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Canonization of Smooth Equivalence Relations on Infinite-Dimensional E_0-Large Products

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Abstract We propose a canonization scheme for smooth equivalence relations on $\mathbb{R}^\omega$ modulo restriction to E_0-large infinite products. It shows that, given a pair of Borel smooth equivalence relations E, F on $\mathbb{R}^\omega$, there is an infinite E_0-large perfect product $P \subseteq \mathbb{R}^\omega$ such that either $\mathsf{F} \subseteq \mathsf{E}$ on P, or, for some $\ell < \omega$, the following is true for all $x, y \in P$: $x \mathrel{\mathsf{E}} y$ implies $x(\ell) = y(\ell)$, and $x \restriction (\omega \smallsetminus \{\ell\}) = y \restriction (\omega \smallsetminus \{\ell\})$ implies $x \mathrel{\mathsf{F}} y$.

1 Introduction

The canonization problem can be broadly formulated as follows. Given a class $\mathscr{E}$ of mathematical structures E, and a collection $\mathscr{P}$ of sets P considered as *large*, or *essential*, find a smaller and better structured subcollection $\mathscr{E}' \subseteq \mathscr{E}$ such that for any structure $E \in \mathscr{E}$ with the domain P there is a smaller set $P' \in \mathscr{P}$, $P' \subseteq P$, such that the restricted substructure $E \restriction P'$ belongs to $\mathscr{E}'$. For instance, the theorem saying that every Borel real map is either a bijection or a constant on a perfect set, can be viewed as a canonization theorem, with $\mathscr{E} = \{\text{Borel maps}\}$, $\mathscr{E}' = \{\text{bijections and constants}\}$, and $\mathscr{P} = \{\text{perfect sets}\}$. We refer to Kanovei, Sabok, and Zapletal [7] as the background of the general canonization problem for Borel and analytic equivalence relations in descriptive set theory.

Among other results, it is established in [7, Section 9.3, Theorems 9.26 and 9.27] that if E belongs to one of two large families of analytic equivalence relations[1] on $(2^\omega)^\omega$, then there is an infinite perfect product $P \subseteq (2^\omega)^\omega$ such that $\mathsf{E} \restriction P$ is *smooth*, that is, there simply exists a Borel map $f : P \to 2^\omega$ satisfying $x \mathrel{\mathsf{E}} y \iff f(x) = f(y)$ for all $x, y \in P$. The canonization problem for smooth equivalence relations themselves was not considered in [7].[2] Theorem 2.1, the main result of this note, contributes to this problem.

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2 Large Products and the Main Theorem

Recall that the equivalence relation E_0 is defined on 2^ω so that $x \mathsf{E}_0 y$ if and only if the equality $x(n) = y(n)$ holds for all but finite n. If $X \subseteq 2^\omega$, then we define the E_0-*hull* $[X]_{\mathsf{E}_0} = \{y \in 2^\omega : y \mathsf{E}_0 x\}$ of X. It is known that E_0 is not *smooth*; that is, there is no Borel map $f : \mathtt{dom}\, f = 2^\omega \to 2^\omega$ satisfying $x \mathsf{E} y \iff f(x) = f(y)$ for all $x, y \in 2^\omega$. A Borel set $X \subseteq 2^\omega$ is E_0-*large* if $\mathsf{E}_0 \restriction X$ is still not smooth. (See more on this in Zapletal [8], [9], Kanovei [6], [7], or elsewhere.)

An infinite *perfect product* is any set $P \subseteq (2^\omega)^\omega$ such that $P = \prod_{\ell<\omega} P(\ell)$, where $P(\ell) = \{x(\ell) : x \in P\}$ is the *projection* on the ℓth coordinate, and it is required that each set $P(\ell)$ be a perfect subset of 2^ω. Let **PP** be the set of all perfect products. If every factor $P(\ell)$ is an E_0-*large* set, then say that P is an E_0-*large* perfect product.

To set up a convenient notation, say that an equivalence relation E on $(2^\omega)^\omega$:

captures $\ell \in \omega$ on $P \in$ PP: if $x \mathsf{E} y$ implies $x(\ell) = y(\ell)$ for all $x, y \in P$;

is reduced to $U \subseteq \omega$ on $P \in$ PP: if $x \restriction U = y \restriction U$ implies $x \mathsf{E} y$ for all $x, y \in P$.

Theorem 2.1 *If E, F are smooth Borel equivalence relations on $(2^\omega)^\omega$, then there is an E_0-large perfect product $P \subseteq (2^\omega)^\omega$ such that either $\mathsf{F} \subseteq \mathsf{E}$ on P, or, for some $\ell < \omega$, E captures ℓ on P and F is reduced to $\omega \smallsetminus \{\ell\}$ on P.*

The two options of the theorem are incompatible with perfect products. The result can be compared to canonization results related to *finite* products and equivalence relations defined on spaces $(2^\omega)^m$, $m < \omega$. Theorem 9.3 in [7, Section 9.1] implies that every analytic equivalence relation on $(2^\omega)^m$ coincides with one of the *multiequalities* D_U, $U \subseteq \{0, 1, \dots, m-1\}$, on some E_0-large perfect product $P \subseteq (2^\omega)^m$, where $x \mathsf{D}_U y$ if and only if $x \restriction U = y \restriction U$. One may ask whether such a result holds for equivalence relations on $(2^\omega)^\omega$ and, accordingly, for infinite perfect products. This gives a negative answer, even for smooth equivalences.

Example 2.2 Let E be defined on $(2^\omega)^\omega$ so that $x \mathsf{E} y$ if and only if $x(0) = y(0)$, and also $x(\ell+1) = y(\ell+1)$ for all numbers ℓ such that $x(0)(\ell) = 0$. That E is smooth can be witnessed by the map sending each $x \in (2^\omega)^\omega$ to $a = \vartheta(x) \in (2^\omega)^\omega$ defined so that $a(k) = x(k)$ whenever $k = 0$ or $k = \ell+1$ and $x(0)(\ell) = 0$, and $a(k)(n) = 0$ for all other k and all $n < \omega$. That E is not equal (and even not Borel bireducible) to any D_U on any perfect product $P \subseteq (2^\omega)^\omega$, is easy.

The proof of Theorem 2.1 is based on a splitting/fusion technique known in the theory of iterations and products of the perfect-set forcing (see, e.g., Baumgartner [1] and Kanovei [4], [5]), although the splitting construction for infinite E_0-large products is different and way more complex than in the case of perfect-set products.

See Section 9 on applications of the theorem to the structure of the constructibility degrees in generic extensions via the forcing by E_0-large products.

3 Large Sets

Here and in the next section, we reproduce some definitions and results from Golshani and the authors in [3] related to perfect and large trees; but here we consider sets rather than trees.

Strings. The set $2^{<\omega}$ contains all strings (finite sequences) of numbers $0,\ 1$, including the *empty string* Λ. If $t \in 2^{<\omega}$ and $i = 0, 1$, then $t^\frown i$ is the extension of t by i as the rightmost term. If $s, t \in 2^{<\omega}$, then $s \subseteq t$ means that the string t extends s (including the case $s = t$), while $s \subset t$ means proper extension. The length of s is $\mathtt{lh}(s)$, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length n).

If $u \in 2^{<\omega}$, then let $\boldsymbol{I}_u = \{a \in 2^\omega : u \subset a\}$, a *Cantor interval* in 2^ω.

Trees and perfect sets. If $X \subseteq \omega^\omega$, then let $\mathtt{tree}(X) = \{u \in 2^{<\omega} : X \cap \boldsymbol{I}_u \neq \varnothing\}$, the *tree* of X. If $u \in \mathtt{tree}(X)$, then define $X \restriction_u = X \cap \boldsymbol{I}_u$, the *truncated* set. If $\mathtt{card}\, X \geq 2$, then there is a longest string $s = \mathtt{stem}(X) \in 2^{<\omega}$ satisfying $X \subseteq \boldsymbol{I}_s$ (the *stem* of T). A string $u \in \mathtt{tree}(X)$ is a *splitnode* if both $u^\frown 0$ and $u^\frown 1$ belong to $\mathtt{tree}(X)$. A closed set $\varnothing \neq X \subseteq 2^\omega$ is *perfect* if and only if every string $u \in \mathtt{tree}(X)$ can be extended into a splitnode $v \in \mathtt{tree}(X)$, $u \subset v$.

Action. Every string $s \in 2^{<\omega}$ *acts* on 2^ω in such a way that if $x \in 2^\omega$, then $(s{\cdot}x)(k) = x(k) + s(k) \pmod 2$ for $k < \mathtt{lh}(s)$, and $(s{\cdot}x)(k) = x(k)$ otherwise. If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$, then let $s{\cdot}X = \{s{\cdot}x : x \in X\}$. Similarly if $s, t \in 2^m$, then define a string $s{\cdot}t \in 2^m$ so that $(s{\cdot}t)(k) = t(k) + s(k) \pmod 2$ for $k < m$.

This action of strings on 2^ω induces the relation E_0, so that if $x, y \in 2^\omega$, then $x \mathrel{\mathsf{E}_0} y$ is equivalent to $y = s{\cdot}x$ for a string $s \in 2^{<\omega}$.

Special E_0-large perfect sets. Following [8, Definition 2.3.28], a perfect set $X \subseteq 2^\omega$ is called *special* E_0-*large* if the following holds: for every splitnode $u \in \mathtt{tree}(X)$, if $u_0, u_1 \in \mathtt{tree}(X)$ are the minimal splitnodes in $\mathtt{tree}(X)$ satisfying $u^\frown 0 \subseteq u_0$ and $u^\frown 1 \subseteq u_1$, then $\mathtt{lh}(u_0) = \mathtt{lh}(u_1)$ and (the *symmetry*) $X \restriction_{u_1} = (u_1{\cdot}u_0){\cdot}X \restriction_{u_0}$. The symmetry condition is equivalent to $u_0{}^\frown a \in X \iff u_1{}^\frown a \in X$ for all $a \in 2^\omega$, and we have $X \restriction_u = X \restriction_{u_0} \cup X \restriction_{u_1} = X \restriction_{u^\frown 0} \cup X \restriction_{u^\frown 1}$ anyway.

Let **SLS** be the collection of all special E_0-large (perfect) sets.

Sets in **SLS** admit a special combinatorial representation. Suppose that $r \in 2^{<\omega}$, and suppose that $\langle q^i_k \rangle_{k<\omega, i=0,1}$ is a system of strings $q^i_k \in 2^{<\omega}$ such that $\mathtt{lh}(q^0_k) = \mathtt{lh}(q^1_k) \geq 1$ and $q^0_k(0) = 0$, $q^1_k(0) = 1$ for all k. Let $[r, \{q^i_k\}]$ be the perfect set of all infinite strings of the form $a = r^\frown q_0^{i_0}{}^\frown q_1^{i_1}{}^\frown q_2^{i_2}{}^\frown \cdots {}^\frown q_n^{i_n}{}^\frown \cdots \in 2^\omega$, where $i_k = 0, 1$ for all k. One easily proves that *every set of this form is special* E_0-*large*, and conversely, *every special* E_0-*large set has the form* $[r, \{q^i_k\}]$ *for suitable strings* $r, q^i_k \in 2^{<\omega}$.

See Conley [2], [7, Section 7.1], and [6, Section 10.9] for details on these categories of sets.

Proposition 3.1 *Every set $X = [r, \{q^i_k\}] \in$ **SLS** is E_0-large. Conversely, every E_0-large Borel set $X \subseteq 2^\omega$ contains a special E_0-large subset.*

Proof To prove the first claim note that the map sending each $a \in 2^\omega$ to $r^\frown q_0^{a(0)}{}^\frown q_1^{a(1)}{}^\frown q_2^{a(2)}{}^\frown \cdots {}^\frown q_n^{a(n)}{}^\frown \cdots \in 2^\omega$ is an isomorphism between $\langle 2^\omega ; \mathsf{E}_0 \rangle$ and $\langle X ; \mathsf{E}_0 \rangle$. Regarding the second claim (which we will not use) see [8, Lemma 2.3.29]. □

We finally define splitting levels of sets $X = [r, \{q^i_k\}] \in$ **SLS**. Then $\mathtt{stem}(X) = r$, and the strings $q^i_k = q^i_k[X]$ are unique. If $n < \omega$, then we let

$$\mathtt{spl}_n(X) = \mathtt{lh}(r) + \mathtt{lh}(q_0^{i_0}) + \mathtt{lh}(q_1^{i_1}) + \cdots + \mathtt{lh}(q_{n-1}^{i_{n-1}})$$

(independent of the values of $i_k = 0,1$). In particular, $\mathtt{spl}_0(X) = \mathtt{lh}(r)$. Thus $\mathtt{spl}(X) = \{\mathtt{spl}_n(X) : n < \omega\} \subseteq \omega$ is the set of all *splitting levels* of X.

Example 3.2 If $s \in 2^{<\omega}$, then $\boldsymbol{I}_s = \{a \in 2^\omega : s \subset a\}$ is special E_0-large; in fact, $\boldsymbol{I}_s = [s, \{q^i_k\}]$, where $q^i_k = q^i_k(\boldsymbol{I}_s) = \langle i \rangle$ for all k.

4 Splitting E_0-Large Sets

The *simple splitting* of a perfect set $X \subseteq 2^\omega$ consists of subsets $X(\to i) = \{x \in X : x(n) = i\}$, $i = 0,1$, where $n = \mathtt{lh}(r)$ (the length of a string $r \in 2^{<\omega}$), and $r = \mathtt{stem}(X)$ is the largest string in $2^{<\omega}$ satisfying $r \subset x$ for all $x \in X$. Then $X = X(\to 0) \cup X(\to 1)$ is a disjoint partition of a perfect set $X \subseteq 2^\omega$ onto two perfect subsets. Splittings can be iterated. We let $X(\to \Lambda) = X$ for the empty string Λ, and if $s \in 2^n$, $s \neq \Lambda$, then we define

$$X(\to s) = X\big(\to s(0)\big)\big(\to s(1)\big)\big(\to s(2)\big)\cdots\big(\to s(n-1)\big).$$

Lemma 4.1 *If $X \subseteq 2^\omega$ is a special E_0-large set, $u \in \mathtt{tree}(X)$, and $s \in 2^n$, then the sets $X(\to s)$ and $X \restriction_u$ belong to $\boldsymbol{SLS}$, too.*

Lemma 4.2 *Let $X = [r, \{q^i_k\}] \in \boldsymbol{SLS}$, and let $s \in 2^{<\omega}$. Then $X(\to s) = X \restriction_{u[s]}$, where $u[s] = u[s,X] = r^\frown q_0^{s(0)\frown} q_1^{s(1)\frown} \cdots {}^\frown q_{n-1}^{s(n-1)} \in T = \mathtt{tree}(X)$. Conversely, if $u \in T$, then there is a string $s = s[u] \in 2^{<\omega}$ such that $X \restriction_u = X(\to s)$.*

Proof To prove the converse, we put $s(k) = u(\mathtt{spl}_k(X))$ for all k such that $\mathtt{spl}_k(X) < \mathtt{lh}(u)$. □

Lemma 4.3 *Let $X \in \boldsymbol{SLS}$, let $n < \omega$, and let $h = \mathtt{spl}_n(X)$. Then*

(i) *if $u, v \in \mathtt{tree}(X) \cap 2^h$, then $X \restriction_u = (u \cdot v) \cdot (X \restriction_v)$;*

(ii) *if $s, t \in 2^n$, then $X(\to s) = \sigma \cdot (X(\to t))$, where $\sigma = u[s,X] \cdot u[t,X]$;*

(iii) *if $u, v \in \mathtt{tree}(X) \cap 2^j$, $j < \omega$, then $X \restriction_u = \sigma \cdot (X \restriction_v)$ for some $\sigma \in 2^{<\omega}$.*

Proof To prove (ii) use Lemma 4.2. To prove (iii) take the least number $h \in \mathtt{spl}(X)$ with $j \le h$. There is a unique pair of strings $u', v' \in 2^h$ satisfying $u \subseteq u'$, $v \subseteq v'$. Then $X \restriction_u = X \restriction_{u'}$, $X \restriction_v = X \restriction_{v'}$, and $X \restriction_{u'} = (u' \cdot v') \cdot (X \restriction_{v'})$. □

Definition 4.4 (Refinement) If $X, Y \subseteq 2^\omega$ are perfect sets and $n < \omega$, then define $X \subseteq_n Y$ if $X(\to s) \subseteq Y(\to s)$ for all $s \in 2^n$; $X \subseteq_0 Y$ is equivalent to $X \subseteq Y$. Clearly, $X \subseteq_{n+1} Y$ implies $X \subseteq_n Y$ (and $X \subseteq Y$).

If X, Y are special E_0-large sets and $n \ge 1$, then the relation $X \subseteq_n Y$ is equivalent to $\mathtt{stem}(X) = \mathtt{stem}(Y)$, $q^i_k[X] = q^i_k[Y]$ for all $i = 0,1$ and $k < n-1$, and $q^i_{n-1}[X] \subseteq q^i_{n-1}[Y]$ for all $i = 0,1$.

Lemma 4.5 *Assume that X, U are perfect sets, that $s_0 \in 2^n$, and that $U \subseteq X(\to s_0)$. Then the set $Y = A \cup \bigcup_{u \in 2^n, u \neq s} X(\to u)$ is perfect, $Y \subseteq_n X$, and $Y(\to s_0) = A$.*

Lemma 4.6 *If $X, U \in \boldsymbol{SLS}$, $s_0 \in 2^n$, and $U \subseteq X(\to s_0)$, then there is a unique special E_0-large set X' satisfying $X' \subseteq_n X$ and $X'(\to s_0) = U$. We have then*

(i) *$X'(\to s) = u[s_0, X] \cdot u[s, X] \cdot X'(\to s_0)$ for all $s \in 2^n$;*

(ii) *if U is clopen in $X(\to s_0)$, then X' is clopen in X.*

Proof If $s \in 2^n$, then $X(\to s) = u[s_0, X]\cdot u[s, X]\cdot X(\to s_0)$ by Lemma 4.3. Put $U_s = u[s_0, X]\cdot u[s, X]\cdot U$ for all $s \in 2^n$, in particular, $U_{s_0} = U$. The set $X' = \bigcup_{u\in 2^n} U_s$ is as required. □

The next lemma is a more complex version of $\subseteq_n$-refinement. For the proof (in terms of trees) see [3, Lemma 4.1(iv)].

Lemma 4.7 *If $X, U, V \in \boldsymbol{SLS}$, $s_0, s_1 \in 2^n$, $U \subseteq X(\to s_0{}^\frown 0)$, $V \subseteq X(\to s_1{}^\frown 1)$, and $[U]_{\mathsf{E}_0} = [V]_{\mathsf{E}_0}$, then there is a special E_0-large set X' satisfying $X' \subseteq_{n+1} X$ and $X'(\to s_0{}^\frown 0) \subseteq U$, $X'(\to s_1{}^\frown 1) \subseteq V$.*

Lemma 4.8 *Let $\cdots \subseteq_4 X_3 \subseteq_3 X_2 \subseteq_2 X_1 \subseteq_1 X_0$ be an infinite sequence of sets in $\boldsymbol{SLS}$. Then $X = \bigcap_n X_n$ is a special E_0-large set and $X \subseteq_{n+1} X_n$, for all n.*

Proof Note that $\mathtt{spl}(X) = \{\mathtt{spl}_n(X_n) : n < \omega\}$; this implies both claims. □

5 Splitting Perfect and Special E_0-Large Products

A perfect product P is a *special* E_0*-large product*, $P \in \mathbf{SLP}$ for brevity, if each factor $P(\ell)$, $\ell < \omega$, belongs to $\mathbf{SLS}$. Thus $\mathbf{SLP} = \mathbf{SLS}^\omega$.

Now we extend the splitting technique to special E_0-large products.

Definition 5.1 Fix once and for all a function $\phi : \omega \xrightarrow{\text{onto}} \omega$ taking each value infinitely many times, so that if $\ell < \omega$, then the following set is infinite:

$$\phi^{-1}(\ell) = \big\{k : \phi(k) = \ell\big\} = \{\mathbf{k}_{0\ell} < \mathbf{k}_{1\ell} < \mathbf{k}_{2\ell} < \cdots < \mathbf{k}_{l\ell} < \cdots\}.$$

If $m < \omega$, then let $\boldsymbol{\nu}_{m\ell}$ be the number of indices $k < m$, $k \in \phi^{-1}(\ell)$.

Let $m < \omega$, and let $\sigma \in 2^m$ (a string of length m). If $\ell \in \phi"m = \{\phi(k) : k < m\}$, then the set $\phi^{-1}(\ell)$ cuts in σ a substring $\sigma[\ell] \in 2^{\boldsymbol{\nu}_{m\ell}}$, of length $\mathtt{lh}(\sigma[\ell]) = \boldsymbol{\nu}_{m\ell}$, defined by $(\sigma[\ell])(j) = \sigma(\mathbf{k}_{j\ell})$ for all $j < \boldsymbol{\nu}_{m\ell}$. Thus the string $\sigma \in 2^m$ splits into an array of strings $\sigma[\ell] \in 2^{\boldsymbol{\nu}_{m\ell}}$ ($\ell \in \phi"m$) of total length $\sum_{\ell\in\phi"m} \boldsymbol{\nu}_{m\ell} = m$.

Let P be a special E_0-large product. If $\sigma \in 2^m$, then define $P(\Rightarrow\sigma) \in \mathbf{SLP}$ so that $P(\Rightarrow\sigma)(\ell) = P(\ell)(\to\sigma[\ell])$ for all ℓ. In particular, if $\ell \notin \phi"m$, then $P(\Rightarrow\sigma)(\ell) = P(\ell)$, because $\mathtt{lh}(\sigma[\ell]) = \boldsymbol{\nu}_{m\ell} = 0$ holds provided $\ell \notin \phi"m$.

Let $P, Q \in \mathbf{SLP}$. Define $P \subseteq_m Q$ if $P(\ell) \subseteq_{\boldsymbol{\nu}_{m\ell}} Q(\ell)$ for all ℓ. This is equivalent to $P(\Rightarrow\sigma) \subseteq Q(\Rightarrow\sigma)$ for all $\sigma \in 2^m$.

If $\sigma, \tau \in 2^m$, then let $\boldsymbol{\Delta}[\sigma, \tau] = \omega \smallsetminus \{\phi(i) : i < m \wedge \sigma(i) \neq \tau(i)\}$.

Lemma 5.2 *Under the conditions of Definition 5.1, let $P \in \boldsymbol{SLP}$. Then*

(i) *if $\sigma \in 2^{<\omega}$, then $P(\Rightarrow\sigma) \in \boldsymbol{SLP}$ and the set $P(\Rightarrow\sigma)$ is clopen in P;*

(ii) *if $m < \omega$ and $\sigma, \tau \in 2^m$, then $P(\Rightarrow\sigma) \restriction \boldsymbol{\Delta}[\sigma, \tau] = P(\Rightarrow\tau) \restriction \boldsymbol{\Delta}[\sigma, \tau]$;*

(iii) *if $x \in P$, and U is an open neighborhood of x, then there exists a string $\sigma \in 2^m$ satisfying $x \in P(\Rightarrow\sigma) \subseteq U$;*

(iv) *if $m < \omega$, $\sigma \in 2^m$, and $U \in \boldsymbol{SLP}$, $U \subseteq P(\Rightarrow\sigma)$, then there exists a unique set $Q \in \boldsymbol{SLP}$ such that $Q \subseteq_m P$ and $Q(\Rightarrow\sigma) = U$, and then if U is clopen in $P(\Rightarrow\sigma)$, then Q is clopen in P.*

Proof (i) and (ii). These are clear. (iii) We have $\{x\} = \bigcap_m [P(\Rightarrow a \restriction m)]$ for a suitable sequence $a \in 2^\omega$. By compactness, there is m such that $P(\Rightarrow a \restriction m) \subseteq U$.

(iv) If $\ell < \omega$, then $U(\ell) \subseteq P(\Rightarrow\sigma)(\ell) = P(\ell)(\to s)$, where $s = \sigma[\ell]$. By Lemma 4.6, there is a set $S_\ell \in \mathbf{SLS}$ satisfying $S_\ell \subseteq_n P(\ell)$, where $n = \boldsymbol{\nu}_{m\ell} = \mathtt{lh}(s)$, and $S_\ell(\to s) = U(\ell)$. Let $Q(\ell) = S_\ell$ for all ℓ. □

A version of Lemma 4.8 for special E_0-large products is as follows.

Lemma 5.3 *Let* $\cdots \subseteq_5 P_4 \subseteq_4 P_3 \subseteq_3 P_2 \subseteq_2 P_1 \subseteq_1 P_0$ *be a sequence of special* E_0*-large products. Then* $Q = \bigcap_n P_n \in \boldsymbol{SLP}$*,* $Q(\ell) = \bigcap_m P_m(\ell)$ *for all* $\ell < \omega$*, and* $Q \subseteq_{m+1} P_m$ *for all* m*.*

Proof Apply Lemma 4.8 componentwise. □

Corollary 5.4 (see [7, Section 9.3, Proposition 9.31]) *If* $P \subseteq (2^\omega)^\omega$ *is a special* E_0*-large product and* $B \subseteq P$ *is a Borel set, then there is a special* E_0*-large product* $Q \subseteq P$ *such that* $Q \subseteq B$ *or* $Q \cap B = \varnothing$*.*

Corollary 5.5 *If* $P \in \boldsymbol{SLP}$ *and* $f : P \to 2^\omega$ *is a Borel map, then there is a special* E_0*-large product* $Q \in \boldsymbol{SLP}$ *such that* $Q \subseteq P$ *and* $f \restriction Q$ *is continuous.*

Proof If $n < \omega$ and $i = 0, 1$, then let $B_{ni} = \{x \in P : f(x)(n) = i\}$. Using Corollary 5.4 and Lemma 5.2(iv), we get a sequence $\cdots \subseteq_3 P_2 \subseteq_2 P_1 \subseteq_1 P_0 \subseteq P$ of special E_0-large products as in Lemma 5.3 such that if $m < \omega$ and $\sigma \in 2^m$, then $P_m(\Rightarrow\sigma) \subseteq B_{m0}$ or $P_m(\Rightarrow\sigma) \subseteq B_{m1}$. Then $Q = \bigcap_m P_m$ is as required. □

6 Proof of the Main Theorem: Beginning

Beginning the proof of Theorem 2.1, we let Borel maps $\boldsymbol{e}, \boldsymbol{f} : 2^\omega \to 2^\omega$ witness the smoothness of the equivalence relations E, F, respectively, so that

$$x \mathrel{\mathsf{E}} y \iff \boldsymbol{e}(x) = \boldsymbol{e}(y) \qquad \text{and} \qquad x \mathrel{\mathsf{F}} y \iff \boldsymbol{f}(x) = \boldsymbol{f}(y).$$

In fact, by Corollary 5.5, we can assume that $\boldsymbol{e}, \boldsymbol{f}$ are *continuous*.

Lemma 6.1 *If* P *is a special* E_0*-large product,* $U_0, U_1, \ldots \subseteq \omega$*, and* E *is reduced to each* U_k *on* P*, then* E *is reduced to* $U = \bigcap_k U_k$ *on* P*. The same for* F*.*

Proof For just two sets, if $U = U_0 \cap U_1$ and $x, y \in P$, $x \restriction U = y \restriction U$, then, using the product structure, find a point $z \in P$ with $z \restriction U_0 = x \restriction U_0$ and $z \restriction U_1 = y \restriction U_1$. Then $\boldsymbol{e}(x) = \boldsymbol{e}(z) = \boldsymbol{e}(y)$, and hence $x \mathrel{\mathsf{E}} y$. The case of finitely many sets follows by induction. Therefore, we can assume that $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$ in the general case. Let $x, y \in P$, and let $x \restriction U = y \restriction U$. There exist points $x_k \in P$ satisfying $x_k \restriction U_k = x \restriction U_k$ and $x_k \restriction (B \smallsetminus U_k) = y \restriction (B \smallsetminus U_k)$. Then immediately $\boldsymbol{e}(x_k) = \boldsymbol{e}(x)$ for all k. On the other hand, clearly $x_k \to y$; hence, $\boldsymbol{e}(x_k) \to \boldsymbol{e}(y)$ as $\boldsymbol{e}$ is continuous. Thus $\boldsymbol{e}(x) = \boldsymbol{e}(y)$, and hence $x \mathrel{\mathsf{E}} y$. □

We argue in terms of Definition 5.1. The plan is to define a sequence of special E_0-large products as in Lemma 5.3, with some extra properties. Let $m < \omega$. A special E_0-large product $R \in \mathbf{SLP}$ is *m-good* if the following hold (see the definitions in Section 2):

(1)E: if $\sigma \in 2^m$, then either (i) E is reduced to $\omega \smallsetminus \{\phi(m)\}$ on $R(\Rightarrow\sigma)$, or (ii) there is no set $R' \in \mathbf{SLP}$, $R' \subseteq R(\Rightarrow\sigma)$ on which E is reduced to $\omega \smallsetminus \{\phi(m)\}$;

(1)F: the same for F;

(2)E: if $\sigma, \tau \in 2^m$, then either (i) E is reduced on $R(\Rightarrow\sigma) \cup R(\Rightarrow\tau)$ to

$$\boldsymbol{\Delta}[\sigma, \tau] = \omega \smallsetminus \{\phi(i) : i < m \wedge \sigma(i) \neq \tau(i)\},$$

or (ii) $\boldsymbol{e}[R(\Rightarrow\sigma)] \cap \boldsymbol{e}[R(\Rightarrow\tau)] = \varnothing$[3]—equivalently, the sets $R(\Rightarrow\sigma)$ and $R(\Rightarrow\tau)$ do not contain E-related points;

(2)F: the same for F.

7 The Key Lemma

Lemma 7.1 *If $m < \omega$ and a special E_0-large product R is m-good, then there is an $(m+1)$-good special E_0-large product $Q \subseteq_{m+1} R$.*

Proof Consider a string $\sigma' \in 2^{m+1}$, and first define a special E_0-large product $Q \subseteq_{m+1} R$, satisfying (1)E relative to this string only. Let $\ell' = \phi(m+1)$. If there exists $R' \in \mathbf{SLP}$, $R' \subseteq R(\Rightarrow\sigma')$ on which E is reduced to $\omega \smallsetminus \{\ell'\}$, then let S_0 be such R'. If there is no such R', then put $S_0 = R(\Rightarrow\sigma')$. By Lemma 5.2(iv), there is a special E_0-large product $Q \subseteq_{m+1} R$ such that $Q(\Rightarrow\sigma') = S_0$. Thus the set Q satisfies (1)E with respect to σ'. Now take Q as the "new" special E_0-large product R, consider another string $\sigma' \in 2^{m+1}$, and do the same as above. Consider all strings in 2^{m+1} consecutively the same way. This ends with a special E_0-large product $Q \subseteq_{m+1} R$, satisfying (1)E for all $\sigma' \in 2^{m+1}$.

Now take care of (2)E. Let $\ell = \phi(m)$, and let $B = \omega \smallsetminus \{\ell\}$.

Step 1. We fulfill (2)E for one particular pair $\sigma' = \sigma^\frown 0$, $\tau' = \sigma^\frown 1$, where $\sigma \in 2^m$. Then $\boldsymbol{\Delta}[\sigma', \tau'] = B$. The goal is to define $P \in \mathbf{SLP}$, $P \subseteq_{m+1} Q$, satisfying (2)E relative to this pair σ', τ'.

If the relation E is reduced to B on $Q(\Rightarrow\sigma)$, then E is reduced to B on the set $Q(\Rightarrow\sigma') \cup Q(\Rightarrow\tau') = Q(\Rightarrow\sigma)$, and we are done. Thus, by (1)E for $Q(\Rightarrow\sigma)$, we assume that *there is no set $Q' \in \mathbf{SLP}$, $Q' \subseteq Q(\Rightarrow\sigma)$ on which E is reduced to B.*

In particular, E is *not* reduced to B on $Q(\Rightarrow\sigma')$. But $Q(\Rightarrow\sigma') \restriction B = Q(\Rightarrow\tau') \restriction B$, since $B = \boldsymbol{\Delta}[\sigma', \tau'] = \omega \smallsetminus \{\ell\}$. It follows that there are points $x_0 \in Q(\Rightarrow\sigma')$ and $y_0 \in Q(\Rightarrow\tau')$ such that $x_0 \restriction B = y_0 \restriction B$ and $\boldsymbol{e}(x_0) \neq \boldsymbol{e}(y_0)$; that is, we have $\boldsymbol{e}(x_0)(k) = p \neq q = \boldsymbol{e}(y_0)(k)$ for some k and $p, q = 0, 1$, $p \neq q$.

As $\boldsymbol{e}$ is continuous, there are strings $u, v \in 2^{<\omega}$ of equal length $\mathtt{lh}(u) = \mathtt{lh}(v)$ such that $\sigma' \subset u$, $\tau' \subset v$, $x_0 \in X = Q(\Rightarrow u)$, $y_0 \in Y = Q(\Rightarrow v)$, and $\boldsymbol{e}(x)(k) = p$, $\boldsymbol{e}(y)(k) = q$ for all $x \in X$, $y \in Y$. We are going to define a special E_0-large product $P \subseteq_{n+1} Q$ such that $P(\Rightarrow\sigma') \subseteq X$ and $P(\Rightarrow\tau') \subseteq Y$. In this case we shall have $\boldsymbol{e}[P(\Rightarrow\sigma')] \cap \boldsymbol{e}[P(\Rightarrow\tau')] = \varnothing$ by construction, as required.

To carry out the construction of P, let $r_j = \sigma[j]$, $s_j = u[j]$, $t_j = v[j]$ for all j.

Consider any index $j \neq \ell$. Then $x_0(j) = y_0(j)$ (as $x_0 \restriction B = y_0 \restriction B$), and then easily $r_j \subset s_j = t_j$. It follows that the set $S_j = X(j) = Y(j) = Q(j)(\to s_j)$ belongs to $\mathbf{SLS}$ and satisfies $S_j \subseteq Q(j)(\to r_j)$. By Lemma 4.6, there is a set $P_j \in \mathbf{SLS}$ satisfying $P_j \subseteq_{\nu_j} Q(j)$ and $P_j(\to r_j) = S_j$, where $\nu_j = \boldsymbol{\nu}_{mj} = \mathtt{lh}(r_j)$.

Now consider the index ℓ itself. The strings s_ℓ and t_ℓ are different (of the same length), but still satisfy $r_\ell{}^\frown 0 = \sigma'[\ell] \subseteq s_\ell$, $r_\ell{}^\frown 1 = \tau'[\ell] \subseteq t_\ell$. It follows that the sets $S_\ell = X(\ell)$, $V_\ell = Y(\ell)$ satisfy $S_\ell = H(\to s_\ell) \subseteq H(\to r_\ell{}^\frown 0)$, $V_\ell = H(\to t_\ell) \subseteq H(\to r_\ell{}^\frown 1)$, where $H = Q(\ell)$. And moreover, $[S_\ell]_{\mathsf{E}_0} = [V_\ell]_{\mathsf{E}_0}$ holds by Lemma 4.3(ii). Lemma 4.7 yields a set $H' \in \mathbf{SLS}$ satisfying $H' \subseteq_{\nu+1} H$, where $\nu_\ell = \boldsymbol{\nu}_{m\ell} = \mathtt{lh}(s)$, and $H'(\to s^\frown 0) \subseteq S_\ell$, $H'(\to s^\frown 1) \subseteq V_\ell$.

We finally define a special E_0-large product P such that $P(\ell) = H'$ and $P(j) = P_j$ for all $j \neq \ell$. Then by construction $P \subseteq_{m+1} Q$, $P(\Rightarrow\sigma') \subseteq X$, and $P(\Rightarrow\tau') \subseteq Y$, as required.

Step 2. Iterating the construction at Step 1, we obtain a special E_0-large product $R \subseteq_{m+1} Q$ which fulfills (2)E for all pairs $\sigma', \tau' \in 2^{m+1}$ of the form $\sigma' = \sigma^\frown 0$, $\tau' = \sigma^\frown 1$, where $\sigma \in 2^m$.

Step 3. We claim that R satisfies (2)E for all pairs $\sigma', \tau' \in 2^{m+1}$ of any form. Indeed, let $\sigma' = \sigma^\frown i$, $\tau' = \tau^\frown k$ be any pair in 2^{m+1}, where $\sigma, \tau \in 2^m$ and $i, k \in \{0,1\}$. By (2)E for the pair σ, τ, either E is reduced to $U = \mathbf{\Delta}[\sigma, \tau]$ on $R(\Rightarrow\sigma) \cup R(\Rightarrow\tau)$, or $\boldsymbol{e}[R(\Rightarrow\sigma)] \cap \boldsymbol{e}[R(\Rightarrow\tau)] = \varnothing$. In the second case, $\boldsymbol{e}[R(\Rightarrow\sigma')] \cap \boldsymbol{e}[R(\Rightarrow\tau')] = \varnothing$. Thus, we can assume without loss of generality that E *is reduced to* U *on* $R(\Rightarrow\sigma) \cup R(\Rightarrow\tau)$. Let $U' = \mathbf{\Delta}[\sigma', \tau']$. If $i = k$ or $\ell \notin U$, then $U = U'$, so that (2)E relative to σ', τ' follows from (2)E relative to σ, τ. Thus, we can assume without loss of generality that $\sigma' = \sigma^\frown 0$, $\tau' = \tau^\frown 1$, *and* $\ell \in U$. Then $U' = U \smallsetminus \{\ell\} = U \cap B$, of course.

Because of the achievement at Step 2, we have two cases.

Case 3.1: E is reduced to B on $R(\Rightarrow\sigma') \cup R(\Rightarrow\sigma'_1)$, where $\sigma'_1 = \sigma^\frown 1$. Prove that E is reduced to U' on $R(\Rightarrow\sigma') \cup R(\Rightarrow\tau')$, so that (2)$\mathsf{E}$(i) holds for σ', τ'. Indeed, assume that $x \in R(\Rightarrow\sigma')$, $y \in R(\Rightarrow\tau')$, $x \restriction U' = y \restriction U'$. Let $x' \in (2^\omega)^\omega$ be defined so that $x' \restriction B = x \restriction B$ but $x'(\ell) = y(\ell)$. Thus, if $j \neq \ell$, then $x'(j) = x(j) \in R(\Rightarrow\sigma')(j) = R(\Rightarrow\sigma'_1)(j)$ (because $R(\Rightarrow\sigma') \restriction B = R(\Rightarrow\sigma'_1) \restriction B$). While for ℓ itself we have $x'(\ell) = y(\ell) \in R(\Rightarrow\tau') = R(\Rightarrow\sigma'_1)$ (because now we have $\ell \in U = \mathbf{\Delta}[\tau', \sigma'_1]$). It follows that $x' \in R(\Rightarrow\sigma'_1)$. Therefore, by the Case 3.1 hypothesis, we have $\boldsymbol{e}(x) = \boldsymbol{e}(x')$. On the other hand, $x' \restriction U = y \restriction U$; therefore, $\boldsymbol{e}(y) = \boldsymbol{e}(x')$ without loss of generality, as assumed above. Thus $\boldsymbol{e}(x) = \boldsymbol{e}(y)$, as required.

Case 3.2: $\boldsymbol{e}[R(\Rightarrow\sigma')] \cap \boldsymbol{e}[R(\Rightarrow\sigma'_1)] = \varnothing$. However, E is reduced to $U = \mathbf{\Delta}[\sigma, \tau]$ on $R(\Rightarrow\sigma) \cup R(\Rightarrow\tau)$ without loss of generality as assumed above and, hence, on the smaller set $R(\Rightarrow\sigma'_1) \cup R(\Rightarrow\tau')$ as well, while $R(\Rightarrow\sigma'_1) \restriction U = R(\Rightarrow\tau') \restriction U$ (since the equality $U = \mathbf{\Delta}[\sigma'_1, \tau'] = \mathbf{\Delta}[\sigma, \tau]$ holds). We conclude that $\boldsymbol{e}[R(\Rightarrow\sigma'_1)] = \boldsymbol{e}[R(\Rightarrow\tau')]$. It follows that $\boldsymbol{e}[R(\Rightarrow\sigma')] \cap \boldsymbol{e}[R(\Rightarrow\tau')] = \varnothing$; hence, R satisfies (2)E(ii) for σ', τ'.

Thus, indeed, we have got a special E_0-large product $R \subseteq_{m+1} Q$ satisfying (2)E for all $\sigma', \tau' \in 2^{m+1}$ (and still satisfying (1)E).

It remains to repeat the same procedure for F. □

8 Proof of the Main Theorem: Conclusion

We come back to the proof of Theorem 2.1. Lemma 7.1 yields an infinite sequence $\cdots \leq_3 Q_2 \leq_2 Q_1 \leq_1 Q_0$ of special E_0-large products Q_m such that each Q_m is m-good. The limit special E_0-large product $P = \bigcup_m Q_m \in \mathbf{SLP}$ satisfies $P \subseteq_{m+1} Q_m$ for all m by Lemma 5.3. Therefore, P is m-good for every m and, hence, we can freely use (1)E, F and (2)E, F for P in the following final argument.

Case 1: if $m < \omega$, $\sigma, \tau \in 2^m$, and $\boldsymbol{e}[P(\Rightarrow\sigma)] \cap \boldsymbol{e}[P(\Rightarrow\tau)] = \varnothing$, then we have $\boldsymbol{f}[P(\Rightarrow\sigma)] \cap \boldsymbol{f}[P(\Rightarrow\tau)] = \varnothing$. Prove that $\mathsf{F} \subseteq \mathsf{F}$ on P in this case, as required by the "either" option of Theorem 2.1. Let $x, y \in P$ and $x \mathrel{\mathsf{E}} y$ *fails*, that is, $\boldsymbol{e}(x) \neq \boldsymbol{e}(y)$; show that $\boldsymbol{f}(x) \neq \boldsymbol{f}(y)$. Pick $a, b \in 2^\omega$ satisfying $\{x\} = \bigcap_m P(\Rightarrow a \restriction m)$ and $\{y\} = \bigcap_m P(\Rightarrow b \restriction m)$. As $x \neq y$, we have $\boldsymbol{e}[Q(\Rightarrow a \restriction m)] \cap \boldsymbol{e}[Q(\Rightarrow b \restriction m)] = \varnothing$ for some m by continuity and compactness. Then by the Case 1 assumption,

$f[P(\Rightarrow a \restriction m)] \cap f[P(\Rightarrow b \restriction m)] = \varnothing$ holds, hence $f(x) \neq f(y)$, and $x \mathrel{\mathsf{F}} y$ *fails*.

Case 2 = not Case 1. Then there is a number $m < \omega$ and a pair of strings $\sigma' = \sigma^\frown i$, $\tau' = \tau^\frown k \in 2^{m+1}$ such that $e[P(\Rightarrow \sigma')] \cap e[P(\Rightarrow \tau')] = \varnothing$, but $f[P(\Rightarrow \sigma')] \cap f[P(\Rightarrow \tau')] \neq \varnothing$; hence, the relation F is reduced to $U' = \boldsymbol{\Delta}[\sigma', \tau']$ on $Z' = P(\Rightarrow \sigma') \cup P(\Rightarrow \tau')$ by (2)F. Assume that m is the least possible witness of this case. We are going to prove that the special E_0-large product $P(\Rightarrow \sigma)$ satisfies the "or" option of Theorem 2.1, with the number $\ell = \phi(m)$; that is, (*) F is reduced to $\omega \smallsetminus \{\ell\}$ on $P(\Rightarrow \sigma)$, and (**) E captures ℓ on $P(\Rightarrow \sigma)$.

Lemma 8.1 *The relation* E *is*

(A) *reduced to* $U = \boldsymbol{\Delta}[\sigma, \tau]$ *on the set* $Z = P(\Rightarrow \sigma) \cup P(\Rightarrow \tau)$,

(B) not *reduced to* $U' = \boldsymbol{\Delta}[\sigma', \tau']$ *on* $Z' = P(\Rightarrow \sigma') \cup P(\Rightarrow \tau')$,

(C) not *reduced to* $\omega \smallsetminus \{\ell\}$ *on any special* E_0*-large product* $P' \subseteq P(\Rightarrow \sigma)$.

In addition, (D) $U \neq U'$, *hence* $\ell \in U$ *and* $U' = U \smallsetminus \{\ell\}$.

Proof (A) Otherwise we have $e[P(\Rightarrow \sigma)] \cap e[P(\Rightarrow \tau)] = \varnothing$ by (2)E, and hence $f[P(\Rightarrow \sigma)] \cap f[P(\Rightarrow \tau)] = \varnothing$ by the choice of m; then $f[P(\Rightarrow \sigma')] \cap f[P(\Rightarrow \tau')] = \varnothing$ as well, contrary to the fact that F is reduced to U' on $P(\Rightarrow \sigma') \cup P(\Rightarrow \tau')$, because $P(\Rightarrow \sigma') \restriction U' = P(\Rightarrow \tau') \restriction U'$ by Lemma 5.2(ii).

(B) The otherwise assumption contradicts $e[P(\Rightarrow \sigma')] \cap e[P(\Rightarrow \tau')] = \varnothing$.

(D) This follows from (A) and (B).

(C) Otherwise E is reduced to $\omega \smallsetminus \{\ell\}$ on $P(\Rightarrow \sigma)$ by (1)E. Then E is reduced to U' on $P(\Rightarrow \sigma)$ by Lemma 6.1 since $U' = U \smallsetminus \{\ell\}$ by (D).

Claim 8.2 *The relation* E *is reduced to* U' *on* Z.

Proof Let $x, y \in Z = P(\Rightarrow \sigma) \cup P(\Rightarrow \tau)$, and let $x \restriction U' = y \restriction U'$. As the equality $P(\Rightarrow \sigma) \restriction U = P(\Rightarrow \tau) \restriction U$ holds by Lemma 5.2(ii), there are $x', y' \in P(\Rightarrow \sigma)$ with $x \restriction U = x' \restriction U$ and $y \restriction U = y' \restriction U$. We have $x \mathrel{\mathsf{E}} x'$ and $y \mathrel{\mathsf{E}} y'$ by (A), and $x' \mathrel{\mathsf{E}} y'$ since E is reduced to U' on $P(\Rightarrow \sigma)$. We conclude that $x \mathrel{\mathsf{E}} y$. □

It follows that E is reduced to U' on $Z' \subseteq Z$ as well. But this contradicts (B). The contradiction proves the lemma. □

Now, as $U' = U \smallsetminus \{\ell\} \subseteq \omega \smallsetminus \{\ell\}$, the special E_0-large product $P(\Rightarrow \sigma')$ witnesses that F is reduced to $\omega \smallsetminus \{\ell\}$ on $P(\Rightarrow \sigma)$ by (1)F. Thus we have (*).

To check (**), let $x, y \in P(\Rightarrow \sigma)$, and let $x \mathrel{\mathsf{E}} y$; prove $x(\ell) = y(\ell)$. Indeed, $\{x\} = \bigcap_n P(\Rightarrow a \restriction n)$ and $\{y\} = \bigcap_n P(\Rightarrow b \restriction n)$, where $a, b \in 2^\omega$, $\sigma \subset a$, $\sigma \subset b$. Let $U = \bigcap_n \boldsymbol{\Delta}[a \restriction n, b \restriction n]$. Then $x \restriction U = y \restriction U$, since

$$P(\Rightarrow a \restriction n) \restriction \boldsymbol{\Delta}[a \restriction n, b \restriction n] = P(\Rightarrow b \restriction n) \restriction \boldsymbol{\Delta}[a \restriction n, b \restriction n]$$

for all n. Thus it suffices to check $\ell \in \boldsymbol{\Delta}[a \restriction n, b \restriction n]$ for all n.

Suppose to the contrary that $\ell = \phi(m) \notin \boldsymbol{\Delta}[a \restriction n, b \restriction n]$ for some n. Then $n > m$ because $a \restriction m = b \restriction m = \sigma$. However, the relation E is reduced to $\boldsymbol{\Delta}[a \restriction n, b \restriction n]$ on $P(\Rightarrow a \restriction n)$ by (2)E, since $x \mathrel{\mathsf{E}} y$. Yet we have $\ell \notin \boldsymbol{\Delta}[a \restriction n, b \restriction n]$; therefore,

$\mathbf{\Delta}[a \restriction n, b \restriction n] \subseteq \omega \smallsetminus \{\ell\}$. It follows that E is reduced to $\omega \smallsetminus \{\ell\}$ on $P(\Rightarrow a \restriction n)$. But this contradicts Lemma 8.1(C) with $P' = P(\Rightarrow a \restriction n)$.

To conclude Case 2, we have checked (*) and (**). □ (Theorem 2.1)

9 An Application to Degrees of Constructibility

Consider the set $\mathbf{SLP} = \mathbf{SLS}^{\omega}$ of all special E_0-large products as a forcing notion, over the background set universe $\mathbf{V}$. Thus $\mathbf{SLP}$ adjoins an $\mathbf{SLP}$-generic sequence $\vec{a} = \langle a_k \rangle_{k<\omega} \in (2^{\omega})^{\omega}$, of $\mathbf{SLS}$-generic reals, to $\mathbf{V}$.

Lemma 9.1 *The forcing* $\mathbf{SLP}$ *preserves* $\aleph_1$ *and admits continuous reading of names for reals.*[4]

Proof Arguing in the background set universe $\mathbf{V}$, note that if sets $D_n \subseteq \mathbf{SLP}$ $(n < \omega)$ are open dense in $\mathbf{SLP}$, then by Lemma 5.2(iv), for any $P \in \mathbf{SLP}$ there is a sequence $\cdots \subseteq_4 P_3 \subseteq_3 P_2 \subseteq_2 P_1 \subseteq_1 P_0$ as in Lemma 5.3 such that $P_0 \subseteq P$ and for all m, if $\sigma \in 2^m$, then $P_m(\Rightarrow\sigma) \in D_m$. This implies both claims of the lemma, by standard arguments. □

Theorem 9.2 *Let a sequence* $\vec{a} = \langle a_k \rangle_{k<\omega} \in (2^{\omega})^{\omega}$ *be* $\mathbf{SLS}$*-generic over* $\mathbf{V}$*. Assume that* $x, y \in 2^{\omega}$ *are reals in* $\mathbf{V}[\vec{a}]$*. Then either* $x \in \mathbf{V}[y]$ *or there is an index* ℓ *such that* $a_\ell \in \mathbf{V}[x]$ *and* $y \in \mathbf{V}[\langle a_k \rangle_{k \neq \ell}]$*.*

Proof By Lemma 9.1, there exist continuous functions $\boldsymbol{e},\boldsymbol{f} : (2^{\omega})^{\omega} \to 2^{\omega}$, coded in $\mathbf{V}$, such that $x = \boldsymbol{e}(\vec{a})$, $y = \boldsymbol{f}(\vec{a})$. *Argue in* $\mathbf{V}$. Define $\vec{x} \;\mathsf{E}\; \vec{y}$ if and only if $\boldsymbol{e}(\vec{x}) = \boldsymbol{e}(\vec{y})$, and $\vec{x} \;\mathsf{F}\; \vec{y}$ if and only if $\boldsymbol{f}(\vec{x}) = \boldsymbol{f}(\vec{y})$, for $\vec{x}, \vec{y} \in (2^{\omega})^{\omega}$. The set D of all special E_0-large products $P \in \mathbf{SLP}$ such that either $\mathsf{F} \subseteq \mathsf{E}$ on P, or, for some $\ell < \omega$, E captures ℓ on P and F is reduced to $\omega \smallsetminus \{\ell\}$ on P, is dense in $\mathbf{SLP}$ by Theorem 2.1. Therefore, $\vec{a}$ belongs to a set $P \in D$ (or, to be more exact, to the topological closure of $P \in \mathbf{V}$ in $\mathbf{V}[\vec{a}]$).

Case 1: $\mathsf{F} \subseteq \mathsf{E}$ on P in $\mathbf{V}$. This means that $\boldsymbol{f}(\vec{x}) = \boldsymbol{f}(\vec{y}) \Longrightarrow \boldsymbol{e}(\vec{x}) = \boldsymbol{e}(\vec{y})$ for all $\vec{x}$, $\vec{y}$ in P, in $\mathbf{V}$, and hence, by Shoenfield, $\boldsymbol{f}(\vec{x}) = \boldsymbol{f}(\vec{y}) \Longrightarrow \boldsymbol{e}(\vec{x}) = \boldsymbol{e}(\vec{y})$ for all $\vec{x}$, $\vec{y}$ in (the closure of) P, in $\mathbf{V}[\vec{a}]$. It follows that there is an analytic function h, coded in $\mathbf{V}$, such that $\boldsymbol{e}(\vec{x}) = h(\boldsymbol{f}(\vec{x}))$ for all $\vec{x}$, $\vec{y}$ in (the closure of) P, in $\mathbf{V}[\vec{a}]$. In particular, $a = h(b)$, and hence $a \in \mathbf{V}[b]$.

Case 2: $\ell < \omega$, and it is true in $\mathbf{V}$ that E captures ℓ on P and F is reduced to $\omega \smallsetminus \{\ell\}$ on P. The first part of this condition ensures us that, in $\mathbf{V}$, $\boldsymbol{e}(\vec{x}) = \boldsymbol{e}(\vec{y}) \Longrightarrow \vec{x}(\ell) = \vec{y}(\ell)$ for all $\vec{x}$, $\vec{y}$ in P. Similarly to Case 1, this leads to an analytic function h, coded in $\mathbf{V}$, such that $\vec{x}(\ell) = h(\boldsymbol{e}(\vec{x}))$ for all $\vec{x} \in P$, in $\mathbf{V}[\vec{a}]$, and hence $a_\ell = \vec{a}(\ell) = h(\boldsymbol{e}(\vec{a})) = h(\vec{a}) \in \mathbf{V}[\vec{a}]$. Similarly using the second part of the Case 2 hypothesis, we get another analytic function g, coded in $\mathbf{V}$, such that $b = g(\langle a_k \rangle_{k \neq \ell}) \in \mathbf{V}[\langle a_k \rangle_{k \neq \ell}]$, as required. □

Corollary 9.3 *Let a sequence* $\vec{a} = \langle a_k \rangle_{k<\omega} \in (2^{\omega})^{\omega}$ *be* $\mathbf{SLS}$*-generic over* $\mathbf{V}$*, and let* $X = \{a_k : k < \omega\}$*. Assume that* $a, b \in 2^{\omega}$ *are reals in* $\mathbf{V}[\vec{a}]$*. Then* $a \in \mathbf{V}[b]$ *if and only if* $X \cap \mathbf{V}[a] \subseteq X \cap \mathbf{V}[b]$*.*

One may ask whether, under the conditions of Corollary 9.3, it is true in $\mathbf{V}[\vec{a}]$ that for every set $U \subseteq \omega$ there is a real $a \in 2^{\omega}$ satisfying $X \cap \mathbf{V}[a] = \{a_k : k \in U\}$. The answer is positive for sets $U \in \mathbf{V}$, but generally the answer is negative; for instance, take $U = \{k + 1 : a_0(k) = 0\}$ (see Example 2.2).

Notes

1. The first family consists of equivalence relations classifiable by countable structures, the second of those Borel reducible to an analytic P-ideal.

2. "We avoid any attempt at organizing the very complicated class of smooth equivalence relations" [7, p. 232].

3. Given a function h and $X \subseteq \operatorname{dom} h$, the set $h[X] = \{h(x) : x \in X\}$ is the h-image of X.

4. As noted by the anonymous referee, the forcing **SLP**, and basically **SLS** itself, does not necessarily preserve cardinals bigger than $\aleph_1$. This is essentially due to the same reasons as for the Sacks forcing and its countable-support products, although the splitting constructions behind the result are different and essentially more complex for **SLS** than for the Sacks forcing.

References

[1] Baumgartner, J. E., and R. Laver, "Iterated perfect-set forcing," *Annals of Mathematical Logic*, vol. 17 (1979), pp. 271–88. Zbl 0427.03043. MR 0556894. DOI 10.1016/0003-4843(79)90010-X. 118

[2] Conley, C. T., "Canonizing relations on nonsmooth sets," *Journal of Symbolic Logic*, vol. 78 (2013), pp. 101–12. Zbl 1361.03045. MR 3087064. DOI 10.2178/jsl.7801070. 119

[3] Golshani, M., V. Kanovei, and V. Lyubetsky, "A Groszek-Laver pair of undistinguishable E_0 classes," *Mathematical Logic Quarterly*, vol. 63 (2017), pp. 19–31. MR 3647830. DOI 10.1002/malq.201500020. 118, 121

[4] Kanovei, V., "Non-Glimm-Effros equivalence relations at second projective level," *Fundamenta Mathematicae*, vol. 154 (1997), pp. 1–35. Zbl 0883.03034. MR 1472849. DOI 10.4064/fm-154-1-1-35. 118

[5] Kanovei, V., "On non-wellfounded iterations of the perfect set forcing," *Journal of Symbolic Logic*, vol. 64 (1999), pp. 551–74. Zbl 0930.03062. MR 1777770. DOI 10.2307/2586484. 118

[6] Kanovei, V., *Borel Equivalence Relations: Structure and Classification*, vol. 44 of *University Lecture Series*, American Mathematical Society, Providence, 2008. Zbl 1155.03035. MR 2441635. DOI 10.1090/ulect/044. 118, 119

[7] Kanovei, V., M. Sabok, and J. Zapletal, *Canonical Ramsey Theory on Polish Spaces*, vol. 202 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 2013. Zbl 1278.03001. MR 3135065. DOI 10.1017/CBO9781139208666. 117, 118, 119, 122, 127

[8] Zapletal, J., "Descriptive set theory and definable forcing," *Memoirs of the American Mathematical Society*, vol. 167 (2004), no. 793. Zbl 1037.03042. MR 2023448. DOI 10.1090/memo/0793. 118, 119

[9] Zapletal, J., *Forcing Idealized*, vol. 174 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, 2008. Zbl 1140.03030. MR 2391923. DOI 10.1017/CBO9780511542732. 118

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