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Nonstandard Representations of Locally Compact Groups

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Abstract—In the note, it is proved that, under natural conditions, any infinite-dimensional unitary representation T of a direct product of groups $G = K \times N$, where K is a compact group and N is a locally compact Abelian group, is imaged by a representation of the nonstandard analog $\widetilde{G}$ of the group G in the group of nonstandard matrices of a fixed nonstandard size.

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1. INTRODUCTION

Boolean-valued analysis, as well as nonstandard analysis in general, is aimed at the *imaging* (a term which arose in a similar situation in Novikov's paper [1]; one can use the more customary but already occupied term *representation*) of more complicated mathematical objects by using simpler ones; however, the latter are considered in the framework of nonstandard set theory. Here this set theory appears in the form of a model, namely, in the form of a Boolean-valued universe V^B (see [2]), where B is an arbitrarily chosen complete Boolean algebra. For B one takes a Boolean algebra related to the problem under consideration (in our case, B is a Boolean algebra of pairwise commuting projections in an operator algebra). For different problems, one takes for B different specific Boolean algebras (another example is given by the Boolean algebra of all central idempotents of a ring) or even Heyting algebras (for instance, the lattice of all open sets of a topological space) and quantum algebras (which are sometimes referred to as quantum logics; for instance, such is the lattice of all linear subspaces in any $\mathbb{R}^n$), etc. More complicated algebras B lead to a universe V^B with more complicated structure, and it is more complicated to study the properties of such a universe; however, the original "standard" objects (i.e., those in V^{Z_2}) are imaged by simpler and still simpler "nonstandard" objects (i.e., those in V^B). In this sense, the "simplest" universe V^B corresponding to traditional "standard" set theory is obtained if B is the simplest algebra $Z_2 = \{0, 1\}$; this gives the ordinary *class of all sets* $V \simeq V^{Z_2}$, which by definition consists of the "standard" mathematical objects. The class V can canonically be embedded in an arbitrary universe V^B by the mapping $x^\vee \colon V^{Z_2} \to V^B$ induced by the embedding of algebras $b^\vee \colon Z_2 \to B$ taking 0 to 0_B and 1 to 1_B, where 0_B is the least element and 1_B is the greatest element of B (see, e.g., [2], [3]). For any formula $\varphi(f_1, \dots, f_n)$ in the ordinary language of set theory with the original binary relations '$\in$' and '$=$', where the parameters $f_1, \dots, f_n$ belong to V^B, one can define an *estimate* (the terms "reliability degree" and "probability" are also in use) for $\varphi(f_1, \dots, f_n)$ to be true in V^B. This estimate is denoted by $[\![\varphi(f_1, \dots, f_n)]\!]$ and is an element of B. In particular, it is assumed that the above objects V^{Z_2} and V^B are factorized by the equivalence relation

$$[\![f = g]\!] = 1_B,$$

where $f, g \in V^B$.

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In this note, we consider infinite-dimensional representations T of the direct product of groups $G = K \times N$, where K is a compact group and N is a locally compact Abelian group. Under natural restrictions, such a representation is imaged by a representation of the nonstandard analog $\widetilde{G}$ of the group G in the group of nonstandard matrices of fixed nonstandard size. The case in which G is a locally compact Abelian group was considered in [4], [5] and even in a more general form than that used here, namely, for representations of the group G in a Banach space $\mathscr{H}$ by spectral operators of scalar type. In the Abelian case, the image of operators in a complex Banach space are nonstandard complex numbers, which (by themselves) form a commutative algebra. In the case considered here, the group G is non-Abelian, and nonstandard square matrices naturally arise instead of nonstandard numbers. Note that these matrices are of size $n \times n$, where n is a nonstandard natural number and their matrix elements are nonstandard complex numbers. Here the nonstandard complex numbers have all properties of the ordinary field of complex numbers, and the same holds for the ring of nonstandard matrices.

A nonstandard group $\widetilde{G}$ is defined as the *nonstandard completion* of a standard group G with respect to the left uniform structure generated by all neighborhoods of the neutral element in G. For the class of groups treated below (which consists of commutative groups, compact groups, and their direct products), the left and right uniform structures coincide. One can also define the *nonstandard completion* $\widetilde{X}$ of a space X which is only uniform; this object was introduced in [4] (see also [2] and [5]).

The imaging is aimed at the *transfer* of properties; in our case, these are properties of a nonstandard homomorphism $\widetilde{T}$ (with values in the algebra of nonstandard complex matrices of fixed size; such an extension $\widetilde{T}$ is constructed in a special way below in the theorem) *transferred* to the original standard homomorphism T. Of course, some properties cannot be transferred in this way; however, the transfer holds, for instance, for properties expressed by Horn formulas in the language of rings [2] and, which is more important, the individual analysis of separate properties makes it possible to make conclusions concerning T by using $\widetilde{T}$ (for instance, as in [2]).

It is assumed that the reader is acquainted with the first notions of Boolean-valued analysis, for instance, according to [2] or [6], and with the corresponding results in functional analysis. The presentation is organized so as to minimize the use of logical notions of nonstandard analysis by themselves, in particular, the notions of estimate and universe V^B. Therefore, the presentation below can be regarded as an example of a Boolean-valued analysis carried out without using logical notions.

2. DEFINITION AND THE FORMULATION OF THE RESULT

Let G be a group which is a direct product of groups, $G = K \times N$, where K is a compact group and N is a locally compact Abelian group. Let a strongly continuous unitary representation T of the group G in a separable Hilbert space $\mathscr{H}$ be given. Let $\boldsymbol{A}$ be a von Neumann algebra of operators in the space $\mathscr{H}$. Suppose that $\boldsymbol{A}$ contains a complete strongly closed Boolean algebra $\boldsymbol{B}$ of pairwise commuting projections. Denote by $M(G)$ the group algebra of the group G, i.e., the algebra of finite Borel measures on G with the ordinary addition and with the convolution as the multiplication; it is sometimes referred to as the *measure algebra*. Suppose that the image of the representation T of G belongs to the algebra $\boldsymbol{A}$. In this case, a homomorphism of the algebra $M(G)$ into $\boldsymbol{A}$ is naturally defined; denote this homomorphism by T. Denote by $\Sigma(\boldsymbol{B})$ the weakly closed subalgebra of $\boldsymbol{A}$ generated by $\boldsymbol{B}$. Obviously, $\Sigma(\boldsymbol{B})$ is a commutative von Neumann algebra.

Suppose that the following conditions hold.

Condition 1. The image of the center $Z(M(G))$ of the algebra $M(G)$ under the homomorphism T belongs to $\Sigma(\boldsymbol{B})$.

Condition 2. The operators $T(g)$ commute with the elements of the algebra $\Sigma(\boldsymbol{B})$.

Conditions 1 and 2 have natural quantum-mechanical interpretation. Namely, $\boldsymbol{A}$ is the algebra of observables of the quantum system, $\Sigma(\boldsymbol{B})$ is the algebra of its "macroscopic observables," G is the "internal symmetry" group (Condition 2), and the "Casimir operators," i.e., the elements of the center of the group algebra, are "macroscopic observables" by themselves (Condition 1). The property of "macroscopic observability" of the Casimir operators is a basis of all existing schemes of classification of elementary particles [7]. For instance, the center of the group algebra of the group $SU(2)$ is generated by the unit element and central idempotents P_n, where $n = 1, 2, \dots$, which correspond (in the state space of the particles) to projections onto subspaces of states with given spin $j = (n-1)/2$.

Denote by S the spectrum of the algebra $\Sigma(\boldsymbol{B})$, which is naturally isomorphic to the Stone space of the Boolean algebra $\boldsymbol{B}$. It follows from the completeness of the algebra $\boldsymbol{B}$ that the space S is extremally disconnected [8]. As is well known [9], the locally compact Abelian group N contains an open subgroup of the form $N_0 \times \mathbb{R}^n$, where N_0 stands for a compact Abelian group and $\mathbb{R}$ for the additive group of real numbers. Write

$$H = K \times N_0 \times \mathbb{R}^n = G_0 \times \prod_{i=1}^{n} G_i,$$

where $G_0 = K \times N_0$ and $G_i = \mathbb{R}$ for $i = 1, \dots, n$. Obviously, H is an open subgroup in G. Denote by T_i the restriction of T to the subgroup G_i, $i = 0, 1, \dots, n$. Since N belongs to the center of G, it follows from Condition 1 on the homomorphism T that the image of G_i, $i \geq 1$, under the mapping T_i belongs to the algebra $\Sigma(\boldsymbol{B})$ which is isomorphic, as is known, to the algebra $C(S)$ of continuous complex-valued functions on the compactum S. Let us take the subgroup Z_i of G_i consisting of the integers. Denote by e_i the element of G_i corresponding to the number 1. In this case, $T_i(e_i)$, when regarded as an element of the algebra $\Sigma(\boldsymbol{B})$, corresponds to some nonvanishing continuous complex-valued functions φ_i on S.

Since the space S is extremally disconnected, every bounded Borel function on S coincides with a continuous function on S up to a meager set (in another terminology, on a first category set). In particular, taking the composition of the standard complex logarithm Ln (with the cut along the negative semiaxis) [10] and the function φ_i, we obtain a bounded Borel function $\operatorname{Ln}\varphi_i$. Therefore, there is a continuous function on S coinciding with $\operatorname{Ln}\varphi_i$ outside a meager set. Denote this function by $\ln\varphi_i$. Obviously, $\exp(\ln\varphi_i) = \varphi_i$ on a complement to a meager set, and hence, by continuity, on the entire set S. If we now set

$$T_i'(t) = T_i(t)\exp(-t\,\ln\varphi_i)$$

for $t \in G_i$, then T_i' is a homomorphism of G_i into $\Sigma(\boldsymbol{B})$ such that $T_i'(e_i) = 1$, and hence one can regard T_i' as a homomorphism of the compact group G_i/Z_i into $\Sigma(\boldsymbol{B})$.

Since the C^*-algebra $\Sigma(\boldsymbol{B}) \simeq C(S)$ acts on the Hilbert space $\mathscr{H}$, one can realize the space $\mathscr{H}$ as a direct integral (for the definition of a direct integral, see [11]),

$$\mathscr{H} = \int_S^{\oplus} H(\zeta)\,d\mu(\zeta),$$

where μ is a finite Borel measure on S and $\mathscr{H}(\zeta)$ is a measurable field of Hilbert spaces. Thus, every element of $\mathscr{H}$ defines a measurable function $h(\zeta)$ with values in $\mathscr{H}(\zeta)$, and the operators in the algebra $\Sigma(\boldsymbol{B})$, when realized as continuous numerical functions $f(\zeta)$, $\zeta \in S$, act by the formula $f \colon h(\zeta) \mapsto f(\zeta)h(\zeta)$, i.e., as multiplication operators. [1]

By Condition 2, the operators $T(g)$ commute with $\Sigma(\boldsymbol{B})$, and therefore they are realized as measurable operator-valued functions $t_g(\zeta)$ that act pointwise, i.e.,

$$T(g)\colon h(\zeta) \mapsto t_g(\zeta)h(\zeta).$$

Here $t_g(\zeta)$ are by no means numbers.

However, the group G is of the form $K \times N$ and, by Condition 1, the images of the elements in the Abelian factor N belong to $\Sigma(\boldsymbol{B})$, i.e., are realized by continuous numerical functions on S. As far as the elements $g \in K$ are concerned, let us use the fact that the center of the group algebra $M(K)$ is mapped into $\Sigma(\boldsymbol{B})$ under the homomorphism T.

Recall that, if Π is a continuous irreducible representation of the compact group K in a finite-dimensional space $H_\Pi = \mathbb{C}^{n_\Pi}$ and $\chi_\Pi(g) = \operatorname{tr}(\Pi(g))$ is the character of this representation, then

$$P_\Pi = n_\Pi \int_K T(g)\overline{\chi_\Pi(g)}\,dg,$$

[1] Here, the general form of the spectral theorem in a separable Hilbert space is used; see, e.g., [12, Chap. II, Sec. 6, Theorem 1, p. 208].

where dg is the normalized Haar measure on K, is a projection and belongs to the image of the center of the algebra $M(K)$. The projections P_Π form a partition of unity in the space $\mathscr{H}$.

Since, as was noted above, the image of the center of $M(K)$ belongs to $\Sigma(\boldsymbol{B})$, the projections P_Π are realized by continuous functions on S taking the values 0 and 1. Thus, to the projection P_Π there corresponds a function i_Π which is the characteristic function of some open-and-closed subset S_Π in S. The sets S_Π are pairwise disjoint, and their union $\widetilde{S}$ is a full-measure set in S. Further, the complement $S \setminus \widetilde{S}$ is a meager set because the supremum of the elements S_Π in the algebra $\boldsymbol{B}$ is the identity element of $\boldsymbol{B}$. The representation T of the group K, when restricted to the image of the projection P_Π, is a primary representation, a multiple of the representation Π. Hence, for $\zeta \in S_\Pi$, the space $\mathscr{H}(\zeta)$ can be represented in the form $\mathbb{C}^{n_\Pi} \otimes L(\zeta)$, where $L(\zeta)$ is a measurable field of Hilbert spaces, and the operator $t_g(\zeta)$ is of the form $\Pi(g) \otimes 1$. Let us define the reduced Hilbert space

$$\mathscr{H}_{\mathrm{red}} = \int_S^{\oplus} \mathbb{C}^{n(\zeta)}\, d\mu(\zeta),$$

where $n(\zeta)$ is an integer-valued function on S which is equal to n_Π on S_Π. In the space $\mathscr{H}_{\mathrm{red}}$, the elements $\Sigma(\boldsymbol{B})$ still act as operators of multiplication by numerical functions, and the operators $T(g)$, $g \in K$, act as matrix-valued functions taking the value $\Pi(g)$ on S_Π. Hence all operators $T(g)$, $g \in G$, are realized as matrix-valued functions on S, and the size of the corresponding matrices $n(\zeta)$ is the same for any $g \in G$. The matrix elements of these matrices are continuous functions on each set S_Π. Denote by $I(g)$ the matrix-valued function corresponding to the element $g \in G$. Write

$$I = \bigl\{\langle g^\vee, I(g)\rangle\_ \mid g \in G\bigr\}\_,$$

where the underlining $U\_$ stands for the function identically equal to one defined on U and, in turn, U consists of some functions with values in $\boldsymbol{B}$ whose domains are formed by functions of the same type. As a result, $I \in V^{\boldsymbol{B}}$ and, certainly, $G^\vee \in V^{\boldsymbol{B}}$.

Theorem. *The object I is a continuous function on the group $G^\vee$, which can thus be extended by continuity to a homomorphism $\widetilde{T}$ of the group $\widetilde{G}$ into the group of complex matrices of nonstandard size $n(\zeta)$.*

Here $\widetilde{G}$ stands for the result of completing the group $G^\vee \in V^{Z_2} \subseteq V^{\boldsymbol{B}}$ in $V^{\boldsymbol{B}}$ with respect to the natural uniform structure in $G^\vee$. Formally, the theorem claims that the assertion stated in it, when expressed by a formula in the language of the Zermelo–Fraenkel set theory, admits a Boolean estimate which corresponds to the universe $V^{\boldsymbol{B}}$ and is equal to $1_{\boldsymbol{B}}$, and this is provable in the ordinary Zermelo–Fraenkel axiomatic set theory ZFC.

Proof. It immediately follows from a theorem appearing in [4] (see [2, Theorem 24]) that $I(g)$ corresponds to a nonstandard complex matrix of nonstandard size $n(\zeta)$, i.e., the values of the function I belong to the above algebra of nonstandard matrices. It suffices to show that I is continuous in some neighborhood of the neutral element of the group $G^\vee$. For such a neighborhood we take the product

$$H^\vee = G_0^\vee \times \prod_{i=1}^{n} G_i^\vee = K^\vee \times N_0^\vee \times \prod_{i=1}^{n} G_i^\vee,$$

where K is a compact group, N_0 is a compact Abelian group, and $G_i = \mathbb{R}$ for $i = 1, \dots, n$. It suffices to show that $I(g)$ is continuous on each of the groups $K^\vee$, $N_0^\vee$, and $G_i^\vee$, $i = 1, \dots, n$, separately. Let us begin with $K^\vee$. Let J_0 be the family of open neighborhoods of the neutral element of the group K, $J_0^\vee = \{a^\vee \mid a \in J_0\}\_$, let E be the identity matrix (of variable size depending on $\zeta \in S$). The condition that I is continuous at the neutral element of $K^\vee$ is as follows:

$$[\![\forall\, \varepsilon \in (\mathbb{Q}_+^\vee)\ \exists\, O \in J_0^\vee\ \forall\, g \in O \qquad (\|I(g) - E\| < \varepsilon)]\!] = 1. \tag{1}$$

An estimate arises here for the first time; this estimate is defined as follows: to the quantifiers '∀' and '∃' there correspond the operators '∧' (the infimum) and '∨' (the supremum) in the algebra $\boldsymbol{B}$ (do not confuse these operations with the intersection and union of sets), and therefore formula (1) is reduced to the equality

$$\bigwedge_{\varepsilon>0}\bigvee_{O\in J_0}\bigwedge_{g\in O}[\![\|I(g^\vee)-E\|<\varepsilon^\vee]\!]=1.$$

This is equivalent to the formula

$$\bigvee_{O\in J_0}\bigwedge_{g\in O}[\![\|I(g^\vee)-E\|<\varepsilon^\vee]\!]=1$$

for any $\varepsilon>0$. For a given $\varepsilon>0$ and a given irreducible representation Π of the group K, we write

$$O_{\Pi,\varepsilon}=\{g\in G \mid \|\Pi(g)-E\|<\varepsilon\}.$$

Let us show that

$$\forall\, g\in O_{\Pi,\varepsilon}\quad \big([\![\|I(g^\vee)-E\|<\varepsilon^\vee]\!]\ge P_\Pi\big).$$

Indeed, since $t_g(\zeta)=\Pi(g)$ for $\zeta\in S_\Pi$, it follows that

$$[\![\|I(g^\vee)P_\Pi-P_\Pi\|<\varepsilon^\vee]\!]=1$$

for $g\in O_{\Pi,\varepsilon}$. Hence

$$[\![\|I(g^\vee)-E\|<\varepsilon^\vee]\!]\ge P_\Pi,$$

where P_Π inside the estimate is the scalar 1 with reliability P_Π and the scalar 0 with reliability $\neg P_\Pi$.

Since the projections P_Π form a partition of unity in the Boolean algebra $\boldsymbol{B}$, i.e., they are disjoint and their supremum is equal to the identity element of $\boldsymbol{B}$, this implies that

$$\bigvee_{O\in J_0}\bigwedge_{g\in O}[\![\|I(g^\vee)-E\|<\varepsilon^\vee]\!]=1.$$

The continuity of $I(g)$ on $N_0^\vee$ can be proved in a similar way. The difference is that, instead of the partition of unity (in the algebra $\boldsymbol{B}$) corresponding to the family of irreducible representations of the group K, we take the partition of unity corresponding to the irreducible representations of the (compact) group N_0.

The situation is somewhat more complicated for the group $G_i^\vee$ because the group $G_i=\mathbb{R}$ is not compact. Consider, say, the group G_1. For $t\in G_1$, we have

$$T(t)=T_1'(t)\exp(t\,\ln\varphi_i).$$

Here we have two representations of the group G_1, namely, T_1' and $\tau_1(t)=\exp(t\,\ln\varphi_i)$. We must verify the continuity of each of these representations. The representation T_1' is the composition of the representation T_1' of the compact group G_1/Z_1 and the natural projection $G_1\to G_1/Z_1$. The above construction can be used for the representation of the compact group G_1/Z_1, and the continuity of the projection is obvious. The continuity of the representation $\tau_1(t)$ results from the following fact: for a continuous function $a(\zeta)$, $\zeta\in S$, the function $\exp(ta(\zeta))$ of the variables (t,ζ) is jointly continuous. Indeed, for any $\varepsilon>0$ one can choose a $t(\varepsilon)>0$ in such a way that the inequality $|\exp(ta(\zeta))-1|<\varepsilon$ holds for any $\zeta\in S$ and any t, $|t|<t(\varepsilon)$. It suffices to take

$$t(\varepsilon)=\frac{1}{\|a\|}\,\ln(1+\varepsilon),\qquad \text{where}\quad \|a\|=\max|a(\zeta)|.$$

This implies that

$$\bigvee_{O\in J_0}\bigwedge_{t\in O}[\![\|\tau_1(t^\vee)-1\|<\varepsilon^\vee]\!]=1.\quad\square$$

$\square$

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