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Perfect Subsets of Invariant CA-Sets

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Abstract—The familiar theorem that any $\Sigma^1_2(a)$-set X of real numbers (where a is a fixed real parameter) not containing a perfect kernel necessarily satisfies the condition $X \subseteq \mathbf{L}[a]$ is extended to a wider class of sets, with countable ordinals allowed as additional parameters in $\Sigma^1_2(a)$-definitions.

Key words: *perfect kernel property, perfect subset, forcing, descriptive set theory,* CA-*set.*

This note continues our paper [1] devoted to classical problems related to the regularity properties of point sets. One of these properties is the *perfect kernel property*, which holds for a given set X whenever it either is finite or countable or contains a perfect subset. For instance, any Borel set and even any Σ^1_1-set in a Polish space[1] have the perfect kernel property (Souslin [2][2]). As distinct, say, from Lebesgue measurability, the perfect kernel property cannot be carried over to complementary sets. Therefore, the problem concerning the perfect kernel property for Π^1_1-sets (i.e., sets complementary to Σ^1_1-sets) was recognized as one of the most important in descriptive theory quite early (for the first time, perhaps, in [3]).

All attempts to solve this problem using methods of classical descriptive set theory had no effect. The reason for which they failed became clear only after investigations in axiomatic set theory showed that this problem is *undecidable*, that is, it cannot be solved if a "solution" is understood as a definite, positive or negative, answer to the posed question. Namely, Novikov [4] showed that it is impossible to *refute* the existence of counterexamples, i.e., uncountable sets without perfect subsets, in the class Π^1_1, and Solovay [5] established that it is impossible to *prove* the existence of counterexamples in the class Π^1_1 (in fact, in the considerably wider class of all projective sets). Here the words to "prove" and "refute" are understood in the sense of a proof or refutation in the Zermelo–Fraenkel axiomatics **ZFC** of set theory, which is now equated to the existence of ordinary mathematical proof or refutation (in the informal sense). Thus, the perfect kernel problem, like many other problems of classical descriptive set theory, turned out to be undecidable in this strongest sense. (The surveys [1, 6, 7] contain various information about undecidable problems of descriptive set theory.)

These undecidability investigations brought about a number of other remarkable results. For instance, Lyubetskii [8, 9] showed that the existence of a Lebesgue nonmeasurable projective set of class Σ^1_2 implies the existence of an uncountable Π^1_1-set without perfect subsets (i.e., a counterexample to the perfect kernel property), but the reverse implication is not true. The following theorem (see [1, Sec. 4.5]) played a crucial role in the research connected with the perfect kernel property of Π^1_1-sets.

[1]A metric space is said to be *Polish* if it is complete and separable. For instance, the real line $\mathbb{R}$ and the Baire space $\mathbb{N}^\omega$ are Polish spaces. Σ^1_1-sets in Polish spaces are the continuous images of Borel sets or, which is the same, projections of Borel sets, e.g., from $\mathbb{R}^{n+1}$ to $\mathbb{R}$. These sets are also called A-*sets*, and their complementary sets are called CA-*sets*.

[2]More historical details concerning this and some other classical theorems of descriptive set theory are given at the end of the first section in [1].

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Theorem 1 (Solovay [10], Lyubetskii [11]). *If $a \in \mathbb{N}^\omega$ and a $\Sigma^1_2(a)$-set $X \subseteq \mathbb{N}^\omega$ does not contain perfect subsets, then $X \subseteq \mathbf{L}[a]$.*[3]

Here $\mathbb{N}^\omega$ is the Baire space, the class $\Sigma^1_2(a)$ is formed by all the sets definable by Σ^1_2-formulas with a single parameter a, and the class of all sets constructive with respect to $a \in \mathbb{N}^\omega$ is denoted by $\mathbf{L}[a]$. (Certain important notations introduced in [1] are used in this paper without special comment.)

Our goal is to prove a strengthening of this theorem, i.e., basically the same result for a considerably wider class of point sets. The idea of this wider class is to allow the usage of countable ordinals as additional definability parameters (used in addition to the parameter a). To realize this idea directly is rather difficult, since the language of analytical formulas (see [1, Sec. 1C]), in terms of which classes of the form $\Sigma^1_2(a)$ are defined, does not involve ordinals.

However, there is a roundabout way based on encoding of ordinals by points of the space $\mathbb{N}^\omega$.

Let us fix, once and for all, a recursive enumeration $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ of the set of rational numbers $\mathbb{Q}$. Set $Q_w = \{q_n : w(n) = 0\}$ for each $w \in \mathbb{N}^\omega$. Now, for any $\xi < \omega_1$, we denote by $\mathbf{WO}_\xi$ the set of all $w \in \mathbb{N}^\omega$ such that Q_w is of the ordinal type ξ in the sense of the natural order on $\mathbb{Q}$. Elements of the set $\mathbf{WO}_\xi$ are regarded as *codes* of the ordinal ξ. Finally, we put $\mathbf{WO} = \bigcup_{\xi<\omega_1} \mathbf{WO}_\xi$. This set belongs to the class Π^1_1 (see [1, Sec. 1E]), and the formula $|w| = \xi$ for $w \in \mathbf{WO}_\xi$ defines a function on this set. Viewing each $\mathbf{WO}_\xi$ as an *image* of the ordinal $\xi < \omega_1$, now available for the language of analytical formulas, we arrive at the following definition:

Definition 2. A formula $\varphi(w, x)$ (in any language) is said to be *w-invariant* if, for any $x \in \mathbb{N}^\omega$ and $w, w' \in \mathbf{WO}$ satisfying the equation $|w| = |w'|$, we have

$$\varphi(w, x) \iff \varphi(w', x).$$

The formula $\varphi(w, x)$ is said to be *absolutely w-invariant* if it remains w-invariant in any generic extension of the universe.

Any set of the form $X = \{x \in \mathbb{N}^\omega : \varphi(w_0, x)\}$, where $w_0 \in \mathbf{WO}$ and $\varphi(w, x)$ is an absolutely w-invariant Σ^1_2-formula with a single parameter $a \in \mathbb{N}^\omega$, is called an *absolutely invariant $\Sigma^1_2(a, \omega_1)$-set.*

For instance, if $\psi(\xi, x)$ is any formula in which $\xi < \omega_1$ and $x \in \mathbb{N}^\omega$, then the formula $w \in \mathbf{WO} \wedge \psi(|w|, x)$ is w-invariant. The latter formula can be analytical, whereas the initial one, ψ, certainly, cannot. For *absolute* invariance, it suffices that the invariance be deducible in **ZFC** from the definition of φ, as is the case, e.g., for formulas of the form $w \in \mathbf{WO} \wedge \psi(|w|, x)$.

Any $\Sigma^1_2(a)$-set is trivially an absolutely invariant $\Sigma^1_2(a, \omega_1)$-set. A less trivial example: obviously, any set of the form $\mathbf{WO}_\xi$ is absolutely invariant and even a $\Delta^1_1(a, \omega_1)$-set, but does not belong to $\Sigma^1_2(a)$ in the case $\omega_1^{\mathbf{L}[a]} \le \xi < \omega_1$. Therefore, the following theorem (the main theorem of this note) strengthens Theorem 1.

Theorem 3. *If $a \in \mathbb{N}^\omega$ and an absolutely invariant $\Sigma^1_2(a, \omega_1)$-set $X \subseteq \mathbb{N}^\omega$ does not contain perfect subsets, then $X \subseteq \mathbf{L}[a]$. In particular, if it is additionally known that $\omega_1^{\mathbf{L}[a]} < \omega_1$, then X is at most countable.*

[3] As is customary in modern descriptive set theory, all results here are formulated and proved for point sets in the Baire space $\mathbb{N}^\omega$. However, since $\mathbb{N}^\omega$ is homeomorphic to the "Baire line," i.e., to the set of all irrational points of $\mathbb{R}$, the majority of descriptive set theory results is automatically carried over from $\mathbb{N}^\omega$ to $\mathbb{R}$ by means of a very simple, in effect, recursive mapping. In particular, this applies to Theorem 1 and to our main result, Theorem 3, although the latter needs some additional work in order to obtain $\mathbb{R}$-versions of the notions connected with Definition 2. A general discussion of the independence of descriptive set theory problems from the choice of the base space can be found in [1, Sec. 1A].

Proof. Let $X = \{x \in \mathbb{N}^\omega : \varphi(w, x)\}$, where $w \in \mathbf{WO}$ and $\varphi(w, x)$ is an absolutely w-invariant Σ^1_2-formula with a single parameter $a \in \mathbb{N}^\omega$. Let $\xi = |w|$.

Further reasoning makes use of the *collapse forcing*[4] $\mathbb{C}(\xi) = \mathrm{Coll}(\mathbb{N}, \xi)$. Thus, $\mathbb{C}(\xi)$ consists of all finite sequences of ordinals $< \xi$, in other words, of all functions

$$p\colon m \to \xi, \qquad \text{where} \quad m = \{0, 1, \dots, m-1\},$$

and $p \le q$ (i.e., p is "stronger") whenever $q \subseteq p$, i.e., p extends q as a function. Any $\mathbb{C}(\xi)$-generic set $G \subseteq \mathbb{C}(\xi)$ generates the function $f[G] = \cup G\colon \mathbb{N} \xrightarrow{\text{onto}} \xi$, a *generic collapse* of ξ. Conversely, we have $G = \{f[G] \restriction n : n \in \mathbb{N}\}$.

A convenient technique is to consider the universe of all sets $\mathbf{V}$ as a model in a certain "virtual" wider universe, in which the desired generic extensions of $\mathbf{V}$ exist. In particular, in such a wider universe, one can consider a $\mathbb{C}(\xi) \times \mathbb{C}(\xi)$-generic extension $\mathbf{V}[G, G']$ of the universe $\mathbf{V}$ generated by a pair of sets $G_1, G_2 \subseteq \mathbb{C}(\xi)$ generic over $\mathbf{V}$ (consequently, over $\mathbf{L}[a]$ as well) in which, by the previous argument, there exist two collapse functions, $f_1 = \cup G_1$ and $f_2 = \cup G_2$, from $\mathbb{N}$ onto ξ. Accordingly, there exist codes $w_1 \in \mathbf{WO}_\xi \cap \mathbf{L}[f_1]$ and $w_2 \in \mathbf{WO}_\xi \cap \mathbf{L}[f_2]$ of the ordinal ξ. Recall that we also have a code $w \in \mathbf{WO}_\xi \cap \mathbf{V}$ which, of course, belongs to the class $\mathbf{V}[f_1, f_2]$.

Notice that in our assumptions the formula φ is w-invariant in $\mathbf{V}[f_1, f_2]$. Thus, in this extended universe $\mathbf{V}[f_1, f_2]$, the sets

$$X' = \{x : \varphi(w, x)\}, \qquad X'_1 = \{x : \varphi(w_1, x)\}, \qquad \text{and} \qquad X'_2 = \{x : \varphi(w_2, x)\}$$

coincide. For convenience, we assume that primed capital letters stand for point sets defined in the extended universe $\mathbf{V}[f_1, f_2]$.

We claim that the *set X' has no perfect kernel in* $\mathbf{V}[f_1, f_2]$. To prove this, we choose, reasoning in $\mathbf{V}[f_1, f_2]$, a uniform $\Pi^1_1(a, w)$-set $P' \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ such that $X' = \operatorname{dom} P' = \{x : \exists y \quad P(x, y)\}$. (We use the Novikov–Kondô–Addison uniformization theorem; see, for instance, [1, Sec. 1.9].)

It follows from the Shoenfield absoluteness theorem[5] that the set $P = P' \cap \mathbf{V}$ lies in $\mathbf{V}$, is a $\Pi^1_1(a)$-set in $\mathbf{V}$, and satisfies the relation $X = \operatorname{dom} P$. Since it is true in $\mathbf{V}$ that X does not contain perfect subsets, the same is valid for P as well. But then, applying the Shoenfield absoluteness theorem again, we see that P' also does not contain perfect subsets in $\mathbf{V}[f_1, f_2]$, because the absence of a perfect subset can be expressed by a Π^1_2-formula saying that none of the constituents of a given $\Pi^1_1(a)$-set contains a perfect kernel. Thus, by Theorem 1, $P' \subseteq \mathbf{L}[a, w]$ in $\mathbf{V}[f_1, f_2]$. It follows that $X' \subseteq \mathbf{L}[a, w]$, and, therefore, X' does not contain a perfect kernel in $\mathbf{V}[f_1, f_2]$ by the Groszek–Slaman theorem,[6] as claimed.

Applying Theorem 1 again and taking into account the absence of a perfect kernel proved above, we see that the same set $X' = X'_1 = X'_2$ satisfies $X'_1 \subseteq \mathbf{L}[a, f_1]$ and $X'_2 \subseteq \mathbf{L}[a, f_2]$ in $\mathbf{V}[f_1, f_2]$, and so

$$X' \subseteq \mathbf{L}[a, f_1] \cap \mathbf{L}[a, f_2] \qquad \text{in} \quad \mathbf{V}[f_1, f_2].$$

However, by the forcing product theorem [1, Theorem 4.2(i)] we have

$$\mathbf{L}[a, f_1] \cap \mathbf{L}[a, f_2] = \mathbf{L}[a] \qquad \text{in} \quad \mathbf{V}[f_1, f_2].$$

Thus, $X' \subseteq \mathbf{L}[a, f_1]$; but then we also have $X \subseteq \mathbf{L}[a]$, since $X = X' \cap \mathbf{V}$ (for instance, because $P = P' \cap \mathbf{V}$). This completes the proof of the theorem. $\square$

[4] We assume a certain familiarity of the reader with forcing. A summary of main theorems of the theory of forcing with references to the corresponding sources can be found in [1, Sec. 4A]. We shall apply these theorems without special explanations.

[5] This theorem, as applied to the situation in question, states that any closed Σ^1_2-formula or Π^1_2-formula with parameters from $\mathbf{V}$ is either simultaneously true in $\mathbf{V}$ and in $\mathbf{V}[f_1, f_2]$ or simultaneously false in $\mathbf{V}$ and in $\mathbf{V}[f_1, f_2]$ (see, e.g., [1, 2.8]).

[6] This theorem states that the set $\mathbb{N}^\omega \cap \mathbf{L}[b]$, $b \in \mathbb{N}^\omega$, cannot have perfect subsets unless $\mathbb{N}^\omega \subseteq \mathbf{L}[b]$. The proof can be found, e.g., in [12].

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