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On polynomials of odd degree over reals

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Abstract. We prove that there exists a large set of real hypersurfaces having elliptic points. In particular, for almost all nonlinear polynomials of odd degree in two or three variables, the graph of the polynomial contains an elliptic point. Some polynomials in many variables are also considered.

Let us denote by $\mathcal{F}$ an affine hypersurface, that is, the vanishing locus of a polynomial over the field of real numbers. A smooth point $P \in \mathcal{F}$ is said to be elliptic if it is the isolated real point of the intersection of the hypersurface $\mathcal{F}$ with the tangent hyperplane $\mathcal{T}_P$ to $\mathcal{F}$ at this point and the second fundamental form is positive definite. Roughly speaking, in a sufficiently small analytic neighborhood of an elliptic point, the hypersurface looks like an ellipsoid. If the affine hypersurface is a graph of a polynomial f, then the matrix of second partial derivatives $\frac{\partial^2 f}{\partial x_j \partial x_k}$ is definite at every elliptic point.

In a sufficiently small analytic neighborhood of an elliptic point, all points of the hypersurface are elliptic. If a polynomial is defined over the field of rational numbers, then its graph contains an everywhere dense set of rational points. Consequently, if there exists an elliptic point on the graph, then there exists a rational elliptic point. It is easy to check whether a given rational point is elliptic. In practice, one can use software for symbolic computations [1].

The term "almost all" means "all but a set covered by a vanishing locus of a nonzero polynomial".

Theorem 1. *Given an odd integer $d \geq 3$. For almost all bivariate polynomials of degree d, the graph of the polynomial contains an elliptic point.*

Remark. In accordance with Theorem 1, for almost all nonlinear bivariate polynomials of odd degree, there exists a point, where the matrix of second partial derivatives is definite. Thus, for each bivariate polynomial of odd degree, there exists a point, where this matrix is semidefinite. In particular, for linear polynomials, the matrix vanishes.

Example. The graph of the polynomial $x_1(x_1^2 - 3x_2^2)$ is the monkey saddle; it has no elliptic point. But there exists a tangent plane that intersect the surface along three straight lines meeting at one point. At the origin all second partial derivatives of the polynomial vanish.

Theorem 2. *Given an odd integer $d \geq 3$. For almost all polynomials of degree d in three variables, the graph of the polynomial contains an elliptic point.*

Remark. On the general three-dimensional cubic hypersurface over the field of complex numbers, straight lines fill the whole hypersurface. If this property holds over the field of real numbers, then the hypersurface has no elliptic point. Nevertheless, there is a large set of real cubic hypersurfaces having an elliptic point.

Theorem 3. *Given both integer $n \geq 1$ and odd integer $d \geq 3$. For almost all $(n+1)$-tuples of linear functions in n variables, the graph of the sum of d-th powers of the linear functions contains an elliptic point.*

Remark. In the general case, a linear function is inhomogeneous.

Example. Let us consider a cubic form of the type $x_1^3 + x_2^3 + \mu(x_1 + x_2)^3$. The matrix of second partial derivatives is equal to

$$6 \begin{pmatrix} (1+\mu)x_1 + \mu x_2 & \mu(x_1 + x_2) \\ \mu(x_1 + x_2) & \mu x_1 + (1+\mu)x_2 \end{pmatrix}.$$

Its determinant is equal to $36\left(\mu x_1^2 + \mu x_2^2 + (1+2\mu)x_1 x_2\right)$. If $x_1 = x_2 > 0$ and $\mu > -1/4$, then the matrix is positive definite. On the other hand, if $\mu = -1$, then the determinant is equal to $-36(x_1^2 + x_1 x_2 + x_2^2) \leq 0$. In this case, for all points, the matrix is neither positive nor negative definite.

Theorems 1–3 can be used to develop new heuristic or generic algorithms (cf. [2]) to check some properties of real hypersurfaces. On the other hand, the recognition of elliptic points on a surface can be used in computer-aided geometric design.

References

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