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Definable Elements of Definable Borel Sets

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Abstract—We prove that it is true in Sacks, Cohen, and Solovay generic extensions that any ordinal definable Borel set of reals necessarily contains an ordinal definable element. This result has previously been known only for countable sets.

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1. INTRODUCTION. THE PROBLEM OF THE CHOICE OF A DEFINABLE ELEMENT

The problem of definability of mathematical objects appeared in the focus of attention of discussions on mathematical foundations in connection with the axiom of choice by Zermelo and its role in the construction of a well-ordering of the continuum and other similar arguments. Thus, it is underlined in the the paper [1], which presents the discussion between Hadamard, Borel, Baire, and Lebesgue on questions of mathematical foundations, that a pure existence proof of an element in a given set and a direct definition (or effective construction) of such an element are different mathematical results, of which the second does not follow from the first. In particular, in his part of [1], Lebesgue pointed out difficulties in the problem of effective choice, that is, the choice of a definable element in a definable set.[1]

It was established during the course of development of set theory, and especially *descriptive* set theory, that sets in the real line of the second projective class admit such an effective choice (see P. Novikov and Luzin's paper [2]). To be more exact, in modern terms, every nonempty Σ^1_2 set in the real line $\mathbb{R}$ contains a point of class Δ^1_2 and hence an effectively definable element (see Theorem 2.6 and Corollary 2.7 below.)

As for higher levels of the projective hierarchy[2], that is, beginning with Π^1_2, no similar theorem can be proved. More exactly, there are only hypotheses, which are true under certain assumptions consistent with the **ZFC** axioms but false under some other assumptions, also consistent with the **ZFC** axioms.

Remark 1.1. In modern works on set theory, by *reals* (real numbers) are usually understood both elements of the real line proper and the points of the Baire space ω^ω or the Cantor discontinuum $2^\omega \subseteq \omega^\omega$. It is in this sense that we understand $\mathbb{R}$ in this section. The exact meaning depends on the context, but everything said here is equally related to 2^ω, ω^ω, or the real line proper because of the existence of definable one-to-one correspondences between these three domains.

Thus, on the one hand, the Gödel axiom of constructibility $\mathbf{V} = \mathbf{L}$, which is consistent with **ZFC**, implies the existence of a well-ordering $\le_{\mathbf{L}}$ of type ω_1 of the whole real line $\mathbb{R}$, which is a Δ^1_2 relation, and this allows us to choose just the $\le_{\mathbf{L}}$-least element in any set $X \subseteq \mathbb{R}$. An accurate estimation of

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[1] "Ainsi je vois déjà une difficulté dans ceci 'dans un M' déterminé je puis choisir un m' déterminé' " in the original paper [1], which means "Thus, I already see a difficulty with the assertion that 'in a determinate M' I can choose a determinate m'' " (see the English translation of [1] in Appendix 1 of the book [3].

[2] See [4, Chap. 6] or [5, Chap. 1] on the classes Σ^1_n, Π^1_n, and Δ^1_n of the effective projective hierarchy.

definability shows that, under the assumption $\mathbf{V} = \mathbf{L}$, if $n \geq 3$, then every Σ^1_n set $\varnothing \neq X \subseteq \mathbb{R}$ contains a Δ^1_n point $x \in X$.

On the other hand, models of **ZFC** set theory are known in which there exist nonempty Π^1_2 sets $X \subseteq \mathbb{R}$ containing no points of class Δ^1_n for any n and, in general, no ordinal definable point[3], so that an effective choice of a point in such a set X is not possible not only in the context of the projective hierarchy, but also in the broadest sense. In the *Solovay model* [6] and some other models, the Π^1_2 set $\omega^\omega \setminus \mathbf{L}$ of all nonconstructible points is such a set, and it is definitely uncountable. This category of models is characterized by the feature that their construction involves sufficiently *homogeneous* forcing notions, that is, those which admit fairly rich systems of order automorphisms.

At the same time, the authors have recently defined some models in [7], [8], in which there exist *countable* Π^1_2 sets containing no definable elements. They are related to *nonhomogeneous* forcing notions, that is, those having poor systems of automorphisms, for instance, only rational shifts. On the other hand, as established in [9], it is true in some models of the first, forcing-homogeneous, category that every *countable* **OD** set $\varnothing \neq X \subseteq \mathbb{R}$ contains **OD** elements. The following theorem extends this result from countable sets to arbitrary Borel sets.

Theorem 1.2. *Let* $\mathbf{L}[a]$ *be one of the following three generic extensions of the constructible set universe* $\mathbf{L}$:

(A) *the extension by one Cohen-generic real* $a \in \mathbb{R}$;

(B) *the extension by one Solovay-random real* $a \in \mathbb{R}$;

(C) *the extension by one Sacks-generic real* $a \in \mathbb{R}$.

Then it is true in $\mathbf{L}[a]$ *that if* $\varnothing \neq X \subseteq \mathbb{R}$ *is a Borel* **OD** *set, then* X *contains* **OD** *elements.*

On these generic extensions, see, e.g., the papers [10]–[12].

2. THE CHOICE OF AN ELEMENT IN Σ^1_1 AND Σ^1_2 SETS

For the convenience of the reader, the proof of Theorem 1.2 is preceded by a brief survey of results on the effective choice of elements in sets of initial levels of the projective hierarchy. In this survey, we give references to the seminal book [13] of Moschovakis and also to the books [4] and [14] for a Russian-speaking reader. The content of this section is not connected with the proof of Theorem 1.2.

We begin with Σ^1_1 sets of general form. The following two results show that the choice of a Δ^1_1 element is not generally possible even in the case of Π^0_1 sets, and yet it is always possible to pick an element only slightly more complex than Δ^1_1. Note that, when the points $x \in \omega^\omega$ are classified, the unilateral classes Σ^1_n, Π^1_n reduce to Δ^1_n because of the equivalence

$$x(k) = n \iff \forall\, n' \neq n \quad (x(k) \neq n').$$

Theorem 2.1 (Kleene's Basis Theorem [13, 4E.8], [14, 7.11]). *There is a* Σ^1_1 *set* $U \subseteq \omega$ *of natural numbers such that every* Σ^1_1 *set* $\varnothing \neq X \subseteq \omega^\omega$ *contains a real recursive relative to* U.

Example 2.2 (Kleene; see [13, 4D.14] or [4, 9.2.4]). The cocountable set X of all reals $x \in \omega^\omega$ not belonging to Δ^1_1 is of class Σ^1_1 and, obviously, contains no Δ^1_1 element. Now consider any Π^0_1 set $P \subseteq \omega^\omega \times \omega^\omega$, which projects onto X, so that

$$x \in X \iff \exists\, y\, P(x, y).$$

Then the set P also contains no Δ^1_1 points $\langle x, y\rangle$. Hence there exists a Π^0_1 set $Q \subseteq \omega^\omega$ which contains no Δ^1_1 reals. Namely, Q is the image of P under any recursive homeomorphism $\omega^\omega \times \omega^\omega$ onto ω^ω.

[3] The class **OD** of all *ordinal definable sets*, or simply **OD**sets, consists of all sets definable by set-theoretic formulas which may contain ordinals as parameters. The classes Σ^1_n, Π^1_n, and Δ^1_n are subsets of **OD**, of course.

Example 2.3. The example of a Σ^1_1 set $X \subseteq \omega^\omega$ in 2.2 containing no Δ^1_1 elements can be strengthened by the requirement of *compactness*. Indeed, consider a pair of disjoint Δ^1_1-inseparable Π^1_1 sets $U, V \subseteq \omega$ (see [13, 4B.12] or [4, 8.1.3]) and the corresponding nonempty Σ^1_1 compact set

$$X = \{x \in 2^\omega : \forall\, n \in U\,(x(n) = 0) \wedge \forall\, n \in V\,(x(n) = 1)\}.$$

The set X contains no Δ^1_1 elements $x_0 \in X$, since any such element implies the separation of U from V by the Δ^1_1 set $S = \{n : x_0(n) = 0\}$.

Theorem 2.4 ([13, 4F.11 and 4F.15], [4, 10.6.4]). *If a Δ^1_1 set $\varnothing \neq X \subseteq \omega^\omega$ is at least σ-compact, then it contains a Δ^1_1 element $x \in X$.*

A fairly elementary proof for the compact case is as follows. First of all, given a compact Δ^1_1 set $X \subseteq \omega^\omega$, we define a Δ^1_1 tree $T \subseteq \omega^{<\omega}$ which determines X in the sense that

$$X = [T] = \{x \in \omega^\omega : \forall\, m\,(x \restriction m \in T)\}.$$

Then, using the Koenig lemma (note that T has finite branching nodes), we prove that the lexicographically leftmost element $x_{\text{left}} \in X$ belongs to Δ^1_1.

Theorem 2.4 holds for countable sets X, as they are σ-compact, but, in this case, the following theorem extends the result to Σ^1_1 sets as well, which is not the case for σ-compact and even compact Σ^1_1 sets due to Example 2.3.

Theorem 2.5 ([13, 4F.5], [4, 10.4.1]). *If a Σ^1_1 set $X \subseteq \omega^\omega$ is at most countable, then it consists only of Δ^1_1 points.*

Theorems of this kind are also known for Δ^1_1 sets large in the sense of measure or category. A standard reference to this case, which will not be discussed here, is the book [13, 4F.19 ff.] by Moschovakis; see also our book [4, Sec. 11.4 and 11.5]. Theorem 2.5 is proved by means of the *Gandy − Harrington topology* generated by nonempty Σ^1_1 sets as the base. This topology is widely used in proofs of dichotomical theorems; see, e.g., [15].

For sets more complex than Σ^1_1, the following fundamental result known as the Novikov–Kondo–Addison theorem holds. It was established by Kondo [16] for the projective class $\mathbf{\Pi}^1_1$ on the basis of Novikov's method published in [2], and the result for the effective class Π^1_1 was presented by Addison [17].

Theorem 2.6 ([13, 4E.4], [4, 8.4.1]). *Every $\mathbf{\Pi}^1_1$ set $P \subseteq \omega^\omega \times \omega^\omega$ can be uniformized by a set of class $\mathbf{\Pi}^1_1$. Every Π^1_1 set $P \subseteq \omega^\omega \times \omega^\omega$ can be uniformized by a set of class Π^1_1.*

Corollary 2.7. *Every Π^1_1 set $\varnothing \neq X \subseteq \omega^\omega$ contains a Π^1_1 singleton $\{x\} \subseteq X$, and $x \in \Delta^1_2$. Every Σ^1_2 set $\varnothing \neq X \subseteq \omega^\omega$ contains a Δ^1_2 element $x \in X$.*

3. BOREL CODES

Theorem 1.2 refers to a standard coding system for Borel sets, as in [18] or [4, Section 9.5], which includes the *set of Borel codes* $\mathbf{BK}$ containing all pairs of the form $c = \langle T_c, f_c\rangle$, where $T_c \subseteq \omega^{<\omega}$ is a nonempty well-founded tree, $f_c\colon \omega^{<\omega} \to \omega^{<\omega}$ is any function, and $\omega^{<\omega}$ is the set of all *strings* (finite sequences) of natural numbers containing the *empty sequence* Λ.

If $c \in \mathbf{BK}$, then each string $s \in T_c$ is assigned a Borel set $\mathbf{B}_c(s) \subseteq \omega^\omega$ so that if $s \in \max T_c$ (that is, s is a terminal node), then

$$\mathbf{B}_c(s) = [F(s)] = \{x \in \omega^\omega : f_c(s) \subset x\},$$

and if $s \notin \max T_c$, then

$$\mathbf{B}_c(s) = \omega^\omega \setminus \bigcup_{s^\frown k \in T_c} \mathbf{B}_c(s^\frown k).$$

We finally set $\mathbf{B}_c = \mathbf{B}_c(\Lambda)$; this is a Borel set in ω^ω coded by $c \in \mathbf{BK}$. For every ordinal $\xi < \omega_1$, let $\mathbf{BK}_\xi$ denote the set of all codes $c \in \mathbf{BK}$ such that T_c is a tree of height ξ.

The main properties of this coding system are as follows:

(1) the set of all codes $\mathbf{BK}$ is a Π^1_1 set in the Polish space

$$\mathbb{B} = \mathscr{P}(\omega^{<\omega}) \times (\omega^{<\omega})^{(\omega^{<\omega})},$$

and every set $\mathbf{BK}_\xi$ is a Borel set in $\mathbb{B}$;

(2) if $c \in \mathbf{BK}_\xi$, then the set $\mathbf{B}_c \subseteq \omega^\omega$ is a Borel set of class $\mathbf{\Pi}^0_\xi$;

(3) conversely, if $X \subseteq \omega^\omega$ is a set of class $\mathbf{\Pi}^0_\xi$, then there is a code $c \in \mathbf{BK}_\xi$ such that $X = \mathbf{B}_c$;

(4) the following sets in the Polish space $\mathbb{B} \times \omega^\omega$ belong to Π^1_1:

$$W = \{\langle c, x\rangle : c \in \mathbf{BK} \wedge x \in \mathbf{B}_c\} \qquad \text{and} \qquad W' = \{\langle c, x\rangle : c \in \mathbf{BK} \wedge x \in \omega^\omega \setminus \mathbf{B}_c\}.$$

For details, see, e.g., [19, Section 5.7] or [5, 2.9].

This coding system naturally extends to Borel sets of the space $\omega^\omega \times \omega^\omega$. Namely, first of all, if $x \in 2^{<\omega}$, then we set

$$F(x) = \langle y, z\rangle,$$

where $y(n) = x(2n)$ and $z(n) = x(2n+1)$ for all n, so that F is a homeomorphism of the space ω^ω onto $\omega^\omega \times \omega^\omega$. Now we define $\mathbf{B}^{(2)}_c = \{F(x) : x \in \mathbf{B}_c\}$; this is a Borel set in $\omega^\omega \times \omega^\omega$ coded by $c \in \mathbf{BK}$.

This coding system also extends to Borel maps. We set $\mathbf{BF} = \mathbf{BK}^{\omega\times\omega}$, and if $c \in \mathbf{BF}$ (that is, c is a map from $\omega^2 = \omega \times \omega$ to $\mathbf{BK}$), then we define a function $\boldsymbol{\vartheta}_c \colon \omega^\omega \to \omega^\omega$ by setting $\boldsymbol{\vartheta}_c(x)(n) = k$ if and only if either $k = 0$ and $x \notin \bigcup_{\ell\ge 1} \mathbf{B}_{c(n,\ell)}$ or $k \ge 1$ and $x \in \mathbf{B}_{c(n,k)} \setminus \bigcup_{1\le \ell<k} \mathbf{B}_{c(n,\ell)}$. Claims similar to (1)–(4) hold for the functional coding system; thus,

(5) $\mathbf{BF}$ is a Π^1_1 set in the Polish space $\mathbb{B}^{\omega\times\omega}$;

(6) the following sets in $\mathbb{B}^{\omega\times\omega} \times \omega^\omega \times \omega^\omega$ belong to Π^1_1:

$$\Phi = \{\langle c, x, y\rangle : c \in \mathbf{BF} \wedge x, y \in \omega^\omega \wedge y = \boldsymbol{\vartheta}_c(x)\},$$
$$\Phi' = \{\langle c, x, y\rangle : c \in \mathbf{BF} \wedge x, y \in \omega^\omega \wedge y \ne \boldsymbol{\vartheta}_c(x)\}.$$

To admit Borel codes as values of coded functions, we fix a recursive homeomorphism $K \colon \omega^\omega \xrightarrow{\text{onto}} \mathbb{B}$. If $c \in \mathbf{BF}$ and $x \in \omega^\omega$, then we set $\boldsymbol{\kappa}_c(x) = K(\boldsymbol{\vartheta}_c(x))$, so that $\boldsymbol{\kappa}_c(x) \in \mathbb{B}$ (but not necessarily $\boldsymbol{\kappa}_c(x) \in \mathbf{BK}$!).

Remark 3.1. Claim (4) can be understood in the sense that the relation $x \in \mathbf{B}_c$ can be expressed both by a Π^1_1 formula $\langle c, x\rangle \in W$, and a Σ^1_1 formula $\langle c, x\rangle \notin W'$, provided that $c \in \mathbf{BK}$ (otherwise, the formulas are nonequivalent). A more complex relation $y \in \mathbf{B}_{\boldsymbol{\kappa}_c(x)}$ can be expressed by the formulas

$$\psi(c, x, y) := \exists\, h(\langle c, x, h\rangle \notin \Phi' \wedge \langle K(h), y\rangle \notin W') \qquad (\text{type } \Sigma^1_1);$$
$$\psi'(c, x, y) := \forall\, h(\langle c, x, h\rangle \notin \Phi' \Longrightarrow \langle K(h), y\rangle \in W) \qquad (\text{type } \Pi^1_1),$$

so that we have

$$y \in \mathbf{B}_{\boldsymbol{\kappa}_c(x)} \iff \psi(c, x, y) \iff \psi'(c, x, y)$$

whenever $c \in \mathbf{BF}$ and $\boldsymbol{\kappa}_c(x) \in \mathbf{BK}$.

4. PROOF FOR COHEN GENERIC EXTENSIONS

Here we prove Theorem 1.2 in Part (A), that is, for Cohen generic extensions. The Cohen forcing notion $\mathbf{Coh} = 2^{<\omega}$ consists of all dyadic strings, that is, finite sequences of the numbers 0 and 1. If $s, t \in 2^{<\omega}$, then $s \subseteq t$ means that t extends the string s, and $s \subset t$ means proper extension. If $t \in 2^{<\omega}$ and $i = 0, 1$, then $t^\frown i$ denotes the extension of t by i as the rightmost term. If $s \in 2^{<\omega}$, then $\mathrm{lh}(s)$ is the length of the string s.

If $u \in \mathbf{Coh}$, then the set $I_u = \{a \in 2^\omega : u \subset a\}$, that is, a *Cantor interval* in 2^ω, is clopen in the Cantor space 2^ω.

Theorem 4.1. *If $a_0 \in 2^\omega$ is a Cohen generic real over a set universe $\mathbf{V}$, then it is true in $\mathbf{V}[a_0]$ that if $\varnothing \neq X \subseteq 2^\omega$ is a Borel set definable by a set-theoretic formula with sets in $\mathbf{V}$ as parameters, then X has an element in $\mathbf{V}$.*

Remark 4.2. The set universe $\mathbf{V}$ in the theorem can be understood both as a fixed (for instance, countable) transitive model of the **ZFC** set theory and as the real set-theoretic universe of all sets. In the latter case, generic extensions, such as, for instance, $\mathbf{V}[a_0]$ in the theorem, are understood as Boolean-valued extensions of the universe $\mathbf{V}$.

Theorem 4.1 immediately implies Theorem 1.2, Part (A). Indeed, it suffices to set $\mathbf{V} = \mathbf{L}$, and make use of the fact that $\mathbf{L} \subseteq \mathbf{OD}$ always holds.

Proof of Theorem 4.1. There is a formula $\varphi(x)$ with sets of the ground universe $\mathbf{V}$ as parameters and a code $p \in \mathbf{BK} \cap \mathbf{V}[a_0]$ such that

$$X = \mathbf{B}_p = \{x \in 2^\omega : \varphi(x)\} \qquad \text{in} \quad \mathbf{V}[a_0].$$

Arguing by contradiction, suppose that $X \cap \mathbf{V} = \varnothing$, that is, the set X contains no point in the ground universe $\mathbf{V}$.

Recall that the Cohen generic extensions satisfy the requirement of *Borel reading of names*, according to which there is a code $c \in \mathbf{BF} \cap \mathbf{V}$ satisfying $p = \boldsymbol{\kappa}_c(a_0)$ (see, e.g., [11, Theorem 2.4(iii)]). Therefore, it is true in the extension $\mathbf{V}[a_0]$ that the Borel set $\mathbf{B}_{\boldsymbol{\kappa}_c(a_0)}$ is equal to the set $X = \{x \in 2^\omega : \varphi(x)\}$, where the formula φ contains only sets in $\mathbf{V}$ as parameters. Accordingly, we have

$$\mathbf{B}_{\boldsymbol{\kappa}_c(a_0)} \cap \mathbf{V} = \varnothing.$$

Hence there exists a Cohen condition (that is, a string) $u \in 2^{<\omega}$ which $\mathbf{Coh}$-forces, over $\mathbf{V}$, the relations

$$\boldsymbol{\kappa}_c(\breve{\mathbf{a}}) \in \mathbf{BK}, \qquad \mathbf{B}_{\boldsymbol{\kappa}_c(\breve{\mathbf{a}})} = \{x : \varphi(x)\} \neq \varnothing, \qquad \mathbf{B}_{\boldsymbol{\kappa}_c(\breve{\mathbf{a}})} \cap \mathbf{V} = \varnothing.$$

Here $\breve{\mathbf{a}}$ is a name for the Cohen generic real.

Lemma 4.3. *It is true in the universe $\mathbf{V}$ that $Y = \{x \in I_u : \boldsymbol{\kappa}_c(x) \in \mathbf{BK}\}$ is a comeager subset of I_u.*

Proof. The set Y belongs to $\boldsymbol{\Pi}^1_1$ along with the set $\mathbf{BK}$, since the former set is equal to the Borel preimage of the latter under the map $\boldsymbol{\kappa}_c$. It follows that Y has the Baire property. Therefore, if Y is not comeager in I_u, then there is a string $v \in \mathbf{Coh}$ extending u and such that $Y \cap I_v$ is, on the contrary, a meager set. Therefore, this set is covered by a meager $\mathbf{F}_\sigma$ set $F \subseteq I_v$. The complementary $\mathbf{G}_\delta$ set $G = I_v \setminus F$ is comeager in I_v, and we have $\boldsymbol{\kappa}_c(x) \notin \mathbf{BK}$ for all $x \in G$. Let us fix any code $g \in \mathbf{BK} \cap \mathbf{V}$ for G, so that it is true in $\mathbf{V}$ that $\forall\, x \in \mathbf{B}_g \ (\boldsymbol{\kappa}_c(x) \notin \mathbf{BK})$.

But this sentence is expressed by a Π^1_2 formula, namely, by the formula

$$\forall\, x\, \forall\, y(\langle g, x\rangle \in W \wedge \langle c, x, y\rangle \in \Phi \quad \Longrightarrow \quad K(y) \notin \mathbf{BK}),$$

where the subformulas $\langle g, x\rangle \in W$ and $\langle c, x, y\rangle \in \Phi$ of type $\mathit{\Pi}^1_1$ express, respectively, the statements $x \in \mathbf{B}_g$ and $y = \boldsymbol{\vartheta}_c(x)$, according to claims (4) and (6) in Sec. 3. Therefore, by the Shoenfield absoluteness theorem, the sentence under consideration is true in any generic extension of the form $\mathbf{V}[a]$, where $a \in I_v$ is an arbitrary Cohen generic real over $\mathbf{V}$. However, the Cohen generic reals are well-known not to belong to meager Borel sets coded in the ground model (see, e.g., Theorem 11.3.3 in [19] for the ideal of meager sets). It follows that $a \in \mathbf{B}_g$ in $\mathbf{V}[a]$ and, subsequently, $\boldsymbol{\kappa}_c(a) \notin \mathbf{BK}$. This is a contradiction, because $u \subseteq v \subset a$, while the string u forces $\boldsymbol{\kappa}_c(\breve{\mathbf{a}}) \in \mathbf{BK}$. $\square$

Arguing in the universe $\mathbf{V}$, we conclude, by the lemma, that there exists a comeager $\mathbf{G}_\delta$ set $D \subseteq I_u$, such that $\boldsymbol{\kappa}_c(x) \in \mathbf{BK}$ for all $x \in D$. Now consider the Borel set $P = \{\langle x, y\rangle : x \in D \wedge y \in \mathbf{B}_{\boldsymbol{\kappa}_c(x)}\}$ and the following equivalence relation on the set D: $x \mathsf{E} x'$ if and only if $x, x' \in D$ and $P_x = P_{x'}$. (As usual, $P_x = \{y : \langle x, y\rangle \in P\}$ is the section.)

Lemma 4.4. *The relation* E *is* $\mathbf{\Pi}^1_1$.

Proof. The equality $P_x = P_{x'}$ is expressed by the Π^1_1 formula

$$\forall y((\neg\psi'(c, x, y) \implies \psi(c, x', y)) \wedge (\neg\psi'(c, x', y) \implies \psi(c, x, y)))$$

(see Remark 3.1) under the condition that $c \in \mathbf{BF}$ and the reals $\boldsymbol{\kappa}_c(x)$ and $\boldsymbol{\kappa}_c(x')$ belong to $\mathbf{BK}$, which is satisfied here for all $x, x' \in D$. □

As a $\mathbf{\Pi}^1_1$ subset of the product space $I_u \times I_u$, E has the Baire property.

Case 1: (it is true in $\mathbf{V}$ that) all E-equivlence classes are meager sets on I_u. Then the $\mathbf{\Pi}^1_1$ set $H = \{\langle x, x'\rangle \in D : \mathbf{B}_{\boldsymbol{\kappa}_c(x)} = \mathbf{B}_{\boldsymbol{\kappa}_c(x')}\}$ is meager in $I_u \times I_u$ by the Ulam–Kuratowski theorem, and hence H can be covered by a meager Borel set $H' = \mathbf{B}^{(2)}_d$, $H \subseteq H' \subseteq I_u \times I_u$, coded by some d.

Now we make use of a method introduced in [9]. We *continue to argue in* $\mathbf{V}$. Let us fix a countable transitive model $\mathfrak{M}$ of a sufficiently large fragment of $\mathbf{ZFC}$ containing the codes c and d and being an elementary submodel of the universe with respect to all analytic formulas.

Lemma 4.5. *There exist reals* $a, b \in I_u$ *Cohen generic over* $\mathbf{V}$ *and such that* $\mathbf{V}[a] = \mathbf{V}[b]$*, and, at the same time, the pair* $\langle a, b\rangle$ *is Cohen generic over the model* $\mathfrak{M}$.

Proof. We let $+_2$ denote the operation of componentwise addition modulo 2 for infinite sequences. In the universe $\mathbf{V}$, choose a real $z \in Z$ Cohen generic over $\mathfrak{M}$ and satisfying $z(k) = 0$ for all $k < m = \mathrm{lh}(u)$. Consider a real $a \in I_u$ Cohen generic over $\mathbf{V}$ and, hence, over $\mathfrak{M}[z]$ as well. The pair $\langle a, z\rangle$ is then Cohen generic over $\mathfrak{M}$. It follows, by the product forcing theorem, that the real z is Cohen generic over $\mathfrak{M}[a]$. But then the real $b = z +_2 a$ is Cohen generic over $\mathfrak{M}[a]$ by the same theorem, since $a \in \mathfrak{M}[a]$. We conclude that the pair $\langle a, b\rangle$ is Cohen generic over $\mathfrak{M}$ for the same reason. And we have $a, b \in I_u$ by construction. However, the real $b = z +_2 a$ is Cohen generic over $\mathbf{V}$ as well, since so is a; at the same time, $z \in \mathbf{V}$, and we have $\mathbf{V}[a] = \mathbf{V}[b]$. □

Recall that Cohen generic reals, as well as pairs of reals, do not belong to meager Borel sets coded in the ground model by the already mentioned Theorem 11.3.3 in [19]. In particular, $\langle a, b\rangle$ do not belong to H' and hence to H, so that we have $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(b)}$.

Remark 4.6. The last argument makes use of the absoluteness of the formula

$$\forall \langle x, x'\rangle(\langle x, x'\rangle \in \mathbf{B}^{(2)}_d \quad \implies \quad \mathbf{B}_{\boldsymbol{\kappa}_c(x)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(x')})$$

in the sense of Shoenfield, which is substantiated by expressing the relation $\langle x, x'\rangle \in \mathbf{B}^{(2)}_d$ in terms of the set W in Claim (4) of Sec. 3 and writing the inequality $\mathbf{B}_{\boldsymbol{\kappa}_c(x)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(x')}$ as the Σ^1_1 formula being the negation of the Π^1_1 formula employed in the proof of Lemma 4.4 above. In this way, we obtain a Π^1_2 formula, to which the Shoenfield theorem applies. □

At the same time, the generic reals a and b belong to the set I_u by construction. It follows from the choice of u that one and the same set $\{x : \varphi(x)\}$ is equal, in the extension $\mathbf{V}[a] = \mathbf{V}[b]$, both to the set $\mathbf{B}_{\boldsymbol{\kappa}_c(a)}$ and to the set $\mathbf{B}_{\boldsymbol{\kappa}_c(b)}$. We conclude that $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} = \mathbf{B}_{\boldsymbol{\kappa}_c(b)}$. However, it was established above that $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(b)}$. The contradiction obtained witnesses that Case 1 is impossible.

Case 2: (it is true in $\mathbf{V}$ that) one of the E-equivalence classes is a comeager set on a set of the form I_v, where $v \in \mathbf{Coh}$, $u \subseteq v$. Then there exists a Borel set $U = \mathbf{B}_f \subseteq I_v \cap D$, $f \in \mathbf{BK}$, such that it is comeager inside the Cantor interval I_v and all reals $x \in U$ are pairwise E-equivalent. In other words, there exists a Borel set $B = \mathbf{B}_e \subseteq 2^\omega$, $e \in \mathbf{BK}$ in $\mathbf{V}$, such that $\mathbf{B}_{\boldsymbol{\kappa}_c(x)} = B \ \forall x \in U = \mathbf{B}_f$.

Now consider an arbitrary real $a \in I_v$ Cohen generic over $\mathbf{V}$. *Arguing in* $\mathbf{V}[a]$, we make use of the fact that any Cohen generic real has to belong to every Borel set $U = \mathbf{B}_f$ comeager on the corresponding Baire interval I_v; see the proof of Lemma 4.3. As above, it follows, by the Shoenfield absoluteness theorem, that $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} = B = \mathbf{B}_e$, where e is a Borel code in the given universe $\mathbf{V}$.

However, we have $a \in I_v \subseteq I_u$ by construction. Therefore, by the choice of u at the beginning of the proof of the theorem, the set $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} = \mathbf{B}_e$ is nonempty. Again by the absoluteness theorem, the set $\mathbf{B}_e$ is nonempty in $\mathbf{V}$ as well (since the code e belongs to $\mathbf{V}$), that is, it contains an element $x \in \mathbf{V}$. Thus, $x \in \mathbf{B}_{\boldsymbol{\kappa}_c(a)} \cap \mathbf{V}$ in $\mathbf{V}[a]$. This yields a conclusive contradiction to the choice of u, because $a \in I_u$. □

5. PROOF FOR SOLOVAY RANDOM EXTENSIONS

Here we prove Theorem 1.2 in Part (B), that is, for Solovay random extensions. The proof largely follows the line of reasoning in the proof of Theorem 4.1, therefore we skip some common details, for instance, those related to absoluteness, but highlight some differences.

A set $T \subseteq 2^{<\omega}$ is called a *tree* if, for any strings $s \subset t$ in $2^{<\omega}$, $t \in T$ implies $s \in T$. The Solovay random forcing notion $\mathbf{Rand}$ consists of all trees $T \subseteq 2^{<\omega}$ containing no terminal nodes and no isolated branches and such that the set

$$[T] = \{x \in 2^\omega : \forall\, n\, (x \restriction n \in T)\}$$

has positive measure $\boldsymbol{\mu}([T]) > 0$, in the sense of the usual probability measure $\boldsymbol{\mu}$ on 2^ω. In contrast to the Cohen forcing $\mathbf{Coh}$, the forcing notion $\mathbf{Rand}$ depends on the choice of the ground model, so that "a real (Solovay) random over a model $\mathfrak{M}$" means "a real $(\mathbf{Rand} \cap \mathfrak{M})$-generic over $\mathfrak{M}$," and this is equivalent to the condition that the real does not belong to any Borel set $\mathbf{B}_c$ of $\boldsymbol{\mu}$-measure 0 with a code $c \in \mathbf{BK} \cap \mathfrak{M}$.

Another difference from the Cohen forcing is the fact that a random pair of reals is **not** a $(\mathbf{Rand} \times \mathbf{Rand})$-generic pair. The notion of a random pair is connected with the forcing by closed sets in $2^\omega \times 2^\omega$ (or trees which generate them), which have strictly positive measure in the sense of the product measure $\boldsymbol{\mu} \times \boldsymbol{\mu}$ on $2^\omega \times 2^\omega$. The following well-known (see, e.g., [11]) characterization of random pairs will be quite important in what follows.

Proposition 5.1. *Let $\mathfrak{M}$ be a transitive model of a sufficiently large subtheory of* $\mathbf{ZFC}$*, and let $a, b \in 2^\omega$. Then the following four claims are equivalent*:

(1) *the pair $\langle a, b\rangle$ is random over $\mathfrak{M}$*;

(2) *$\langle a, b\rangle$ does not belong to any Borel set $\mathbf{B}^{(2)}_c$ of $(\boldsymbol{\mu} \times \boldsymbol{\mu})$-measure 0 having a code $c \in \mathbf{BK} \cap \mathfrak{M}$*;

(3) *a is random over $\mathfrak{M}$ and b is random over $\mathfrak{M}[a]$*;

(4) *b is random over $\mathfrak{M}$ and a is random over $\mathfrak{M}[b]$*.

Theorem 5.2. *Let $a_0 \in 2^\omega$ be a real Solovay random over a set universe* $\mathbf{V}$*. Then it is true in* $\mathbf{V}[a_0]$ *that if $\varnothing \ne X \subseteq 2^\omega$ is a Borel set definable by a set-theoretic formula with parameters in* $\mathbf{V}$*, then X contains a real in* $\mathbf{V}$.

This theorem immediately implies Theorem 1.2, Part (B), as above.

Proof. There is a formula $\varphi(x)$ with sets in the ground model $\mathbf{V}$ as parameters and a code $p \in \mathbf{BK} \cap \mathbf{V}[a_0]$ such that $X = \mathbf{B}_p = \{x \in 2^\omega : \varphi(x)\}$ in $\mathbf{V}[a_0]$. Arguing by contradiction, suppose that $X \cap \mathbf{V} = \varnothing$.

Similarly to Cohen extensions, Solovay random ones satisfy the condition of *Borel reading of names*, so that there exists a code $c \in \mathbf{BF} \cap \mathbf{V}$ such that $p = \boldsymbol{\kappa}_c(a_0)$. Thus, in the extension $\mathbf{V}[a_0]$, the Borel set $\mathbf{B}_{\boldsymbol{\kappa}_c(a_0)}$ is equal to the $\mathbf{OD}$ set $X = \{x \in 2^\omega : \varphi(x)\}$, and hence $\mathbf{B}_{\boldsymbol{\kappa}_c(a_0)} \cap \mathbf{V} = \varnothing$. There is a tree $T \in \mathbf{Rand} \cap \mathbf{V}$ which $\mathbf{Rand}$-forces, over $\mathbf{V}$, the sentence

$$\boldsymbol{\kappa}_c(\breve{\mathbf{a}}) \in \mathbf{BK}, \qquad \mathbf{B}_{\boldsymbol{\kappa}_c(\breve{\mathbf{a}})} = \{x : \varphi(x)\} \ne \varnothing, \qquad \mathbf{B}_{\boldsymbol{\kappa}_c(\breve{\mathbf{a}})} \cap \mathbf{V} = \varnothing.$$

The set $[T] = \{x \in 2^\omega : \forall\, m\, (x \restriction m \in T)\}$ is closed, and $\boldsymbol{\mu}([T]) = M > 0$.

Lemma 5.3. *It is true in the universe* $\mathbf{V}$ *that the set* $Y = \{x \in [T] : \boldsymbol{\kappa}_c(x) \in \mathbf{BK}\}$ *satisfies* $\boldsymbol{\mu}(Y) = M$.

Proof. We follow the proof of Lemma 4.3. The set Y belongs to $\mathbf{\Pi}^1_1$. Therefore, it is measurable. Thus, if $\boldsymbol{\mu}(Y) < M$, then there exists a tree $U \in \mathbf{Rand} \cap \mathbf{V}$ satisfying $U \subseteq T \setminus Y$ and $\boldsymbol{\mu}([U]) > 0$. Then the sentence $\forall x \in [U]\,(\boldsymbol{\kappa}_c(x) \notin \mathbf{BK})$ is true in $\mathbf{V}$. But this sentence is expressible by a Π^1_2 formula. Therefore, by the Shoenfield absoluteness theorem, it is true in any generic extension of the form $\mathbf{V}[a]$, where $a \in [U]$ is an arbitrary Cohen generic real over $\mathbf{V}$. It follows that $\boldsymbol{\kappa}_c(a) \notin \mathbf{BK}$. This implies a contradiction, because $a \in [U] \subseteq [T]$, while T forces $\boldsymbol{\kappa}_c(\breve{\mathbf{a}}) \in \mathbf{BK}$. □

Arguing in the universe $\mathbf{V}$, we conclude, by the lemma, that there exists a tree $S \in \mathbf{Rand}$ such that $[S] \subseteq Y$, so that $\boldsymbol{\kappa}_c(x) \in \mathbf{BK}$ for all $x \in [S]$. Now consider the Borel set

$$P = \{\langle x, y\rangle : x \in [S] \wedge y \in \mathbf{B}_{\boldsymbol{\kappa}_c(x)}\}$$

and the following $\mathbf{\Pi}^1_1$ equivalence relation on the set $[S]$: $x \mathsf{E} x'$ if and only if $x, x' \in [S]$ and $P_x = P_{x'}$, where $P_x = \{y : \langle x, y\rangle \in P\}$. Being a $\mathbf{\Pi}^1_1$ subset in the product $[S] \times [S]$, E is $(\boldsymbol{\mu} \times \boldsymbol{\mu})$-measurable. Therefore, either all E-classes have $\boldsymbol{\mu}$-measure 0 on the set $[S]$ or else one of the E-classes has nonzero measure on $[S]$. We consider these two cases separately.

Case 1: (it is true in $\mathbf{V}$ that) all E-classes have $\boldsymbol{\mu}$-measure 0 on the set $[S]$. Then the $\mathbf{\Pi}^1_1$ set

$$H = \{\langle x, x'\rangle \in [S] \times [S] : \mathbf{B}_{\boldsymbol{\kappa}_c(x)} = \mathbf{B}_{\boldsymbol{\kappa}_c(x')}\}$$

has $(\boldsymbol{\mu} \times \boldsymbol{\mu})$-measure 0 by the Fubini theorem. Then H can be covered by a Borel set $H' = \mathbf{B}^{(2)}_d$, $H \subseteq H' \subseteq I_v \times I_v$, coded by d and having $(\boldsymbol{\mu} \times \boldsymbol{\mu})$-measure 0.

We *continue to argue in* $\mathbf{V}$. Fix a countable transitive model $\mathfrak{M}$ of a sufficiently large fragment of $\mathbf{ZFC}$ containing the codes c and d and the trees T and S and being an elementary submodel of the universe with respect to all analytic formulas (in order not to take special care about absoluteness).

Lemma 5.4 (Lemma 3.3 in [9])**.** *There exist reals* $a, b \in [S]$ *such that they are Solovay random over* $\mathbf{V}$ *and* $\mathbf{V}[a] = \mathbf{V}[b]$*, but the pair* $\langle a, b\rangle$ *is random over the model* $\mathfrak{M}$.

Proof. The argument is more complicated than in the proof of Lemma 4.5. Consider the set

$$P = \{\langle x, x +_2 y\rangle : x, y \in [S]\}.$$

If $x \in [S]$, then the section $P_x = \{z : \langle x, z\rangle \in P\}$ has the same measure as the set $[S]$, because $P_x = \{x +_2 y : y \in [S]\}$. Therefore, by the Fubini theorem, P has the same $(\boldsymbol{\mu} \times \boldsymbol{\mu})$-measure as $[S] \times [S]$, which is nonzero. Again by the Fubini theorem, the projection $Z = \{z \in 2^\omega : \boldsymbol{\mu}(P^z) > 0\}$ also satisfies $\boldsymbol{\mu}(Z) > 0$.

In the universe $\mathbf{V}$, consider any real $z \in Z$ random over $\mathfrak{M}$. In this case, we have $\boldsymbol{\mu}(P^z) > 0$, and there is a real $a \in P^z$ random over $\mathbf{V}$ and hence over $\mathfrak{M}[z]$. The pair of reals $\langle a, z\rangle$ is then random over $\mathfrak{M}$ and belongs to the set P. It follows by Proposition 5.1 that the real z is random over $\mathfrak{M}[a]$. But then the real $b = z +_2 a$ is obviously random over $\mathfrak{M}[a]$, because $a \in \mathfrak{M}[a]$. It follows by Proposition 5.1 that the pair $\langle a, b\rangle$ is random over $\mathfrak{M}$. Note that $a, b \in [S]$ by construction. Finally, the real $b = z +_2 a$ is random over $\mathbf{V}$, since so is a; at the same time, $z \in \mathbf{V}$, and we obviously have $\mathbf{V}[a] = \mathbf{V}[b]$. □

Following the proof of Lemma 4.4, we conclude that $\langle a, b\rangle$ does not belong to H', because this pair is random over $\mathfrak{M}$, and hence is does not belong to $\notin H$, so that $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(b)}$.

In the same time, the generic reals a and b belong to the set $[S] \subseteq [T]$ by construction. It follows, by the choice of T, that one and the same set $\{x : \varphi(x)\}$ is equal, in the extension $\mathbf{V}[a] = \mathbf{V}[b]$, both to the set $\mathbf{B}_{\boldsymbol{\kappa}_c(a)}$ and to the set $\mathbf{B}_{\boldsymbol{\kappa}_c(b)}$; hence $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} = \mathbf{B}_{\boldsymbol{\kappa}_c(b)}$. However, it was established above that $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(b)}$. This contradiction shows that Case 1 is impossible.

Case 2: (it is true in $\mathbf{V}$ that) one of the E-classes has positive $\boldsymbol{\mu}$-measure on $[S]$. Hence there is a tree $Q \in \mathbf{Rand}$, such that $Q \subseteq S$ and all reals $x \in [Q]$ are pairwise E-equivalent. In other words, there exists a Borel set $B = \mathbf{B}_e \subseteq 2^\omega$, $e \in \mathbf{BK}$ in $\mathbf{V}$, such that $\mathbf{B}_{\boldsymbol{\kappa}_c(x)} = B$ for all $x \in [Q]$.

Now consider any real $a \in [Q]$ Solovay random over $\mathbf{V}$. The Shoenfield absoluteness theorem implies that $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} = B = \mathbf{B}_e$. The concluding part of the proof is the same as the end of the proof of Theorem 4.1. □

6. PROOF FOR SACKS EXTENSIONS

Here we prove Theorem 1.2 in Part (C), that is, for Sacks extensions. Recall that the Sacks forcing notion is the set $\mathbf{PT}$ of all perfect trees $\varnothing \neq T \subseteq 2^{<\omega}$. In other words, a tree $T \subseteq 2^{<\omega}$ belongs to $\mathbf{PT}$ if it has no terminal nodes and no isolated branches. For instance, the complete tree $2^{<\omega}$ belongs to $\mathbf{PT}$, and $[2^{<\omega}] = 2^{\omega}$.

Theorem 6.1. *Let $a_0 \in 2^{\omega}$ be a Sacks (that is, $\mathbf{PT}$-generic) real over a set universe $\mathbf{V}$. It is true in $\mathbf{V}[a_0]$ that if $\varnothing \neq X \subseteq 2^{\omega}$ is a Borel $\mathbf{OD}$ set, then X contains a real in $\mathbf{V}$.*

This theorem implies Theorem 1.2 in part (C).

Proof. As at the beginning of the proofs of Theorems 4.1 and 5.2, the contrary assumption leads us to a formula $\varphi(x)$ with ordinals as parameters, a code $c \in \mathbf{BF} \cap \mathbf{V}$, and a tree $T \in \mathbf{PT} \cap \mathbf{V}$ which $\mathbf{PT}$-forces, over $\mathbf{V}$, the sentence

$$\boldsymbol{\kappa}_c(\breve{\mathbf{a}}) \in \mathbf{BK}, \qquad \mathbf{B}_{\boldsymbol{\kappa}_c(\breve{\mathbf{a}})} = \{x : \varphi(x)\} \neq \varnothing, \qquad \mathbf{B}_{\boldsymbol{\kappa}_c(\breve{\mathbf{a}})} \cap \mathbf{V} = \varnothing.$$

We argue in the universe $\mathbf{V}$. Consider the Borel set

$$P = \{\langle x, y\rangle : x \in [T] \wedge y \in \mathbf{B}_{\boldsymbol{\kappa}_c(x)}\}$$

and the following equivalence relation: $x \mathsf{E} x'$ if and only if

$$x, x' \in [T], \quad P_x = P_{x'}, \qquad \text{where} \quad P_x = \{y : \langle x, y\rangle \in P\}.$$

Clearly, E is a $\mathbf{\Pi}^1_1$ relation. Therefore, by the Silver theorem (see [5, 10.1.1] or [4, 12.1.1]), there exists a tree $U \in \mathbf{PT}$ such that $U \subseteq T$ and either $[U]$ consists of pairwise E-equivalent reals or $[U]$ consists of pairwise E-inequivalent reals. Accordingly, we have two cases.

Case 1: (it is true in $\mathbf{V}$ that) $[U]$ consists of pairwise E-inequivalent reals, that is, $\mathbf{B}_{\boldsymbol{\kappa}_c(x)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(x')}$ for all pairs of elements $x \neq x'$ in $[U]$. Arguing in $\mathbf{V}$, we consider any homeomorphism $h \colon [U] \xrightarrow{\text{onto}} [U]$ satisfying $h(x) \neq x$ for all $x \in [U]$ and set $d \in \mathbf{BF} \cap \mathbf{V}$ and $h = \boldsymbol{\vartheta}_d \restriction [U]$. Then it is true in $\mathbf{V}$ that

$$\forall\, x \in [U]\, (\mathbf{B}_{\boldsymbol{\kappa}_c(x)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(\boldsymbol{\vartheta}_d(x))}).$$

Now consider any real $a \in 2^{\omega}$ Sacks generic over the set universe $\mathbf{V}$. *We argue in the generic extension* $\mathbf{V}[a]$. The formula saying that the map $\boldsymbol{\vartheta}_d \restriction [U]$ is a homeomorphism of the set $[U]$ onto itself satisfying $\mathbf{B}_{\boldsymbol{\kappa}_c(x)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(\boldsymbol{\vartheta}_d(x))}$ for all $x \in [U]$ is true in $\mathbf{V}$, and it is absolute by the Shoenfield theorem. Therefore, it is true in $\mathbf{V}[G]$ as well. It follows that $b = \boldsymbol{\vartheta}_d(a) \in [U]$ and $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} \neq \mathbf{B}_{\boldsymbol{\kappa}_c(b)}$.

However the real $b = \boldsymbol{\vartheta}_d(a)$ is $\mathbf{PT}$-generic over $\mathbf{V}$ along with a, because $\boldsymbol{\vartheta}_d \restriction [U]$ is a homeomorphism of the set $[U]$ coded by $d \in \mathbf{V}$, and the Sacks forcing notion $\mathbf{PT}$ is invariant with respect to such homeomorphisms. Moreover, we have $b \in \mathbf{V}[a]$ and $a = \boldsymbol{\vartheta}_d^{-1}(b) \in \mathbf{V}[b]$; hence $\mathbf{V}[a] = \mathbf{V}[a']$ is one and the same model. It follows by the choice of $[T]$ and the fact that $U \subseteq T$ that it is true in $\mathbf{V}[a] = \mathbf{V}[b]$ that

$$\forall\, x \in 2^{\omega} (\varphi(x) \Longleftrightarrow x \in \mathbf{B}_{\boldsymbol{\kappa}_c(a)} \Longleftrightarrow x \in \mathbf{B}_{\boldsymbol{\kappa}_c(b)}).$$

We conclude that $\mathbf{B}_{\boldsymbol{\kappa}_c(a)} = \mathbf{B}_{\boldsymbol{\kappa}_c(b)}$, contrary to the above. The contradiction obtained shows that, in fact, Case 1 is impossible.

Case 2: (it is true in $\mathbf{V}$ that) $[U]$ consists of pairwise E-equivalent reals. Then there exists a code $e \in \mathbf{BK} \cap \mathbf{V}$, such that $\mathbf{B}_{\boldsymbol{\kappa}_c(x)} = \mathbf{B}_e$ for all $x \in [U]$. We consider any real $a \in [Q]$ Sacks random over $\mathbf{V}$ and conclude the proof by the same contradiction as at the end of the proof of Theorem 4.1. □

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