PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 151, Number 5, May 2023, Pages 2201–2210
https://doi.org/10.1090/proc/16232
Article electronically published on February 28, 2023

ON RUSSELL TYPICALITY IN SET THEORY

VLADIMIR KANOVEI AND VASSILY LYUBETSKY

(Communicated by Vera Fischer)

Abstract. According to Tzouvaras, a set is nontypical in the Russell sense if it belongs to a countable ordinal definable set. The class **HNT** of all hereditarily nontypical sets satisfies all axioms of **ZF** and the double inclusion **HOD** $\subseteq$ **HNT** $\subseteq$ **V** holds. Several questions about the nature of such sets, recently proposed by Tzouvaras, are solved in this paper. In particular, a model of **ZFC** is presented in which **HOD** $\subsetneqq$ **HNT** $\subsetneqq$ **V**, and another model of **ZFC** in which **HNT** does not satisfy the axiom of choice.

1. Introduction

One of the fundamental directions in modern set theory is the study of important classes of sets in the set theoretic universe **V**, which themselves satisfy the axioms of set theory. The Gödel class L of all *constructible* sets traditionally belongs to such classes, as well as the class **HOD** of all *hereditarily ordinal definable* sets, see [12]. Both L and **HOD** are transitive classes of sets in which all the axioms of the **ZFC** set theory, with the axiom of choice **AC**, are fulfilled (even if the universe **V** itself only satisfies **ZF** without the axiom of choice). These classes satisfy L $\subseteq$ **HOD** $\subseteq$ **V**, and as it was established in early works on modern axiomatic set theory, the class **HOD** can be strictly between the classes L $\subsetneq$ **V** in suitable generic extensions of L.

Recent studies have shown considerable interest in other classes of sets based on the key concept of ordinal definability, which also satisfy set-theoretic axioms. In particular, the classes of *nontypical* and *hereditarily nontypical* sets are considered; Tzouvaras [27, 28] connects this terminology with philosophical and mathematical studies of Bertrand Russell and the works of van Lambalgen [25] et al. on the axiomatization of the concept of randomness.

Definition 1.1. The set x is *nontypical*, for short $x \in$ **NT**, if it belongs to a countable **OD** (ordinal definable) set. The set x is *hereditarily nontypical*, for short $x \in$ **HNT**, if it itself, all its elements, elements of elements, and so on, are all nontypical, in other words the *transitive closure* TC(x) satisfies TC(x) $\subseteq$ **NT**.

The classes **NT** and **HNT** in this definition correspond to $\mathbf{NT}_{\aleph_1}$ and $\mathbf{HNT}_{\aleph_1}$ in the basic definition system of [28]. Similarly defined narrower classes $\mathbf{NT}_{\aleph_0}$ (elements of *finite* ordinal definable sets) and $\mathbf{HNT}_{\aleph_0}$ in [28] are precisely the *algebraically definable* and *hereditarily algebraically definable* sets that have been investigated in recent papers [6, 7, 10, 11] and are not considered in this article.

Received by the editors November 15, 2021, and, in revised form, August 20, 2022.
2020 *Mathematics Subject Classification.* Primary 03E35; Secondary 03E15.
The authors were partially supported by RFBR grant 20-01-00670.

The class **NT** is not necessarily transitive, but the possibly smaller class $\mathbf{HNT} \subseteq \mathbf{NT}$ is transitive and, as shown in [28], satisfies all axioms of **ZF** (the axiom of choice **AC** not included), and also satisfies the relation $\mathbf{HOD} \subseteq \mathbf{HNT} \subseteq \mathbf{V}$. As for the axiom of choice, if $\mathbf{V} = \mathrm{L}$ (the constructibility axiom), then $\mathbf{HNT} = \mathbf{HOD} = \mathrm{L}$ obviously holds, so in this case **AC** holds in **HNT**.

Problem 1.2 (Essentially Tzouvaras [28, §2])**.** Is it compatible with **ZFC** that the axiom of choice **AC does not hold** in **HNT**?

The next problem of Tzouvaras [28, 2.15] aims to clarify the possibility of the precise *equalities* in the relation $\mathbf{HOD} \subseteq \mathbf{HNT} \subseteq \mathbf{V}$.

Problem 1.3. Are the next sentences compatible with **ZFC**?

(I) $\mathbf{HOD} = \mathbf{HNT} \subsetneqq \mathbf{V}$;
(II) $\mathbf{HOD} \subsetneqq \mathbf{HNT} = \mathbf{V}$;
(III) $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$.

We answer all these questions in the positive. This is the main result of this article. It is contained in Theorems 2.1, 3.1, 4.1, 5.1. The answer will be given through the construction of four corresponding models of **ZFC** by the method of generic extensions of the constructible universe L.

We begin with a model for Problem 1.3(I). Theorem 2.1 proves that it is true in the extension $\mathrm{L}[a]$ of L by a single Cohen generic real a that $\mathrm{L} = \mathbf{HOD} = \mathbf{HNT} \subsetneqq \mathrm{L}[a]$. This is based on our earlier result [19] that the Cohen real a does not belong to a countable **OD** set in $\mathrm{L}[a]$. (Corollary 5 in [6] by G. Fuchs gives a more general result.)

As for Problem 1.3(II), we make use (Section 3) of a forcing notion $\mathbb{P}$ introduced in [14] in order to define a generic real $a \in 2^\omega$ whose E_0-equivalence class $[a]_{\mathsf{E}_0}$ is a lightface Π^1_2 set with no **OD** element.

A positive answer to Problem 1.3(III) is given by means of the finite-support product $\mathbf{P}^{<\omega}$ of essentially Jensen's forcing notion $\mathbf{P} \in \mathrm{L}$ as in [13]. We prove in Section 4 that $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$ holds in $\mathbf{P}^{<\omega}$-generic extensions of L.

Further in-depth study of such $\mathbf{P}^{<\omega}$-generic extensions in Section 5 demonstrate that the class **HNT** in such an extension fails to satisfy the axiom of choice. This gives a positive solution to Problem 1.2.

2. Model I in which not all sets are nontypical

The following theorem solves Problem 1.3(I) in the positive. We make use of a well-known forcing notion.

Theorem 2.1. *If $a \in 2^\omega$ is a Cohen generic real over* L *then it is true in* $\mathrm{L}[a]$ *that* $\mathrm{L} = \mathbf{HOD} = \mathbf{HNT} \subsetneqq \mathrm{L}[a]$.

Recall that Cohen generic extensions involve the forcing notion $\mathbb{C} = 2^{<\omega}$ (all finite dyadic sequences). Countable **OD** sets in Cohen extensions are investigated in our papers [19, 22]. In particular, we'll use the following result here.

Lemma 2.2 ([19, Thm 1.1])**.** *Let $a \in 2^\omega$ be Cohen-generic over the set universe* $\mathbf{V}$*. Then it holds in* $\mathbf{V}[a]$ *that if $Z \subseteq 2^\omega$ is a countable* ***OD*** *set then* $Z \in \mathbf{V}$*.*

This result admits the following extension for the case $\mathbf{V} = \mathrm{L}$:

Corollary 2.3. *Let $a \in 2^\omega$ be Cohen-generic over the constructive universe* L*. Then it holds in* $\mathrm{L}[a]$ *that if* $X \in \mathrm{L}$ *and* $C \subseteq 2^X$ *is countable* ***OD*** *then* $C \subseteq \mathrm{L}$*.*

Proof. As $\mathbb{C}$ is countable, there is a set $Y \subseteq XY \in \mathrm{L}$, countable in L and such that if $c \neq d$ belong to C then $c(x) \neq d(x)$ for some $x \in Y$. Then Y is countable and **OD** in $\mathrm{L}[a]$, so the *projection* $D = \{c \restriction Y : c \in C\}$ of the set C will also be countable and **OD** in $\mathrm{L}[a]$. We have $D \subseteq \mathrm{L}$ by the lemma. (The set Y here can be identified with ω.) Hence, each $d \in D$ is **OD** in $\mathrm{L}[a]$. However, if $c \in C$ and $d = c \restriction Y$, then by the choice of Y it holds in $\mathrm{L}[a]$ that c is the only element in C satisfying $c \restriction Y = d$. Hence, $c \in \mathbf{OD}$. So $c \in \mathrm{L}$, by the homogeneity of Cohen forcing. □

Proof (Theorem 2.1*).* The fact that $\mathrm{L} = \mathbf{HOD}$ in $\mathrm{L}[a]$ is a standard consequence of the homogeneity of the Cohen forcing $\mathbb{C}$. Further, it is clear that $\mathbf{HOD} \subseteq \mathbf{HNT}$. Let's prove the inverse relation $x \in \mathbf{HNT} \implies x \in \mathrm{L}$ in $\mathrm{L}[a]$ by induction on the set-theoretic rank $\mathtt{rk}\, x$ of sets $x \in \mathrm{L}[a]$. Since each set consists only of sets of strictly lower rank, it is sufficient to check that if a set $H \in \mathrm{L}[a]$ satisfies $H \subseteq \mathrm{L}$ and $H \in \mathbf{HNT}$ in $\mathrm{L}[a]$ then $H \in \mathrm{L}$. Here we can assume that in fact $H \subseteq \mathbf{Ord}$, since L allows an **OD** well-ordering. Thus, let $H \subseteq \lambda \in \mathbf{Ord}$. Additionally, since $H \in \mathbf{HNT}$, we have, in $\mathrm{L}[a]$, a countable **OD** set $A \subseteq \mathscr{P}(\lambda)$ containing H. However, $A \in \mathrm{L}$ by Corollary 2.3. This implies $H \in \mathrm{L}$. □

3. Model II in which there are more nontypical sets than HOD sets

The following theorem solves Problem 1.3(II) positively.

Theorem 3.1. *There is a generic extension of the constructible universe* L*, in which it is true that* $\mathbf{HOD} \subsetneq \mathbf{HNT} = \mathbf{V}$*.*

Before the proof starts, we recall some definitions related to perfect and Silver trees. By $2^{<\omega}$ we denote the set of all *tuples* (finite sequences) of terms $0, 1$, including the empty tuple Λ. The length of a tuple s is $\mathtt{lh}\, s$, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}\, s = n\}$ (all tuples of length n). A tree $\varnothing \neq T \subseteq 2^{<\omega}$ is *perfect*, symbolically $T \in \mathbf{PT}$, if it has no endpoints and no isolated branches. In this case, the set

$$[T] = \{a \in 2^\omega : \forall n\, (a \restriction n \in T)\}$$

of all *branches* of T is a perfect set in 2^ω.

- If $u \in T \in \mathbf{PT}$, then a *portion* (or a *pruned tree*) $T\restriction_u \in \mathbf{PT}$ is defined by $T\restriction_u = \{s \in T : u \subset s \vee s \subseteq u\}$.
- Let $\sigma \in 2^{<\omega}$. If $v \in 2^{<\omega}$ is another tuple of length $\mathtt{lh}\, v \geq \mathtt{lh}\, \sigma$, then the tuple $v' = \sigma \bullet v$ of the same length $\mathtt{lh}\, v' = \mathtt{lh}\, v$ is defined by $v'(i) = v(i) +_2 \sigma(i)$ (addition modulo 2) for all $i < \mathtt{lh}\, \sigma$, but $v'(i) = v(i)$ whenever $\mathtt{lh}\, \sigma \leq i < \mathtt{lh}\, v$. If $\mathtt{lh}\, v < \mathtt{lh}\, \sigma$, then we just define $\sigma \bullet v = (\sigma \restriction \mathtt{lh}\, v) \bullet v$.

If $T \subseteq 2^{<\omega}$, then the set $\sigma \bullet T = \{\sigma \bullet v : v \in T\}$ is a *shift* of T.

A tree $T \subseteq 2^{<\omega}$ is a *Silver tree*, symbolically $T \in \mathbf{ST}$, if there is an infinite sequence of tuples $u_k = u_k(T) \in 2^{<\omega}$, such that T consists of all tuples of the form

$$s = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \ldots {}^\frown u_n{}^\frown i_n$$

and their subtuples, where $n < \omega$ and $i_k = 0, 1$. In this case $[T]$ consists of all infinite sequences $a = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \cdots \in 2^\omega$, where $i_k = 0, 1$, $\forall k$.

Recall that the equivalence relation E_0 is defined on 2^ω so that $a \mathrel{\mathsf{E}_0} b$ iff the set $a \mathbin{\Delta} b = \{k : a(k) \neq b(k)\}$ is finite. To prove Theorem 3.1, we will use an **OD** E_0-equivalence class

$$[a]_{\mathsf{E}_0} = \{b \in 2^\omega : a \mathrel{\mathsf{E}_0} b\} = \{\sigma \bullet a : \sigma \in 2^{<\omega}\}$$

of a non-**OD** generic real $a \in 2^\omega$, introduced in [14] and also applied in [8, 20, 21]. This is done by a forcing notion $\mathbb{P}$ having the following key properties, see [14].

(1*) $\mathbb{P} \in \mathrm{L}$, $\mathbb{P} \subseteq \mathbf{ST}$ consists of Silver trees and is ordered by inclusion: smaller trees are stronger conditions.
(2*) If $u \in T \in \mathbb{P}$ and $\sigma \in 2^{<\omega}$ then $T{\restriction}_u \in \mathbb{P}$ and $\sigma \bullet T \in \mathbb{P}$—this is the property of *invariance* w.r.t. shifts and portions.
(3*) $\mathbb{P}$ satisfies the countable antichain condition CCC in L.
(4*) The forcing $\mathbb{P}$ adjoins a generic real $a \in 2^\omega$ to L, whose E_0-class $[a]_{\mathsf{E}_0} = \{b \in 2^\omega : b \mathrel{\mathsf{E}_0} a\}$ is a (countable) **OD**, and even Π^1_2 (lightface) set in $\mathrm{L}[a]$.
(5*) If a real $a \in 2^\omega$ is $\mathbb{P}$-generic over L, then a is not **OD** in the generic extension $\mathrm{L}[a]$. (This property is an elementary consequence of the invariance property as in (2*), see Lemma 7.5 in [14].)

Proof (Theorem 3.1*).* Let a real $a \in 2^\omega$ be $\mathbb{P}$-generic over L. According to (4*) the real a belongs to **HNT** in $\mathrm{L}[a]$. Moreover let $z \in \mathrm{L}[a]$ be arbitrary. Then by Gödel we have $z = F(a)$, where F is an **OD** function defined on 2^ω, in $\mathrm{L}[a]$. It follows that $z \in Z = \{F(b) : b \in [a]_{\mathsf{E}_0}\}$, where Z is a countable **OD** set along with $[a]_{\mathsf{E}_0}$, hence $z \in \mathbf{HNT}$ in $\mathrm{L}[a]$. We conclude that the equality $\mathbf{HNT} = \mathbf{V}$ holds in $\mathrm{L}[a]$. On the other hand, $a \notin \mathbf{OD}$ in $\mathrm{L}[a]$ by (5*), thus $\mathbf{HOD} \subsetneqq \mathbf{HNT}$ in $\mathrm{L}[a]$, as required. (A more thorough analysis based on (2*) shows that $\mathbf{HOD} = \mathrm{L}$ in $\mathrm{L}[a]$.) □

4. Model III: Nontypical sets in general position

The following theorem positively solves Problem 1.3(III), providing a model in which hereditarily nontypical sets are strictly between **HOD** and **V**.

Theorem 4.1. *There is a generic extension of* L*, in which* $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$.

We make use of a forcing notion $\mathbf{P} \in \mathrm{L}$ defined in [17, §7] in order to obtain a model with a nonempty countable **OD** set of pairwise generic reals, containing no **OD** reals.[1] Modulo technical details, this forcing coincides with the Jensen forcing from [13] (also presented in [12, 28.A]). The crucial step in [17] was the proof that those key properties of Jensen's forcing responsible for the uniqueness and definability of generic reals, previously established for $\mathbf{P}$ and its finite products $\mathbf{P}^n$, for example, in [2], also hold for the countable product $\mathbf{P}^{<\omega}$. This forcing and its derivatives were used in [1] and recently in [5, 16, 24] for various purposes. This forcing notion $\mathbf{P}$ has the following main properties (1°)–(5°), see [17].

(1°) $\mathbf{P} \in \mathrm{L}$, $\mathbf{P} \subseteq \mathbf{PT}$, $\mathbf{P}$ contains the full tree $2^{<\omega}$, and $\mathbf{P}$ is ordered by inclusion, so that smaller trees are stronger conditions.
(2°) If $u \in T \in \mathbf{P}$, then the *portion* $T{\restriction}_u$ also belongs to $\mathbf{P}$.
(3°) $\mathbf{P}$ satisfies CCC in L: each antichain $A \subseteq \mathbf{P}$ is at most countable.

[1] Note the difference with the model used in Section 3. A countable **OD** set $[a]_{\mathsf{E}_0} \subseteq 2^\omega$ sans **OD** reals, used there, is an E_0-equivalence class whose elements by necessity are in close connection with each other. Now we are getting a countable **OD** set $X \subseteq 2^\omega$ sans **OD** elements such that its elements are mutually generic, that is, extremely far from each other.

(4°) The set $\mathbf{P}^{<\omega}$, that is, the *weak product*, or *product with finite support*, also satisfies CCC. To be precise, here $\mathbf{P}^{<\omega}$ consists of all functions $\tau : \mathtt{dom}\,\tau \to \mathbf{P}$, where $\mathtt{dom}\,\tau \subseteq \omega$ is finite, say $\mathtt{dom}\,\tau = \{0, 1, \ldots, n\}$ for some n.

(5°) Forcing $\mathbf{P}^{<\omega}$ naturally adjoins a generic sequence of the form $\mathbf{a} = \langle a_n \rangle_{n<\omega} \in (2^\omega)^\omega$ of $\mathbf{P}$-generic reals $a_n \in 2^\omega$ to L. The corresponding set $W(\mathbf{a}) = \{a_n : n < \omega\} \subseteq 2^\omega$ is a (countable) **OD**, and even lightface Π^1_2 (without parameters) set in the generic extension L[**a**].

Proof (Theorem 4.1*).* We consider a $\mathbf{P}^{<\omega}$-generic extension L[**a**] as in (5°), where $\mathbf{a} = \langle a_n \rangle_{n<\omega} \in (2^\omega)^\omega$ is a $\mathbf{P}^{<\omega}$-generic, over L, sequence of $\mathbf{P}$-generic reals $a_n \in 2^\omega$. Thus, in L[**a**], $W(\mathbf{a}) = \{a_n : n < \omega\} \subseteq 2^\omega$ is a countable **OD** set containing no **OD** elements. In other words, if $n < \omega$ then $a_n \in \mathbf{HNT} \smallsetminus \mathbf{HOD}$, and hence we have $\mathbf{HOD} \subsetneqq \mathbf{HNT}$ in L[**a**]. (This observation is due to Tzouvaras [28, 2.13].)

Now we prove that $\mathbf{HNT} \subsetneqq \mathbf{V}$ in L[**a**]. Utilizing another argument by Tzouvaras [28, 2.12], we show that in fact $\mathbf{a} \notin \mathbf{HNT}$ in L[**a**]. Suppose to the contrary that $X \in$ L[**a**], $\mathbf{a} \in X \subseteq (2^\omega)^\omega$, and it holds in L[**a**] that X is a countable **OD** set, say $X = \{x \in (2^\omega)^\omega : \varphi(x, \boldsymbol{\gamma})\}$ in L[**a**], where $\boldsymbol{\gamma} = \langle \gamma_0, \ldots, \gamma_m \rangle$ is a tuple of ordinals. This is forced by a condition $\tau = \langle T_0, \ldots, T_n \rangle \in \mathbf{P}^{<\omega}$ extended by **a**, so that

(∗) $T_i \in \mathbf{P}$, $a_i \in [T_i]$ for $i \le n$, and if a sequence $\mathbf{b} \in (2^\omega)^\omega$ is $\mathbf{P}^{<\omega}$-generic over L and extends τ then it is true in the extension L[**b**] that the set $X_\mathbf{b} = \{x \in (2^\omega)^\omega : \varphi(x, \boldsymbol{\gamma})\}$ is countable and contains **b**.

To get a contradiction let $B \in$ L be the set of all bijections $\beta : \omega \xrightarrow{\text{onto}} \omega$, $\beta \in$ L, such that $\beta(j) = j$ for all $j \le n$. Arguing with the generic sequence $\mathbf{a} = \langle a_n \rangle_{n<\omega} \in (2^\omega)^\omega$ as above, if $\beta \in B$ then define the superposition $\mathbf{a} \circ \beta \in (2^\omega)^\omega$ by $(\mathbf{a} \circ \beta)(j) = \mathbf{a}(\beta(j))$ for all j. Consider the set $A = \{\mathbf{a} \circ \beta : \beta \in B\}$ in L[**a**].

First of all A is uncountable in L[**a**] since B is obviously uncountable in L and cardinals do not collapse by (3°).

Secondly if $\mathbf{b} = \mathbf{a} \circ \beta \in A$ then **b** is $\mathbf{P}^{<\omega}$-generic over L since so is **a** and $\beta \in B \in$ L, and obviously L[**b**] = L[**a**], so that $X_\mathbf{b} = X_\mathbf{a}$. And finally in this case we have $\mathbf{b}(j) = \mathbf{a}(j)$ for all $j \le n$ by the definition of B, therefore **b** still extends the condition τ, and we have $\mathbf{b} \in X_\mathbf{b}$ by 4. To conclude,

$$A = \{\mathbf{a} \circ \beta : \beta \in B\} \subseteq \bigcup_{\beta \in B} X_\mathbf{b} = X_\mathbf{a},$$

but A is uncountable in L[**a**] while $X_\mathbf{a} = X$ is countable. This contradiction completes the proof that $\mathbf{a} \notin \mathbf{HNT}$ in L[**a**], and ends the proof of Theorem 4.1. □

Remark 4.2. Further studies in the next section will show that in fact **HNT** fails to satisfy **AC** in $\mathbf{P}^{<\omega}$-generic extensions of L. One may ask whether the relation $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$ can be realized by a generic model in which the class **HNT** still satisfies the full **ZFC** with the axiom of choice. The positive answer has recently been obtained in [23] by means of a ($\mathbb{P} \times$ the Cohen forcing)-generic extension of L, where $\mathbb{P}$ is the forcing notion used in Section 3. This construction involves rather lengthy investigation of different properties of Borel maps defined on $2^\omega \times \omega^\omega$.

5. Model IV: Nontypical sets sans the axiom of choice

The following theorem solves Problem 1.2 in the positive.

Theorem 5.1. *It is true in* $\mathbf{P}^{<\omega}$*-generic extensions* L[**a**] *as in* (5°) *of Section* 4 *that the class* **HNT** *does not satisfy* **AC**.

The theorem will be a simple consequence of the next lemma.

In the remainder, if $W \subseteq 2^\omega$ is infinite then $\mathbb{C}(W)$ will denote the Cohen forcing for adding a generic 1–1 function $f : \omega \xrightarrow{\text{onto}} W$. Thus, $\mathbb{C}(W)$ consists of all 1–1 functions $p : \mathtt{dom}\, p \to W$, where $\mathtt{dom}\, p \subseteq \omega$ is finite.

Lemma 5.2. *Let $\mathbf{a} = \langle a_n \rangle_{n<\omega}$ be a $\mathbf{P}^{<\omega}$-generic sequence over* L. *Then:*

(i) *the set $W = W(\mathbf{a})$ (defined as in (5°)) is not well-orderable in* $\mathrm{L}(W)$*;*
(ii) $\mathrm{L}(W) \subseteq (\mathbf{HNT})^{\mathrm{L}[\mathbf{a}]}$*;*
(iii) $\mathbf{a}$ *is a $\mathbb{C}(W)$-generic function over* $\mathrm{L}(W)$[2]*;*
(iv) *if $\mathbf{b} \in W^\omega$ is a $\mathbb{C}(W)$-generic function over $\mathrm{L}(W)$ then $\mathbf{b}$ is a $\mathbf{P}^{<\omega}$-generic sequence over* L *in the sense of* (5°) *and* $\mathrm{L}(W)[\mathbf{b}] = \mathrm{L}[\mathbf{b}]$*;*
(v) *if $\mathbf{b} \in W^\omega$, the pair $\langle \mathbf{a}, \mathbf{b} \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathrm{L}(W)$, and $Z \in \mathrm{L}(W)[\mathbf{a}] \cap \mathrm{L}(W)[\mathbf{b}]$, $Z \subseteq \mathrm{L}(W)$, then $Z \in \mathrm{L}(W)$;*
(vi) *if $Z \in \mathrm{L}[\mathbf{a}]$, $Z \subseteq W^\omega$ is a countable* **OD** *set in $\mathrm{L}[\mathbf{a}]$, then $Z \subseteq \mathrm{L}(W)$.*[3]

Proof (Lemma). To prove (ii), note that W is a countable **OD** set in $\mathrm{L}[\mathbf{a}]$ by (5°) of Section 4, therefore W belongs to **HNT** in $\mathrm{L}[\mathbf{a}]$.

Further, (i) is a common property of permutation models.

To prove (iii), assume towards the contrary that there is a set $D \in \mathrm{L}(W) D \subseteq \mathbb{C}(W)$, dense in $\mathbb{C}(W)$, and such that no condition $q \in D$ is extended by $\mathbf{a}$. As an element of $\mathrm{L}(W)$, the set D is definable in $\mathrm{L}(W)$ in the form:

$$D = \{q \in \mathbb{C}(W) : \varphi(q, W, a_0, \dots, a_m, x)\},$$

where $x \in \mathrm{L}$, $m < \omega$, and $a_0, \dots, a_m$ are the initial terms of the sequence $\mathbf{a}$. There is a condition $\tau \in \mathbf{P}^{<\omega}$ extended by $\mathbf{a}$, which forces our assumption over L with $\mathbf{P}^{<\omega}$ as the forcing notion. That is, if a $\mathbf{P}^{<\omega}$-generic sequence $\mathbf{b} = \langle b_n \rangle_{n<\omega}$ extends τ then the set

$$D(\mathbf{b}) = \{q \in \mathbb{C}(W(\mathbf{b})) : \varphi(q, W(\mathbf{b}), b_0, \dots, b_m, x) \text{ holds in } \mathrm{L}(W(\mathbf{b}))\}$$

is dense in $\mathbb{C}(W(\mathbf{b}))$, but no condition $q \in D(\mathbf{b})$ is extended by $\mathbf{b}$.

We can w.l.o.g. assume that $\mathtt{dom}\, \tau = \{0, 1, \dots, m\}$.

Now consider a condition $p \in \mathbb{C}(W)$ defined by $p(j) = a_j$ for all $j = 0, 1, \dots, m$. As D is dense, there exists a condition $q \in D$ extending p. Then $\mathtt{dom}\, q = \{0, 1, \dots, m\} \cup U$, where $U \subseteq \{m+1, m+2, \dots\}$ is a finite set. If $i \in U$ then by definition $q(i) = a_{k_i}$, where $k_i \geq m+1$ and the map $i \longmapsto k_i$ is injective.

There is a bijection $\pi : \omega \xrightarrow{\text{onto}} \omega$ satisfying $\pi(j) = j$ for all $j \leq m$, $\pi(i) = k_i$ for all $i \in U$, and $\pi(\ell) = \ell$ generally for all but finitely many numbers $\ell < \omega$, in particular, $\pi \in \mathrm{L}$. The sequence $\mathbf{b} = \langle b_n \rangle_{n<\omega}$, defined by $b_i = a_{\pi(i)}$ for all $i < \omega$, is $\mathbf{P}^{<\omega}$-generic by the choice of π, and obviously $W(\mathbf{b}) = W(\mathbf{a}) = W$. In addition, $b_j = a_j$ for all $j = 0, 1, \dots, m$, thus $\mathbf{b}$ extends τ. We also have $D(\mathbf{b}) = D(\mathbf{a}) = D$, and hence the above-defined condition q belongs to $D(\mathbf{b})$. *We finally claim that* $\mathbf{b}$ *extends* q. This contradicts the contrary assumption above and completes the proof of (iii).

[2]It is an important point here that the same function or sequence $\mathbf{a} \in W^\omega$ can act as both a $\mathbf{P}^{<\omega}$-generic object over L and as a $\mathbb{C}(W)$-generic object over $\mathrm{L}(W)$. Moreover, the extensions $\mathrm{L}[\mathbf{a}]$ and $\mathrm{L}(W)[\mathbf{a}]$ coincide. Such representations of a one-step generic extension as a multi-step extension (here two-step) are well known, see, for example, [9, 26], [15, §7], [18].

[3]This claim can be compared with a result obtained in Section 4 in the course of the proof of Theorem 4.1. There we proved that, in $\mathrm{L}[\mathbf{a}]$, no countable **OD** set contains $\mathbf{a}$. Here we establish by (vi) that no countable **OD** set $Z \subseteq W^\omega$ contains an element not in $\mathrm{L}(W)$.

To prove the extension claim, one has to check that $q(i) = b_i$ for all $i \in U$. If $i \in U$ then $b_i = a_{\pi(i)} = a_{k_i} = q(i)$ by construction, as required.

To prove claim (iv) of the lemma, suppose otherwise. This is forced by a condition $p \in \mathbb{C}(W)$, such that no function $\mathbf{b} \in W^\omega$, $\mathbb{C}(W)$-generic over $\mathrm{L}(W)$ and extending p, is $\mathbf{P}^{<\omega}$-generic over L. Arguing as in the proof of (iii), we get a suitable permutation π that yields a function $\mathbf{b} \in W^\omega$, $\mathbb{C}(W)$-generic over $\mathrm{L}(W)$ and at the same time $\mathbf{P}^{<\omega}$-generic over L along with $\mathbf{a}$, and satisfies $W(\mathbf{b}) = W(\mathbf{a}) = W$ (as a finite permutation of $\mathbf{a}$), and extends the condition p. Therefore $\mathbf{b}$ is $\mathbb{C}(W)$-generic over $\mathrm{L}(W)$ by claim (iii) already established. This is a contradiction.

(v) This is a generally known fact, yet we add a typical proof. As $Z \subseteq \mathrm{L}(W)$, there is a set $X \in \mathrm{L}(W)$ with $Z \subseteq X$. Consider $\mathbb{C}(W)$-names $s, t \in \mathrm{L}(W)$ such that $Z = s[\![\mathbf{a}]\!] = t[\![\mathbf{b}]\!]$, where $s[\![\mathbf{a}]\!]$ denotes the $\mathbf{a}$-interpretation of any given $\mathbb{C}(W)$-name s. By genericity, the equality $s[\![\mathbf{a}]\!] = t[\![\mathbf{b}]\!]$ is forced by a pair of conditions $p, q \in \mathbb{C}(W)$, i.e. $\mathbf{a}$ extends p, $\mathbf{b}$ extends q, and if a pair $\langle \mathbf{a}', \mathbf{b}' \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathrm{L}(W)$ and $\mathbf{a}'$ extends p, $\mathbf{b}'$ extends q, then $s[\![\mathbf{a}']\!] = t[\![\mathbf{b}']\!]$. *We claim that the condition p $\mathbb{C}(W)$-decides over $\mathrm{L}(W)$ every sentence of the form $x \in s[\![\dot{\mathbf{a}}]\!]$*, where $\dot{\mathbf{a}}$ is a canonical $\mathbb{C}(W)$-name for the principal generic function in W^ω.

Indeed otherwise there exist functions $\mathbf{a}', \mathbf{a}'' \in W^\omega$, $\mathbb{C}(W)$-generic over $\mathrm{L}(W)$ and extending the condition p, and an element $x \in X$, such that $x \in s[\![\mathbf{a}']\!]$ but $x \notin s[\![\mathbf{a}'']\!]$. Consider a function $\mathbf{b}' \in W^\omega$, $\mathbb{C}(W)$-generic both over $\mathrm{L}(W)[\mathbf{a}']$ and over $\mathrm{L}(W)[\mathbf{a}'']$, and extending the condition q. Then either pair $\langle \mathbf{a}', \mathbf{b}' \rangle$, $\langle \mathbf{a}'', \mathbf{b}' \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathrm{L}(W)$, but at least one of the two equalities $s[\![\mathbf{a}']\!] = t[\![\mathbf{b}']\!]$, $s[\![\mathbf{a}'']\!] = t[\![\mathbf{b}']\!]$ definitely fails, which is a contradiction.

Thus p indeed $\mathbb{C}(W)$-decides over $\mathrm{L}(W)$ every sentence $x \in s[\![\dot{\mathbf{a}}]\!]$. This implies

$$Z = \{x \in X : \} p \ \mathbb{C}(W)\text{-forces } x \in s[\![\dot{\mathbf{a}}]\!] \text{ in } \mathrm{L}(W) \in \mathrm{L}(W).$$

(vi) To prove this key claim we apply a method introduced in [19]. Consider a countable *OD* set $Z \subseteq W^\omega$ in $\mathrm{L}[\mathbf{a}]$. **Suppose towards the contrary that** $Z \not\subseteq \mathrm{L}(W)$.

There is a formula $\varphi(z)$ with an ordinal γ_0 as a parameter, such that we have $Z = \{z : \varphi(z)\}$ in $\mathrm{L}[\mathbf{a}]$. There also exists a condition $p_0 \in \mathbb{C}(W)$, $p_0 \subset \mathbf{a}$, which forces our assumptions, that is

(1) p_0 $\mathbb{C}(W)$-forces, over $\mathrm{L}(W)$, that the set $\{z \in W^\omega : \varphi(z)\}$ is countable and is not included in $\mathrm{L}(W)$, or equivalently, if $\mathbf{b} \in W^\omega$ is $\mathbb{C}(W)$-generic over $\mathrm{L}(W)$ and extends p_0 then it is true in the extension $\mathrm{L}(W)[\mathbf{b}] = \mathrm{L}[\mathbf{b}]$ that the set $\Phi_{\mathbf{b}} = \{z \in W^\omega : \varphi(z)\}$ is countable and $\exists\, z\, (z \notin \mathrm{L}(W) \wedge \varphi(z))$. It follows from the countability that there is a map $f_{\mathbf{b}} : \omega \xrightarrow{\text{onto}} \Phi_{\mathbf{b}}$, $f_{\mathbf{b}} \in \mathrm{L}[\mathbf{b}]$.

Let $T \in \mathrm{L}(W)$ be a canonical $\mathbb{C}(W)$-name for $f_{\mathbf{b}}$, so $f_{\mathbf{b}} = T[\![\mathbf{b}]\!]$. Then (1) implies:

(2) p_0 $\mathbb{C}(W)$-forces $\mathtt{ran}\, T[\![\dot{\mathbf{a}}]\!] = \{T[\![\dot{\mathbf{a}}]\!](n) : n < \omega\} = \{z \in W^\omega : \varphi(z)\} \not\subseteq \mathrm{L}(W)$ over $\mathrm{L}(W)$, or equivalently, if $\mathbf{b} \in W^\omega$ is $\mathbb{C}(W)$-generic over $\mathrm{L}(W)$ and $p_0 \subset \mathbf{b}$ then it is true in $\mathrm{L}[\mathbf{b}]$ that

$$\mathtt{ran}\, T[\![\mathbf{b}]\!] = \{T[\![\mathbf{b}]\!](n) : n < \omega\} = \{z \in W^\omega : \varphi(z)\} \not\subseteq \mathrm{L}(W).$$

Now our goal will be to **get a contradiction from** (2). Consider an uncountable cardinal $\kappa > \gamma_0$, such that the set L_κ is an elementary submodel of L w.r.t. a fragment of **ZFC** sufficiently large to prove the part of Lemma 5.2 already established including both (1) and (2). Then the set $\mathrm{L}_\kappa(W)$ contains γ_0 and the name T. As elements of the model $\mathrm{L}_\kappa(W) \subseteq \mathrm{L}_\kappa[\mathbf{a}]$, the sets W, T admit canonical $\mathbf{P}^{<\omega}$-names in L_κ. Consider a countable elementary submodel $\mathfrak{M} \in \mathrm{L}$ of L_κ, containing those names and γ_0. Then the sets W, T and the forcing notion $\mathbb{C}(W)$ belong to $\mathfrak{M}(W)$. Consider the Mostowski collapse map $\pi : \mathfrak{M}(W) \xrightarrow{\text{onto}} \mathrm{L}_\lambda(W)$ onto a transitive set of the form $\mathrm{L}_\lambda(W)$, countable in $\mathrm{L}[\mathbf{a}]$, where $\lambda < \omega_1^{\mathrm{L}}$. As W is countable, we have $\pi(W) = W$, $\pi(T) = T$, and hence $T \in \mathrm{L}_\lambda(W)$, $\mathbb{C}(W) \in \mathrm{L}_\lambda(W)$.

We assert that there is $\mathbf{b} \in W^\omega$ satisfying

(3) $\mathrm{L}[\mathbf{b}] = \mathrm{L}[\mathbf{a}]$, $\mathbf{b}$ is a $\mathbb{C}(W)$-generic function over $\mathrm{L}(W)$, $p_0 \subset \mathbf{b}$, and the pair $\langle \mathbf{a}, \mathbf{b} \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathrm{L}_\lambda(W)$.

Indeed, as the set L_λ is countable in L, there exists a bijection $h : \omega \xrightarrow{\text{onto}} \omega$, $h \in \mathrm{L}$, equal to the identity on the (finite) domain $\mathtt{dom}\, p_0$ of the condition $p_0 \in \mathbb{C}(W)$ (see above on p_0), and generic over L_λ in the sense of the Cohen-style forcing notion $\mathbb{B}$ which consists of all injective tuples $u \in \omega^{<\omega}$. Let $\mathbf{b}(n) = \mathbf{a}(h(n))$ for all n, i.e. $\mathbf{b} = \mathbf{a} \circ h$ is a superposition. Let's check that $\mathbf{b}$ satisfies (3).

Indeed, the function $\mathbf{a}$ of Lemma 5.2 is generic over L, hence it is generic over $\mathrm{L}_\lambda[h] \in \mathrm{L}$, and hence the bijection h is $\mathbb{B}$-generic over $\mathrm{L}_\lambda[\mathbf{a}]$ by the product forcing theorem. Therefore h is generic over $\mathrm{L}_\lambda(W)$, a smaller model. However $\mathbf{a}$ is $\mathbb{C}(W)$-generic over $\mathrm{L}_\lambda(W)$ by (iii) of the lemma. It follows that the pair $\langle \mathbf{a}, h \rangle$ is $(\mathbb{C}(W) \times \mathbb{B})$-generic over $\mathrm{L}_\lambda(W)$ still by the product forcing theorem. One easily proves then that $\langle \mathbf{a}, \mathbf{b} \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathrm{L}_\lambda(W)$.

We further have $\mathrm{L}[\mathbf{b}] = \mathrm{L}[\mathbf{a}]$, because $h \in \mathrm{L}$. Moreover $\mathbf{b}$ is $\mathbb{C}(W)$-generic over $\mathrm{L}(W)$, since $h \in \mathrm{L}$ induces an order isomorphism of $\mathbb{C}(W)$ in $\mathrm{L}(W)$. Finally h is compatible with p_0 because h is the identity on $\mathtt{dom}\, p_0$ by construction. This completes the proof that $\mathbf{b} = \mathbf{a} \circ h$ satisfies (3). In particular $W(\mathbf{b}) = W(\mathbf{a}) = W$ holds, and $\mathtt{ran}\, T[\![\mathbf{a}]\!] = \mathtt{ran}\, T[\![\mathbf{b}]\!] \not\subseteq \mathrm{L}(W)$ by (2).

On the other hand, the set $Z = \mathtt{ran}\, T[\![\mathbf{a}]\!] = \mathtt{ran}\, T[\![\mathbf{b}]\!]$ is an element of the intersection $\mathrm{L}_\lambda(W)[\mathbf{a}] \cap \mathrm{L}_\lambda(W)[\mathbf{b}]$ by construction. We conclude that $Z \in \mathrm{L}(W)$ by (v) of the lemma. (The above proof of (v) is valid for L_λ instead of L as the ground model.) Therefore definitely $Z \subseteq \mathrm{L}(W)$, contrary to the above. The contradiction obtained completes the proof of (vi). □

Proof (Theorem 5.1). The set $W = W(\mathbf{a}) = \{a_n : n < \omega\}$ belongs to **HNT** in $\mathrm{L}[\mathbf{a}]$ by (5°) of Section 4. It remains to prove that W is not well-orderable in $\mathbf{HNT}^{\mathrm{L}[\mathbf{a}]}$. Suppose to the contrary that such a well-ordering exists. Then there is also a bijection $f \in \mathbf{HNT}^{\mathrm{L}[\mathbf{a}]}$, $f : \omega \xrightarrow{\text{onto}} W$. By definition, such a bijection belongs to a countable **OD** set $Z \in \mathrm{L}[\mathbf{a}]$, $Z \subseteq W^\omega$. According to claim (vi) of the lemma, we have $Z \subseteq \mathrm{L}(W)$, so $f \in \mathrm{L}(W)$, i.e. W is well-ordered in $\mathrm{L}(W)$, which gives a contradiction with claim (i) of the lemma. □

6. Comments and questions

We should point to a recent paper [6] by G. Fuchs, containing a comprehensive further study of the class of nontypical sets and its generalizations in the direction of sets that do not belong to OD sets of a given cardinality κ, collected under a common title of *blurry definability*.

Coming back to the Cohen-generic extensions, recall that if a is a Cohen generic real over L then $\mathbf{HNT} = \mathrm{L}$ in $\mathrm{L}[a]$ by Theorem 2.1.

Problem 6.1. Is it true in generic extensions of L by a single Cohen real that any countable **OD** set consists of **OD** elements?

We cannot solve this even for *finite* **OD** sets. By the way it is not that obvious to expect the *positive* answer. Indeed, the problem solves in the *negative* for Sacks and some other generic extensions even for *pairs*, see [3, 4]. For instance, if a is a Sacks-generic real over L then it is true in L[a] that there is an **OD** unordered pair $\{X, Y\}$ of sets of reals $X, Y \subseteq \mathscr{P}(2^\omega)$ such that X, Y themselves are non-**OD** sets. See [3] for a proof of this rather surprising result originally by Solovay.

See also Remark 4.2 as a comment.

Acknowledgments

The authors thank Gunter Fuchs for very interesting discussions related to the content of this article. They thank the anonymous referee for a number of valuable remarks and corrections which greatly helped to improve the text.

References

[1] Uri Abraham, *A minimal model for* ¬CH*: iteration of Jensen's reals*, Trans. Amer. Math. Soc. **281** (1984), no. 2, 657–674, DOI 10.2307/2000078. MR722767

[2] Ali Enayat, *On the Leibniz-Mycielski axiom in set theory*, Fund. Math. **181** (2004), no. 3, 215–231, DOI 10.4064/fm181-3-2. MR2099601

[3] Ali Enayat and Vladimir Kanovei, *An unpublished theorem of Solovay on OD partitions of reals into two non-OD parts, revisited*, J. Math. Log. **21** (2021), no. 3, Paper No. 2150014, 22, DOI 10.1142/S0219061321500148. MR4330522

[4] Ali Enayat, Vladimir Kanovei, and Vassily Lyubetsky, *On effectively indiscernible projective sets and the Leibniz-Mycielski axiom*, Mathematics **9** (2021), no. 14, 1–19 (English), Article no. 1670, DOI 10.3390/math9141670.

[5] Sy-David Friedman, Victoria Gitman, and Vladimir Kanovei, *A model of second-order arithmetic satisfying AC but not DC*, J. Math. Log. **19** (2019), no. 1, 1850013, 39, DOI 10.1142/S0219061318500137. MR3960895

[6] Gunter Fuchs, *Blurry definability*, Mathematics **10** (2022), no. 3, Article no. 452, DOI 10.3390/math10030452.

[7] Gunter Fuchs, Victoria Gitman, and Joel David Hamkins, *Ehrenfeucht's lemma in set theory*, Notre Dame J. Form. Log. **59** (2018), no. 3, 355–370, DOI 10.1215/00294527-2018-0007. MR3832085

[8] Mohammad Golshani, Vladimir Kanovei, and Vassily Lyubetsky, *A Groszek-Laver pair of undistinguishable* E_0*-classes*, MLQ Math. Log. Q. **63** (2017), no. 1-2, 19–31, DOI 10.1002/malq.201500020. MR3647830

[9] Serge Grigorieff, *Intermediate submodels and generic extensions in set theory*, Ann. of Math. (2) **101** (1975), 447–490, DOI 10.2307/1970935. MR373889

[10] Marcia J. Groszek and Joel David Hamkins, *The implicitly constructible universe*, J. Symb. Log. **84** (2019), no. 4, 1403–1421, DOI 10.1017/jsl.2018.57. MR4045982

[11] Joel David Hamkins and Cole Leahy, *Algebraicity and implicit definability in set theory*, Notre Dame J. Form. Log. **57** (2016), no. 3, 431–439, DOI 10.1215/00294527-3542326. MR3521491

[12] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded. MR1940513

[13] Ronald Jensen, *Definable sets of minimal degree*, Mathematical logic and foundations of set theory (Proc. Internat. Colloq., Jerusalem, 1968), North-Holland, Amsterdam, 1970, pp. 122–128, DOI 10.1016/S0049-237X(08)71934-7. MR0306002

[14] Vladimir Kanovei and Vassily Lyubetsky, *A definable* E_0 *class containing no definable elements*, Arch. Math. Logic **54** (2015), no. 5-6, 711–723, DOI 10.1007/s00153-015-0436-9. MR3372617

[15] Vladimir Kanovei and Vassily Lyubetsky, *Generalization of one construction by Solovay*, Sib. Math. J. **56** (2015), no. 6, 1072–1079 (English), DOI 10.1134/S0037446615060117.

[16] Vladimir Kanovei and Vassily Lyubetsky, *Counterexamples to countable-section* Π^1_2 *uniformization and* Π^1_3 *separation*, Ann. Pure Appl. Logic **167** (2016), no. 3, 262–283, DOI 10.1016/j.apal.2015.12.002. MR3437647

[17] Vladimir Kanovei and Vassily Lyubetsky, *A countable definable set containing no definable elements*, Math. Notes **102** (2017), no. 3, 338–349 (English), arXiv:1408.3901.

[18] Vladimir Kanovei and Vassily Lyubetsky, *A generic property of the Solovay set* Σ, Sib. Math. J. **58** (2017), no. 6, 1012–1014 (English), DOI 10.1134/S0037446617060106.

[19] Vladimir Kanovei and Vassily Lyubetsky, *Countable OD sets of reals belong to the ground model*, Arch. Math. Logic **57** (2018), no. 3-4, 285–298, DOI 10.1007/s00153-017-0569-0. MR3778960

[20] Vladimir Kanovei and Vassily Lyubetsky, *Definable* E_0 *classes at arbitrary projective levels*, Ann. Pure Appl. Logic **169** (2018), no. 9, 851–871, DOI 10.1016/j.apal.2018.04.006. MR3808398

[21] Vladimir Kanovei and Vassily Lyubetsky, *Non-uniformizable sets of second projective level with countable cross-sections in the form of Vitali classes*, Izv. Math. **82** (2018), no. 1, 61–90, DOI 10.1070/IM8521.

[22] Vladimir Kanovei and Vassily Lyubetsky, *Borel OD sets of reals are OD-Borel in some simple models*, Proc. Amer. Math. Soc. **147** (2019), no. 3, 1277–1282, DOI 10.1090/proc/14286. MR3896073

[23] Vladimir Kanovei and Vassily Lyubetsky, *A generic model in which the Russell-nontypical sets satisfy ZFC strictly between HOD and the universe*, Mathematics **10** (2022), no. 3, Article no. 491, DOI 10.3390/math10030491.

[24] Vladimir Kanovei and Ralf Schindler, *Definable Hamel bases and* $\mathsf{AC}_\omega(\mathbb{R})$, Fund. Math. **253** (2021), no. 3, 239–256, DOI 10.4064/fm909-6-2020. MR4205974

[25] Michiel van Lambalgen, *The axiomatization of randomness*, J. Symbolic Logic **55** (1990), no. 3, 1143–1167, DOI 10.2307/2274480. MR1071321

[26] Robert M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. of Math. (2) **92** (1970), 1–56, DOI 10.2307/1970696. MR265151

[27] Athanassios Tzouvaras, *Russell's typicality as another randomness notion*, MLQ Math. Log. Q. **66** (2020), no. 3, 355–365, DOI 10.1002/malq.202000038. MR4174113

[28] Athanassios Tzouvaras, *Typicality á la Russell in set theory*, Notre Dame J. Form. Log. **63** (2021), no. 2, 185–196, DOI 10.1215/00294527-2022-0011.

Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute), Moscow, Russia

Email address: kanovei@iitp.ru

Institute for Information Transmission Problems of the Russian Academy of Sciences (Kharkevich Institute), Moscow, Russia

Email address: lyubetsk@iitp.ru