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On the Uniform Projection and Covering Problems in Descriptive Set Theory under the Axiom of Constructibility

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Abstract: The following two consequences of the axiom of constructibility $\mathbf{V} = \mathbf{L}$ will be established for every $\mathbb{n} \geq 3$: 1. Every linear $\mathbf{\Sigma}^1_{\mathbb{n}}$ set is the projection of a uniform planar $\mathbf{\Pi}^1_{\mathbb{n}-1}$ set. 2. There is a planar $\mathbf{\Pi}^1_{\mathbb{n}-1}$ set with countable cross-sections, not covered by a union of countably many uniform $\mathbf{\Sigma}^1_{\mathbb{n}}$ sets. If $\mathbb{n} = 2$ then claims 1,2 hold in **ZFC** alone, without the assumption of $\mathbf{V} = \mathbf{L}$.

Keywords: constructibility; projective hierarchy; uniform sets; projections; covering

MSC: 03E15; 03E45

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1. Introduction

The following theorem is **the main result** of this paper. It relates to the problems of *uniform projection* and *countable uniform covering* in descriptive set theory.

Theorem 1. *Assume that* $\mathbb{n} \geq 2$*, and either* (I) *the axiom of constructibility* $\mathbf{V} = \mathbf{L}$ *holds, or* (II) $\mathbb{n} = 2$*. Then*

(a) (uniform projection) *any* $\mathbf{\Sigma}^1_{\mathbb{n}}$ *set* $X \subseteq \omega^\omega$ *is the projection of a uniform* $\mathbf{\Pi}^1_{\mathbb{n}-1}$ *set* $P \subseteq (\omega^\omega)^2$;
(b) (countable uniform non-covering) *there is a* $\mathbf{\Pi}^1_{\mathbb{n}-1}$ *set* $P \subseteq (\omega^\omega)^2$ *with countable cross-sections,* **not** *covered by a union of countably many uniform* $\mathbf{\Sigma}^1_{\mathbb{n}}$ *sets.*

Uniform projection problem. By definition [1,2], a set X in *the Baire space* ω^ω belongs to $\mathbf{\Sigma}^1_{n+1}$ iff it is equal to the *projection* $\mathtt{dom}\, P = \{x : \exists\, y\, P(x,y)\}$ of a planar $\mathbf{\Pi}^1_n$ set $P \subseteq (\omega^\omega)^2 = \omega^\omega \times \omega^\omega$, hence in symbol $\mathbf{\Sigma}^1_{n+1} = \mathbf{proj}\, \mathbf{\Pi}^1_n$. The picture drastically changes if we consider only *uniform* sets $P \subseteq (\omega^\omega)^2$, *i.e.*, those satisfying $P(x,y) \wedge P(x,z) \implies y = z$.

Remark 1. *As it is customary in texts on modern set theory, we use* $\mathtt{dom}\, P$ *for the projection* $\mathtt{dom}\, P = \{x : \exists\, y\, P(x,y)\}$ *of a planar set* P *to the first coordinate, and we use more compact relational expressions like* $P(x,y)$, $Q(x,y,z)$ *etc. instead of* $\langle x,y \rangle \in P$, $\langle x,y,z \rangle \in Q$ *etc.*. □

Proposition 1 (Luzin [3,4], see also [1,2])**.** *The following three classes coincide:*

– *class* $\mathbf{\Delta}^1_1$ *of all Borel sets in* ω^ω;
– *class* $\mathbf{proj\, unif}\, \mathbf{\Delta}^1_1$ *of projections of uniform* $\mathbf{\Delta}^1_1$ *(that is, Borel) sets in* $(\omega^\omega)^2$;
– *class* $\mathbf{proj\, unif}\, \mathbf{\Pi}^1_0$ *of projections of uniform* $\mathbf{\Pi}^1_0$ *(that is, closed) sets in* $(\omega^\omega)^2$.

Thus symbolically, $\mathbf{proj\, unif}\, \mathbf{\Pi}^1_0 = \mathbf{proj\, unif}\, \mathbf{\Delta}^1_1 = \mathbf{\Delta}^1_1 \subsetneqq \mathbf{\Sigma}^1_1 = \mathbf{proj}\, \mathbf{\Pi}^1_0$. □

In Luzin's monograph [4], it is indicated that after constructing the projective hierarchy, "we immediately meet" with a number of questions, the general meaning of which is: can some properties of the first level of the hierarchy be transferred to the following levels? Luzin raised several concrete

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problems of this kind in [4, pp. 274-276,285], related to different results on Borel ($\mathbf{\Delta}^1_1$), analytic ($\mathbf{\Sigma}^1_1$), and coanalytic ($\mathbf{\Pi}^1_1$) sets, already known by that time. In particular, in connection with the results of Proposition 1, Luzin asked a few questions in [4], the common content of which can be formulated as follows:

Problem 1 (Luzin [4]). *For any given* $\mathbb{n} \geq 2$, *figure out the relations between the classes* $\mathbf{\Delta}^1_{\mathbb{n}} \subsetneqq \mathbf{\Sigma}^1_{\mathbb{n}} = \textbf{proj}\, \mathbf{\Pi}^1_{\mathbb{n}-1}$ *and* $\textbf{proj unif}\, \mathbf{\Pi}^1_{\mathbb{n}-1} \subseteq \textbf{proj unif}\, \mathbf{\Delta}^1_{\mathbb{n}}$.

Proposition 1 handles case $\mathbb{n} = 1$ of the problem, of course.

Case $\mathbb{n} = 2$ in Problem 1 was solved by *the Novikov – Kondo uniformization theorem* [5,6], which asserts that every $\mathbf{\Pi}^1_1$ set $P \subseteq (\omega^\omega)^2$ is *uniformizable* by a $\mathbf{\Pi}^1_1$ set Q, that is, $Q \subseteq P$ is uniform and $\mathtt{dom}\, Q = \mathtt{dom}\, P$, and hence

$$\textbf{proj unif}\, \mathbf{\Pi}^1_1 = \textbf{proj unif}\, \mathbf{\Delta}^1_2 = \mathbf{\Sigma}^1_2 = \textbf{proj}\, \mathbf{\Pi}^1_1 , \tag{1}$$

which by the way implies Theorem 1(a) in case $\mathbb{n} = 2$.

Thus we have pretty different state of affairs in cases $\mathbb{n} = 1$ and $\mathbb{n} = 2$. In this context, the result of our Theorem 1(a) answers Luzin's problem, under Gödel's axiom of constructibility, in such a way that $\mathbf{V} = \mathbf{L}$ implies

$$\textbf{proj unif}\, \mathbf{\Pi}^1_{\mathbb{n}-1} = \textbf{proj unif}\, \mathbf{\Delta}^1_{\mathbb{n}} = \mathbf{\Sigma}^1_{\mathbb{n}} = \textbf{proj}\, \mathbf{\Pi}^1_{\mathbb{n}-1} . \tag{2}$$

for all $\mathbb{n} \geq 3$, pretty similar to the solution in case $\mathbb{n} = 2$ given by (1).

Countable uniform non-covering problem. Assertion (b) of Theorem 1 also has its origins in some results of classical descriptive set theory. It concerns the following important result.

Proposition 2 (Luzin [3,4], Novikov [7], see also [1,2] for modern treatment). *Every "planar"* $\mathbf{\Sigma}^1_1$ *set* $P \subseteq (\omega^\omega)^2$ *with all cross-sections* $P_x = \{y : \langle x, y\rangle \in P\}$ *(where* $x \in \omega^\omega$*) being at most countable, is covered by the union of a countable number of uniform* $\mathbf{\Delta}^1_1$ *sets.* □

Luzin was also interested to know whether this result transfers to levels $\mathbb{n} \geq 2$.

Problem 2 (Luzin [4]). *For any given* $\mathbb{n} \geq 2$, *find out if it is true that every* $\mathbf{\Sigma}^1_{\mathbb{n}}$ *set* $P \subseteq (\omega^\omega)^2$ *with countable cross-sections* P_x *is covered by the union of countably many uniform* $\mathbf{\Delta}^1_{\mathbb{n}}$ *sets.*

Our Theorem 1(b) solves this problem *in the negative*, outright for $\mathbb{n} = 2$ and under the assumption of the axiom of constructibility for $\mathbb{n} \geq 3$. We may note that this solution seems to be strongest possible under the assumption (I) ∨ (II) of Theorem 1, since this assumption implies that every planar $\mathbf{\Pi}^1_{\mathbb{n}-1}$ set, and even $\mathbf{\Sigma}^1_{\mathbb{n}}$ set, with countable cross-sections **can be** covered by a union of countably many uniform $\mathbf{\Delta}^1_{\mathbb{n}+1}$ sets.

On the other hand, even much stronger non-covering results are known in generic models of **ZFC**. For instance it is true in the Solovay model [8,9] that the Σ^1_2 set $P = \{\langle x, y\rangle \in (\omega^\omega)^2 : y \in \mathbf{L}[x]\}$ is a set with countable cross-sections not covered by a countable union of uniform projective sets of any class, and even real-ordinal definable sets. Different models containing a $\mathit{\Pi}^1_2$ set with the same properties, were defined in [10,11], and, unlike the Solovay model, without the assumption of the existence of an inaccessible cardinal.

The axiom of constructibility and consistency. As for the axiom of constructibility in Theorem 1, it was proved by Gödel [12] that $\mathbf{V} = \mathbf{L}$ is consistent with **ZFC**, therefore all its consequences like (a), (b) of Theorem 1, are consistent as well. We have recently succeeded [13, Theorem 74.1] to prove that the negations of (a), in the forms $\mathbf{\Sigma}^1_{\mathbb{n}} \not\subseteq \textbf{proj unif}\, \mathbf{\Pi}^1_{\mathbb{n}}$ and $\mathbf{\Delta}^1_{\mathbb{n}} \not\subseteq \textbf{proj unif}\, \mathbf{\Pi}^1_{\mathbb{n}-1}$, for any given $\mathbb{n} \geq 3$, hold in appropriate generic models of **ZFC**.

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Corollary 1. *If* $\mathbb{n} \geq 3$ *then each of the following three statements is consistent with, and independent of* **ZFC** : $\mathbf{\Sigma}^1_{\mathbb{n}} = \mathbf{proj\ unif}\,\mathbf{\Pi}^1_{\mathbb{n}-1}$, $\mathbf{\Sigma}^1_{\mathbb{n}} \not\subseteq \mathbf{proj\ unif}\,\mathbf{\Pi}^1_{\mathbb{n}}$, $\mathbf{\Delta}^1_{\mathbb{n}} \not\subseteq \mathbf{proj\ unif}\,\mathbf{\Pi}^1_{\mathbb{n}-1}$. □

No consistency result related to a positive solution of Problem 2 is known so far; in particular both $\mathbf{V} = \mathbf{L}$ and generic models tend to solve the problem in the negative. This raises *the problem of the consistency of the positive solution* (Problem 5 in the final Section), which can definitely inspire further research.

Outline of the proof. We'll make use of a wide range of methods related to constructibility and effective descriptive set theory. Section 2 contains a brief introduction to universal sets and constructibility and presents some known results used in the proof of Theorem 1; it is written for the convenience of the reader.

Section 3 contains a proof of Claim (a) of Theorem 1. To prove the result we define the class Γ as the closure of $\mathbf{\Sigma}^1_{\mathbb{n}-1} \cup \mathbf{\Pi}^1_{\mathbb{n}-1}$ under finite intersections and countable pairwise disjoint unions. Then we prove, under $\mathbf{V} = \mathbf{L}$, that every set in Γ is a uniform projection of a $\Pi^1_{\mathbb{n}-1}$ set (Lemma 1, an easy result), and that every set in $\mathbf{\Sigma}^1_{\mathbb{n}}$ is a uniform projection of a set in Γ. To prove the latter result (Lemma 2), we make use of such a consequence of $\mathbf{V} = \mathbf{L}$ as a Δ^1_2 well-ordering $<_{\mathrm{L}}$ of the reals. However this method (sketched *e.g.* in [2, Chapter 5]) does not seem to immediately work. Therefore we have to combine it with an elaborate technique of effective descriptive set theory due to Harrington [14], which is not a trivial and easily seen modification.

Section 4 contains a proof of Claim (b) of Theorem 1. The proof evolves around the set $U = U[\mathbb{n}]$ of all pairs $\langle x, f\rangle \in \omega^\omega \times 2^\omega$ such that f is the indicator function of a $\Sigma^1_{\mathbb{n}}(x)$ set $u \subseteq \omega$. We prove that U is not covered by countably many uniform $\mathbf{\Sigma}^1_{\mathbb{n}}$ sets (Lemma 3, rather elementary), and then prove that U is $\Sigma^1_{\mathbb{n}}$ (Lemma 4) by quite a complex argument. Finally a $\Pi^1_{\mathbb{n}-1}$ set with necessary properties is obtained from U by Claim (a) of Theorem 1.

Section 5 contains some conclusions and offers several problems for further study.

2. Preliminaries

We make use of the modern notation [1,2,15] $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$ for classes of the projective hierarchy (*boldface* classes), and Σ^1_n, Π^1_n, Δ^1_n for the corresponding effective (or *lightface*) classes, of sets in the spaces of the form $(\omega^\omega)^m \times \omega^k$, $m, k < \omega$ — which we'll call *product spaces*. As usual, elements $a, b, \dots \in \omega^\omega$ will be called *reals*. If $a, b, \dots \in \omega^\omega$ is a finite list of reals then $\Sigma^1_n(a, b, \dots)$, $\Pi^1_n(a, b, \dots)$, $\Delta^1_n(a, b, \dots)$ are the effective classes *relative to* $a, b, \dots$. Every real $x \in \omega^\omega$ is formally a subset of ω^2, hence it can belong to one of the effective classes say Δ^1_n or $\Delta^1_n(a)$.

Proposition 3 (universal sets). (i) *If* $n \geq 1$, $\mathscr{X}$ *is a product space, and* K *is a class of the form* Σ^1_n *or* $\Sigma^1_n(a)$, $a \in \omega^\omega$, *then there is a set* $U \subseteq \omega \times \mathscr{X}$, universal *in the sense that if* $X \subseteq \mathscr{X}$ *belongs to* K *then there exists* m *such that* $X = U_m = \{x : \langle m, x\rangle \in U\}$.

(ii) *If* $n \geq 1$ *then there is a* Σ^1_n *set* $W \subseteq \omega^\omega \times \omega \times \omega$, *such that if* $a \in \omega^\omega$ *and a set* $x \subseteq \omega$ *belongs to* $\Sigma^1_n(a)$ *then there is* $m < \omega$ *satisfying* $X = W_{am} = \{k : \langle a, m, k\rangle \in W\}$.

Proof (sketch). (i) is a well-known standard fact, see *e.g.* [2] or [16, Theorem 4.9 in Chapter C.8]. To prove (ii) let $U \subseteq \omega \times (\omega^\omega \times \omega)$ be a universal Σ^1_n set as in (i) for $\mathscr{X} = \omega^\omega \times \omega$. Then put $W = \{\langle a, m, k\rangle : \langle m, a, k\rangle \in U\}$. □

Constructible sets were introduced by Gödel [12] as those which can be obtained by a certain transfinite construction. The axiom of constructibility claims that all sets are constructible, symbolically $\mathbf{V} = \mathbf{L}$, where $\mathbf{V}$ = all sets, $\mathbf{L}$ = all constructible sets. See [15,17] as modern reference on theory of constructibility. Analytical representation of Gödel's constructibility is well-known since 1950s, see *e.g.* Addison [18,19], and Simpson's book [20]. The next proposition gathers some major facts:

Proposition 4 (see [2,15] for proofs and an extended survey). *Assume* $\mathbf{V} = \mathbf{L}$. *Then*:

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(i) *there exists a* Δ^1_2 *well-ordering* $<_{\mathrm{L}}$ *of the set* ω^ω *of order type* ω_1*;*
(ii) *if* $n \ge 2$*,* K *is a class of the form* $\Sigma^1_n(b)$*,* $b \in \omega^\omega$*, and* $P \subseteq (\omega^\omega)^3$ *is a set in* K*, then*

$$U = \{\langle y,z\rangle : \forall\, x <_{\mathrm{L}} y\, P(x,y,z)\} \quad \text{and} \quad V = \{\langle y,z\rangle : \exists\, x <_{\mathrm{L}} y\, P(x,y,z)\}$$

are still sets in K*. The same for* $K = \Pi^1_n(b)$ *and* $\Delta^1_n(b)$*.* □

Corollary 2 (essentially Addison [18,19])**.** *Let* $n \ge 2$ *and* $a \in \omega^\omega$*. Then*

(i) *if* K *is a class of the form* Δ^1_n*,* Σ^1_n*,* $\Delta^1_n(a)$*, or* $\Sigma^1_n(a)$*, then every set* $P \subseteq \omega^\omega \times \omega^\omega$ *in* K *is uniformizable by a set* $Q \subseteq P$ *still in* K*;*
(ii) *any* $\mathbf{\Sigma}^1_n$ *set* $X \subseteq \omega^\omega$ *is the projection of a uniform* $\mathbf{\Delta}^1_n$ *set;*
(iii) *any non-empty* Σ^1_n*, resp.,* $\Sigma^1_n(a)$ *set* $X \subseteq \omega^\omega$ *contains a* Δ^1_n*, resp.,* $\Delta^1_n(a)$ *real* $x \in X$*;*
(iv) *if* $x, y \in \omega^\omega$ *and* $x <_{\mathrm{L}} y$ *then* $x \in \Delta^1_2(y)$*.*

Proof. (i) If $P \in \Delta^1_n(a)$ then the set $Q = \{\langle x,y\rangle \in P : \forall y' <_{\mathrm{L}} y \neg P(x,y')\}$ obviously uniformizes P, whereas $Q \in \Delta^1_n(a)$ follows from Proposition 4(ii). Now suppose that $P \in \Sigma^1_n(a)$. There is a Π^1_{n-1} set $C \subseteq (\omega^\omega)^3$ satisfying $P = \{\langle x,y\rangle : \exists z\, C(x,y,z)\}$. Using a canonical homeomorphism $H : (\omega^\omega)^2 \xrightarrow{\text{onto}} \omega^\omega$, and the result for $\Delta^1_n(a)$ already established, we can uniformize C, as a $\Delta^1_n(a)$ subset of $\omega^\omega \times (\omega^\omega)^2$, via a $\Delta^1_n(a)$ set $D \subseteq C$, so that, for any $x \in \omega^\omega$, $\exists y,z\, C(x,y,z) \implies \exists!\, y,z\, D(x,y,z)$. It remains to define $Q = \{\langle x,y\rangle \in P : \exists z\, D(x,y,z)\}$.

(ii) If $X \in \mathbf{\Sigma}^1_n$ then $X \in \Sigma^1_n(a)$ for some $a \in \omega^\omega$. By definition, $X = \mathtt{dom}\, P$ for some Π^1_{n-1} set $P \subseteq \omega^\omega \times \omega^\omega$. Let $Q \subseteq P$ be a $\Delta^1_n(a)$ set that uniformizes P, by (i).

(iii) Define $\mathbf{0} \in \omega^\omega$ by $\mathbf{0}(k) = 0$, $\forall k$. If $X \in \Sigma^1_n(a)$ then the set $P = \{\mathbf{0}\} \times X = \{\langle \mathbf{0}, x\rangle : x \in X\}$ is $\Sigma^1_n(a)$ as well, and hence by (i) it can be uniformized by a $\Sigma^1_n(a)$ set $Q \subseteq P$. In fact $Q = \{\langle \mathbf{0}, x_0\rangle\}$ for some $x_0 \in X$. To see that x_0 is $\Delta^1_n(a)$ use the equivalence

$$x_0(j) = k \iff \exists\, x\, (Q(\mathbf{0},x) \wedge x(j) = k) \iff \forall\, x\, (Q(\mathbf{0},x) \implies x(j) = k).$$

(iv) If $f \in \omega^\omega$ and $m < \omega$ then define $(f)_m \in \omega^\omega$ by $(f)_m(k) = f(2^m(2k+1) - 1)$, $\forall k$. The set $X = \{f \in \omega^\omega : \forall x' <_{\mathrm{L}} y\, \exists m\, (x' = (f)_m)\}$ belongs to $\Delta^1_2(y)$ by Proposition 4(ii). Thus X contains a $\Delta^1_2(a)$ element $f \in X$ by (iii). Then $x = (f)_m \in \Delta^1_2(y)$ for some m. □

3. Proof of the uniform projection theorem

Here we prove Theorem 1(a). We may note that Case (II) ($\mathfrak{n} = 2$) of this statement is covered by the Novikov–Kondo uniformization theorem, and hence we can assume that $\mathfrak{n} \ge 3$ and Case (I): the axiom of constructibility $\mathbf{V} = \mathbf{L}$ holds.

Thus we fix a number $\mathfrak{n} \ge 3$ and assume $\mathbf{V} = \mathbf{L}$ in the course of the proof.

Note that the result will be achieved **not** by a reference to the $\mathbf{\Pi}^1_{\mathfrak{n}-1}$ uniformization claim, which actually fails for $\mathfrak{n} \ge 3$ under $\mathbf{V} = \mathbf{L}$.

Definition 1. Let Γ be the closure of the union $\mathbf{\Sigma}^1_{\mathfrak{n}-1} \cup \mathbf{\Pi}^1_{\mathfrak{n}-1}$ under the operations 1) of finite intersections and 2) of countable unions of pairwise disjoint sets — both operations for sets in one and the same space, of course. □

The proof of Theorem 1(a) consists of two lemmas related to this intermediate class.

Lemma 1. *Every* Γ *set* $X \subseteq \omega^\omega$ *is the projection of a uniform* $\mathbf{\Pi}^1_{\mathfrak{n}-1}$ *set.*

Proof. The proof goes on by induction on the construction of sets in Γ from initial sets in $\mathbf{\Sigma}^1_{\mathfrak{n}-1} \cup \mathbf{\Pi}^1_{\mathfrak{n}-1}$. The result for $\mathbf{\Pi}^1_{\mathfrak{n}-1}$ sets is obvious, and for $\mathbf{\Sigma}^1_{\mathfrak{n}-1}$ sets it follows from Corollary 2(ii). Now the step.

Assume that sets $X_0, X_1, X_2, \ldots \subseteq \omega^\omega$ are pairwise disjoint, and, by the inductive hypothesis, $X_k = \operatorname{dom} P_k$ and $P_k \in \mathbf{\Pi}^1_{\mathtt{n}-1}$, $P_k \subseteq \omega^\omega \times \omega^\omega$ is uniform for each $k < \omega$. Then the set $X = \bigcup_k X_k$ satisfies $X = \operatorname{dom} P$, where $P = \bigcup P_k$ is uniform and belongs to $\mathbf{\Pi}^1_{\mathtt{n}-1}$. (Since the class $\mathbf{\Pi}^1_{\mathtt{n}-1}$ is closed under the countable operations $\bigcup$ and $\bigcap$.)

Now assume that $X_0, X_1 \ldots \subseteq \omega^\omega$, and, by the inductive hypothesis, $X_k = \operatorname{dom} P_k$ and $P_k \in \mathbf{\Pi}^1_{\mathtt{n}-1}$, $P_k \subseteq \omega^\omega \times \omega^\omega$ is uniform for each $k = 0, 1$. We put

$$P = \{\langle x, y, z\rangle : \langle x, y\rangle \in P_0 \wedge \langle x, z\rangle \in P_1\} \quad \text{and} \quad Q = \{\langle x, G(y,z)\rangle : \langle x, y, z\rangle \in P\},$$

where $G : \omega^\omega \times \omega^\omega \xrightarrow{\text{onto}} \omega^\omega$ is a homeomorphism. Then the set $X = X_0 \cap X_1$ satisfies $X = \operatorname{dom} Q$, where Q is uniform and belongs to $\mathbf{\Pi}^1_{\mathtt{n}-1}$. □

Lemma 2. *Every $\mathbf{\Sigma}^1_{\mathtt{n}}$ set $X \subseteq \omega^\omega$ is the projection of a uniform Γ set.*

Proof. This is a much more involved argument. Consider a $\mathbf{\Sigma}^1_{\mathtt{n}}$ set $X \subseteq \omega^\omega$, so that $X = \operatorname{dom} P$, where $P \subseteq \omega^\omega \times \omega^\omega$ is $\mathbf{\Pi}^1_{\mathtt{n}-1}$. We can w.l.o.g. assume that in fact $P \subseteq \omega^\omega \times 2^\omega$, where $2^\omega \subseteq \omega^\omega$ (all infinite dyadic sequences) is the Cantor discontinuum. (If this is not the case then replace P with $P' = \{\langle x, F(y)\rangle : P(x,y)\}$, where $F : \omega^\omega \to 2^\omega$ is the injection defined by $F(y) = 1^\frown 0^{y(0)\frown} 1^\frown 0^{y(1)\frown} 1^\frown 0^{y(2)\frown} \ldots$.)

Note that P belongs to $\Pi^1_{\mathtt{n}-1}(a)$ for some $a \in \omega^\omega$. We assume that in fact P is lightface $\Pi^1_{\mathtt{n}-1}$, and hence X is $\Sigma^1_{\mathtt{n}}$; the general case does not differ. Then there exists a $\Sigma^1_{\mathtt{n}-2}$ set $C \subseteq (2^\omega)^3$ satisfying $P = \{\langle x, y\rangle \in (\omega^\omega)^2 : \forall z\, C(x,y,z)\}$.

Note that $x \in X \iff \exists y\, \forall z\, C(x,y,z)$. Consider the set

$$W = \{\langle x, w\rangle \in (\omega^\omega)^2 : \forall y <_{\mathrm{L}} w\, \exists z <_{\mathrm{L}} w\, \neg C(x,y,z)\}.$$

Quite obviously if $x \in \omega^\omega$ then the cross-section $W_x = \{w : \langle x, w\rangle \in W\}$ is non-empty (contains the $<_{\mathrm{L}}$-least element of ω^ω), is closed in ω^ω in the sense of the order $<_{\mathrm{L}}$, and satisfies $\langle x, y\rangle \in P \wedge w \in W_x \implies w \leqslant_{\mathrm{L}} y$. We conclude that if $x \in X$ then there exists a $<_{\mathrm{L}}$-largest element $w_x \in W_x$. Saying it differently,

(A) if $\langle x, y\rangle \in P$ then w_x exists and $w_x \leqslant_{\mathrm{L}} y$.

Now define the relation $B(x,y,w) := w \in W_x \wedge \forall w' \leqslant_{\mathrm{L}} y\, (w <_{\mathrm{L}} w' \implies w' \notin W_x)$. It follows from (A) that

(B) $B(x,y,w) \iff w = w_x$, whenever $\langle x, y\rangle \in P$.

The next claim makes use of an idea presented in Harrington's paper [14].

(C) if $x \in X$ then there is $y \in \Delta^1_{\mathtt{n}-1}(x, w_x)$ such that $\langle x, y\rangle \in P$.

To prove this crucial claim, *we fix* $x \in X$, and let $f \in \omega^\omega$ be the $<_{\mathrm{L}}$-least element of the difference $\omega^\omega \smallsetminus \Delta^1_{\mathtt{n}-1}(x, w_x)$. We assert that

(D) if $z \in \omega^\omega$ then the equivalence $z <_{\mathrm{L}} f \iff z \in \Delta^1_{\mathtt{n}-1}(x, w_x)$ holds.

Indeed, in the nontrivial direction, suppose that the left-hand side fails, *i.e.*, $f \leqslant_{\mathrm{L}} z$. Then $f \in \Delta^1_2(z)$ by Corollary 2(iv). We conclude that $z \notin \Delta^1_{\mathtt{n}-1}(x, w_x)$. (Indeed, otherwise $f \in \Delta^1_{\mathtt{n}-1}(x, w_x)$, contrary to the choice of f.) This completes the proof of (D).

Taking $z = w_x$ in (D), we obtain $w_x <_{\mathrm{L}} f$, and hence $f \notin W_x$. By definition, there exists $y <_{\mathrm{L}} f$ satisfying

(E) $\forall z <_{\mathrm{L}} f\, C(x,y,z)$.

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Fix such a real y. We assert that $\langle x,y\rangle \in P$. Suppose otherwise. Then the $\mathit{\Pi}^1_{\mathtt{n}-2}(x,y)$ set $Z=\{z : \langle x,y,z\rangle \notin C\}$ is non-empty, and hence there is a $\Delta^1_{\mathtt{n}-1}(x,y)$ real $z\in Z$ by Corollary 2(iii). However $y <_{\mathrm{L}} f$ by construction. We conclude by (D) that $y \in \Delta^1_{\mathtt{n}-1}(x,w_x)$. This implies $z\in \Delta^1_{\mathtt{n}-1}(x,w_x)$, which contradicts (D), (E) an the choice of z. The contradiction ends the proof of $\langle x,y\rangle\in P$, and thereby completes the proof of (C) as well since $y\in\Delta^1_{\mathtt{n}-1}(x,w_x)$ is already established. Recall the following technical notation.

Definition 2. The *indicator function* $\chi_u \in 2^\omega$ of a set $u\subseteq\omega$ is defined by $\chi_u(k)=1$ in case $k\in u$, and $\chi_u(k)=0$ in case $k\notin u$.

If $h\in\omega^\omega$, $m<\omega$, then define $(h)_m\in\omega^\omega$ by $(h)_m(j)=h(2^m(2j+1)-1)$, $\forall j$. □

In continuation of the proof of Lemma 2, we recall that Proposition 3(ii) yields a $\Sigma^1_{\mathtt{n}-1}$ set $D\subseteq(\omega^\omega)^2\times\omega$, universal in the sense that

(F) if $x,w\in\omega^\omega$ and a real $y\in 2^\omega$ belongs to $\Sigma^1_{\mathtt{n}-1}(x,w)$, then there is $m<\omega$ such that $y=(f[x,w])_m$, where $f[x,w]=\chi_{D[x,w]}$ and $D[x,w]=\{k : D(x,w,k\}$.

The set $Q=\{\langle x,f[x,w_x]\rangle : x\in X\}$ is obviously uniform, and $\mathtt{dom}\,Q=X$ by (A). Thus it remains to prove that $Q\in\Gamma$. This is the last step in the proof of Lemma 2. We claim that

(G) $$Q=\{\langle x,f\rangle : f\in 2^\omega \wedge \exists\, m\, P(x,(f)_m)\wedge \wedge\,\forall\, j\,\big(f(j)=1 \iff \exists\, w\,(B(x,(f)_m,w)\wedge D(x,w,j))\big)\}.$$

Direction $\subseteq$ *in* (G). Suppose that $x\in X$ and $f=f[x,w_x]$. By (C), take $y\in\Delta^1_{\mathtt{n}-1}(x,w_x)$ such that $\langle x,y\rangle\in P$. Note that $y\in 2^\omega$ as $P\subseteq\omega^\omega\times 2^\omega$ was assumed in the beginning of the proof. Then by (F) we have $y=(f)_m$ for some m.

Finally, to check the equivalence $\forall\, j\,(\dots)$ in (G), let $j<\omega$. Assume that $f(j)=1$ (*direction* $\Longrightarrow$). Take $w=w_x$. Then $j\in D[x,w_x]$, that is, $D(x,w_x,j)$ holds, whereas $B(x,(f)_m,w)$ holds by (B) in the presence of $P(x,(f)_m)$. Now assume that some w witnesses $B(x,(f)_m,w)\wedge D(x,w,j)$ (*direction* $\Longleftarrow$). Then $w=w_x$ yet again by (B), hence $j\in D[x,w_x]$ and $f(j)=1$ by construction. This ends the proof of $\forall\, j\,(\dots)$ and completes the direction $\subseteq$ in (G).

Direction $\supseteq$ *in* (G). Let $\langle x,f\rangle$ belong the right-hand side of the equality (G); we have to prove that $\langle x,f\rangle\in Q$, that is, $f=f[x,w_x]$. As $P(x,(f)_m)$ holds for some m, (B) implies $B(x,(f)_m,w)\iff w=w_x$ once again, and hence the second line in (G) takes the form $\forall\, j\,(f(j)=1\Leftrightarrow D(x,w_x,j))$, obviously meaning that $f=f[x,w_x]$, as required.

The proof of (G) is accomplished. It remains to prove that Q is a set in Γ. We recall that C is $\mathit{\Pi}^1_{\mathtt{n}-2}$, hence W is $\mathit{\Pi}^1_{\mathtt{n}-2}$ as well by Proposition 4(ii), and then B is $\Delta^1_{\mathtt{n}-1}$ still by Proposition 4(ii). Finally D is $\Sigma^1_{\mathtt{n}-1}$. Therefore we can rewrite the subformula $\forall\, j\,(\cdots\iff\dots)$ in (G) as $\forall\, j\,(\cdots\Longrightarrow\dots)\wedge\forall\, j\,(\cdots\Longleftarrow\dots)$, which yields the conjunction of a $\Sigma^1_{\mathtt{n}-1}$ formula and a $\mathit{\Pi}^1_{\mathtt{n}-1}$ formula. Finally P is $\mathit{\Pi}^1_{\mathtt{n}-1}$. Thus Q can be represented in the form (*) $Q=\bigcup_{m<\omega}(S_m\cap T_m)$, where $S_m\in\Sigma^1_{\mathtt{n}-1}$, $T_m\in\mathit{\Pi}^1_{\mathtt{n}-1}$, $\forall\, m$.

To get a representation in Γ, we let $S^-_m=\omega^\omega\smallsetminus S_m$ and $T^-_m=\omega^\omega\smallsetminus T_m$. Then (*) implies $Q=\bigcup_{m<\omega}\big((S_m\cap T_m)\cap\big[\bigcap_{j<m}(S^-_j\cup(S_j\cap T^-_j))\big]\big)$, where all unions in the right-hand side are pairwise-disjoint unions. This $Q\in\Gamma$, as required. □

Proof of Theorem 1(a), case (I). Immediately from Lemma 1 and Lemma 2. □

4. Proof of the uniform covering theorem

Here we prove Theorem 1(b). An essential part of the arguments will be common for both case (I) and case (II) of the theorem.

Note that unlike Theorem 1(a), no classical result is known to immediately imply the result for $\mathtt{n}=2$. Our plan is to first define a $\mathbf{\Sigma}^1_{\mathtt{n}}$ set $U\subseteq(\omega^\omega)^2$ with the required properties, and then convert it to a $\mathbf{\Pi}^1_{\mathtt{n}-1}$ set using claim (a) already proved.

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Thus we fix $\Bbbk \geq 2$ and assume that either (I) $\Bbbk = 2$ or (II) $\mathbf{V} = \mathbf{L}$ holds.

Let $\vartheta(k, x, j)$ be a $\Sigma^1_\Bbbk$ formula universal in the sense that for any $\Sigma^1_\Bbbk$ formula $\psi(x, j)$ there is $k < \omega$ such that $\vartheta(k, x, j) \iff \psi(x, j)$ for all $x \in \omega^\omega$ and $j < \omega$.

Let $f_{kx} \in 2^\omega$ be the indicator function (see Definition 2) of the set $u_{kx} = \{j : \vartheta(k, x, j)\}$.

Definition 3. We define $U = U[\Bbbk] := \{\langle x, f_{kx}\rangle : x \in \omega^\omega \wedge k < \omega\}$. $\square$

Thus, by the universality of ϑ, we have

$(*)$ $U = \{\langle x, f\rangle \in \omega^\omega \times 2^\omega : f = \chi_u$ is the indicator function of a set $u \in \Sigma^1_\Bbbk(x)$, $u \subseteq \omega\}$.

Lemma 3. *$U \subseteq \omega^\omega \times 2^\omega$ is a set with countable cross-sections,* **not** *covered by a union of countably many uniform $\mathbf{\Sigma}^1_\Bbbk$ sets.*

Proof. Suppose towards the contrary that $U \subseteq \bigcup_m U_m$, where all sets $U_m \subseteq \omega^\omega \times 2^\omega$ are $\mathbf{\Sigma}^1_\Bbbk$ and uniform. There is $x \in \omega^\omega$ such that every U_m belongs to $\Sigma^1_\Bbbk(x)$. Then every non-empty cross-section $U_{mx} = \{f : \langle x, f\rangle \in U_m\}$ is a $\Sigma^1_\Bbbk(x)$ singleton whose only element is $\Delta^1_\Bbbk$. Thus the whole cross-section $U_x = \{f : \langle x, f\rangle \in U\}$ contains only $\Delta^1_\Bbbk$ elements. Thic contradicts $(*)$ above because there exist sets $u \subseteq \omega$ in $\Sigma^1_\Bbbk(x) \smallsetminus \Delta^1_\Bbbk(x)$. $\square$

Lemma 4. *U is a $\Sigma^1_\Bbbk$ set.*

Proof. The argument is somewhat different in the two cases considered.

Case (I): $\mathbf{V} = \mathbf{L}$. First of all, if φ is an analytic formula and $z \in \omega^\omega$ then let φ^z be the formal relativization of φ to $\{y \in \omega^\omega : y <_{\mathrm{L}} z\}$, so that all quantifiers $\exists y$, $\forall y$ over ω^ω are replaced by resp. $\exists y <_{\mathrm{L}} z$, $\forall y <_{\mathrm{L}} z$.

Let $f^z_{kx} \in 2^\omega$ be the indicator function of $\{j : \vartheta^z(k, x, j)\}$. Proposition 4(ii) implies:

(1) The set $\{\langle k, x, z, f^z_{kx}\rangle : k < \omega \wedge x, z \in \omega^\omega\}$ is Δ^1_2.

The $\Sigma^1_\Bbbk$ formula $\vartheta(k, x, j)$ has the form $\exists y\, \psi(y, k, x, j)$, where ψ is a $\Pi^1_{\Bbbk-1}$ formula. The following set E belongs to $\Delta^1_\Bbbk$ by (1), the choice of ψ and Proposition 4(ii):

$$E = \{z \in \omega^\omega : \forall k, j\, \forall x, y <_{\mathrm{L}} z\, (\psi^z(y, k, x, j) \iff \psi(y, k, x, j))\}.$$

Corollary 2(iii) implies the next claim:

(2) If $k < \omega$, $z \in E$, $x <_{\mathrm{L}} z$, and $\Delta^1_\Bbbk(x) \cap \omega^\omega \subseteq C_z = \{c \in \omega^\omega : c <_{\mathrm{L}} z\}$ then $f^z_{kx} = f_{kx}$.

In addition, we have the following claim by standard model-theoretic arguments:

(3) If $C \subseteq \omega^\omega$ is countable then there is $z \in \omega^\omega$ with $C \subseteq C_z = \{c \in \omega^\omega : c <_{\mathrm{L}} z\}$.

We now prove that

(4) $U = \{\langle x, f\rangle : \exists k\, \exists z\, (z \in E \wedge x <_{\mathrm{L}} z \wedge f <_{\mathrm{L}} z \wedge f = f^z_{kx})\}$.

Indeed suppose that $\langle x, f\rangle \in U$, so that $f = f_{kx}$ for some k. Let, by (3), $z \in \omega^\omega$ satisfy $\{f\} \cup (\Delta^1_\Bbbk(x) \cap \omega^\omega) \subseteq C_z$. Then $x, f <_{\mathrm{L}} z$, and hence we have $f = f^z_{kx}$ by (2).

Conversely suppose that $x, f <_{\mathrm{L}} z \in E$ and $f = f^z_{kx}$. We have two cases, A and B:

A: $\Delta^1_\Bbbk(x) \cap \omega^\omega \subseteq C_z$. Then $f^z_{kx} = f_{kx}$ by (2), as above, hence $f = f_{kx}$ and $\langle x, f\rangle \in U$.

B: there is a $\Delta^1_\Bbbk(x)$ real y satisfying $z \leqslant_{\mathrm{L}} y$. Then $f, x <_{\mathrm{L}} y$, hence $f \in \Delta^1_2(y)$ by Corollary 2(iv). We conclude that $f \in \Delta^1_\Bbbk(x)$ by the choice of y. Now $\langle x, f\rangle \in U$ easily follows from $(*)$. This ends the proof of (4).

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We finally note that the right-hand side of (4) is definitely a $\Sigma^1_{\mathbb{n}}$ set because E is $\Delta^1_{\mathbb{n}}$, $<_{\mathbf{L}}$ is Σ^1_2, and the equality $f = f^z_{kx}$ is Δ^1_2 by (1). Thus U is $\Sigma^1_{\mathbb{n}}$, and we are done with case $\mathbf{V} = \mathbf{L}$ in Lemma 4.

Case (II): $\mathbb{n} = 2$, sketch. As the axiom of constructibility is not assumed any more in this case, we are going to use the technique of *relative constructibility*. For any real $a \in \omega^\omega$ (and in principle for any set x, but we don't need such a generality here) the class $\mathbf{L}[a]$ is defined similarly to $\mathbf{L}$ itself, see [15, Chapter 12]. All major consequences of $\mathbf{V} = \mathbf{L}$ are preserved *mutatis mutandis* under the relative constructibility $\mathbf{V} = \mathbf{L}[a]$. In particular:

1°. There exists a Σ^1_2 formula $\zeta(a, x)$ such that for all $a, x \in \omega^\omega$: $x \in \mathbf{L}[a] \iff \zeta(a, x)$.
2°. For any $a \in \omega^\omega$ there is a well-ordering $<_{\mathbf{L}[a]}$ of $\omega^\omega \cap \mathbf{L}[a]$ of order type $\omega_1^{\mathbf{L}[a]}$ such that the ternary relation $x, y \in \mathbf{L}[a] \wedge x <_{\mathbf{L}[a]} y$ on $(\omega^\omega)^3$ is Σ^1_2.
3°. If $a, b \in \omega^\omega$, $\mathbf{V} = \mathbf{L}[a]$ holds, $m \geq 2$, K is a class of the form $\Sigma^1_m(a, b)$, $b \in \omega^\omega$, and $P \subseteq (\omega^\omega)^3$ is a set in K, then similar to Proposition 4(ii) the sets

$$U = \{\langle y, z\rangle : \forall\, x <_{\mathbf{L}} y\, P(x, y, z)\} \quad \text{and} \quad V = \{\langle y, z\rangle : \exists\, x <_{\mathbf{L}} y\, P(x, y, z)\}$$

are still sets in K. The same for $K = \Pi^1_m(a, b)$ and $\Delta^1_m(a, b)$.

After these remarks, let's prove that the set $U = U[2]$ (Definition 3) belongs to Σ^1_2 without any reference to the axiom of constructibility or anything beyond **ZFC**.

Indeed the proof of Lemma 4 in Case (I): $\mathbf{V} = \mathbf{L}$ and with $\mathbb{n} = 2$ can be compressed to the existence of a Σ^1_2 formula $\mathbf{u}(x, f)$ such that $U = \{\langle x, f\rangle : \mathbf{u}(x, f)\}$ under $\mathbf{V} = \mathbf{L}$. The relativized version, essentially with nearly the same proof based on 2° and 3°, yields a Σ^1_2 formula $\mathbf{u}'(a, x, f)$ such that:

4°. If $a \in \omega^\omega$ and $\mathbf{V} = \mathbf{L}[a]$ then $U = \{\langle x, f\rangle : \mathbf{u}'(a, x, f)\}$.

Now let $\mathbf{u}''(x, f)$ be the formula: $x, f \in \omega^\omega \wedge f \in \mathbf{L}[x] \wedge \mathbf{u}'(x, x, f)$. Clearly $\mathbf{u}''$ is Σ^1_2 by 1° and the choice of $\mathbf{u}'$. Thus it suffices to prove that $U = \{\langle x, f\rangle : \mathbf{u}''(x, f)\}$ (in **ZFC** with no extra assumptions).

Suppose that $\langle x, f\rangle \in U$. Then $f \in \mathbf{L}[x]$ by the Shoenfield absoluteness theorem [21]. It follows by 4° (with $a = x$) that $\mathbf{u}'(x, x, f)$ holds in $\mathbf{L}[x]$, and hence holds in the universe by the same Shoenfield's absoluteness. Thus we have $\mathbf{u}''(x, f)$ as required.

Conversely assume $\mathbf{u}''(x, f)$, so that $f \in \mathbf{L}[x]$ and we have $\mathbf{u}'(x, x, f)$. Then $\mathbf{u}'(x, x, f)$ holds in $\mathbf{L}[x]$ by Shoenfield, and hence $\langle x, f\rangle \in U$ still by 4° (with $a = x$), as required. □

Proof of Theorem 1(b). As U is $\Sigma^1_{\mathbb{n}}$ by Lemma 4, Theorem 1(a) implies that there exists a $\Pi^1_{\mathbb{n}-1}$ set $Q \subseteq (\omega^\omega)^3$ such that $U = \mathtt{dom}_2\, Q := \{\langle x, y\rangle : \exists\, z\, Q(x, y, s)\}$ (the projection on $(\omega)^2$), and Q is uniform in $(\omega)^2 \times \omega$, *i.e.*, $Q(x, y, z) \wedge Q(x, y, z') \implies z = z'$. Then each cross-section $Q_x = \{\langle y, z\rangle : Q(x, y, z)\}$ is at most countable by the choice of U and Q.

We claim that Q is not covered by a countable union of $\boldsymbol{\Sigma}^1_{\mathbb{n}}$ sets uniform in $\omega^\omega \times (\omega^\omega)^2$. Indeed assume to the contrary that $Q \subseteq \bigcup_m Q_m$, where each Q_m is $\boldsymbol{\Sigma}^1_{\mathbb{n}}$ and uniform in $\omega^\omega \times (\omega^\omega)^2$, *i.e.*, $Q(x, y, z) \wedge Q(x, y', z') \implies y = y' \wedge z = z'$. Then each set $U_m = \mathtt{dom}_2\, Q_m$ is still $\boldsymbol{\Sigma}^1_{\mathbb{n}}$ and is uniform in $\omega^\omega \times \omega^\omega$ by the uniformity of Q_m. On the other hand, $U \subseteq \bigcup_m U_m$ by construction, which contradicts Lemma 3.

Finally let $P = \{\langle x, H(y, z)\rangle : Q(x, y, z)\}$, where $H : (\omega^\omega)^2 \xrightarrow{\text{onto}} \omega^\omega$ is an arbitrary homeomorphism. Then P witnesses (b) of Theorem 1. □

5. Conclusions and problems

In this study, methods of effective descriptive set theory and constructibility theory are employed to the solution of two old problems of classical descriptive set theory, raised by Luzin in 1930, under the assumption of the axiom of constructibility $\mathbf{V} = \mathbf{L}$ (Theorem 1). In addition, we established

(Corollary 1) an ensuing consistency and independence result. These are new results, and they make a significant contribution to descriptive set theory in the constructible universe. The technique developed in this paper may lead to further progress in studies of different aspects of the projective hierarchy under the axiom of constructibility.

The following problems arise from our study.

Problem 3. Find a "classical" proof of Theorem 1(b) in case $\mathbb{n} = 2$ without any reference to "effective" descriptive set theory and constructibility.

Problem 4. Instead of the set $U = U[\mathbb{n}]$ as in Definition 3, one may want to consider a somewhat simpler set $U'[\mathbb{n}] = \{\langle x, f\rangle \in (\omega^\omega)^2 : f \text{ is } \Delta^1_n(x)\}$. Does it prove Theorem 1(b) ?

Problem 5. Find a model of **ZFC** in which Problem 2 In Section 1 is solved in the positive, at least in the following form: for a given $\mathbb{n} \geq 3$, every $\mathbf{\Pi}^1_{\mathbb{n}-1}$ set $P \subseteq (\omega^\omega)^2$ with countable cross-sections is covered by a union of countably many uniform $\mathbf{\Sigma}^1_{\mathbb{n}}$ sets.

As for the Problem 5, we hope that it can be solved by the method of definable generic forcing notions, introduced by Harrington [22,23]. This method has been recently applied for some definability problems in modern set theory, including the following applications:

- a generic model of **ZFC**, with a Groszek–Laver pair (see [24]), which consists of two OD-indistinguishable E_0 classes $X \neq Y$, whose union $X \cup Y$ is a Π^1_2 set, in [25];
- a generic model of **ZFC**, in which, for a given $\mathbb{n} \geq 3$, there is a $\Delta^1_{\mathbb{n}}$ real coding the collapse of $\omega_1^{\mathbf{L}}$, whereas all $\Delta^1_{\mathbb{n}}$ reals are constructible, in [26];
- a generic model of **ZFC**, which solves the Alfred Tarski [27] 'definability of definable' problem, in [28].

We hope that this study of generic models will eventually contribute to a solution of the following well-known key problem by S. D. Friedman, see [29, P. 209] and [30, P. 602]: *find a model of* **ZFC**, *for a given* $\mathbb{n}$, *in which all* $\mathbf{\Sigma}^1_{\mathbb{n}}$ *sets of reals are Lebesgue measurable and have the Baire and perfect set properties, and in the same time there exists a* $\mathbf{\Delta}^1_{\mathbb{n}+1}$ *well-ordering of the reals.*

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