PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, Number 3, March 2019, Pages 1277–1282
https://doi.org/10.1090/proc/14286
Article electronically published on December 3, 2018

BOREL OD SETS OF REALS ARE OD-BOREL IN SOME SIMPLE MODELS

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(Communicated by Heike Mildenberger)

Abstract. It is true in the Cohen, Solovay-random, and Sacks generic extensions that every ordinal-definable Borel set of reals has a Borel code in the ground model, and hence if non-empty, then has an element in the ground model.

1. Introduction

It is known from [9] that for each lightface $\varDelta^1_1$ set X, its Borel class is witnessed by a lightface $\varDelta^1_1$ code. This *effective Borel coding* property is not necessarily true for more general definability classes instead of $\varDelta^1_1$. For instance there are models of **ZFC** in which there exists a countable, hence $\boldsymbol{F}_\sigma$, lightface $\varPi^1_2$ non-empty set X of reals with no OD[1] elements [3, 4], and such a set X definitely has no OD $\boldsymbol{F}_\sigma$ code. These models make use of very non-homogeneous forcing notions.[2] Therefore one may expect that homogeneous forcing notions generally yield opposite results. Working in this direction, we prove here the following theorem.

Theorem 1. *Let a be either* (A) *a Cohen-generic real or* (B) *a Solovay-random real or* (C) *a Sacks real over the set universe* $\mathbf{V}$. *Then it is true in* $\mathbf{V}[a]$ *that if* $X \subseteq 2^\omega$ *is a Borel OD set, then it has a Borel code in* $\mathbf{V}$ *of the same ordinal level.*

One may expect such a theorem to be true for other suitably homogeneous generic models like e.g. the dominating forcing extensions. However it does not seem to be an easy task to manufacture a proof of sufficient generality because of various ad hoc arguments in the proofs below, lacking a common denominator.

2. Borel coding

We fix any reasonable system of Borel coding, which involves a $\varPi^1_1$ set $\mathbf{BC} \subseteq \omega^\omega$ of *Borel codes* and an assignment of a Borel set $\boldsymbol{B}_c \subseteq \omega^\omega$ for each $c \in \mathbf{BC}$, as e.g. in [6, 2.9] or [10, 3H]. This also includes a pair of $\varPi^1_1$ sets $S, S' \subseteq \omega^\omega \times \omega^\omega$ such that we have $x \in \boldsymbol{B}_c \iff \langle c, x\rangle \in S \iff \langle c, x\rangle \notin S'$ whenever $c \in \mathbf{BC}$ and $x \in \omega^\omega$ are

Received by the editors March 10, 2018, and, in revised form, June 14, 2018.
2010 *Mathematics Subject Classification.* Primary 03E35, 03E45.
The first author acknowledges partial support of RFFI grant 17-01-00705.
Vladimir Kanovei served as corresponding author.
The second author acknowledges partial support of Russian Scientific Fund grant 14-50-00150.

[1]OD means *ordinal-definable*, that is, a set which can be defined by an $\in$-formula with ordinals as parameters. See Jech [1, Section 13] on this notion.

[2]The model in [4] involves the countable product of Jensen's minimal $\varDelta^1_3$ real forcing [2]. The model in [3] involves a shift-invariant version of Jensen's forcing, and it contains a countable $\varPi^1_2$ set $X \subseteq \omega^\omega$ of reals with no OD elements, and X is an E_0-class.

arbitrary. If $1 \le \xi < \omega_1$, then there is a Borel subset $\mathbf{BC}_\xi \subseteq \mathbf{BC}$ which canonically produces $\mathbf{\Pi}^0_\xi$ sets, so that $\{\boldsymbol{B}_c : c \in \mathbf{BC}_\xi\}$ is equal to the set of all boldface $\mathbf{\Pi}^0_\xi$ sets $X \subseteq \omega^\omega$.

To accordingly code *Borel maps* $F : \omega^\omega \to \omega^\omega$, we let $\mathbf{FC}$ be the lightface Π^1_1 set of all reals $h \in \omega^\omega$ such that $(h)_n \in \mathbf{BC}$, $\forall n$, where $(h)_n \in \omega^\omega$ is defined by $(h)_n(k) = h(2^n(2k+1)-1)$ for all k. If $h \in \mathbf{FC}$, then a Borel map $\boldsymbol{\vartheta}_h : \omega^\omega \to \omega^\omega$ (a total map with the full domain ω^ω) is defined so that $\boldsymbol{\vartheta}_h(x)(n) = k$ iff either $k \ge 1$ and $x \in \boldsymbol{B}_{(h)_n(k)} \smallsetminus \bigcup_{1 \le \ell < k} \boldsymbol{B}_{(h)_n(\ell)}$ or $k = 0$ and $x \notin \bigcup_{1 \le \ell} \boldsymbol{B}_{(h)_n(\ell)}$.

Remark 2. There is another system of Borel codes of the form $c = \langle T_c, f_c \rangle$, where T_c is a well-founded tree and f_c maps terminal nodes of T into *Baire intervals* in ω^ω; see e.g. [12]. If one assumes that T_c is a tree in $\omega^{<\omega}$, then this is fully equivalent to the above system of coding by $\boldsymbol{B}_c$, $c \in \mathbf{BC}$.

But assuming that $T_c \subseteq \lambda^{<\omega}$, $\lambda < \omega_1$ leads to new insights, and then, as essentially proved in [12], our Theorem 1 is true also in the Solovay model (the Levy-collapse extension of $\mathbf{L}$) in such a way that the codes $c = \langle T_c, f_c \rangle$ which witness the Borel class of Borel OD sets belong to $\mathbf{L}$, but the trees T_c may not be countable in $\mathbf{L}$.

As for the coding system by $\boldsymbol{B}_c$, $c \in \mathbf{BC} \subseteq \omega^\omega$, Theorem 1 obviously fails in the Solovay model, the countable set $X = \omega^\omega \cap \mathbf{L}$ being a counterexample.

3. Cohen-generic reals, case A of Theorem 1

Let $\mathbf{Coh} = 2^{<\omega}$ be the Cohen forcing.

Proof of Theorem 1*, case A.* Let $a_0 \in 2^\omega$ be a real $\mathbf{Coh}$-generic over the background set universe $\mathbf{V}$. Assume that $1 \le \xi < \omega_1$ $(= \omega_1^{\mathbf{V}})$, and know that in $\mathbf{V}[a_0]$ it is true that $X = \{x \in 2^\omega : \varphi(x)\} \subseteq 2^\omega$ is a $\mathbf{\Pi}^0_\xi$ set definable by a formula φ with sets in $\mathbf{V}$ as parameters; this includes the OD case. Suppose to the contrary that there is no Borel code $c \in \mathbf{BC}_\xi$ in $\mathbf{V}$ satisfying $X = \boldsymbol{B}_c$.

As X is $\mathbf{\Pi}^0_\xi$, there is a code $d \in \mathbf{BC}_\xi$ in $\mathbf{V}[a_0]$ satisfying $X = \boldsymbol{B}_d$. Cohen extensions are known to satisfy the property of *Borel reading of names*; hence $d = \boldsymbol{\vartheta}_h(a_0)$, where $h \in \mathbf{FC}$ in $\mathbf{V}$. Thus $X = \boldsymbol{B}_{\boldsymbol{\vartheta}_h(a_0)}$. It follows that there is a condition $u \in \mathbf{Coh}$ which forces $\boldsymbol{\vartheta}_h(\dot{\mathbf{a}}) \in \mathbf{BC}_\xi$ and $\{x \in 2^\omega : \varphi(x)\} = \boldsymbol{B}_{\boldsymbol{\vartheta}_h(a_0)}$, where $\dot{\mathbf{a}}$ is a canonical name for the Cohen generic real in 2^ω, and also forces that there is no Borel code $c \in \mathbf{BC}_\xi$ in $\mathbf{V}$ satisfying $\{x \in 2^\omega : \varphi(x)\} = \boldsymbol{B}_c$.

Argue in the universe V. The set $I_u = \{x \in 2^\omega : u \subset x\}$ is a *Cantor interval*, clopen in 2^ω. The set $\{x \in I_u : \boldsymbol{\vartheta}_h(x) \in \mathbf{BC}_\xi\}$ is comeager in I_u by the choice of u. It follows that there is a dense $\boldsymbol{G}_\delta$ set $D \subseteq I_u$ such that $\boldsymbol{\vartheta}_h(x) \in \mathbf{BC}_\xi$ for all $x \in D$. Consider the Borel set

$$P = \{\langle x, y \rangle : x \in D \wedge y \in \boldsymbol{B}_{\boldsymbol{\vartheta}_h(x)}\} \subseteq 2^\omega \times \omega^\omega$$

and the $\mathbf{\Pi}^1_1$ equivalence relation $x \mathsf{E} x'$ iff $P_x = P_{x'}$, on D, where as usual $P_x = \{y : P(x,y)\}$. As a subset of $I_u \times I_u$, E has the Baire property, and so do all E-equivalence classes. Thus there is a condition $v \in \mathbf{Coh}$ which satisfies the requirements of one of the two cases below.

Case 1 (in $\mathbf{V}$). All E-equivalence classes are meager on $I_v = \{x \in 2^\omega : v \subset x\}$. Then the $\mathbf{\Pi}^1_1$-set $W = \{\langle x, x' \rangle : x, x' \in I_v \cap D \wedge x \mathsf{E} x'\}$ is meager in $I_v \times I_v$ by Ulam–Kuratowski. Therefore W is covered by an $\boldsymbol{F}_\sigma$ meager set $F \subseteq I_v \times I_v$.

Fix a transitive model $\mathfrak{M}$ of a sufficiently large fragment of **ZFC** which contains the code h and codes for D, F and is an elementary submodel of the universe $\mathbf{V}$ w.r.t. all analytic formulas.

Lemma 3. *There are reals $a, b \in I_v$, Cohen generic over $\mathbf{V}$, such that $\mathbf{V}[a] = \mathbf{V}[b]$ and the pair $\langle a, b\rangle$ is Cohen$\times$Cohen generic over $\mathfrak{M}$.*

Proof. The set $P = \{\langle x, x +_2 y\rangle : x, y \in I_v\}$ is non-meager, hence so is the projection $Z = \{z \in 2^\omega : P^z \text{ non-meager}\}$ by Ulam–Kuratowski, where $P^z = \{x : \langle x, z\rangle \in P\}$ and $+_2$ is the componentwise addition mod2. Let, in $\mathbf{V}$, $z \in Z$ be Cohen generic over $\mathfrak{M}$. Then P^z is non-meager. Pick a real $a \in P^z$ Cohen over $\mathbf{V}$, hence over $\mathfrak{M}[z]$. The pair $\langle a, z\rangle$ belongs to P and is Cohen over $\mathfrak{M}$; hence z is Cohen over $\mathfrak{M}[a]$. It follows that $b = z +_2 a$ is Cohen over $\mathfrak{M}[a]$; thus $\langle a, b\rangle$ is Cohen over $\mathfrak{M}$, and $a, b \in I_v$ by construction. Finally $b = z +_2 a$ is Cohen over $\mathbf{V}$ since so is a while $z \in \mathbf{V}$, and clearly $\mathbf{V}[a] = \mathbf{V}[b]$. □ (Lemma) □

Consider such a pair of reals $a, b \in I_v$. Then $\langle a, b\rangle \notin F$ since F is a Borel meager set coded in $\mathfrak{M}$. It follows that $\langle a, b\rangle \notin W$, hence, $a \not\mathsf{E} b$, meaning that $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(b)} \neq \boldsymbol{B}_{\boldsymbol{\vartheta}_h(a)}$. But on the other hand a, b are Cohen generic over $\mathbf{V}$, $\mathbf{V}[b] = \mathbf{V}[a]$, and we have $a, b \in I_v$ by construction. It follows that both $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(b)}$ and $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(a)}$ coincide in $\mathbf{V}[a]$ with one and the same (since φ has only parameters in $\mathbf{V}$) set $\{x : \varphi(x)\}$, which contradicts the above. Thus Case 1 is impossible.

Case 2 (in $\mathbf{V}$). At least one of the E-equivalence classes is not meager on I_v. Then, in $\mathbf{V}$, there is a condition $w \in \mathbf{Coh}$ such that $v \subseteq w$ and comeager-many points in I_w are E-equivalent. In other words there exists a particular $\boldsymbol{\Pi}^0_\xi$ set $A = \boldsymbol{B}_c$ with a code $c \in \mathbf{BC}_\xi$ in $\mathbf{V}$ such that $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(x)} = \boldsymbol{B}_c$ for comeager-many $x \in I_w$. Then w Cohen-forces over $\mathbf{V}$ that $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(\mathring{\mathbf{a}})} = \boldsymbol{B}_c$. But this contradicts the contrary assumption in the beginning of the proof, since $u \subseteq w$.

□ (Theorem 1, case A) □

4. Solovay-random reals, case B of Theorem 1

Let $\boldsymbol{\lambda}$ be the standard probability Lebesgue measure on 2^ω. The Solovay-random forcing **Rand** consists of all trees $T \subseteq 2^{<\omega}$ with no endpoints and no isolated branches and such that the set $[T] = \{x \in 2^\omega : \forall\, n\, (x \restriction n \in T)\}$ has positive measure $\boldsymbol{\lambda}([T]) > 0$. The forcing **Rand** depends on the ground model, so that "random over a model $\mathfrak{M}$" will mean "(**Rand** $\cap\, \mathfrak{M}$)-generic over $\mathfrak{M}$".

Unlike the Cohen-generic case, a random pair of reals is **not** a (**Rand** $\times$ **Rand**)-generic pair. The notion of a random pair is rather related to forcing by closed sets in $2^\omega \times 2^\omega$ (or trees which generate them or equivalently Borel sets) of positive product measure (non-null). We'll make use of the following well-known characterization of Solovay-random pairs.

Proposition 4. *Let $\mathfrak{M}$ be a transitive model of a large fragment of* **ZFC**, *and let $a, b \in 2^\omega$. Then the following three assertions are equivalent*:

(1) *the pair $\langle a, b\rangle$ is a random pair over $\mathfrak{M}$;*
(2) *a is random over $\mathfrak{M}$, and b is random over $\mathfrak{M}[a]$;*
(3) *b is random over $\mathfrak{M}$, and a is random over $\mathfrak{M}[b]$.*
(4) *$\langle a, b\rangle$ avoids any $(\boldsymbol{\lambda} \times \boldsymbol{\lambda})$-null Borel set $Q \subseteq 2^\omega \times 2^\omega$ coded in $\mathfrak{M}$.*

Sketch of proof. Regarding the equivalence $1 \iff 4$, see e.g. [1, Lemma 26.4] or V.4.19, V.4.20 in [8], where the 1-dimensional version of the equivalence is established, saying that $a \in 2^\omega$ is random over $\mathfrak{M}$ iff a avoids any $\boldsymbol{\lambda}$-null Borel set $Q \subseteq 2^\omega$ coded in $\mathfrak{M}$.

To prove that (1) implies (2), suppose that (2) fails.

If a is not random over $\mathfrak{M}$, then by the same Lemma 26.4 in [1], a belongs to a null Borel set X coded in $\mathfrak{M}$. Then $\langle a, b\rangle$ belongs to the $(\boldsymbol{\lambda} \times \boldsymbol{\lambda})$-null Borel set $X \times 2^\omega$ still coded in $\mathfrak{M}$, so $\langle a, b\rangle$ is not random.

If b is not random over $\mathfrak{M}[a]$, then b belongs to a $\boldsymbol{\lambda}$-null set X coded in $\mathfrak{M}[a]$. By Borel reading of names, X has a Borel code of the form $f(a)$, where $f : 2^\omega \to 2^\omega$ is a Borel map coded by some $p \in 2^\omega \cap \mathfrak{M}$, that is, a $\varDelta^1_1(p)$ map. This results in a $\varDelta^1_1(p)$ set $P \subseteq 2^\omega \to 2^\omega$ such that $\langle a, b\rangle \in P$ and the cross-section $P_a = \{b' : \langle a, b'\rangle \in P\}$ (it contains b!) is $\boldsymbol{\lambda}$-null. Therefore $\langle a, b\rangle$ belongs to $P' = \{\langle a', b'\rangle \in P : \boldsymbol{\lambda}(P_{a'}) = 0\}$, which is a $\varPi^1_1(p)$ set (because being null is a $\varPi^1_1$ property in this context by e.g. 2.2.3 in [7]) and a $(\boldsymbol{\lambda} \times \boldsymbol{\lambda})$-null set by Fubini. Covering P' by a Borel null set coded in $\mathfrak{M}$, we conclude that $\langle a, b\rangle$ is not random.

To prove that conversely (2) implies (1), suppose that (1) fails, that is, by (4), $\langle a, b\rangle \in P$, where $P \subseteq 2^\omega \times 2^\omega$ is a $(\boldsymbol{\lambda} \times \boldsymbol{\lambda})$-null $\varDelta^1_1(p)$ set, $p \in 2^\omega \cap \mathfrak{M}$. Consider the partition $P = P' \cup P''$ of P into the $\varPi^1_1(p)$ set $P' = \{\langle a', b'\rangle \in P : \boldsymbol{\lambda}(P_{a'}) = 0\}$ and the $\varSigma^1_1(p)$ set $P'' = P \smallsetminus P'$. Now if $\langle a, b\rangle \in P'$, then b belongs to the $\boldsymbol{\lambda}$-null $\varPi^1_1(p, a)$ set P'_a, and hence b is not random over $\mathfrak{M}[a]$ (by covering P'_a by a Borel null set coded in $\mathfrak{M}[a]$). If $\langle a, b\rangle \in P''$, then a belongs to the projection

$$\mathtt{dom}(P'') = \{a' : \exists\, b'\, (\langle a', b'\rangle \in P)\} = \{a' : \boldsymbol{\lambda}(P_{a'}) > 0\},$$

which is a $\varSigma^1_1(p)$ set, $\boldsymbol{\lambda}$-null by Fubini (as P is null), so a is not random. □

Proof of Theorem 1*, case B.* Similarly to case A, the contrary assumption leads to an ordinal $1 \le \xi < \omega_1$, a code $h \in \mathbf{FC}$ in $\mathbf{V}$, and a condition $T_0 \in \mathbf{Rand}$ in $\mathbf{V}$ which $\mathbf{Rand}$-forces, over $\mathbf{V}$, that $\boldsymbol{\vartheta}_h(\dot{\mathbf{a}}) \in \mathbf{BC}_\xi$ and $\{x \in 2^\omega : \varphi(x)\} = \boldsymbol{B}_{\boldsymbol{\vartheta}_h(\dot{\mathbf{a}})}$, where $\dot{\mathbf{a}}$ is a canonical name for the random real and also forces that there is no Borel code $c \in \mathbf{BC}_\xi$ in $\mathbf{V}$, satisfying $\{x \in 2^\omega : \varphi(x)\} = \boldsymbol{B}_c$ in $\mathbf{V}[\dot{\mathbf{a}}]$.

Argue in the universe V. There exists a closed non-null set $D \subseteq [T_0]$ such that $\boldsymbol{\vartheta}_h(x) \in \mathbf{BC}_\xi$ for all $x \in D$. Consider the Borel set

$$P = \{\langle x, y\rangle : x \in D \wedge y \in \boldsymbol{B}_{\boldsymbol{\vartheta}_h(x)}\} \subseteq 2^\omega \times \omega^\omega$$

and the $\mathbf{\Pi}^1_1$ equivalence relation $x \mathrel{\mathsf{E}} x'$ iff $P_x = P_{x'}$, on D, where $P_x = \{y : P(x, y)\}$. Then E is $\boldsymbol{\lambda}$-measurable, and so are all E-equivalence classes. Thus there is a condition $T_1 \in \mathbf{Rand}$ in $\mathbf{V}$ which satisfies $[T_1] \subseteq D$ and satisfies the requirements of one of the two cases:

Case 1 (in $\mathbf{V}$). All E-equivalence classes in $[T_1]$ are $\boldsymbol{\lambda}$-null sets. Then the $\mathbf{\Pi}^1_1$-set $W = \{\langle x, x'\rangle : x, x' \in [T_1] \wedge x \mathrel{\mathsf{E}} x'\}$ is $\boldsymbol{\lambda}^2$-null by Fubini. Therefore W is covered by a $\boldsymbol{G}_\delta$ null set $G \subseteq [T_1] \times [T_1]$. Fix a transitive model $\mathfrak{M} \in \mathbf{V}$ of a sufficiently large fragment of $\mathbf{ZFC}$ which contains h, T_1, and a code G and is an elementary submodel of the universe $\mathbf{V}$ w.r.t. all analytic formulas.

Lemma 5 (= Lemma 3.3 in [5])**.** *There are reals $a, b \in [T_1]$, separately random over $\mathbf{V}$, such that $\mathbf{V}[a] = \mathbf{V}[b]$ and the pair $\langle a, b\rangle$ is random over $\mathfrak{M}$.*

Proof. The set $P = \{\langle x, x +_2 y\rangle : x, y \in [T_1]\}$ is non-null; hence, by Fubini, so is the projection $Z = \{z \in 2^\omega : \boldsymbol{\lambda}(P^z) > 0\}$. Then follow the proof of Lemma 3, using Proposition 4 in treatment of the random pairs involved. □ (Lemma) □

The lemma leads to a contradiction similarly to Case 1 in Section 3.

Case 2 (in $\mathbf{V}$). At least one of the E-equivalence classes in $[T_1]$ is not $\boldsymbol{\lambda}$-null. Then, in $\mathbf{V}$, there is a condition $T \subseteq T_1$ such that all points $x \in [T]$ are E-equivalent. In other words there exists a particular $\boldsymbol{\Pi}^0_\xi$ set $A = \boldsymbol{B}_c$ with a code $c \in \mathbf{BC}_\xi$ in $\mathbf{V}$ such that $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(x)} = \boldsymbol{B}_c$ for all $x \in [T]$. Then T forces over $\mathbf{V}$ that $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(\dot{\mathbf{a}})} = \boldsymbol{B}_c$. But this contradicts the contrary assumption in the beginning of the proof, since $T \subseteq T_0$ by construction.

□ (Theorem 1, case B) □

5. Sacks reals, case C of Theorem 1

The Sacks forcing **Perf** consists of all perfect trees $T \subseteq 2^{<\omega}$ (no endpoints and no isolated branches).

Proof of Theorem 1*, case C.* As above, the contrary assumption leads to an ordinal $1 \le \xi < \omega_1$, a code $h \in \mathbf{FC}$ in $\mathbf{V}$, and a condition $T_0 \in \mathbf{Perf}$ in $\mathbf{V}$ which **Perf**-forces, over $\mathbf{V}$, that $\boldsymbol{\vartheta}_h(\dot{\mathbf{a}}) \in \mathbf{BC}_\xi$ and $\{x \in 2^\omega : \varphi(x)\} = \boldsymbol{B}_{\boldsymbol{\vartheta}_h(\dot{\mathbf{a}})}$, where $\dot{\mathbf{a}}$ is a canonical name for the Sacks-generic real, and also forces that there is no Borel code $c \in \mathbf{BC}_\xi$ in $\mathbf{V}$ satisfying $\{x \in 2^\omega : \varphi(x)\} = \boldsymbol{B}_c$ in $\mathbf{V}[\dot{\mathbf{a}}]$.

Argue in the universe V. There exists a condition $T_1 \in \mathbf{Perf}$, $T_1 \subseteq T_0$, such that $\boldsymbol{\vartheta}_h(x) \in \mathbf{BC}_\xi$ for all $x \in [T_1]$. Consider the Borel set

$$P = \{\langle x, y\rangle : x \in [T_1] \wedge y \in \boldsymbol{B}_{\boldsymbol{\vartheta}_h(x)}\} \subseteq 2^\omega \times \omega^\omega$$

and the $\boldsymbol{\Pi}^1_1$ equivalence relation $x \mathrel{\mathsf{E}} x'$ iff $P_x = P_{x'}$, on $[T_1]$. By the Silver dichotomy theorem,[3] there is a condition $T \in \mathbf{Perf}$ in $\mathbf{V}$ which satisfies $T \subseteq T_1$ and satisfies the requirements of one of the two cases:

Case 1 (in $\mathbf{V}$). The reals in $[T]$ are pairwise E-inequivalent. Then using any homeomorphism $g : [T] \xrightarrow{\text{onto}} [T]$ coded in $\mathbf{V}$ and satisfying $g(x) \ne x$ for all x, we easily get a pair of reals $a \ne b$ in $[T]$ Sacks generic over $\mathbf{V}$ and satisfying $\mathbf{V}[a] = \mathbf{V}[b]$; basically, $b = g(a)$. Then on the one hand, $a \not\mathrel{\mathsf{E}} b$ (since $a \ne b$), thus $P_a \ne P_b$, that is, $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(a)} \ne \boldsymbol{B}_{\boldsymbol{\vartheta}_h(b)}$, but on the other hand it is true in $\mathbf{V}[a] = \mathbf{V}[b]$ that the sets $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(a)}$ and $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(b)}$ are equal to one and the same set $\{x : \varphi(x)\}$, which is a contradiction. Thus Case 1 is impossible.

Case 2 ($\mathbf{V}$). The reals in $[T]$ are pairwise E-equivalent. Then, in $\mathbf{V}$, there is a particular $\boldsymbol{\Pi}^0_\xi$ set $A = \boldsymbol{B}_c$ with a code $c \in \mathbf{BC}_\xi$ in $\mathbf{V}$ such that $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(x)} = \boldsymbol{B}_c$ for all $x \in [T]$. Then T forces over $\mathbf{V}$ that $\boldsymbol{B}_{\boldsymbol{\vartheta}_h(\dot{\mathbf{a}})} = \boldsymbol{B}_c$. But this contradicts the contrary assumption in the beginning of the proof, since $T \subseteq T_0$ by construction.

□ (Theorem 1, case C) □

[3]Silver's theorem [11] claims that if E is a $\boldsymbol{\Pi}^1_1$ equivalence relation on a Borel set X, then either there exist at most countably many E-equivalence classes inside X or there is a perfect set $Y \subseteq X$ of pairwise E-inequivalent elements.

Acknowledgment

The authors thank the anonymous referee for a number of valuable remarks which helped to improve the text.

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