$-formula $A(\,\cdot\,, \dots, \,\cdot\,)$ (*formulas* are understood as finite sequences of a certain form in this proof), $A(x_1, \dots, x_n)$ is true in $\langle X\,;\in\rangle$ if and only if $A(r_1, \dots, r_n)$ is true in $\langle {}^*X\,;\in\rangle$.

According to the SS-choice axiom, there exists a one-to-one length-preserving map of finite sequences $f\colon X^{<\omega} \to ({}^*X)^{<\omega}$ such that (A) holds for $\langle x_1', \dots, x_n'\rangle = f(\langle x_1, \dots, x_n\rangle)$, whatever the n-tuple $\langle x_1, \dots, x_n\rangle \in X^{<\omega}$. Clearly, we have $f(\langle x\rangle) = \langle \phi(x)\rangle$, where $\phi\colon X \to {}^*X$ is a one-to-one function.

If $D \subseteq X$ is finite and F is a finite set of $\in$-formulas, then let $\Pi_{DF} \in {}^q\mathbb{I}$ be the set of all one-to-one maps $\pi \in {}^q\mathbb{I}$, $\pi\colon {}^*X$ onto ${}^*X$ such that for any $\in$-formula $A(v_1, \dots, v_n) \in F$ and all $x_1, \dots, x_n \in D$ we have:

(B) it is true in ${}^*X$ that $A(\pi(\phi(x_1)), \dots, \pi(\phi(x_n))) \iff A(r_1, \dots, r_n)$, where $\langle r_1, \dots, r_n\rangle = f(\langle x_1, \dots, x_n\rangle)$.

Notice that the sets Π_{DF} are nonempty by the choice of ${}^*X$ and f. (For example, if F contains only one formula A, we simply take the bijection $\pi\colon {}^*X$ onto ${}^*X$ such that $\pi(\phi(x_i)) = r_i$ for all i.) In addition, the family of all sets Π_{DF} is of standard size and satisfies the finite intersection property. (Indeed, $\Pi_{D_1F_1} \cap \Pi_{D_2F_2} \supseteq \Pi_{D_1\cup D_2, F_1\cup F_2}$.) This means that there exists a one-to-one map $\pi \in {}^q\mathbb{I}$, $\pi\colon {}^*X \to {}^*X$ contained in each of our sets Π_{DF} such that (B) holds for all $x_1, \dots, x_n \in X^{<\omega}$ and all $\in$-formulas A. It readily follows that $p(x) = \pi(\phi(x))$ is an elementary embedding of $\langle X\,;\in\rangle$ in $\langle {}^*X\,;\in\rangle$. $\square$

3. DEVELOPING NONSTANDARD ANALYSIS IN SHST

Informally, the class $\mathbb{V}$ of all well-founded sets is identified with the "standard" mathematical universe. Then, since **SHST** satisfies Theorem 1 (as a subtheory of **HST**), the universe **SHST** can be regarded as a quite well defined extension of the "true" universe $\mathbb{V}$ just like $\mathbb{C}$ is an extension of $\mathbb{R}$. Hence, **SHST** is not just a syntactic tool: we have a complete interpretation in **ZFC**.

It is known that the set $X = V_{\omega+\omega}$, defined in $\mathbb{V}$, suffices for constructing almost all mathematical structures in $\mathbb{V}$. In particular, the sets $\mathbb{N} = \omega$ (natural numbers) and $\mathbb{R}$ belong to $\mathbb{V}$.

Lemma 1 yields the transitive set ${}^*X = {}^*V_{\omega+\omega} \in {}^q\mathbb{I}$ and elementary embedding $x \mapsto {}^*x$ of the structure $\langle V_{\omega+\omega}\,;\in\rangle$ into $\langle {}^*V_{\omega+\omega}\,;\in\rangle$. (Notice that ${}^*V_{\omega+\omega}$ is a ${}^q\mathbb{I}$-analog of $V_{\omega+\omega}$: in fact, ${}^*V_{\omega+\omega} = V_{{}^*\omega+{}^*\omega}$ in ${}^q\mathbb{I}$.) It is easy to show that ${}^*n = n \in {}^*\mathbb{N}$ for any $n \in \mathbb{N}$ (for instance, by induction on n); so $\mathbb{N}$ is the initial segment of ${}^*\mathbb{N}$. Moreover, ${}^*\mathbb{N} \setminus \mathbb{N}$ is nonempty by saturation applied to the family of sets $S_n = \{k \in {}^*\mathbb{N} : k > n\}$, $n \in \mathbb{N}$. Elements of ${}^*\mathbb{N}$ are exactly the ${}^q\mathbb{I}$-natural numbers, which can be called, as usual, *hypernatural.* The numbers in ${}^*\mathbb{N} \setminus \mathbb{N}$ are called *infinitely large.*

As regards the real numbers (again, in the sense of $\mathbb{V}$), we have $\mathbb{R} \in V_{\omega+\omega}$ and $\mathbb{R} \subseteq V_{\omega+\omega}$ in $\mathbb{V}$; hence, ${}^*\mathbb{R} \in {}^*V_{\omega+\omega}$ in ${}^q\mathbb{I}$, and ${}^*x \in {}^*\mathbb{R}$ is well defined for all $x \in \mathbb{R}$. Elements of ${}^*\mathbb{R}$, i.e., ${}^q\mathbb{I}$-real numbers, can be called *hyperreal.* Now we can introduce the notions of *infinitely large, infinitesimal, bounded* hyperreal numbers and the ratio $\approx$ of *infinite closeness* in the ordinary way.

Lemma 2. *If $x \in {}^*\mathbb{R}$ is bounded, then $x \approx {}^*z$ for a certain $z \in \mathbb{R}$.*

Proof. Notice that the sets $A = \{y \in \mathbb{R} : {}^*y \le x\}$ and $B = \{y \in \mathbb{R} : {}^*y > x\}$ are nonempty by the boundedness of x. These sets belong to $\mathbb{V}$, because this class is closed with respect to taking subsets. Reasoning in $\mathbb{V}$, we find a number z which is either the greatest in A or the smallest in B. $\square$

This simple argument demonstrates the potential of **SHST**. As to more complicated examples, such as the Loeb measure and "hyperfinite" descriptive set theory, we refer the reader to [3, 2.2 and 2.3], where it is explained how to perform typical "nonstandard" calculations in the context of similar systems.

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