On Cubic Hypersurfaces with Involutions

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Abstract. Multidimensional generalizations of the Weierstrass normal form are considered, depending on the Waring decomposition. The straightforward generalization exists for Fermat-type cubic forms, but does not exist for the general cubic forms in four variables. On the other hand, if a cubic form has a sufficiently small rank, then the corresponding hypersurface is invariant under a nonidentity birational involution of the complex projective space. The involution can be calculated in terms of radicals.

Let us focus on cubic hypersurfaces that are invariant under a nonidentity birational involution of the complex projective space. Throughout the paper all coefficients are denoted by small Greek letters. A form means a homogeneous polynomial over the field of complex numbers. A hypersurface means a projective variety of codimension one. A hypersurface given by the form f is smooth if its gradient ∇f is nonzero outside of the origin; otherwise it is singular. Two forms f and g are equivalent to each other if there exists a nondegenerate linear transformation J such that $f(\mathbf{x}) = g(J\mathbf{x})$. A cubic form in three variables is equivalent to the Weierstrass normal form $y_0^2 y_2 + y_1^3 + \alpha y_1 y_2^2 + \beta y_2^3$. It is invariant under the linear involution $(y_0, y_1, y_2) \mapsto (-y_0, y_1, y_2)$. The rank of a form f of degree d is the minimal number of linear forms needed to represent f as a sum of d-powers. This sum is known as the Waring decomposition. For example, each ternary cubic form can be decomposed as the sum of five cubes (Sylvester Pentahedral Theorem). The next example shows the relationship between the Weierstrass normal form and the Waring decomposition. Let us consider the linear transformation given by two equations $x_0 = \frac{1}{6}y_1 + y_0$ and $x_1 = \frac{1}{6}y_1 - y_0$. Then $x_0^3 + x_1^3 = y_0^2 y_1 + \frac{1}{108}y_1^3$.

Theorem 1. *The general cubic form in four variables is not equivalent to any form of the type $y_0^2 y_3 + g(y_1, y_2, y_3)$.*

Proof. Let us suppose the general cubic form f in four variables is equivalent to a form of the type $y_0^2 y_3 + g(y_1, y_2, y_3)$. One can assume that the surface given by the equation $f = 0$ is smooth. The requirement of smoothness does not reduce the dimension of the set of forms. The curve given by the equation $g = 0$ is smooth. Thus, the form g is equivalent to the second normal form $g = z_1^3 + z_2^3 + z_3^3 - 3\lambda z_1 z_2 z_3$

with replacement of three variables y_1, y_2, and y_3 by linear forms in three variables z_1, z_2, and z_3. Then $f = y_0^2(\rho_1^2 z_1 + \rho_2^2 z_2 + \rho_3^2 z_3) + z_1^3 + z_2^3 + z_3^3 - 3\lambda z_1 z_2 z_3$, where at least one of the coefficients ρ_1, ρ_2, or ρ_3 is nonzero. Otherwise, the form would not depend on the variable y_0; therefore the point with homogeneous coordinates $[1 : 0 : 0 : 0]$ would be a singular point of the surface. One can assume that $\rho_3 \neq 0$. Replacing the variable $y_0 = \rho_3 z_0$ yields an equivalent cubic form of the type $f = z_0^2(\mu_1 z_1 + \mu_2 z_2 + z_3) + z_1^3 + z_2^3 + z_3^3 - 3\lambda z_1 z_2 z_3$. So a cubic form in four variables with at most 20 monomials is defined by a matrix with 16 entries and three parameters λ, μ_1, and μ_2. Mapping of the pair consisting of the form $f(\mathbf{x})$ and the matrix J to another form $f(J\mathbf{x})$ obtained by the linear transformation of coordinates defines a regular surjection from the 19-dimensional affine complex space onto an open set of the 20-dimensional complex space. There is a small polydisc such that the map is bijective. This contradicts Brouwer's theorem. □

Theorem 2. *Given the cubic form $f = x_0^3 + \cdots + x_n^3 + (\alpha_0 x_0 + \cdots + \alpha_n x_n)^3$ in at least three variables $x_0, \ldots, x_n$. There exists a transformation of coordinates such that f is equal to the rational function $y_0^2 y_n + g(y_1, \ldots, y_n)$ in the complement of a hyperplane given by the linear equation $y_n = 0$ in at most three variables x_0, x_1, and x_n. The transformation is the identity map for all coordinates except three; moreover it can be calculated in terms of radicals.*

Proof. Let us consider the linear form $\ell = \alpha_0 x_0 + \cdots + \alpha_n x_n$ and the Hessian matrix H, whose entries are equal to $\frac{\partial^2 f}{\partial x_i \partial x_j}$. The matrix H is equal to the sum of the diagonal matrix $\operatorname{diag}(6x_0, \ldots, 6x_n)$ and the matrix with entries $6\alpha_i\alpha_j\ell$. Let us consider a point $\mathbf{u}$ with coordinates $u_i = 0$ for all $2 \leq i \leq n-1$ such that it is not the origin, and both $\ell(\mathbf{u})$ and $f(\mathbf{u})$ vanish. Its coordinates can be calculated in terms of radicals. The rank of the matrix $H(\mathbf{u})$ is at most three and does not increase under a linear transformation of the coordinates. Let us consider the quadratic form $h = u_0 x_0^2 + u_1 x_1^2 + u_n x_n^2$ with the matrix $H(\mathbf{u})$. It vanishes at the point $\mathbf{u}$ because $h(\mathbf{u}) = f(\mathbf{u}) = 0$; likewise both gradients $\nabla f(\mathbf{u})$ and $\nabla h(\mathbf{u})$ are collinear and nonzero. Both quadric $h = 0$ and cubic $f = 0$ have a common tangent hyperplane with defining linear form $z_n = u_0^2 x_0 + u_1^2 x_1 + u_n^2 x_n$ up to a nonzero factor. A linear subspace of codimension two lies on the quadric $h = 0$. It is defined by two linear equations $z_1 = z_n = 0$ for some linear form z_1 in three variables x_0, x_1, and x_n. Let us choose an independent linear form $z_0(x_0, x_1, x_n)$ such that $z_0(u_0, u_1, u_n) \neq 0$. Let us set at last $z_i = x_i$ for all indices $2 \leq i \leq n-1$. The linear transformation is nondegenerate. Thus, the set $\{z_i\}$ is a basis for the dual space. If both u_0 and u_n are nonzero, then one can choose the forms $z_0 = x_0$ and $z_1 = u_1 x_0 - u_0 x_1$.

The restriction of h to the subspace vanishes identically. Thus, the cubic form is equal to $f = \rho_0^2 z_0^2 z_n + 2\rho_0 z_0(\rho_1 z_1 + \cdots + \rho_n z_n) z_n + 2\tau\rho_0 z_0 z_1^2 + s(z_1, \ldots, z_n)$, where ρ_k and τ are complex numbers. As the cubic is not a cone, $\rho_0 \neq 0$.

In case $\tau = 0$, the cubic form can be transformed to $f = y_0^2 y_n + g(y_1, \ldots, y_n)$, where $y_0 = \rho_0 z_0 + \rho_1 z_1 + \cdots + \rho_n z_n$ and for all indices $i \neq 0$ we set $y_i = z_i$.

In case $\tau \neq 0$, if $z_n \neq 0$, then f is equal to

$$f = \rho_0^2 z_0^2 z_n + 2\rho_0 z_0 \left(\rho_1 z_1 + \cdots + \rho_n z_n + \tau \frac{z_1^2}{z_n} \right) z_n + s(z_1, \ldots, z_n).$$

Let us set $y_0 = \rho_0 z_0 + \rho_1 z_1 + \cdots + \rho_n z_n + \tau \frac{z_1^2}{z_n}$ and for all indices $i \neq 0$ we set $y_i = z_i$. Then $f = y_0^2 y_n + g(y_1, \ldots, y_n)$, but y_0 and g are rational functions. Their denominators are powers of a linear form $u_0^2 x_0 + u_1^2 x_1 + u_n^2 x_n$. □

Remark. There are $\frac{1}{6}(n^3 - n)$ choices of three coordinates x_i, x_j, and x_k instead of x_0, x_1, and x_n.

Theorem 3. *Given the cubic form $f = x_0^3 + \cdots + x_n^3 + (\alpha_0 x_0 + \cdots + \alpha_n x_n)^3$ in at least three variables $x_0, \ldots, x_n$. The corresponding cubic hypersurface is invariant under a nonidentity birational involution of the ambient projective space.*

Proof. According to Theorem 2, there is a birational map φ from the cubic hypersurface $f = 0$ to a hypersurface, which is invariant under the action of the linear involution $[y_0 : y_1 : \cdots : y_n] \mapsto [-y_0 : y_1 : \cdots : y_n]$. The composition of the map φ, the involution, and φ^{-1} yields a sought involution. □

Remark. All cubic surfaces are rational. Thus, a large set of birational involutions exists for any cubic surface. If there is a regular involution of an open set of the surface with a unique singular point, then the point is fixed under the involution. In this way, one can either localize the singular point, or verify smoothness of a cubic surface having at most one singular point. The requirement for uniqueness of the singular point is significant. Otherwise, two singular points can be mapped one into another under the involution.

The following theorem improves the result from [1] in case of cubic hypersurfaces. The homogeneous coordinates of $(-1, 1)$-points are equal to $[\pm 1 : \cdots : \pm 1 : 1]$ up to a common nonzero factor.

Theorem 4. *Given the cubic form $f = x_0^3 + \cdots + x_n^3 + (\alpha_0 x_0 + \cdots + \alpha_n x_n)^3$ in at least three variables $x_0, \ldots, x_n$, where all the coefficients α_k are nonzero. There exists a one-to-one correspondence between singular points of the cubic hypersurface $f = 0$ and $(-1, 1)$-points belonging to the hyperplane defined by the linear form $h = \beta_0 y_0 + \cdots + \beta_n y_n + y_{n+1}$ in $n + 2$ variables with the coefficients $\beta_k = \sqrt{-\alpha_k^3}$.*

Proof. Let us consider the cubic form $g = \beta_0 y_0^3 + \cdots + \beta_n y_n^3 + y_{n+1}^3$. Since all the coefficients β_k are nonzero, the hypersurface $g = 0$ is smooth. Its hyperplane section is projectively equivalent to the hypersurface $f = 0$. If both forms h and g vanish simultaneously at a $(-1, 1)$-point, then the hyperplane is tangent to the hypersurface $g = 0$ at this point. Thus, the section is singular.

At a singular point of the section, the hyperplane $h = 0$ coincides with the tangent hyperplane to the hypersurface $f = 0$. Since all the coefficients β_k are nonzero, both gradients ∇h and ∇g can be collinear only at the points whose coordinates satisfy the system of the equations $x_k^2 = x_j^2$ for all indices k and j. All the points are $(-1, 1)$-points. □

In accordance with the Alexander–Hirschowitz theorem [2], the rank of the general cubic form in four variables is equal to five. It is exactly one more than the number of variables. If the Waring decomposition is known, then Theorem 4 solves the system for cubic surface by means of an auxiliary combinatorial task that is equivalent to the set partition problem. Unfortunately, it is hard to find a $(-1,1)$-point belonging to the hyperplane in high dimensions [3]. On the other hand, one can find $(-1,1)$-points belonging to the hyperplane given by a linear form with integer coefficients near zero, using dynamic programming.

References

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